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THE ADAMS FORMULAS FOR NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS FROM 1ST TO 20TH ORDER

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16. Abstract

The Adams-Bashforth predictor coefficients and the Adams-Moulton corrector coefficients for the integration of differential equations are presented for methods of 1st to 20th order. The order of the method as presented in this report refers to the highest order difference formula used in Newton's backward difference interpolation formula, on which the Adams method is based. The Adams method is a polynomial approximation method derived from Newton's backward difference interpolation formula. The Newton formula is derived and expanded to 20th order. The Adams predictor and corrector formulas are derived and expressed in terms of differences of the derivatives, as well as in terms of the derivatives themselves. All coefficients are given to 18 significant digits. For the difference formula only, the ratio coefficients are given to 10th order.

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CONTENTS

THE ADAMS FORMULAS FOR NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS FROM 1ST TO 20TH ORDER

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SUMMARY

The Adams integration method is a fast, accurate polynomial approximation method for integrating differential equations numerically. The method, which was derived from Newton's backward difference interpolation formula, requires only two derivative evaluations for each integration step. The formulas are derived in terms of differences of the derivatives, as well as in terms of the derivatives themselves. All coefficients are given to 18 significant digits. For the difference formula only, the ratio coefficients are given to 10th order. The binomial and factorial coefficients associated with the difference formulas are also given to 20th order.

The accuracy of the Adams integration method is a function of (1) the accuracy of the first $n + 1$ derivative values, which must be initially supplied, and (2) the order of the method chosen in relationship to the word length of the computer. (The higher the order of the method used, the smaller the size of the integration step that can be used; however, an order that is too high or too low can lead to serious errors and cause the solution to diverge.)

INTRODUCTION

The Adams formulas constitute one of the fastest and most accurate methods for numerical integration of differential equations; however, there are two major drawbacks: the formulas are not self-starting, and the amount of labor associated with the computation of the higher order formulas is nearly prohibitive. The first drawback can be overcome by using a self-starting method to initiate the solution. Of the self-starting methods, the fourth-order Runge-Kutta method (with the Gill modification) and the Fehlberg 7/8 method are the most common. The second drawback can be overcome by judicious programing. It is hoped that the results presented in this report will eliminate the need for programing the solution for the computation of required coefficients.

The Adams method is a polynomial approximation method derived from Newton's backward difference interpolation formula. The Newton formula is derived in the analysis section, and the results are expanded to 20th order in appendix A. The Adams formulas are also derived in the analysis section, and those results are expanded for use in 1st- to 20th-order systems in appendix B. Appendixes C and D provide, respectively, the expansions of the Newton formula used in the derivation of the Adams formulas and the difference formulas in terms of the tabulated function values. Both expansions are carried out to 20th order.

ANALYSIS

Newton's backward difference formula is used for interpolating a value of a function $y = f(x)$ near the end of an ordered set of tabular values y_i. Thus, if $y = f(x)$ denotes a function that takes the values $y_n, y_{n-1}, y_{n-2}, \ldots, y_1$ for the $n + 1$ equidistant values $x_n, x_{n-1}, y_{n-2}, \ldots, x_n$ of the independent variable x, then finding a suitable polynomial $F(x)$ to replace the tabulated function over a given interval can be accomplished (ref. 1) by

$$\begin{aligned} F(x) &= a_0 + \sum_{i=0}^{n-1} a_{i+1} \prod_{k=n-i}^{n} \left(x - x_k\right) \\ &= a_0 + a_1\left(x - x_n\right) + a_2\left(x - x_{n-1}\right)\left(x - x_n\right) \\ &\quad + a_3\left(x - x_{n-2}\right)\left(x - x_{n-1}\right)\left(x - x_n\right) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad + a_r\left(x - x_{n-r+1}\right)\left(x - x_{n-r+2}\right)\cdots\left(x - x_n\right) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad + a_n\left(x - x_1\right)\left(x - x_2\right)\cdots\left(x - x_n\right) \end{aligned} \tag{1}$$

The coefficients $a_0, a_1, a_2, \ldots, a_n$ will be determined to make $F(x_n) = y_n$, $F(x_{n-1}) = y_{n-1}$, $F(x_{n-2}) = y_{n-2}, \ldots, F(x_1) = y_1$.

Substituting successive values $x_n, x_{n-1}, \ldots, x_1$ for x, together with the corresponding tabular function values $y_n, y_{n-1}, \ldots, y_1$, in equation (1) gives

$$y_n = a_0 \quad \text{or} \quad a_0 = y_n \tag{2}$$

$$y_{n-1} = a_0 + a_1\left(x_{n-1} - x_n\right)$$
$$= y_n + a_1\left(x_{n-1} - x_n\right) \tag{3}$$

Because the x_i are equidistant, $x_{i+1} - x_i = h$ and

$$a_1 = \frac{\left(y_n - y_{n-1}\right)}{h} = \frac{\Delta_1 y_n}{h} \tag{4}$$

Similarly,

$$\left.\begin{aligned} y_{n-2} &= a_0 + a_1\left(x_{n-2} - x_n\right) + a_2\left(x_{n-2} - x_{n-1}\right)\left(x_{n-1} - x_n\right) \\ &= y_n + \frac{y_n - y_{n-1}}{h}(-2h) + a_2(-h)(-2h) \\ &= -y_n + 2y_{n-1} + 2a_2h^2 \\ a_2 &= \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\Delta_2 y_n}{2h^2} \end{aligned}\right\} \tag{5}$$

Equation (5) may be verified by recalling that

$$\Delta_2 y_n = \Delta_1 y_n - \Delta_1 y_{n-1}$$
$$= \left(y_n - y_{n-1}\right) - \left(y_{n-1} - y_{n-2}\right)$$
$$= y_n - 2y_{n-1} + y_{n-2} \tag{6}$$

A list of the higher order differences in terms of the tabulated function values is given in appendix D.

$$
\left.
\begin{aligned}
y_{n-3} &= a_0 + a_1(x_{n-3} - x_n) + a_2(x_{n-3} - x_{n-1})(x_{n-3} - x_n) \\
&\quad + a_3(x_{n-3} - x_{n-2})(x_{n-3} - x_{n-1})(x_{n-3} - x_n) \\
&= y_n + \frac{y_n - y_{n-1}}{h}(-3h) + \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2}(-2h)(-3h) \\
&\quad + a_3(-h)(-2h)(-3h) \\
&= y_n(1 - 3 + 3) + y_{n-1}(3 - 6) + y_{n-2}(3) - 6h^3 a_3 \\
&= y_n - 3y_{n-1} + 3y_{n-2} - 6h^3 a_3 \\
a_3 &= \frac{y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}}{6h^3} = \frac{\Delta_3 y_n}{3!h^3}
\end{aligned}
\right\} \quad (7)
$$

In general, for $1 \leq i \leq n$,

$$
a_i = \frac{\Delta_i y_n}{i!h^i} \quad (8)
$$

Equation (1) can now be written as follows.

$$
\begin{aligned}
F(x) &= y_n + \sum_{i=0}^{n-1} \frac{\Delta_{i+1} y_n}{(i+1)!} \prod_{k=n-i}^{n} \left(\frac{x - x_k}{h}\right) \\
&= y_n + \Delta_1 y \left(\frac{x - x_n}{h}\right) + \frac{\Delta_2 y_n}{2}\left(\frac{x - x_{n-1}}{h}\right)\left(\frac{x - x_n}{h}\right) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + \frac{\Delta_r y_n}{r!}\left(\frac{x - x_{n-r+1}}{h}\right)\left(\frac{x - x_{n-r+2}}{h}\right)\cdots\left(\frac{x - x_n}{h}\right) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + \frac{\Delta_n y_n}{n!}\left(\frac{x - x_1}{h}\right)\left(\frac{x - x_2}{h}\right)\cdots\left(\frac{x - x_n}{h}\right)
\end{aligned}
\quad (9)
$$

Let

$$u = \frac{x - x_n}{h} \quad \text{or} \quad x = x_n + hu \tag{10}$$

Then, because $x_{n-1} = x_n - h$, $x_{n-2} = x_n - 2h$, . . ., the above factors can be written as follows.

$$\left.\begin{aligned}
\frac{x - x_{n-1}}{h} &= \frac{x - (x_n - h)}{h} = \frac{x - x_n}{h} + \frac{h}{h} = u + 1 \\
\frac{x - x_{n-2}}{h} &= \frac{x - (x_n - 2h)}{h} = \frac{x - x_n}{h} + \frac{2h}{h} = u + 2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{x - x_1}{h} &= \frac{x - x_{n-(n-1)}}{h} = u + n - 1
\end{aligned}\right\} \tag{11}$$

Substituting equation (11) in equation (9) gives

$$\begin{aligned}
F(x) &= y_n + \sum_{k=1}^{n} \frac{\Delta_k y_n}{k!} \prod_{i=0}^{k-1} (u + i) \\
&= y_n + u\,\Delta_1 y_n + \frac{u(u + 1)}{2!} \Delta_2 y_n \\
&\quad + \frac{u(u + 1)(u + 2)}{3!} \Delta_3 y_n + \ldots \\
&\quad + \frac{u(u + 1)(u + 2) \ldots (u + n - 1)}{n!} \Delta_n y_n
\end{aligned} \tag{12}$$

Equation (12) is Newton's backward difference formula. It has been expanded to 20th order in appendix C. The formula can be placed in terms of the function values if the difference formulas in appendix D are used.

If the first n + 1 derivative values of the derivatives of a differential equation are known at n + 1 consecutive, equally spaced values of the independent variable, then a polynomial can be fitted to these data and the equation can be integrated by integrating the polynomial. (The appropriate polynomial for this purpose is that given by equation (12)). Integration can be done because, in the numerical integration of differential equations, the solution is always begun from a set of given values (initial conditions) and constructed from that point forward. Hence, the values of the function immediately behind at a given point of the solution are always known, but the values ahead are unknown. The problem of finding the next value ahead is solved by the Adams predictor and corrector formulas, which are derived as follows.

By applying Newton's formula to a set of ordered values of the derivatives of a function to be integrated (e.g., writing $y' = dy/dx$ for y in equation (12) and using the expanded terms from appendix C),

$$y' = y'_n + u\,\Delta_1 y'_n + \left(u^2 + u\right)\frac{\Delta_2 y'_n}{2} + \left(u^3 + 3u^2 + 2u\right)\frac{\Delta_3 y'_n}{6}$$

$$+ \left(u^4 + 6u^3 + 11u^2 + 6u\right)\frac{\Delta_4 y'_n}{24} + \ldots \tag{13}$$

Because the change in y for any interval is given by the formula

$$\Delta y = \int_{x_k}^{x_{k+1}} \left(\frac{dy}{dx}\right) dx = \int_{x_k}^{x_{k+1}} y'\,dx \tag{14}$$

equation (13) can be used to compute the change in y over any interval wherein dy/dx is continuous. Therefore, over any interval $x_{k+1} - x_k$,

$$\Delta y = \int_{x_k}^{x_{k+1}} \left[y'_n + u\,\Delta_1 y'_n + \left(u^2 + u\right)\frac{\Delta_2 y'_n}{2} \right.$$

$$+ \left(u^3 + 3u^2 + 2u\right)\frac{\Delta_3 y'_n}{6}$$

$$\left. + \left(u^4 + 6u^3 + 11u^2 + 6u\right)\frac{\Delta_4 y'_n}{24} + \ldots \right] dx \tag{15}$$

Because

$$\left.\begin{aligned} x &= x_n + hu \\ dx &= h\,du \end{aligned}\right\} \tag{16}$$

equation (15) becomes

$$\Delta y = h \Bigg[u y'_n + \frac{u^2}{2} \Delta_1 y'_n + \left(\frac{u^3}{3} + \frac{u^2}{2}\right)\frac{\Delta_2 y'_n}{2}$$

$$+ \left(\frac{u^4}{4} + u^3 + u^2\right)\frac{\Delta_3 y'_n}{6}$$

$$+ \left(\frac{u^5}{5} + \frac{3u^4}{2} + \frac{11u^3}{3} + 3u^2\right) + \ldots \Bigg]_{u_k}^{u_{k+1}} \tag{17}$$

after integration with the change in limits $x_{k+1} \to u_{k+1}$ and $x_k \to u_k$. In the interval $x_{n+1} - x_n$,

$$\left.\begin{aligned} u_{k+1} &= \frac{x_{n+1} - x_n}{h} = \frac{h}{h} = 1 \\ u_k &= \frac{x_n - x_n}{h} = \frac{0}{h} = 0 \end{aligned}\right\} \tag{18}$$

Integrating equation (17) between the limits 0 and 1 gives the following predictor formula.

$$\Delta y = \Big]_n^{n+1} = h \Bigg[y'_n + \frac{1}{2} \Delta_1 y'_n + \frac{5}{12} \Delta_2 y'_n + \frac{3}{8} \Delta_3 y'_n + \frac{251}{720} \Delta_4 y'_n$$

$$+ \frac{95}{288} \Delta_5 y'_n + \frac{19087}{60480} \Delta_6 y'_n + \frac{5257}{17280} \Delta_7 y'_n + \frac{1070017}{3628800} \Delta_8 y'_n$$

$$+ \frac{25713}{89600} \Delta_9 y'_n + \frac{26842253}{95800320} \Delta_{10} y'_n + \ldots \Bigg] \tag{19}$$

In the interval $x_{n-1} - x_n$,

$$\left.\begin{aligned} u_{k+1} &= \frac{x_n - x_n}{h} = \frac{0}{h} = 0 \\ u_k &= \frac{x_{n-1} - x_n}{h} = \frac{-h}{h} = -1 \end{aligned}\right\} \tag{20}$$

Integrating equation (17) between the limits -1 and 0 gives the following corrector formula.

$$\Delta y = \Big]_{n-1}^{n} = h\left[y'_n - \frac{1}{2}\Delta_1 y'_n - \frac{1}{12}\Delta_2 y'_n - \frac{1}{24}\Delta_3 y'_n - \frac{19}{720}\Delta_4 y'_n \right.$$

$$- \frac{3}{160}\Delta_5 y'_n - \frac{863}{60480}\Delta_6 y'_n - \frac{275}{24192}\Delta_7 y'_n - \frac{33953}{3628800}\Delta_8 y'_n$$

$$\left. - \frac{8183}{1036800}\Delta_9 y'_n - \frac{32504331}{479001600}\Delta_{10} y'_n - \cdot \cdot \cdot \right] \tag{21}$$

Equations (19) and (21) are the predictor and corrector formulas of the Adams method and are to be used as follows. Use equation (19) to compute a predicted value,

$$\bar{y}_{n+1} = y_n + \Big]_{n}^{n+1} \tag{22}$$

Place a bar over y_{n+1} to indicate that it is a first approximation obtained by extrapolation. Use this value of y_{n+1} to compute the value of the derivative y'_{n+1}. With the value of the y'_{n+1} derivative thus obtained, compute a corrected value of y_{n+1} by using equation (21) with the n and n-1 replaced by k and k-1, wherein k = n + 1.

$$y_{n+1} = y_n + \Big]_{k-1}^{k}$$

$$= y_n + h\left[y'_{n+1} - \frac{1}{2}\Delta_1 y'_{n+1} - \frac{1}{12}\Delta_2 y'_{n+1} - \frac{1}{24}\Delta_3 y'_{n+1} - \cdot \cdot \cdot \right] \tag{23}$$

Equations (22) and (23) have been placed in an easy-to-use form in appendix B. The equations given in appendix B are written in terms of the derivative values at the various intervals ranging from a 1st- to 20th-order system. For an nth-order system, n + 1 derivative values are required.

CONCLUDING REMARKS

The Adams integration method is a fast, accurate method for integrating differential equations numerically. It requires only two derivative evaluations for each integration step, but additional time and storage must be dedicated to continually updating or reordering the table of past derivative values. When large sets of differential equations are integrated by using a high-order method, the amount of time required to update the derivative table can become appreciable. When this occurs, the Adams formula is preferred for its speed.

The accuracy of the Adams integration method is a function of (1) the accuracy of the first $n + 1$ derivative values, which must be initially supplied, and (2) the order of the method chosen in relation to the word length of the computer. The higher the order of the method used, the smaller the size of the integration step that can be used; however, an order that is too high or too low can lead to serious errors and cause the solution to diverge. For single-precision work, no order higher than seventh or eighth should be used with a nine-digit word length. For double-precision work, 13th is probably the highest order that should be attempted with a 20-digit word length. A 10th-order method can be used successfully with a 12-digit word length, and an order up to 20th can be used successfully with a 24-digit word length.

Because the data in this report were computed using floating-point arithmetic, only the first 16 digits of each 18-digit number are recommended without reservation.

Lyndon B. Johnson Space Center
National Aeronautics and Space Administration
Houston, Texas, April 20, 1976
986-16-00-00-72

REFERENCE

1. Scarborough, J. B.: Numerical Mathematical Analysis. The Johns Hopkins Press, 1966, pp. 59-60 and 320-322.

APPENDIX A

NEWTON'S BACKWARD DIFFERENCE FORMULA EXPANDED TO 20TH ORDER

The expansion of Newton's backward difference formula to 20th order is as follows.

$$y' = y'_n + \sum_{k=1}^{n} \frac{\Delta_k y'_n}{k!} \prod_{i=0}^{k-1} (u + i) = y'_n + u\Delta_1 y'_n$$

$$+ \left(u^2 + u\right)\frac{\Delta_2 y'_n}{2}$$

$$+ \left(u^3 + 3u^2 + 2u\right) \frac{\Delta_3 y'_n}{6}$$

$$+ \left(u^4 + 6u^3 + 11u^2 + 6u\right) \frac{\Delta_4 y'_n}{24}$$

$$+ \left(u^5 + 10u^4 + 35u^3 + 50u^2 + 24u\right) \frac{\Delta_5 y'_n}{120}$$

$$+ \left(u^6 + 15u^5 + 85u^4 + 225u^3 + 274u^2 + 120u\right) \frac{\Delta_6 y'_n}{720}$$

$$+ \left(u^7 + 21u^6 + 175u^5 + 735u^4 + 1624u^3 + 1764u^2 + 720u\right) \frac{\Delta_7 y'_n}{5040}$$

$$+ \left(u^8 + 28u^7 + 322u^6 + 1960u^5 + 6769u^4 + 13132u^3 + 13068u^2 + 5040u\right) \frac{\Delta_8 y'_n}{40320}$$

$$+ \left(u^9 + 36u^8 + 546u^7 + 4536u^6 + 22449u^5 + 67284u^4 + 118124u^3 + 109584u^2\right.$$

$$\left. + 40320u\right) \frac{\Delta_9 y'_n}{362880}$$

$$+ \left(u^{10} + 45u^9 + 870u^8 + 9450u^7 + 63273u^6 + 269325u^5 + 723680u^4 + 1172700u^3\right.$$

$$\left. + 1026576u^2 + 362880u\right) \frac{\Delta_{10} y'_n}{3628800}$$

$$+\left(u^{11} + 55u^{10} + 1320u^{9} + 18150u^{8} + 157773u^{7} + 902055u^{6} + 3416930u^{5} + 8409500u^{4}\right.$$

$$\left. + 127535760u^{3} + 10628640u^{2} + 3628800u\right)\frac{\Delta_{11}y'_n}{39916800}$$

$$+\left(u^{12} + 66u^{11} + 1925u^{10} + 32670u^{9} + 357423u^{8} + 2637558u^{7} + 13339535u^{6}\right.$$

$$\left. + 45995730u^{5} + 105258076u^{4} + 150917965u^{3} + 120543840u^{2} + 39916800u\right)\frac{\Delta_{12}y'_n}{479001600}$$

$$+\left(u^{13} + 78u^{12} + 2717u^{11} + 55770u^{10} + 749463u^{9} + 6926634u^{8} + 44990231u^{7}\right.$$

$$+ 206070150u^{6} + 657206836u^{5} + 1414014888u^{4} + 1931559552u^{3} + 1486442880u^{2}$$

$$\left. + 479001600u\right)\frac{\Delta_{13}y'_n}{6227020800}$$

$$+\left(u^{14} + 91u^{13} + 3731u^{12} + 91091u^{11} + 1474473u^{10} + 16669653u^{9} + 135036473u^{8}\right.$$

$$+ 790943153u^{7} + 3336118786u^{6} + 9957703756u^{5} + 20313753096u^{4} + 26596717056u^{3}$$

$$\left. + 19802759040u^{2} + 6227020800u\right)\frac{\Delta_{14}y'_n}{87178291200}$$

$$+\left(u^{15} + 105u^{14} + 5005u^{13} + 143325u^{12} + 2749747u^{11} + 37312275u^{10} + 368411615u^{9}\right.$$

$$+ 2681453775u^{8} + 14409322928u^{7} + 56663366760u^{6} + 159721605680u^{5} + 310989260400u^{4}$$

$$\left. + 392156797824u^{3} + 283465647360u^{2} + 87178291200u\right)\frac{\Delta_{15}y'_n}{1307674368000}$$

$$+\left(u^{16} + 120u^{15} + 6580u^{14} + 218400u^{13} + 4899622u^{12} + 78558480u^{11} + 928095740u^{10}\right.$$

$$+ 8207628000u^{9} + 54631129553u^{8} + 272803210680u^{7} + 1009672107080u^{6}$$

$$+ 2706813345600u^{5} + 5056995703824u^{4} + 6165817614720u^{3} + 4339163001600u^{2}$$

$$\left. + 1307674368000u\right)\frac{\Delta_{16}y'_n}{20922789888000}$$

$$+ \Big(u^{17} + 136u^{16} + 8500u^{15} + 323680u^{14} + 8394022u^{13} + 156952432u^{12} + 2185031420u^{11}$$
$$+ 23057159840u^{10} + 185953177553u^{9} + 1146901283528u^{8} + 5374523477960u^{7}$$
$$+ 18861567058880u^{6} + 48366009233424u^{5} + 87077748875904u^{4} + 102992244837120u^{3}$$
$$+ 70734282393600u^{2} + 20922789888000u\Big) \frac{\Delta_{17} y'_n}{355687428096000}$$

$$+ \Big(u^{18} + 153u^{17} + 10812u^{16} + 468180u^{15} + 13896582u^{14} + 299650806u^{13} + 4853222764u^{12}$$
$$+ 60202693980u^{11} + 577924894833u^{10} + 4308105301929u^{9} + 24871845297936u^{8}$$
$$+ 110228466184200u^{7} + 369012649234384u^{6} + 909299905844112u^{5} + 1583313975727488u^{4}$$
$$+ 1821602444624640u^{3} + 1223405590579200u^{2} + 355687428096000u\Big) \frac{\Delta_{18} y'_n}{6402373705728000}$$

$$+ \Big(u^{19} + 171u^{18} + 13566u^{17} + 662796u^{16} + 22323822u^{15} + 549789282u^{14} + 10246937272u^{13}$$
$$+ 147560703732u^{12} + 1661573386473u^{11} + 14710753408923u^{10} + 102417740732658u^{9}$$
$$+ 557921681547048u^{8} + 2353125040549984u^{7} + 7551527592063024u^{6}$$
$$+ 17950712280921504u^{5} + 30321254007719424u^{4} + 34012249593822720u^{3}$$
$$+ 22376988058521600u^{2} + 6402373705728000u\Big) \frac{\Delta_{19} y'_n}{121645100408832000}$$

$$+ \Big(u^{20} + 190u^{19} + 16815u^{18} + 920550u^{17} + 34916946u^{16} + 973941900u^{15} + 20692933630u^{14}$$
$$+ 342252511900u^{13} + 4465226757381u^{12} + 46280647751910u^{11} + 381922055502195u^{10}$$
$$+ 2503858755467550u^{9} + 12953636989943896u^{8} + 52260903362512720u^{7}$$
$$+ 161429736530118960u^{6} + 371384787345228000u^{5} + 610116075740491775u^{4}$$
$$+ 668609730341153279u^{3} + 431565146817638400u^{2}$$
$$+ 121645100408832000u\Big) \frac{\Delta_{20} y'_n}{2432902008176640000}$$

APPENDIX B

ADAMS PREDICTOR AND CORRECTOR FORMULAS FOR 1ST- TO 20TH-ORDER SYSTEMS IN TERMS OF DERIVATIVE VALUES

The predictor and corrector methods using derivative values directly are as follows.

1ST-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\left(-0.5y'_{n-1} + 1.5y'_n\right)$$

Corrector

$$y_{n+1} = y_n + \Delta T\left(0.5y'_n + 0.5y'_{n+1}\right)$$

2ND-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\left(0.416666666666666667y'_{n-2} - 0.133333333333333333 + 001y'_{n-1} + 0.191666666666666667 + 001y'_n\right)$$

Corrector

$$y_{n+1} = y_n + \Delta T\left(-0.833333333333333335 - 001y'_{n-1} + 0.666666666666666667y'_n + 0.416666666666666668y'_{n+1}\right)$$

3RD-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.375y'_{n-3} + 0.154166666666666666 + 001y'_{n-2}$$

$$- 0.245833333333333333 + 001y'_{n-1} + 0.229166666666666666 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.416666666666665 - 001y'_{n-2} - 0.208333333333333333y'_{n-1}$$

$$+ 0.791666666666666666y'_{n} + 0.375y'_{n+1}\Big)$$

4TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.348611111111111111y'_{n-4} - 0.176944444444444444 + 001y'_{n-3}$$

$$+ 0.363333333333333333 + 001y'_{n-2} - 0.385277777777777777 + 001y'_{n-1}$$

$$+ 0.264027777777777777 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.263888888888888892 - 001y'_{n-3} + 0.147222222222222224y'_{n-2}$$

$$- 0.366666666666666669y'_{n-1} + 0.897222222222222223y'_{n}$$

$$+ 0.348611111111111113y'_{n+1}\Big)$$

5TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.32986111111111111y'_{n-5} + 0.19979166666666666 + 001y'_{n-4}$$

$$- 0.50680555555555554 + 001y'_{n-3} + 0.69319444444444443 + 001y'_{n-2}$$

$$- 0.55020833333333331 + 001y'_{n-1} + 0.29701388888888888 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.1875 - 001y'_{n-4} - 0.12013888888888889y'_{n-3}$$

$$+ 0.33472222222222223y'_{n-2} - 0.55416666666666668y'_{n-1}$$

$$+ 0.99097222222222222y'_{n} + 0.32986111111111114y'_{n+1}\Big)$$

6TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.315591931216931216y'_{n-6} - 0.22234126984126984 0 + 001y'_{n-5}$$

$$+ 0.67317956349206349 0 + 001y'_{n-4} - 0.11379894179894179 9 + 002y'_{n-3}$$

$$+ 0.11665823412698412 7 + 002y'_{n-2} - 0.73956349206349206 2 + 001y'_{n-1}$$

$$+ 0.32857308201058201 0 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.142691798941798941 - 001y'_{n-5} + 0.104365079365079364y'_{n-4}$$

$$- 0.334176587301587300y'_{n-3} + 0.620105820105820105y'_{n-2}$$

$$- 0.768204365079365079y'_{n-1} + 0.107658730158730159 + 001y'_{n}$$

$$+ 0.315591931216931219y'_{n+1}\Big)$$

7TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.304224537037037036y'_{n-7} + 0.244516369047619047 + 001y'_{n-6}$$

$$- 0.861212797619047617 + 001y'_{n-5} + 0.173796544312169311 + 002y'_{n-4}$$

$$- 0.220277529761904761 + 002y'_{n-3} + 0.180545386904761904 + 002y'_{n-2}$$

$$- 0.952520667989417986 + 001y'_{n-1} + 0.358995535714285713 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.113673941798941799 - 001y'_{n-6} - 0.938409391534391534 - 001y'_{n-5}$$

$$+ 0.343080357142857143y'_{n-4} - 0.732035383597883596y'_{n-3}$$

$$+ 0.101796461640211640 + 001y'_{n-2} - 0.100691964285714286 + 001y'_{n-1}$$

$$+ 0.115615906084656084 + 001y'_{n} + 0.304224537037037040y'_{n+1}\Big)$$

8TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.294868000440917107y'_{n-8} - 0.266316854056437389 + 001y'_{n-7}$$

$$+ 0.107014677028218694 + 002y'_{n-6} - 0.251247360008818341 + 002y'_{n-5}$$

$$+ 0.380204144620811286 + 002y'_{n-4} - 0.385403610008818341 + 002y'_{n-3}$$

$$+ 0.263108427028218694 + 002y'_{n-2} - 0.118841506834215167 + 002y'_{n-1}$$

$$+ 0.388482335758377424 + 001y'_{n}\Big)$$

Corrector

$$y'_{n+1} = y_n + \Delta T\Big(-0.5653659611992939 - 002y'_{n-7} + 0.8621968694885361 50 - 001y'_{n-6}$$

$$- 0.355823963844797176y'_{n-5} + 0.867046406525573189y'_{n-4}$$

$$- 0.138699294532627865 + 001y'_{n-3} + 0.154193066578483245 + 001y'_{n-2}$$

$$- 0.126890266754850088 + 001y'_{n-1} + 0.123101135361552028 + 001y'_{n}$$

$$+ 0.294868000440917110y'_{n+1}\Big)$$

9TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.286975446428571428y'_{n-9} + 0.287764701829805996 + 001y'_{n-8}$$

$$- 0.129942846119929453 + 002y'_{n-7} + 0.348074052028218694 + 002y'_{n-6}$$

$$- 0.612836422508818339 + 002y'_{n-5} + 0.741793207120811284 + 002y'_{n-4}$$

$$- 0.626462985008818339 + 002y'_{n-3} + 0.366419587742504408 + 002y'_{n-2}$$

$$- 0.144669297012786595 + 002y'_{n-1} + 0.417179880401234566 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.789255401234567893 - 002y'_{n-8} - 0.803895227072310397 - 001y'_{n-7}$$

$$+ 0.370351631393298056y'_{n-6} - 0.101879850088183421 + 001y'_{n-5}$$

$$+ 0.186150821208112873 + 001y'_{n-4} - 0.238145475088183419 + 001y'_{n-3}$$

$$+ 0.220490520282186947 + 001y'_{n-2} - 0.155303461199294532 + 001y'_{n-1}$$

$$+ 0.130204433972663139 + 001y'_{n} + 0.286975446428571432y'_{n+1}\Big)$$

10TH-ORDER SYSTEM

Predictor

$$
\begin{aligned}
y_{n+1} = y_n + \Delta T\Big(&0.280189596443936721y'_{n-10} - 0.308887141086793863 + 001y'_{n-9} \\
&+ 0.154861788582752124 + 002y'_{n-8} - 0.466170361852653517 + 002y'_{n-7} \\
&+ 0.936472204560485808 + 002y'_{n-6} - 0.131891420554753887 + 003y'_{n-5} \\
&+ 0.133019135965307840 + 003y'_{n-4} - 0.962690500741542404 + 002y'_{n-3} \\
&+ 0.492504906142275932 + 002y'_{n-2} - 0.172688256657180267 + 002y'_{n-1} \\
&+ 0.445198840045628238 + 001y'_{n}\Big)
\end{aligned}
$$

Corrector

$$
\begin{aligned}
y_{n+1} = y_n + \Delta T\Big(&-0.678584998463470693 - 002y'_{n-9} + 0.757510538586927482 - 001y'_{n-8} \\
&- 0.385752772015792851y'_{n-7} + 0.118465362954946289 + 001y'_{n-6} \\
&- 0.244382699765512266 + 001y'_{n-5} + 0.357154240820907487 + 001y'_{n-4} \\
&- 0.380648324765512264 + 001y'_{n-3} + 0.301920720097803430 + 001y'_{n-2} \\
&- 0.185839786130150713 + 001y'_{n-1} + 0.136990283957297846 + 001y'_{n} \\
&+ 0.280189596443936725y'_{n+1}\Big)
\end{aligned}
$$

11TH-ORDER SYSTEM

Predictor

$$
\begin{aligned}
y_{n+1} = y_n + \Delta T\Big(&-0.274265540031599059y'_{n-11} + 0.329711053679152637 + 001y'_{n-10} \\
&- 0.181734761126058868 + 002y'_{n-9} + 0.607399929634890570 + 002y'_{n-8} \\
&- 0.137124664395693041 + 003y'_{n-7} + 0.220357899950647346 + 003y'_{n-6}
\end{aligned}
$$

$$- 0.258602100049352652 + 003y'_{n-5} + 0.223526764175735529 + 003y'_{n-4}$$

$$- 0.141522864179368085 + 003y'_{n-3} + 0.643350953159655414 + 002y'_{n-2}$$

$$- 0.202857466060656163 + 002y'_{n-1} + 0.472625394048788144 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.592405641233766230 - 002y'_{n-10} - 0.719504705203489922 - 001y'_{n-9}$$

$$+ 0.401574156537264175y'_{n-8} - 0.136322208005150713 + 001y'_{n-7}$$

$$+ 0.313959224562089145 + 001y'_{n-6} - 0.518074106015512264 + 001y'_{n-5}$$

$$+ 0.630845647070907485 + 001y'_{n-4} - 0.576142186372655120 + 001y'_{n-3}$$

$$+ 0.399667650901374858 + 001y'_{n-2} - 0.218422096398007856 + 001y'_{n-1}$$

$$+ 0.143506746010869274 + 001y'_{n} + 0.274265540031599064y'_{n+1}\Big)$$

12TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.269028846773648774y'_{n-12} - 0.350261170131538434 + 001y'_{n-11}$$

$$+ 0.210530144238523454 + 002y'_{n-10} - 0.773598224028086170 + 002y'_{n-9}$$

$$+ 0.193909272116445200 + 003y'_{n-8} - 0.350195511040422869 + 003y'_{n-7}$$

$$+ 0.468940554369498813 + 003y'_{n-6} - 0.471672946694082481 + 003y'_{n-5}$$

$$+ 0.356696043328691671 + 003y'_{n-4} - 0.200709210469570815 + 003y'_{n-3}$$

$$+ 0.820909992030263604 + 002y'_{n-2} - 0.235140927673494016 + 002y'_{n-1}$$

$$+ 0.499528278726153021 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.523669325795028499 - 002y'_{n-11} + 0.687643755077410822 - 001y'_{n-10}$$

$$- 0.417572225545067802y'_{n-9} + 0.155364667328632687 + 001y'_{n-8}$$

$$- 0.395538524273689820 + 001y'_{n-7} + 0.728705330591751715 + 001y'_{n-6}$$

$$- 0.100194456305011860 + 002y'_{n-5} + 0.104559175310057006 + 002y'_{n-4}$$

$$- 0.835358502641194226 + 001y'_{n-3} + 0.514874902576281128 + 001y'_{n-2}$$

$$- 0.252984271900479736 + 001y'_{n-1} + 0.149790777920409616 + 001y'_{n}$$

$$+ 0.269028846773648779y'_{n+1}\Big)$$

13TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.264351348366606509y'_{n-13} + 0.370559637553953339 + 001y'_{n-12}$$

$$- 0.241220168739106920 + 002y'_{n-11} + 0.966575000567018070 + 002y'_{n-10}$$

$$- 0.266371036484932270 + 003y'_{n-9} + 0.534129457464267776 + 003y'_{n-8}$$

$$- 0.803822424837519638 + 003y'_{n-7} + 0.922567468166595582 + 003y'_{n-6}$$

$$- 0.811893132041905058 + 003y'_{n-5} + 0.545707257410815325 + 003y'_{n-4}$$

$$- 0.276313696102420276 + 003y'_{n-3} + 0.102710404375621668 + 003y'_{n-2}$$

$$- 0.269506602961152862 + 002y'_{n-1} + 0.525963413562813672 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.467749840704226446 - 002y'_{n-12} - 0.660441725494997230 - 001y'_{n-11}$$

$$+ 0.433609251257037710y'_{n-10} - 0.175533676995915544 + 001y'_{n-9}$$

$$+ 0.489805803432154596 + 001y'_{n-8} - 0.997532539260029256 + 001y'_{n-7}$$

$$+ 0.153136405724020429 + 002y'_{n-6} - 0.180460328969357118 + 002y'_{n-5}$$

$$+ 0.164758579808690949 + 002y'_{n-4} - 0.116979963874471613 + 002y'_{n-3}$$

$$+ 0.648651357017689891 + 001y'_{n-2} - 0.289468759475409399 + 001y'_{n-1}$$

$$+ 0.155871525849564560 + 001y'_{n} + 0.264351348366606514y'_{n+1}\Big)$$

14TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.260136396127601036y'_{n-14} - 0.390626089415302101 + 001y'_{n-13}$$

$$+ 0.273780084231512276 + 002y'_{n-12} - 0.118811665064357469 + 003y'_{n-11}$$

$$+ 0.357054032580430444 + 003y'_{n-10} - 0.787164101532389544 + 003y'_{n-9}$$

$$+ 0.131531905503545369 + 004y'_{n-8} - 0.169661053634744639 + 004y'_{n-7}$$

$$+ 0.170375706573778149 + 004y'_{n-6} - 0.133268619708936233 + 004y'_{n-5}$$

$$+ 0.806103789934543962 + 003y'_{n-4} - 0.371003344292867054 + 003y'_{n-3}$$

$$+ 0.126382816423233362 + 003y'_{n-2} - 0.305925698419017007 + 002y'_{n-1}$$

$$+ 0.551977053175573776 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.421495223900547286 - 002y'_{n-13} + 0.636868297531188845 - 001y'_{n-12}$$

$$- 0.449604826298997753y'_{n-11} + 0.196785186625502983 + 001y'_{n-10}$$

$$- 0.597450396120363375 + 001y'_{n-9} + 0.133363924168105026 + 002y'_{n-8}$$

$$- 0.226328272663337275 + 002y'_{n-7} + 0.297793566566688258 + 002y'_{n-6}$$

$$- 0.307035344707191467 + 002y'_{n-5} + 0.249141923633580516 + 002y'_{n-4}$$

$$- 0.159171635786916397 + 002y'_{n-3} + 0.802075618517489103 + 001y'_{n-2}$$

$$- 0.327824824850359202 + 001y'_{n-1} + 0.161772458984172222 + 001y'_{n}$$

$$+ 0.260136396127601041y'_{n+1}\Big)$$

15TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.256309496574389152y'_{n-15} + 0.410477884474343831 + 001y'_{n-14}$$

$$- 0.308187580344638819 + 002y'_{n-13} + 0.143998829364498292 + 003y'_{n-12}$$

$$- 0.468674127888398661 + 003y'_{n-11} + 0.112675145079332106 + 004y'_{n-10}$$

$$- 0.206999313188720724 + 004y'_{n-9} + 0.296467066549164788 + 004y'_{n-8}$$

$$- 0.334596214680364058 + 004y'_{n-7} + 0.298658609609259920 + 004y'_{n-6}$$

$$- 0.210238361530225295 + 004y'_{n-5} + 0.115596625275858515 + 004y'_{n-4}$$

$$- 0.487624165234214118 + 003y'_{n-3} + 0.153295313563544223 + 003y'_{n-2}$$

$$- 0.344372122905175380 + 002y'_{n-1} + 0.577608002833012691 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.382689955321188443 - 002y'_{n-14} - 0.616184455371837393 - 001y'_{n-13}$$

$$+ 0.465511282840366749y'_{n-12} - 0.219084412301040517 + 001y'_{n-11}$$

$$+ 0.719156975638925207 + 001y'_{n-10} - 0.174666833194989227 + 002y'_{n-9}$$

$$+ 0.324900246806359841 + 002y'_{n-8} - 0.472589258912522038 + 002y'_{n-7}$$

$$+ 0.544054552815873020 + 002y'_{n-6} - 0.498571667345446283 + 002y'_{n-5}$$

$$+ 0.3640637172165334 05 + 002y'_{n-4} - 0.211408814688258619 + 002y'_{n-3}$$

$$+ 0.976199548188629844 + 001y'_{n-2} - 0.368007270159083988 + 001y'_{n-1}$$

$$+ 0.167512808313990048 + 001y'_{n} + 0.256309496574389157y'_{n+1}\Big)$$

16TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.2528121467290039235y'_{n-16} - 0.430130384423901690 + 001y'_{n-15}$$

$$+ 0.3444223645222281464 + 002y'_{n-14} - 0.172393560202725853 + 003y'_{n-13}$$

$$+ 0.601116936411349698 + 003y'_{n-12} - 0.157295758480084203 + 004y'_{n-11}$$

$$+ 0.315127112179946725 + 004y'_{n-10} - 0.496216409046741608 + 004y'_{n-9}$$

$$+ 0.621836299389438282 + 004y'_{n-8} - 0.623813310538384942 + 004y'_{n-7}$$

$$+ 0.501110576709874538 + 004y'_{n-6} - 0.320666707221469633 + 004y'_{n-5}$$

$$+ 0.161608435980543656 + 004y'_{n-4} - 0.629198967402476089 + 003y'_{n-3}$$

$$+ 0.183632771171028931 + 003y'_{n-2} - 0.384822066381821657 + 002y'_{n-1}$$

$$+ 0.602889217505916614 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.349734984534991753 - 002y'_{n-15} + 0.597844970788105648 - 001y'_{n-14}$$

$$- 0.4813004269791738 43y'_{n-13} + 0.242402719623632056 + 001y'_{n-12}$$

$$- 0.855602084154725506 + 001y'_{n-11} + 0.224679938808776918 + 002y'_{n-10}$$

$$- 0.454734608810610621 + 002y'_{n-9} + 0.724997069114390405 + 002y'_{n-8}$$

$$- 0.922698184009056423 + 002y'_{n-7} + 0.944151375123903584 + 002y'_{n-6}$$

$$- 0.778639442961067678 + 002y'_{n-5} + 0.516827958461417802 + 002y'_{n-4}$$

$$- 0.275060581873627117 + 002y'_{n-3} + 0.117205113952822522 + 002y'_{n-2}$$

$$- 0.409975468303282999 + 001y'_{n-1} + 0.173108568066549916 + 001y'_{n}$$

$$+ 0.252812146729039240y'_{n+1}\Big)$$

17TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.249597650297715667y'_{n-17} + 0.449597220179020557 + 001y'_{n-16}$$

$$- 0.382465842847283475 + 002y'_{n-15} + 0.204168638654674800 + 003y'_{n-14}$$

$$- 0.766435967911289140 + 003y'_{n-13} + 0.214862719645361424 + 004y'_{n-12}$$

$$- 0.466197810488537112 + 004y'_{n-11} + 0.800544622478944153 + 004y'_{n-10}$$

$$- 0.110298829692048839 + 005y'_{n-9} + 0.122860818726318507 + 005y'_{n-8}$$

$$- 0.110923082083738237 + 005y'_{n-7} + 0.810012628718327447 + 004y'_{n-6}$$

$$- 0.475117733225696087 + 004y'_{n-5} + 0.221012676751399984 + 004y'_{n-4}$$

$$- 0.798925369604922742 + 003y'_{n-3} + 0.217578051611518262 + 003y'_{n-2}$$

$$- 0.427253666932433320 + 002y'_{n-1} + 0.627848982535688180 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.321449643132356720 - 002y'_{n-16} - 0.581437891778505569 - 001y'_{n-15}$$

$$+ 0.496956011738815704y'_{n-14} - 0.266715800027919954 + 001y'_{n-13}$$

$$+ 0.100745287027864105 + 002y'_{n-12} - 0.284473247585774889 + 002y'_{n-11}$$

$$+ 0.622506017149331594 + 002y'_{n-10} - 0.107988987477441797 + 003y'_{n-9}$$

$$+ 0.150644115156914959 + 003y'_{n-8} - 0.170414226646381561 + 003y'_{n-7}$$

$$+ 0.156930664108771093 + 003y'_{n-6} - 0.117646552130167235 + 003y'_{n-5}$$

$$+ 0.715740997631720140 + 002y'_{n-4} - 0.351565596939128017 + 002y'_{n-3}$$

$$+ 0.139063689685822779 + 002y'_{n-2} - 0.453692619769283512 + 001y'_{n-1}$$

$$+ 0.178573211999799980 + 001y'_{n} + 0.249597650297715674y'_{n+1}\Big)$$

18TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.246628202582257457y'_{n-18} - 0.468890529677834989 + 001y'_{n-17}$$

$$+ 0.422300871968755965 + 002y'_{n-16} - 0.239495197591850432 + 003y'_{n-15}$$

$$+ 0.958850938556382618 + 003y'_{n-14} - 0.287954640763607103 + 004y'_{n-13}$$

$$+ 0.672703314919064167 + 004y'_{n-12} - 0.125106740238631324 + 005y'_{n-11}$$

$$+ 0.187974031133838633 + 005y'_{n-10} - 0.230209461787542415 + 005y'_{n-9}$$

$$+ 0.230780387612262725 + 005y'_{n-8} - 0.189410041273515850 + 005y'_{n-7}$$

$$+ 0.126785322399203019 + 005y'_{n-6} - 0.686428777198174276 + 004y'_{n-5}$$

$$+ 0.296480906741570766 + 004y'_{n-4} - 0.100017398291204483 + 004y'_{n-3}$$

$$+ 0.255312166606603652 + 003y'_{n-2} - 0.471646743397239662 + 002y'_{n-1}$$

$$+ 0.652511802793913926 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.296944771545820966 - 002y'_{n-17} + 0.566645553095713411 - 001y'_{n-16}$$

$$- 0.512469289642956638y'_{n-15} + 0.292002534755271478 + 001y'_{n-14}$$

$$- 0.117536680095813211 + 002y'_{n-13} + 0.355167567288323508 + 002y'_{n-12}$$

$$- 0.835721521483436930 + 002y'_{n-11} + 0.156750305811680223 + 003y'_{n-10}$$

$$- 0.237926080610462135 + 003y'_{n-9} + 0.295018663082493113 + 003y'_{n-8}$$

$$- 0.300351319779401899 + 003y'_{n-7} + 0.251430368205513157 + 003y'_{n-6}$$

$$- 0.172771379519933439 + 003y'_{n-5} + 0.970163277892179544 + 002y'_{n-4}$$

$$- 0.442430697032149232 + 002y'_{n-3} + 0.163294383043961770 + 002y'_{n-2}$$

$$- 0.499125169815794119 + 001y'_{n-1} + 0.183918217887624758 + 001y'_{n}$$

$$+ 0.246628202582257464y'_{n+1}\Big)$$

19TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(-0.243872812282820742y'_{n-19} + 0.488021163595585154 + 001y'_{n-18}$$

$$- 0.463911561971406966 + 002y'_{n-17} + 0.278542842298928894 + 003y'_{n-16}$$

$$- 0.118474621800006362 + 004y'_{n-15} + 0.379460399978102219 + 004y'_{n-14}$$

$$- 0.949630255049356339 + 004y'_{n-13} + 0.190152964144974132 + 005y'_{n-12}$$

$$- 0.309430689218232897 + 005y'_{n-11} + 0.413258857664462777 + 005y'_{n-10}$$

$$- 0.455494288318166560 + 005y'_{n-9} + 0.415104336591864297 + 005y'_{n-8}$$

$$- 0.312292673926583565 + 005y'_{n-7} + 0.192952893827777942 + 005y'_{n-6}$$

$$- 0.970004083320638234 + 004y'_{n-5} + 0.391006008782392086 + 004y'_{n-4}$$

$$- 0.123648673801409812 + 004y'_{n-3} + 0.297014417506965999 + 003y'_{n-2}$$

$$- 0.517982577730975603 + 002y'_{n-1} + 0.676899084022196000 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(0.275539029943671580 - 002y'_{n-18} - 0.553218634047558098 - 001y'_{n-17}$$

$$+ 0.527836296513249742y'_{n-16} - 0.318244248979713424 + 001y'_{n-15}$$

$$+ 0.135999181481694252 + 002y'_{n-14} - 0.437933464114314523 + 002y'_{n-13}$$

$$+ 0.110276006333149324 + 003y'_{n-12} - 0.222410758556360928 + 003y'_{n-11}$$

$$+ 0.365008215423706076 + 003y'_{n-10} - 0.492463525691827066 + 003y'_{n-9}$$

$$+ 0.549556108163858044 + 003y'_{n-8} - 0.508609229391427752 + 003y'_{n-7}$$

$$+ 0.390268974613530393 + 003y'_{n-6} - 0.247530629124250412 + 003y'_{n-5}$$

$$+ 0.129056006191068085 + 003y'_{n-4} - 0.549229625038316336 + 002y'_{n-3}$$

$$+ 0.189994115045503546 + 002y'_{n-2} - 0.546242343936161959 + 001y'_{n-1}$$

$$+ 0.189153459456554518 + 001y'_{n} + 0.243872812282820749y'_{n+1}\Big)$$

20TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T\Big(0.241305789737813504y'_{n-20} - 0.506998860703909081 + 001y'_{n-19}$$

$$+ 0.507283116861404171 + 002y'_{n-18} - 0.321479756498248090 + 003y'_{n-17}$$

$$+ 0.144766939357863532 + 004y'_{n-16} - 0.492595118209512418 + 004y'_{n-15}$$

$$+ 0.131476164100186736 + 005y'_{n-14} - 0.282023283709688662 + 005y'_{n-13}$$

$$+ 0.494125867477697802 + 005y'_{n-12} - 0.714727893661864457 + 005y'_{n-11}$$

$$+ 0.859085782552457494 + 005y'_{n-10} - 0.860791492761798119 + 005y'_{n-9}$$

$$+ 0.719077239924587968 + 005y'_{n-8} - 0.499352922131336593 + 005y'_{n-7}$$

$$+ 0.286483017930154456 + 005y'_{n-6} - 0.134412457973014429 + 005y'_{n-5}$$

$$+ 0.507918663910362727 + 004y'_{n-4} - 0.151157533831520551 + 004y'_{n-3}$$

$$+ 0.342862517557150564 + 003y'_{n-2} - 0.566243735678538303 + 002y'_{n-1}$$

$$+ 0.701029662995977350 + 001y'_{n}\Big)$$

Corrector

$$y_{n+1} = y_n + \Delta T\Big(-0.256702254500723787 - 002y'_{n-19} + 0.540958411995814733 - 001y'_{n-18}$$

$$- 0.543056146956131006y'_{n-17} + 0.345424199782150092 + 001y'_{n-16}$$

$$- 0.156196667203572017 + 002y'_{n-15} + 0.533990356859616411 + 002y'_{n-14}$$

$$- 0.143291140255911992 + 003y'_{n-13} + 0.309271594022110403 + 003y'_{n-12}$$

$$- 0.545778588550922682 + 003y'_{n-11} + 0.796165322083121750 + 003y'_{n-10}$$

$$- 0.966736343017184307 + 003y'_{n-9} + 0.980713214823273717 + 003y'_{n-8}$$

$$- 0.831977059385989507 + 003y'_{n-7} + 0.589264562302491472 + 003y'_{n-6}$$

$$- 0.347028422968730952 + 003y'_{n-5} + 0.168855123728860301 + 003y'_{n-4}$$

$$- 0.673601867343917011 + 002y'_{n-3} + 0.219258172058586058 + 002y'_{n-2}$$

$$- 0.595015772291299479 + 001y'_{n-1} + 0.194287504546568993 + 001y'_{n}$$

$$+ 0.241305789737813511y'_{n+1}\Big)$$

APPENDIX C

ADAMS PREDICTOR AND CORRECTOR DIFFERENCE FORMULAS

The Adams predictor and corrector difference formulas are expressed as follows.

PREDICTOR DIFFERENCE FORMULA

$$
\begin{aligned}
y_{n+1} = y_n + \Delta T\big(y'_n &+ 0.5\ \Delta_1 y'_n + 0.416666666666666667\ \Delta_2 y'_n + 0.375\ \Delta_3 y'_n \\
&+ 0.348611111111111111\ \Delta_4 y'_n + 0.329861111111111111\ \Delta_5 y'_n \\
&+ 0.315591931216931216\ \Delta_6 y'_n + 0.304224537037037036\ \Delta_7 y'_n \\
&+ 0.294868000440917107\ \Delta_8 y'_n + 0.286975446428571428\ \Delta_9 y'_n \\
&+ 0.280189596443936721\ \Delta_{10} y'_n + 0.274265540031599059\ \Delta_{11} y'_n \\
&+ 0.269028846773648774\ \Delta_{12} y'_n + 0.264351348366606509\ \Delta_{13} y'_n \\
&+ 0.260136396127601036\ \Delta_{14} y'_n + 0.256309496574389152\ \Delta_{15} y'_n \\
&+ 0.252812146729039235\ \Delta_{16} y'_n + 0.249597650297715667\ \Delta_{17} y'_n \\
&+ 0.246628202582257457\ \Delta_{18} y'_n + 0.243872812282820742\ \Delta_{19} y'_n \\
&+ 0.241305789737813504\ \Delta_{20} y'_n\big)
\end{aligned}
$$

CORRECTOR DIFFERENCE FORMULA

$$
\begin{aligned}
y_{n+1} = y_n + \Delta T\big(y'_{n+1} &- 0.5\ \Delta_1 y'_{n+1} - 0.833333333333333335 - 001\ \Delta_2 y'_{n+1} \\
&- 0.416666666666666665 - 001\ \Delta_3 y'_{n+1} - 0.263888888888888892 - 001\ \Delta_4 y'_{n+1} \\
&- 0.1875 - 001\ \Delta_5 y'_{n+1} - 0.142691798941798941 - 001\ \Delta_6 y'_{n+1} \\
&- 0.113673941798941799 - 001\ \Delta_7 y'_{n+1} - 0.935653659611992939 - 002\ \Delta_8 y'_{n+1}
\end{aligned}
$$

$$- 0.789255401234567893 - 002\ \Delta_9 y'_{n+1} - 0.678584998463470693 - 002\ \Delta_{10} y'_{n+1}$$

$$- 0.592405641233766230 - 002\ \Delta_{11} y'_{n+1} - 0.523669325795028499 - 002\ \Delta_{12} y'_{n+1}$$

$$- 0.467749840704226446 - 002\ \Delta_{13} y'_{n+1} - 0.421495223900547286 - 002\ \Delta_{14} y'_{n+1}$$

$$- 0.382689955321188443 - 002\ \Delta_{15} y'_{n+1} - 0.349734984534991753 - 002\ \Delta_{16} y'_{n+1}$$

$$- 0.321449643132356720 - 002\ \Delta_{17} y'_{n+1} - 0.296944771545820966 - 002\ \Delta_{18} y'_{n+1}$$

$$- 0.275539029943671580 - 002\ \Delta_{19} y'_{n+1} - 0.256702254500723787 - 002\ \Delta_{20} y'_{n+1} \Big)$$

APPENDIX D

EXPANSION OF DIFFERENCE FORMULAS IN TERMS OF TABULATED VALUES

The expansion of the backward difference formula is as follows.

$$\Delta_1 y'_n = y'_n - y'_{n-1}$$

$$\Delta_2 y'_n = y'_n - 2y'_{n-1} + y'_{n-2}$$

$$\Delta_3 y'_n = y'_n - 3y'_{n-1} + 3y'_{n-2} - y'_{n-3}$$

$$\Delta_4 y'_n = y'_n - 4y'_{n-1} + 6y'_{n-2} - 4y'_{n-3} + y'_{n-4}$$

$$\Delta_5 y'_n = y'_n - 5y'_{n-1} + 10y'_{n-2} - 10y'_{n-3} + 5y'_{n-4} - y'_{n-5}$$

$$\Delta_6 y'_n = y'_n - 6y'_{n-1} + 15y'_{n-2} - 20y'_{n-3} + 15y'_{n-4} - 6y'_{n-5} + y'_{n-6}$$

$$\Delta_7 y'_n = y'_n - 7y'_{n-1} + 21y'_{n-2} - 35y'_{n-3} + 35y'_{n-4} - 21y'_{n-5} + 7y'_{n-6} - y'_{n-7}$$

$$\Delta_8 y'_n = y'_n - 8y'_{n-1} + 28y'_{n-2} - 56y'_{n-3} + 70y'_{n-4} - 56y'_{n-5} + 28y'_{n-6} - 8y'_{n-7} + y'_{n-8}$$

$$\Delta_9 y'_n = y'_n - 9y'_{n-1} + 36y'_{n-2} - 84y'_{n-3} + 126y'_{n-4} - 126y'_{n-5} + 84y'_{n-6} - 36y'_{n-7} + 9y'_{n-8} - y'_{n-9}$$

$$\Delta_{10} y'_n = y'_n - 10y'_{n-1} + 45y'_{n-2} - 120y'_{n-3} + 210y'_{n-4} - 252y'_{n-5} + 210y'_{n-6} - 120y'_{n-7} + 45y'_{n-8} - 10y'_{n-9} + y'_{n-10}$$

$$\Delta_{11}y'_n = y'_n - 11y'_{n-1} + 55y'_{n-2} - 165y'_{n-3} + 330y'_{n-4} - 462y'_{n-5} + 462y'_{n-6} - 330y'_{n-7} + 165y'_{n-8} - 55y'_{n-9} + 11y'_{n-10} - y'_{n-11}$$

$$\Delta_{12}y'_n = y'_n - 12y'_{n-1} + 66y'_{n-2} - 220y'_{n-3} + 495y'_{n-4} - 792y'_{n-5} + 924y'_{n-6} - 792y'_{n-7} + 495y'_{n-8} - 220y'_{n-9} + 66y'_{n-10} - 12y'_{n-11} + y'_{n-12}$$

$$\Delta_{13}y'_n = y'_n - 13y'_{n-1} + 78y'_{n-2} - 286y'_{n-3} + 715y'_{n-4} - 1287y'_{n-5} + 1716y'_{n-6} - 1716y'_{n-7} + 1287y'_{n-8} - 715y'_{n-9} + 286y'_{n-10} - 78y'_{n-11} + 13y'_{n-12} - y'_{n-13}$$

$$\Delta_{14}y'_n = y'_n - 14y'_{n-1} + 91y'_{n-2} - 364y'_{n-3} + 1001y'_{n-4} - 2002y'_{n-5} + 3003y'_{n-6} - 3432y'_{n-7} + 3003y'_{n-8} - 2002y'_{n-9} + 1001y'_{n-10} - 364y'_{n-11} + 91y'_{n-12} - 14y'_{n-13} + y'_{n-14}$$

$$\Delta_{15}y'_n = y'_n - 15y'_{n-1} + 105y'_{n-2} - 455y'_{n-3} + 1365y'_{n-4} - 3003y'_{n-5} + 5005y'_{n-6} - 6435y'_{n-7} + 6435y'_{n-8} - 5005y'_{n-9} + 3003y'_{n-10} - 1365y'_{n-11} + 455y'_{n-12} - 105y'_{n-13} + 15y'_{n-14} - y'_{n-15}$$

$$\Delta_{16}y'_n = y'_n - 16y'_{n-1} + 120y'_{n-2} - 560y'_{n-3} + 1820y'_{n-4} - 4368y'_{n-5} + 8008y'_{n-6} - 11440y'_{n-7} + 12870y'_{n-8} - 11440y'_{n-9} + 8008y'_{n-10} - 4368y'_{n-11} + 1820y'_{n-12} - 560y'_{n-13} + 120y'_{n-14} - 16y'_{n-15} + y'_{n-16}$$

$$\Delta_{17}y'_n = y'_n - 17y'_{n-1} + 136y'_{n-2} - 680y'_{n-3} + 2380y'_{n-4} - 6188y'_{n-5} + 12376y'_{n-6} - 19448y'_{n-7} + 24310y'_{n-8} - 24310y'_{n-9} + 19448y'_{n-10} - 12376y'_{n-11} + 6188y'_{n-12} - 2380y'_{n-13} + 680y'_{n-14} - 136y'_{n-15} + 17y'_{n-16} - y'_{n-17}$$

$$\begin{aligned}\Delta_{18}y_n' = {} & y_n' - 18y_{n-1}' + 153y_{n-2}' - 816y_{n-3}' + 3060y_{n-4}' - 8568y_{n-5}' + 18564y_{n-6}' \\ & - 31824y_{n-7}' + 43758y_{n-8}' - 48620y_{n-9}' + 43758y_{n-10}' - 31824y_{n-11}' + 18564y_{n-12}' \\ & - 8568y_{n-13}' + 3060y_{n-14}' - 816y_{n-15}' + 153y_{n-16}' - 18y_{n-17}' + y_{n-18}'\end{aligned}$$

$$\begin{aligned}\Delta_{19}y_n' = {} & y_n' - 19y_{n-1}' + 171y_{n-2}' - 969y_{n-3}' + 3876y_{n-4}' - 11628y_{n-5}' + 27132y_{n-6}' - 50388y_{n-7}' \\ & + 75582y_{n-8}' - 92378y_{n-9}' + 92378y_{n-10}' - 75582y_{n-11}' + 50388y_{n-12}' - 27132y_{n-13}' \\ & + 11628y_{n-14}' - 3876y_{n-15}' + 969y_{n-16}' - 171y_{n-17}' + 19y_{n-18}' - y_{n-19}'\end{aligned}$$

$$\begin{aligned}\Delta_{20}y_n' = {} & y_n' - 20y_{n-1}' + 190y_{n-2}' - 1140y_{n-3}' + 4845y_{n-4}' - 15504y_{n-5}' + 38760y_{n-6}' \\ & - 77520y_{n-7}' + 125970y_{n-8}' - 167960y_{n-9}' + 184756y_{n-10}' - 167960y_{n-11}' \\ & + 125970y_{n-12}' - 77520y_{n-13}' + 38760y_{n-14}' - 15504y_{n-15}' + 4845y_{n-16}' \\ & - 1140y_{n-17}' + 190y_{n-18}' - 20y_{n-19}' + y_{n-20}'\end{aligned}$$