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Abstract

A procedure is presented for the active control of a spinning flexible spacecraft. Such a system exhibits gyroscopic effects. The design of the controller is based on modal decomposition of the gyroscopic system. This modal decoupling procedure leads to a control mechanism implemented in modular form, which represents a distinct computational advantage over the control of the coupled system. Design procedures are demonstrated for two types of control algorithms, linear and nonlinear. The first represents classical linear feedback approach and the second represents an application of on-off control, both types made feasible by the modal decomposition scheme.

1. Introduction

As spacecraft structures increase in size, weight limitations demand that various substructures be made as light as possible, which in turn requires that they be highly flexible. On the other hand, greater pointing accuracy necessitates finer attitude control, which can be achieved only through active control of the spacecraft. In controlling a flexible spacecraft, the problem of simulating the flexibility is often critical, as the number of degrees of freedom of the simulation can become so large as to render various mathematical techniques unfeasible. Of course, quite often proper modelling of the spacecraft can result in a model possessing relatively few degrees of freedom and yet retaining all the essential dynamic characteristics of the system. Even then some truncation may be necessary. In the case of control of nonrotating spacecraft, it is common practice to use the system natural modes to decouple the system and control only a limited number of lower modes. This procedure can be implemented with relative ease because natural modes of nonrotating structures can be readily computed. Recent advances in the analysis of gyroscopic systems, however, makes a modal approach possible also for rotating spacecraft. Following is a brief literature survey of related work.

In an attempt to control a flexible space booster, Gevarter (Ref. 1) presents a procedure whereby the response can be represented in terms of the rigid-body modes and the bending modes of the missile. This approach permits a description of the system in terms of transfer functions for the uncoupled system. When the spacecraft is spinning, or when it possesses spinning parts, the classical modal decomposition is no longer possible because the modal matrix will not diagonalize the gyroscopic matrix. Kuo et al (Ref. 2) have presented a technique for the design of a digital controller for spinning flexible spacecraft using a redesign of a preliminary continuous-data control system. No attempt was made in Ref. 2 to uncouple the system equations. Control of flexible spacecraft by modal synthesis is discussed by Poelaert (Ref. 3), but the procedure presented is valid only for nongyroscopic systems. In fact, the mathematical model considered, used first in Ref. 4, consists of three disks mounted on a flexible shaft and rotating about a common symmetry axis. Such a model does not exhibit the gyroscopic effect typical of a spinning flexible structure capable of nutation.

This paper develops a method for the design of a controller based on modal decomposition of spinning structures developed in Refs. 5 and 6. For high-order systems, this approach offers substantial computational advantages. In the first place, the modal decoupling procedure leads to a control mechanism which can be implemented in modular form. Moreover, one can use decoupled dynamics to design an observer also. Following decoupling, each control-group is governed by a set of two first-order differential equations with a skew symmetric matrix of coefficients. These sets of equations can be integrated readily, thus permitting independent control of spacecraft modes. Design procedures are demonstrated for two types of control algorithms, linear and nonlinear. The first represents a classical linear feedback approach in the form of proportional control and the second represents an application of on-off control, both types made feasible by the modal decomposition scheme.

2. Kinematical Considerations

Let us consider a general spacecraft consisting of a central body with an arbitrary number of appendages. The central body will be referred to as the "platform" and it can be rigid or elastic. Quantities pertaining to the platform will be designated by the subscript P. The appendages can be of three types: rigid and rotating relative to the platform, elastic and nonrotating relative to the platform, and elastic and rotating relative to the platform. Quantities pertaining to the types of appendages listed will be denoted by the subscripts R, E, and A, respectively. An example of the first type is a rigid rotor, examples of the second are flexible solar panels or flexible antennas cantilevered from the platform, and an illustration of the third is a flexible rotor. Clearly, there can be more than one appendage of a given type. We shall confine our discussion to one of each type, however, with a summation implied over appendages of the same type.

To describe the motion of the spacecraft, it will prove convenient to introduce various sets of axes. In the first place, we wish to identify an inertial system of axes XYZ with the origin at a point O. Then, we shall identify a system of axes $x_p y_p z_p$ with the origin at the center of mass P of

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the platform and coinciding with the principal axes of the undeformed platform. Similarly, we shall consider a set of axes $x_R y_R z_R$ attached to the rotor and with the origin at the mass center R of the rotor. The motion of an elastic member nonrotating relative to the platform can be described by means of a set $x_E y_E z_E$ attached to the member in undeformed state and with the origin E at the point of attachment of the member. In the same manner, we can define a set of axes $x_A y_A z_A$ with the origin at A and coinciding with a given set of axes in the rotating elastic member when in undeformed state. The spacecraft and the various sets of axes are shown in Fig. 1.

The position of the point P relative to the inertial space is given by the radius vector $\underline{R}_{OP}$ from O to P. The rotation of axes $x_P y_P z_P$ relative to axes XYZ is given by the angular velocity vector $\underline{\Omega}_P$. In addition, any point of the platform can undergo elastic motion relative to $x_P y_P z_P$. For simplicity, however, we shall assume that the platform is rigid. The positions of the interconnecting points R, E, and A relative to P are denoted by the radius vectors $\underline{R}_{PR}$, $\underline{R}_{PE}$, and $\underline{R}_{PA}$, respectively, and the rotations of axes $x_R y_R z_R$ and $x_A y_A z_A$ relative to $x_P y_P z_P$ are denoted by the angular velocity vectors $\underline{\omega}_R$ and $\underline{\omega}_A$, respectively. It follows that the positions of R, E, and A relative to O are $\underline{R}_{OP} + \underline{R}_{PR}$, $\underline{R}_{OP} + \underline{R}_{PE}$, and $\underline{R}_{OP} + \underline{R}_{PA}$, respectively. The position of an arbitrary point in the platform relative to P is given by the radius vector $\underline{r}_P$. Similarly, the position of an arbitrary point in a spinning rotor relative to R is given by $\underline{r}_R$. On the other hand, points in the flexible appendages E and A are described by $\underline{r}_E + \underline{u}_E$ and $\underline{r}_A + \underline{u}_A$, where $\underline{r}_E$ and $\underline{r}_A$ are nominal positions of the points when the appendages are undeformed and $\underline{u}_E$ and $\underline{u}_A$ are elastic displacements. It follows that the absolute positions of arbitrary points in the various spacecraft members are

$$\underline{R}_P = \underline{R}_{OP} + \underline{r}_P \tag{1a}$$

$$\underline{R}_A = \underline{R}_{OP} + \underline{R}_{PR} + \underline{r}_R \tag{1b}$$

$$\underline{R}_E = \underline{R}_{OP} + \underline{R}_{PE} + \underline{r}_E + \underline{u}_E \tag{1c}$$

$$\underline{R}_A = \underline{R}_{OP} + \underline{R}_{PA} + \underline{r}_A + \underline{u}_A \tag{1d}$$

It should be pointed out that $\underline{R}_{OP}$ is generally given in terms of inertial components, $\underline{r}_P$, $\underline{R}_{PR}$, $\underline{R}_{PE}$, and $\underline{R}_{PA}$ in terms of components along axes $x_P y_P z_P$, $\underline{r}_R$ in terms of components along $x_R y_R z_R$, $\underline{r}_E + \underline{u}_E$ in terms of components along $x_E y_E z_E$, and $\underline{r}_A + \underline{u}_A$ in terms of components along $x_A y_A z_A$.

The angular velocity vectors of axes $x_R y_R z_R$, $x_E y_E z_E$, and $x_A y_A z_A$ relative to the inertial space are

$$\underline{\Omega}_R = \underline{\Omega}_P + \underline{\omega}_R \ , \ \underline{\Omega}_E = \underline{\Omega}_P \ , \ \underline{\Omega}_A = \underline{\Omega}_P + \underline{\omega}_A \tag{2}$$

respectively. Note that $\underline{\Omega}_P$ in terms of components along XYZ and $\underline{\omega}_R$ and $\underline{\omega}_A$ are in terms of components along $x_R y_R z_R$ and $x_A y_A z_A$, respectively. This permits us to calculate absolute velocity vectors for arbitrary points in the various members in the form

$$\underline{v}_P = \underline{v}_{OP} + \underline{\Omega}_P \times \underline{r}_P \tag{3a}$$

$$\underline{v}_R = \underline{v}_{OP} + \underline{\Omega}_P \times \underline{R}_{PR} + \underline{\Omega}_R \times \underline{r}_R \tag{3b}$$

$$\underline{v}_E = \underline{v}_{OP} + \underline{\Omega}_P \times (\underline{R}_{PE} + \underline{r}_E + \underline{u}_E) + \dot{\underline{u}}_E \tag{3c}$$

$$\underline{v}_A = \underline{v}_{OP} + \underline{\Omega}_P \times \underline{R}_{PA} + \underline{\Omega}_A \times (\underline{r}_A + \underline{u}_A) + \dot{\underline{u}}_A \tag{3d}$$

where $\dot{\underline{u}}_E$ and $\dot{\underline{u}}_A$ are velocities of the points in question relative to the moving frames. We note once again that the various terms in Eqs. (3) are in terms of different sets of axes.

It will prove convenient to work with velocity components in terms of member axes. For example, we shall express $\underline{\Omega}_P$ and $\underline{v}_P$ in terms of components along axes $x_P y_P z_P$, etc. To this end it is more natural to work with matrix notation, which necessitates the introduction of the matrix form of the vector cross product. Hence, let us define the following skew symmetric matrices

$$\tilde{\Omega} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} , \quad \tilde{r} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$\tilde{u} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \tag{4}$$

Then the cross products $\underline{\Omega} \times \underline{r}$, $\underline{\Omega} \times \underline{u}$, $\underline{r} \times \underline{\Omega}$, and $\underline{u} \times \underline{\Omega}$ have the matrix counterparts $\tilde{\Omega}\underline{r}$, $\tilde{\Omega}\underline{u}$, $\tilde{r}\underline{\Omega}$, and $\tilde{u}\underline{\Omega}$, respectively. In addition, we must introduce the matrices of direction cosines between various systems of axes. For example, the matrix of direction cosines between axes XYZ and axes $x_P y_P z_P$ will be denoted by L_{OP}, so that

$$[x_P \ y_P \ z_P]^T = L_{OP}[X \ Y \ Z] \tag{5}$$

Similarly, the matrix of direction cosines between axes $x_P y_P z_P$ and axes $x_R y_R z_R$ will be denoted by L_{PR}, etc. With this notation, Eqs. (2) can be replaced by

$$\underline{\Omega}_R = L_{PR}\underline{\Omega}_P + \underline{\omega}_R \ , \ \underline{\Omega}_E = L_{PE}\underline{\Omega}_P, \ \underline{\Omega}_A = L_{PA}\underline{\Omega}_P + \underline{\omega}_A \tag{6}$$

and Eqs. (3) can be replaced by

$$\begin{aligned} \underline{v}_P &= L_{OP}\underline{v}_{OP} - \tilde{r}_P\underline{\Omega}_P \\ \underline{v}_R &= L_{PR}L_{OP}\underline{v}_{OP} - L_{PR}\tilde{R}_{PR}\underline{\Omega}_P - \tilde{r}_R\underline{\Omega}_R \\ \underline{v}_E &= L_{PE}L_{OP}\underline{v}_{OP} - L_{PE}\tilde{R}_{PE}\underline{\Omega}_P - (\tilde{r}_E + \tilde{u}_E)L_{PE}\underline{\Omega}_P + \dot{\underline{u}}_E \\ \underline{v}_A &= L_{PA}L_{OP}\underline{v}_{OP} - L_{PA}\tilde{R}_{PA}\underline{\Omega}_P - (\tilde{r}_A + \tilde{u}_A)\underline{\Omega}_A + \dot{\underline{u}}_A \end{aligned} \tag{7}$$

3. Kinetic Energy, Potential Energy and Nonconservative Virtual Work

The kinetic energy of any member can be written in the general form

$$T = \frac{1}{2}\int_{m_P} \underline{v}_P^T\underline{v}_P dm_P + \frac{1}{2}\int_{m_R} \underline{v}_R^T\underline{v}_R dm_R + \frac{1}{2}\int_{m_E} \underline{v}_E^T\underline{v}_E dm_E + \frac{1}{2}\int_{m_A} \underline{v}_A^T\underline{v}_A dm_A \tag{8}$$

Using Eqs. (7), the kinetic energy becomes

$$T = \frac{1}{2} m\, v_{OP}^2 + \frac{1}{2} \Omega_P^T J\, \Omega_P + \frac{1}{2} \omega_R^T J_R\, \omega_R + \frac{1}{2} \omega_A^T J_A\, \omega_A$$
$$+ \Omega_P^T L_{PR}^T J_R\, \omega_R + \Omega_P^T L_{PA}^T J_A\, \omega_A + \frac{1}{2} \int_{m_E} \dot{u}_E^T \dot{u}_E\, dm_E$$
$$+ \frac{1}{2} \int_{m_A} \dot{u}_A^T \dot{u}_A\, dm_A - v_{OP}^T L_{OP}^T (m_E \tilde{R}_{PE} + m_A \tilde{R}_{PA}$$
$$+ L_{PE}^T \tilde{r}_{EG} L_{PE} + L_{PA}^T \tilde{r}_{AG} L_{PA}) \Omega_P$$
$$- v_{OP}^T L_{OP}^T L_{PA}^T \tilde{r}_{AG}\, \omega_A + v_{OP}^T L_{OP}^T (L_{PE}^T p_E + L_{PA}^T p_A)$$
$$+ \Omega_P^T (\tilde{R}_{PE}^T L_{PE}^T \tilde{r}_{EG} L_{PE} + \tilde{R}_{PA}^T L_{PA}^T \tilde{r}_{AG} L_{PA}) \Omega_P$$
$$+ \Omega_P^T \tilde{R}_{PA}^T L_{PA}^T \tilde{r}_{AG}\, \omega_A - \Omega_P^T (\tilde{R}_{PE}^T L_{PE}^T p_E + \tilde{R}_{PA}^T L_{PA}^T p_A)$$
$$- \Omega_P^T (L_{PE}^T h_E + L_{PA}^T h_A) - \omega_A^T h_A \qquad (9)$$

where m is the total mass of the spacecraft, v_{OP} is the magnitude of v_{OP}, and J is the total inertia matrix of the entire spacecraft in deformed state about P in terms of components along $x_P y_P z_P$. Moreover

$$J_R = \int_{m_R} \tilde{r}_R^T \tilde{r}_R\, dm_R$$
$$J_A = \int_{m_A} (\tilde{r}_A^T + \tilde{u}_A^T)(\tilde{r}_A + \tilde{u}_A) dm_A \qquad (10a)$$

$$\tilde{r}_{EG} = \int_{m_E} (\tilde{r}_E + \tilde{u}_E) dm_E$$
$$\tilde{r}_{AG} = \int_{m_A} (\tilde{r}_A + \tilde{u}_A) dm_A \qquad (10b)$$

$$p_E = \int_{m_E} \dot{u}_E\, dm_E \ , \quad p_A = \int_{m_A} \dot{u}_A\, dm_A \qquad (10c)$$

$$h_E = \int_{m_E} (\tilde{r}_E^T + \tilde{u}_E^T) \dot{u}_E\, dm_E$$
$$h_A = \int_{m_A} (\tilde{r}_A^T + \tilde{u}_A^T) \dot{u}_A\, dm_A \qquad (10d)$$

Generally, the potential energy is of two types, namely, gravitational and elastic. Because of the high altitude of geosynchronous satellites, the differential gravity effect is negligibly small, so that the potential energy will be assumed to be entirely due to flexibility. We shall express the potential energy in the form

$$V = \int_{D_E} \hat{V}_E\, dD_E + \int_{D_A} \hat{V}_A\, dD_A \qquad (11)$$

where $\hat{V}_E$ and $\hat{V}_A$ are potential energy densities associated with the elastic appendages and D_E and D_A are the domains of extension of these appendages. The densities $\hat{V}_E$ and $\hat{V}_A$ depend on spatial derivatives of the components of the elastic displacement vectors u_A and u_A, respectively. We shall not give explicit expressions for $\hat{V}_E$ and $\hat{V}_A$ at this point, but return to this subject in the next section.

The potential energy can be used to derive the conservative forces acting on the system. In addition, there can be nonconservative forces present. Such forces can arise from various sources such as solar radiation pressure, meteorite impact, etc. Letting f be the nonconservative force vector per unit area at a given point on the surface S of the spacecraft and δR the virtual displacement of that point, the nonconservative virtual work for the entire spacecraft can be written in the form

$$\delta W = \int_{S_P} f_P \cdot \delta R_P dS_P + \int_{S_R} f_R \cdot \delta R_R dS_R$$
$$+ \int_{S_E} f_E \cdot \delta R_E dS_E + \int_{S_A} f_A \cdot \delta R_A dS_A \qquad (12)$$

where δR_P, δR_R, δR_E, and δR_A can be obtained from Eqs. (1). Note that concentrated forces can be treated as distributed by using spatial delta functions.

The kinetic energy, potential energy, and virtual work can be used in conjunction with Lagrange's equations to derive the system equations of motion. This system of equations is of the hybrid type, i.e., some of the equations are ordinary differential equations and the balance are partial differential equations. The first are associated with the rigid body motions of the spacecraft whereas the second are associated with the elastic displacements. It will prove most convenient, however, to work with a completely discrete system, which requires the transformation of the partial differential equations into sets of ordinary differential equations. This will be done in the next section.

4. System Discretization

To eliminate the spatial dependence from the formulation, let us assume that the displacement vectors u_E and u_A can be written in the form

$$u_E = \Phi_E \zeta_E \ , \quad u_A = \Phi_A \zeta_A \qquad (13)$$

where Φ_E and Φ_A are rectangular matrices of space-dependent admissible functions and ζ_E and ζ_A are time-dependent vectors of generalized coordinates. If ζ_E and ζ_A have dimensions n_E and n_A, then Φ_E and Φ_A are $3 \times n_E$ and $3 \times n_A$ matrices, respectively.

Using Eqs. (28), we can write

$$\int_{m_E} \dot{u}_E^T \dot{u}_E\, dm_E = \dot{\zeta}_E^T M_E \dot{\zeta}_E$$
$$\int_{m_A} \dot{u}_A^T \dot{u}_A\, dm_A = \dot{\zeta}_A^T M_A \dot{\zeta}_A \qquad (14)$$

where

$$M_E = \int_{m_E} \phi_E^T \phi_E \, dm_E \, , \; M_A = \int_{m_A} \phi_A^T \phi_A \, dm_A \qquad (15)$$

are symmetric positive definite matrices of order n_E and n_A, respectively. Moreover, the matrices J, J_E, J_A, r_{EG}, and r_{AG} now depend on ζ_E and ζ_A instead of u_E and u_A. Similarly, we have $p_E = p_E(\dot{\zeta}_E)$, $p_A = p_A(\dot{\zeta}_A)$, $h_E = h_E(\zeta_E, \dot{\zeta}_E)$, and $h_A = h_A(\zeta_A, \dot{\zeta}_A)$, so that the kinetic energy, Eq. (9), can be regarded as being entirely free of spatial dependence.

Assuming linear elasticity, the spacecraft potential energy can be written in the discretized form

$$V = \frac{1}{2} \zeta_E^T K_E \zeta_E + \frac{1}{2} \zeta_A^T K_A \zeta_A \qquad (16)$$

where K_E and K_A are symmetric positive definite stiffness matrices of order n_E and n_A, respectively. The matrices K_E and K_A represent integrals over the domain D_E and D_A of functions involving spatial derivatives of Φ_E and Φ_A, respectively.

The nonconservative virtual work, Eq. (12), can be discretized in a similar fashion. We shall not proceed with the discretization process at this time, but defer the question for a later section.

5. Lagrange's Equations of Motion

Let us consider a discrete (or discretized) system and denote by $q(t)$ the configuration vector of the entire system. Then, the system Lagrangian can be written in the general functional form

$$L = T - V = L(\dot{q}, q) \qquad (17)$$

where it was assumed that L does not depend on time explicitly. The system admits equilibria at constant solutions of the equations

$$\partial L/\partial q = 0 \qquad (18)$$

where $\partial L/\partial q$ denotes symbolically a vector with the components $\partial L/\partial q_i$ $(i = 1,2,\ldots,n)$. Without loss of generality, we can assume that the trivial solution $q = 0$ is a solution of Eq. (18). This is so because one can always shift the origin of the configuration space to make it coincide with an equilibrium point.

Expanding about the trivial solution and linearizing, the Lagrangian can be written in the quadratic form

$$L = \frac{1}{2} \dot{q}^T m \dot{q} + q^T f \dot{q} + \frac{1}{2} q^T k q \qquad (19)$$

where m, f, and k are constant square matrices of order n. Moreover, m and k are symmetric. Lagrange's equations of motion can be written in the symbolic form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q \qquad (20)$$

where Q is the n-dimensional generalized force vector, which can be obtained from the virtual work expression

$$\delta W = Q^T \delta q \qquad (21)$$

in which δq is the generalized virtual displacement vector. Introducing Eq. (20) into Eq. (19), we obtain the equations of motion

$$m\ddot{q} + g\dot{q} + kq = Q \qquad (22)$$

where $g = f^T - f$ is a skew symmetric matrix of order n. The solution of Eq. (22) can be obtained in closed form, as shown in the next section.

6. Modal Analysis for the Response

The solution of Eq. (22) can be obtained by the modal analysis of Refs. 5 and 6. To this end, we transform the set of n second-order differential equations, Eqs. (22), into a set of 2n first-order equations, which amounts to working with the state space instead of the configuration space.

Hence, let us define the 2n-dimensional state vector $x(t)$ and the associated force vector $X(t)$ as follows:

$$x(t) = [\dot{q}^T(t) \quad q^T(t)]^T \, , \; X(t) = [Q^T(t) \quad 0^T]^T \qquad (23)$$

where 0 is the n-dimensional null vector. Moreover, let us introduce the 2n x 2n matrices

$$I = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix} \, , \quad G = \begin{bmatrix} g & k \\ -k & 0 \end{bmatrix} \qquad (24)$$

where we note that I is symmetric and G is skew symmetric. This permits us to replace Eq. (22) by

$$I\dot{x}(t) + Gx(t) = X(t) \qquad (25)$$

We shall assume that both m and k are positive definite, so that I is positive definite.

To obtain a closed-form solution of Eq. (25), we first consider the eigenvalue problem

$$\lambda I x + G x = 0 \qquad (26)$$

The fact that I is positive definite guarantees that the eigenvalues are pure imaginary complex conjugates, $\lambda_r = i\omega_r$, $\bar{\lambda}_r = -i\omega_r$, and that the associated eigenvectors are also complex conjugates, $x_r = y_r + iz_r$, $\bar{x}_r = y_r - iz_r$. Instead of working with complex quantities, it is shown in Ref. 5 that the eigenvalue problem (26) can be replaced by the real symmetric eigenvalue problem

$$\omega_r^2 I y_r = K y_r \, , \; \omega_r^2 I z_r = K z_r \, , \; r = 1,2,\ldots,n \qquad (27)$$

where $K = G^T I^{-1} G$ is not only symmetric but also positive definite, because I is positive definite.

The eigenvalues ω_r^2 of the problem (27) have multiplicity two. The corresponding eigenvectors are y_r and z_r. Because I and K are positive definite, the set of 2n eigenvectors y_r and z_r $(r = 1,2,\ldots,n)$ are orthogonal (with respect to the matrix I). They can be normalized so as to satisfy

$$y_r^T I y_s = z_r^T I z_s = \delta_{rs} \, , \; y_r^T I z_s = z_r^T I y_s = 0$$

$$z_s^T G y_r = -y_s^T G z_r = \omega_r \delta_{rs} \, , \; y_s^T G y_r = z_s^T G z_r = 0$$

$$r,s = 1,2,\ldots,n \qquad (28)$$

Because the set of vectors y_r and z_r is orthonormal (with respect to the matrix I), it constitutes a basis in a 2n-dimensional vector space. Hence, the state vector $x(t)$ can be represented as a linear combination of these eigenvectors in the form

$$x(t) = \sum_{r=1}^{n} [\xi_r(t) y_r + \eta_r(t) \quad (29)$$

where $\xi_r(t)$ and $\eta_r(t)$ are generalized coordinates associated with the vectors y_r and z_r, respectively. Introducing Eq. (29) into Eq. (25), multiplying by y_s and z_s in sequence, and using the orthogonality relations (28), we obtain the independent set of pairs of equations

$$\begin{aligned} \dot{\xi}_r(t) - \omega_r \eta_r(t) &= Y_r(t) \\ \dot{\eta}_r(t) + \omega_r \xi_r(t) &= Z_r(t) \end{aligned} \quad r = 1,2,\ldots,n \quad (30)$$

where

$$Y_r(t) = y_r^T X(t) \ , \ Z_r(t) = z_r^T X(t)$$

$$r = 1,2,\ldots,n \quad (31)$$

are generalized forces associated with the generalized coordinates $\xi_r(t)$ and $\eta_r(t)$ respectively.

Equations (30) can be solved for the pair of generalized coordinates $\xi_r(t)$ and $\eta_r(t)$ independently of any other pair. Introducing this solution into Eq. (29), we obtain the complete response

$$\begin{aligned} x(t) = \sum_{r=1}^{n} \{ \int_0^t & [(y_r y_r^T + z_r z_r^T) X(\tau) \cos \omega_r(t-\tau) \\ & + (y_r z_r^T - z_r y_r^T) X(\tau) \sin \omega_r(t-\tau)] d\tau \\ & + (y_r y_r^T + z_r z_r^T) I x(0) \cos \omega_r t \\ & + (y_r z_r^T - z_r y_r^T) I x(0) \sin \omega_r t \} \end{aligned} \quad (32)$$

where $x(0)$ is the initial state vector.

The decoupling procedure can be written in matrix form. To this end, let us introduce the 2n-dimensional generalized coordinate vector

$$w = [\xi_1 \ \eta_1 \ \xi_2 \ \eta_2 \ \cdots \ \xi_n \ \eta_n]^T \quad (33)$$

as well as the corresponding modal matrix

$$P = [y_1 \ z_1 \ y_2 \ z_2 \ \cdots \ y_n \ z_n] \quad (34)$$

so that Eq. (29) can be written in the compact form

$$x = Pw \quad (35)$$

Moreover, the orthogonality relations can be combined into

$$P^T I P = 1_{2n} \quad (36)$$

where 1_{2n} is the unit matrix of order 2n. Premultiplying both sides of Eq. (35) by $P^T I$ and using Eq. (36), we conclude that

$$w = P^T I x \quad (37)$$

Next, let us define a block diagonal matrix Λ as follows:

$$\Lambda = \begin{bmatrix} \omega_1 i & 0 & --- & 0 \\ 0 & \omega_2 i & --- & 0 \\ --- & --- & --- & --- \\ 0 & 0 & --- & \omega_n i \end{bmatrix}, \ i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (38)$$

With this notation, Eqs. (30) and (31) can be written in the compact form

$$\dot{w} = \Lambda w + \ldots \quad (39)$$

7. Response of Uncontrolled System to Impulsive Excitation

Let us consider the case in which system (25) is subjected to an impulsive force at time 0+ while in equilibrium, $x(0) = 0$. The force vector can be written in the form

$$X(t) = \hat{X} \delta(t) \quad (40)$$

where $\hat{X}$ is the magnitude of the impulsive force. Introducing Eq. (40) into solution (49), we obtain

$$\begin{aligned} x(t) &= \sum_{r=1}^{n} \int_0^t [(y_r y_r^T + z_r z_r^T) \hat{X} \delta(\tau) \cos \omega_r(t-\tau) \\ & \quad + (y_r z_r^T - z_r y_r^T) \hat{X} \delta(\tau) \sin \omega_r(t-\tau)] d\tau \\ &= \sum_{r=1}^{n} [(y_r y_r^T + z_r z_r^T) \hat{X} \cos \omega_r t \\ & \quad + (y_r z_r^T - z_r y_r^T) \hat{X} \sin \omega_r t] \end{aligned} \quad (41)$$

which shows that the motion of the system consists of a superposition of harmonic motions at the natural frequencies ω_r.

The impulsive force $\hat{X}\delta(t)$ can be shown to cause a motion analogous to that caused by an initial excitation. Indeed, let us introduce Eq. (40) into Eq. (41) and write

$$I\dot{x}(t) + Gx(t) = \hat{X}\delta(t) \quad (42)$$

Letting the duration of the impulse be Δt and integrating Eq. (42) with respect to time, we obtain

$$I \int_0^{\Delta t} \dot{x}(t) dt + G \int_0^{\Delta t} x(t) dt = \hat{X} \int_0^{\Delta t} \delta(t) dt = \hat{X} \quad (43)$$

For small Δt, the second integral on the left side of Eq. (43) is negligible, so that

$$\lim_{\Delta t \to 0} I \int_0^{\Delta t} \dot{x}(t) dt = I[x(0+) - x(0)] = Ix(0+) = \hat{X} \quad (44)$$

From which we conclude that the impulsive force produces the equivalent initial excitation

$$\underline{x}(0+) = I^{-1}\hat{\underline{X}} \tag{45}$$

It can be easily verified that if the initial excitation (45) is inserted into Eq. (32) instead of the force (40) the result would be the same.

8. System with Proportional Control

Let us assume that the system under consideration was subjected to an impulsive force resulting in oscillation according to Eq. (41). Let us further assume that the response exceeds a given amplitude. Because the oscillation persists without attenuation, the response must be considered unsatisfactory, so that it is deemed necessary to attenuate it by means of active controls. In this section, we shall use proportional controls and in the next we shall use on-off controls.

Denoting by $\underline{U}$ the control vector, the system differential equations of motion can be written in the form

$$I\dot{\underline{x}}(t) + G\underline{x}(t) = \underline{U}(t) \tag{46}$$

which is subject to the initial conditions $\underline{x}(0)$. Using the approach of Sec. 6, we can reduce the simultaneous set (46) to the independent set

$$\begin{aligned} \dot{\xi}_r(t) - \omega_r y_r(t) &= Y_r(t) \\ \dot{y}_r(t) + \omega_r \xi_r(t) &= Z_r(t) \end{aligned} \qquad r = 1,2,\ldots,n \tag{47}$$

where

$$Y_r(t) = \underline{y}_r^T\underline{U}(t)\ ,\ Z_r(t) = \underline{z}_r^T\underline{U}(t),\ r = 1,2,\ldots,n \tag{48}$$

Next, let us assume proportional controls

$$Y_r(t) = -c\xi_r(t)\quad ,\quad Z_r(t) = -c\eta_r(t)\quad ,\qquad r = 1,2,\ldots,n \tag{49}$$

so that Eqs. (47) can be rewritten in the form

$$\begin{aligned} \dot{\xi}_r(t) - \omega_r\eta_r(t) + c\xi_r(t) &= 0 \\ \dot{\eta}_r(t) + \omega_r\xi_r(t) + c\eta_r(t) &= 0 \end{aligned} \qquad r = 1,2,\ldots,n \tag{50}$$

The solution of Eqs. (50), obtained by the Laplace transform method, is

$$\begin{aligned} \xi_r(t) &= e^{-ct}[\xi_r(0)\cos\omega_r t + \eta_r(0)\sin\omega_r t] \\ \eta_r(t) &= e^{-ct}[-\xi_r(0)\sin\omega_r t + \eta_r(0)\cos\omega_r t] \end{aligned} \qquad r = 1,2,\ldots,n \tag{51}$$

so that the response dies out with time.

The question remains as to how the proportional control on the decoupled coordinates $\xi_r(t)$ and $\eta_r(t)$ relate to that of the state vector $\underline{x}(t)$. To this end, we multiply Eq. (29) by $\underline{y}_r^T I$ and $\underline{z}_r^T I$, in sequence, consider Eqs. (28) and obtain

$$\xi_r(t) = \underline{y}_r^T I\underline{x}(t),\ \eta_r(t) = \underline{z}_r^T I\underline{x}(t),\ r = 1,2,\ldots,n \tag{52}$$

Combining Eqs. (48), (49), and (52), we can write

$$\begin{aligned} Y_r(t) &= -c\xi_r(t) = -c\underline{y}_r^T I\underline{x}(t) = \underline{y}_r^T\underline{U}(t) \\ Z_r(t) &= -c\eta_r(t) = -c\underline{z}_r^T I\underline{x}(t) = \underline{z}_r^T\underline{U}(t) \end{aligned} \qquad r = 1,2,\ldots,n \tag{53}$$

from which we conclude that

$$\underline{U}(t) = -cI\underline{x}(t) \tag{54}$$

or, the control vector $\underline{U}(t)$ is proportional to the vector $I\underline{x}(t)$.

9. System with On-Off Control

Proportional control has one drawback, namely, it must operate continuously. A scheme without this drawback is on-off control. The control law assumes a region of deadband based on the recognition that within some tolerance small oscillations are acceptable.

Let us consider once again the system (46) and the decoupling procedure (47) and (48), but, by contrast, we assume a control in the form

$$U = \sum_{s=1}^{n} \frac{1}{\omega_s} I\underline{z}_s u_s \tag{55}$$

where u_s is a nonlinear function of η_s to be specified shortly. Introducing Eq. (55) into Eqs. (48) and considering the orthonormality relations (28), we conclude that

$$\begin{aligned} Y_r(t) &= \sum_{s=1}^{n} \frac{1}{\omega_s} \underline{y}_r^T I\underline{z}_s u_s = 0 \\ Z_r(t) &= \sum_{s=1}^{n} \frac{1}{\omega_s} \underline{z}_r^T I\underline{z}_s u_s = \frac{1}{\omega_r} u_r \end{aligned} \tag{56}$$

so that Eqs. (47) reduce to

$$\begin{aligned} \dot{\xi}_r(t) - \omega_r\eta_r(t) &= 0 \\ \dot{\eta}_r(t) + \omega_r\xi_r(t) - \frac{1}{\omega_r}u_r &= 0 \end{aligned} \qquad r = 1,2,\ldots,n \tag{57}$$

Next, let us specify that the function u_r is given explicitly by

$$u_r = \begin{cases} -k_r\ , & \eta_r > d_r \\ 0\ , & -d_r \le \eta_r \le d_r \\ k_r\ , & \eta_r < -d_r \end{cases} \tag{58}$$

Hence, the solution of Eqs. (57) must be obtained separately for the three intervals indicated above:

i. For $|\eta_r| \le d_r$, $u_r = 0$, Eqs. (57) reduce to

$$\begin{aligned} \dot{\xi}_r(t) - \omega_r\eta_r(t) &= 0 \\ \dot{\eta}_r(t) + \omega_r\xi_r(t) &= 0 \end{aligned} \qquad r = 1,2,\ldots,n \tag{59}$$

which can be shown to have the solution

$$\xi_r(t) = \xi_r(0)\cos\omega_r t + \eta_r(0)\sin\omega_r t$$

$$\eta_r(t) = -\xi_r(0)\sin\omega_r t + \eta_r(0)\cos\omega_r t \qquad r = 1,2,\ldots,n \tag{60}$$

ii. For $\eta_r > d_r$, $u_r = -k_r$, Eqs. (57) become

$$\dot{\xi}_r(t) - \omega_r\eta_r(t) = 0$$
$$\dot{\eta}_r(t) + \omega_r\xi_r(t) + \frac{k_r}{\omega_r} = 0 \qquad r = 1,2,\ldots,n \tag{61}$$

which have the solution

$$\xi_r(t) = \frac{k_r}{\omega_r^2} + \left[\xi_r(0) - \frac{k_r}{\omega_r^2}\right]\cos\omega_r t + \eta_r(0)\sin\omega_r t$$

$$\eta_r(t) = -\left[\xi_r(0) + \frac{k_r}{\omega_r^2}\right]\sin\omega_r t + \eta_r(0)\cos\omega_r t \qquad r = 1,2,\ldots,n \tag{62}$$

iii. For $\eta_r < -d_r$, $u_r = k_r$ the response is obtained by simply replacing $-k_r$ by $+k_r$ in Eqs. (62). Hence,

$$\xi_r(t) = \frac{k_r}{\omega_r^2} + \left[\xi_r(0) - \frac{k_r}{\omega_r^2}\right]\cos\omega_r t + \eta_r(0)\sin\omega_r t$$

$$\eta_r(t) = -\left[\xi_r(0) - \frac{k_r}{\omega_r^2}\right]\sin\omega_r t + \eta_r(0)\cos\omega_r t \qquad r = 1,2,\ldots,n \tag{63}$$

The behavior of the solution, Eqs. (60), (62), and (63), can be discussed most conveniently in the phase plane η_r vs. ξ_r. From Eqs. (60), we conclude that if $\sqrt{\xi_r^2(0) + \eta_r^2(0)} < d_r$, then the trajectories represent circles with the centers at the origin and with radii $\sqrt{\xi_r^2(0) + \eta_r^2(0)}$. If the motion is initiated in the region $\eta_r > d_r$, then from Eqs. (62) we conclude that the trajectories are circles centered at $\xi_r = -k_r/\omega_r^2$ and with radii $\sqrt{[\xi_r(0) + k_r/\omega^2]^2 + \eta_r^2(0)}$. On the other hand, from Eqs. (63) we conclude that the trajectories initiated in the region $\eta_r < d_r$ are circles centered at $\xi_r = k_r/\omega_r^2$ and with radii $\sqrt{[\xi_r(0) - k_r/\omega^2]^2 + \eta_r^2(0)}$. A given motion initiated at some point $\eta_r(0) > d_r$, $\xi_r(0) < 0$ will follow a circular trajectory until it reaches the horizontal line $\eta_r = d_r$, when the control is removed. If at this point $\xi_r < 0$, then the trajectory will tend to move clockwise on a circle centered at the origin, which will take the motion back into the region $\eta_r > d_r$ causing the control to be actuated again. Repetition of this motion pattern results in chattering along the line $\eta_r = d_r$. If the trajectory initiated at $\eta_r(0) > d_r$ hits the line $\eta_r = d_r$ at a point for which $\xi_r > 0$, then the motion will continue on a circle with the center at the origin until it reaches $\eta_r = -d_r$, so that now chattering occurs along the line $\eta_r = -d_r$. Figure 2 shows these trajectories along with some other possible cases.

To prevent chattering, one may wish to delay the time so that the controls are removed when the trajectories are inside the deadband interval. This is equivalent to introducing a phase angle in the solutions (62) and (63). Figure 2 shows in dashed line the system behavior for a time delay corresponding to a phase lag of 10°.

The on-off control with deadband and time delay corresponding to a decoupled mode is illustrated in the block diagram shown in Fig. 3.

It should be pointed out that the above analysis, including the phase plane representation of the motion, would not have been possible without the decoupling procedure.

10. Reconstruction of the State Vector from Available Outputs

Regarding the control as an external excitation and using the analogy with Eq. (41), the equations of motion of a controlled spacecraft subjected to external excitation can be written in the form

$$I\dot{x} + Gx = X + U \tag{64}$$

Unlike the force vector X, however, which depends on time alone, the control vector U depends specifically on the state variables. In Sec. 9, we studied two cases, namely, that in which U is a linear function of the state vector and that in which U is a nonlinear function. Because now the input to the system is $X + U$ instead of X, the decoupled equations of motion can be written in the symbolic form

$$\dot{w} = \Lambda w + P^T(X + U) \tag{65}$$

The above control is predicated upon the knowledge of the state vector $x(t)$. Quite often, however, the state vector is not completely known, so that a method for its estimation is highly desirable. Such a method uses another dynamical system known as an observer.

The discussion of the observer can be conveniently presented in terms of the uncoupled system. Let $\dot{w}_m$ denote a vector of measurements corresponding to the time derivative of the uncoupled state vector w. The object is to construct an observer capable of yielding a good estimate of the state vector. Such an observer should be a dynamical system resembling the dynamical system (65) and should depend both on the input $X + U$ and the measurement $\dot{w}_m$. In general the measurements need not be the complete set of states. The observer approach can easily be extended to include the effect of measurement errors thereby resulting in a filter approach. Hence, let us consider an observer described by the following vector differential equation

$$\dot{\hat{w}} = \Lambda_O\hat{w} + B_O\dot{w}_m + N_O P^T(X + U) \tag{66}$$

where $\hat{w}$ denotes the observer-constructed state vector and Λ_O, B_O, and N_O are block-diagonal matrices to be determined so that the observer exhibits the desired behavior. Moreover, the measurements vector $\dot{w}_m$ is related to the state vector w by

$$\dot{w}_m = C_O\dot{w} \tag{67}$$

where C_0 is generally a rectangular matrix which depends on the characteristics of the measuring devices. Introducing Eqs. (65) and (67) into Eq. (66) and subtracting the result from Eq. (65), we obtain

$$\dot{\underline{w}} - \dot{\hat{\underline{w}}} = (1 - B_0C_0)\Lambda\underline{w} - \Lambda_0\hat{\underline{w}} + (1 - C_0B_0 - N_0)P^T(\underline{X} + \underline{U}) \quad (68)$$

where 1 is the identity matrix. Letting the matrices Λ_0, B_0, and N_0 satisfy the equations

$$(1 - C_0B_0)\Lambda = \Lambda_0 \quad , \quad 1 - C_0B_0 = N_0 \quad (69)$$

Eq. (68) reduces to

$$\dot{\underline{w}} - \dot{\hat{\underline{w}}} = \Lambda_0(\underline{w} - \hat{\underline{w}}) \quad (70)$$

The observer dynamics is simulated in Fig. 4.

Next, let us introduce the notation

$$\underline{\varepsilon}(t) = \underline{w}(t) - \hat{\underline{w}} \quad (71)$$

where $\underline{\varepsilon}(t)$ is known as the estimation error, i.e., the difference between the actual state vector and the reconstructed state vector. Introducing Eq. (71) into Eq. (70), we obtain

$$\dot{\underline{\varepsilon}}(t) = \Lambda_0\underline{\varepsilon}(t) \quad (72)$$

so that if the eigenvalues of Λ_0 have negative real parts, the error decays with time. There are no apparent restrictions on the eigenvalues of Λ_0 until measurement errors are taken into account. Typically, the eigenvalues of Λ_0 would be "faster" than the system eigenvalues by a factor of 5 to 10.

Generally, one chooses the matrix B_0 so that the matrix Λ_0 has the desired eigenvalues. In particular, B_0 is chosen as a block diagonal matrix. We recall that the matrix Λ is itself block-diagonal, as can be seen from Eq. (38).

Next, let us assume that some components of the state vector $\underline{x}(t)$ have been measured by means of on-board sensors, such as rate gyros, accelerometers, etc. Denoting the measured state vector rate by $\dot{\underline{x}}_m(t)$, where

$$\dot{\underline{x}}_m(t) = [\ddot{\underline{q}}_m^T(t) \quad \dot{\underline{q}}_m^T(t)]^T \quad (73)$$

we can use Eq. (37) and write

$$\dot{\underline{w}}_m(t) = P^TI\dot{\underline{x}}_m(t) \quad (74)$$

Assuming for simplicity that C_0 is the identity matrix, which implies complete observability, and introducing Eqs. (69) and (74) into Eq. (66), we obtain the observer equation

$$\dot{\hat{\underline{w}}} = \Lambda_0\hat{\underline{w}} + B_0P^TI\dot{\underline{x}}_m + (1 - B_0)P^T(\underline{X} + \underline{U}) \quad (75)$$

The matrix B_0 can be chosen so that the matrix Λ_0 is the diagonal, i.e.,

$$\Lambda_0 = \text{diag}[\alpha_1 \; \beta_1 \; \dots \; \alpha_r \; \beta_r \; \dots \; \alpha_n \; \beta_n] \quad (76)$$

It follows immediately that

$$1 - B_0 = \Lambda_0\Lambda^{-1} = \text{block-diag}\begin{bmatrix} 0 & -\alpha_r/\omega_r \\ \beta_r/\omega_r & 0 \end{bmatrix} \quad (77a)$$

$$B_0 = \text{block-diag}\begin{bmatrix} 1 & \alpha_r/\omega_r \\ -\beta_r/\omega_r & 1 \end{bmatrix} \quad (77b)$$

Introducing the notation

$$\hat{\underline{Q}} = B_0P^TI\dot{\underline{x}}_m + (1 - B_0)P^T(\underline{X} + \underline{U}) \quad (78)$$

where

$$\hat{\underline{Q}} = [\hat{Q}_{\xi 1} \; \hat{Q}_{\eta 1} \; \dots \; \hat{Q}_{\xi r} \; \hat{Q}_{\eta r} \; \dots \; \hat{Q}_{\xi n} \; \hat{Q}_{\eta n}]^T \quad (79)$$

and considering Eq. (76), Eq. (75) can be written in the form

$$\dot{\hat{\xi}}_r = \alpha_r\hat{\xi}_r + \hat{Q}_{\xi r}, \quad \dot{\hat{\eta}}_r = \beta_r\hat{\eta}_r + \hat{Q}_{\eta r}, \quad r = 1,2,\dots,n \quad (80)$$

which has the general solution

$$\hat{\xi}_r(t) = \hat{\xi}_r(0)e^{\alpha_r t} + \int_0^t e^{\alpha_r(t-\tau)}\hat{Q}_{\xi r}(\tau)d\tau$$

$$\hat{\eta}_r(t) = \hat{\eta}_r(0)e^{\beta_r t} + \int_0^t e^{\beta_r(t-\tau)}\hat{Q}_{\eta r}(\tau)d\tau$$

$$r = 1,2,\dots,n \quad (81)$$

Choosing α_r and β_r $(r = 1,2,\dots,n)$ as complex numbers with negative real parts, we conclude from Eq. (72) that the error $\underline{\varepsilon}(t)$ reduces to zero with time, so that the decoupled observer state vector $\hat{\underline{w}}$ can be used to determine the behavior of the decoupled system state vector $\underline{w}$. Note that to obtain the actual observer state vector we can write

$$\hat{\underline{x}}(t) = P\hat{\underline{w}}(t) \quad (82)$$

The vector $\hat{Q}$ consists of one part due to the measured state vector and external disturbances another part due to the control

$$\hat{\underline{Q}} = \hat{\underline{Q}}_x + \hat{\underline{Q}}_u \quad (83)$$

where

$$\hat{\underline{Q}}_x = B_0P^TI\dot{\underline{x}}_m + (1 - B_0)P^T\underline{X}, \quad \hat{\underline{Q}}_u = (1 - B_0)P^T\underline{U} \quad (84)$$

Hence, the classical separation principle is illustrated in that the observer can be used to actually control the system. If $\hat{Q}_u$ is taken to simulate the proportional control of Sec. 8 or the on-off control of Sec. 9, then the actual control vector is obtained from the second of Eqs. (84) in the form

$$\underline{U} = IP(1 - B_0)^{-1}\hat{\underline{Q}}_u \quad (85)$$

where

$$(1 - B_0)^{-1} = \text{block-diag}\begin{bmatrix} 0 & \omega_r/\beta_r \\ -\omega_r/\alpha_r & 0 \end{bmatrix} \quad (86)$$

11. Control of a Spinning Flexible Spacecraft

Before proceeding with the derivation of Lagrange's equations of motion it will prove convenient to identify the system generalized coordinates. Assuming that the satellite moves in a circular orbit around the earth and that the center of mass of the satellite coincides with the center of mass P of the platform for all practical purposes, the vector v_{OP} can be considered as being known. Moreover, we shall assume that there are no members rotating relative to the platform and that there are two elastic panels attached to the platform. These panels are symmetric with respect to the point P and their motion is antisymmetric. It follows that all quantities with subscripts R and A can be ignored in the kinetic energy Eq. (9). In addition, we have

$$m_E \tilde{R}_{PE} = \tilde{r}_{EG} = 0 \quad , \quad \underline{p}_E = 0 \tag{87}$$

Considering the above assumptions, as well as Eqs. (14), and ignoring constant terms, Eqs. (9) reduces to

$$T = \frac{1}{2}\, \underline{\Omega}_P^T\, J\, \underline{\Omega}_P + \frac{1}{2}\, \dot{\underline{\zeta}}_E^T M_E\, \dot{\underline{\zeta}}_E - \underline{\Omega}_P^T L_{PE}^T\, \underline{h}_E \tag{88}$$

Next, let us assume that in equilibrium the platform axes $x_P y_P z_P$ rotate relative to the inertial space XYZ with the uniform angular velocity Ω about z_P, where z_P is parallel to Z. Then, if we consider a set of auxiliary axes $x_{P'} y_{P'} z_{P'}$ rotating relative to the inertial space with the angular velocity Ω about $z_{P'}$, where $z_{P'}$ is parallel to Z, and if we denote the angular velocity of axes $x_P y_P z_P$ relative to $x_{P'} y_{P'} z_{P'}$ by ω_P, then the angular velocity of the platform axes $x_P y_P z_P$ relative to the inertial space can be written in the form

$$\underline{\Omega}_P = \Omega \underline{\ell} + \underline{\omega}_P \tag{89}$$

where $\underline{\ell}$ is the vector of direction cosines between axis $z_{P'}$ and axes $x_P y_P z_P$. Assuming that axes $x_P y_P z_P$ are obtained from axes $x_{P'} y_{P'} z_{P'}$ by means of three rotations θ_2 about y_P, θ_1 about x_P, and θ_3 about z_P, in that order, then

$$\underline{\ell} = [s\theta_1 s\theta_3 - c\theta_1 s\theta_2 c\theta_3 \quad s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 \quad c\theta_1 c\theta_2]^T \tag{90}$$

and

$$\underline{\omega}_P = \begin{bmatrix} c\theta_3 & c\theta_1 s\theta_3 & 0 \\ -s\theta_3 & c\theta_1 c\theta_3 & 0 \\ 0 & -s\theta_1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{91}$$

where $s\theta_i = \sin\theta_i$, $c\theta_i = \cos\theta_i$ $(i = 1,2,3)$.

Representing the elastic displacements of the panels as follows:

$$u_E = 0 \;,\; v_E = \phi_1 \zeta_1 \;,\; w_E = \phi_2 \zeta_2 + \phi_3 \zeta_3 \tag{92}$$

where ϕ_1 is the first in-plane mode, ϕ_2 is the first our-of-plane mode, and ϕ_3 is the first torsional mode about x_E, introducing Eqs. (89)-(92) into Eq. (88), and linearizing, we obtain

$$\begin{aligned} T = {} & \frac{1}{2}\Omega^2[(B - C)\theta_1^2 + (A - C)\theta_2^2 - 2a\theta_1\zeta_3 + 2b\theta_2\zeta_2 \\ & + m_1\zeta_1^2] + \Omega[-A\dot{\theta}_1\theta_2 + (B - C)\dot{\theta}_2\theta_1 - b\dot{\theta}_1\zeta_2 \\ & - a\dot{\theta}_3\zeta_3 - b\theta_1\dot{\zeta}_2] + \frac{1}{2}(A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + C\dot{\theta}_3^2) + \frac{1}{2}(m_1\dot{\zeta}_1^2 \\ & + m_2\dot{\zeta}_2^2 + m_3\dot{\zeta}_3^2) - b\dot{\theta}_2\dot{\zeta}_2 + a\dot{\theta}_3\dot{\zeta}_1 + a\dot{\theta}_1\dot{\zeta}_3 \end{aligned} \tag{93}$$

where A, B, C are the moments of inertia of the entire undeformed spacecraft about axes x_P, y_P, z_P, respectively, and

$$\begin{aligned} a &= \int_{m_E} x_E \phi_1 dm_E \;,\; b = \int_{m_E} x_E \phi_2 dm_E \;, \\ c &= \int_{m_E} y_E \phi_3 dm_E \;,\; m_i = \int_{m_E} \phi_i^2 dm_E \;,\; i = 1,2,3 \end{aligned} \tag{94}$$

In terms of the modes indicated by Eqs. (92), the potential energy is

$$V = \frac{1}{2} \sum_{i=1}^{3} m_i \Lambda_i^2 \zeta_i^2 \tag{95}$$

where Λ_i $(i = 1,2,3)$ are the natural frequencies of the panel associated with the modes ϕ_i $(i = 1, 2,3)$.

The nonconservative virtual work can be written in terms of the generalized forces and virtual displacements as follows:

$$\delta W = \sum_{i=1}^{3} (F_{\theta i}\, \delta\theta_i + F_{\zeta i}\, \delta\zeta_i) \tag{96}$$

where $F_{\theta i}$ and $F_{\zeta i}$ $(i = 1,2,3)$ are the generalized forces.

Lagrange's equations of motion can be written in the general form

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} &= F_{\theta i} \\ & \qquad\qquad i = 1,2,3 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\zeta}_i}\right) - \frac{\partial L}{\partial \zeta_i} &= F_{\zeta i} \end{aligned} \tag{97}$$

From the third of Eqs. (97), we observe that if $F_{\theta 3} = 0$, then θ_3 is an ignorable coordinate, so that

$$\frac{\partial L}{\partial \dot{\theta}_3} = C\dot{\theta}_3 + a\dot{\zeta}_1 = \beta = \text{const} \tag{98}$$

Hence, introducing

$$\dot{\theta}_3 = \frac{\beta - a\dot{\zeta}_1}{C} \tag{99}$$

into the kinetic energy, we obtain

$$\begin{aligned} T = {} & \frac{1}{2}\Omega^2[(B - C)\theta_1^2 + (A - C)\theta_2^2 - 2c\theta_1\zeta_3 + 2b\theta_2\zeta_2 \\ & + m_1\zeta_1^2] + \Omega[-A\dot{\theta}_1\theta_2 + (B - C)\dot{\theta}_2\theta_1 - b(\dot{\theta}_1\zeta_2 + \theta_1\dot{\zeta}_2) \\ & - c\dot{\theta}_2\zeta_3] + \frac{1}{2}(A\dot{\theta}_1^2 + B\dot{\theta}_2^2) + \frac{1}{2}\,[(m_1 - \frac{a^2}{2C})\dot{\zeta}_1^2 \end{aligned}$$

$$+ m_2\dot{\zeta}_2^2 + m_3\dot{\zeta}_3^2] - b\dot{\theta}_2\dot{\zeta}_2 + e\dot{\theta}_1\dot{\zeta}_3 \quad (100)$$

Next, let us introduce the configuration vector

$$q = [\theta_1\ \theta_2\ \zeta_1\ \zeta_2\ \zeta_3]^T \quad (101)$$

and the associated nonconservative force vector

$$Q = [F_{\theta 1}\ F_{\theta 2}\ F_{\zeta 1}\ F_{\zeta 2}\ F_{\zeta 3}]^T \quad (102)$$

Then Lagrange's equations, Eq. (20), assumes the form (22), in which

$$m = \begin{bmatrix} A & 0 & 0 & 0 & e \\ 0 & B & 0 & -b & 0 \\ 0 & 0 & m_1 - \frac{a^2}{2C} & 0 & 0 \\ 0 & -b & 0 & m_2 & 0 \\ e & 0 & 0 & 0 & m_3 \end{bmatrix}$$

$$g = \Omega \begin{bmatrix} 0 & C-A-B & 0 & 0 & 0 \\ -(C-A-B) & 0 & 0 & 0 & -e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \end{bmatrix} \quad (103)$$

$$k = \Omega^2 \begin{bmatrix} C-B & 0 & 0 & 0 & a \\ 0 & C-A & 0 & -b & 0 \\ 0 & 0 & m_1(\Lambda_1^{*2}-1) & 0 & 0 \\ 0 & -b & 0 & m_2\Lambda_2^{*2} & 0 \\ e & 0 & 0 & 0 & m_3\Lambda_3^{*2} \end{bmatrix}$$

where $\Lambda_i^* = \Lambda_i/\Omega$ $(i = 1,2,3)$.

We observe that the equation for the coordinate ζ_1 is independent of the other ones, so that its solution can be obtained independently. For simplicity of computer programming, however, we choose not to treat ζ_1 separately but as part of the formulation (103).

The solution of Eq. (22) with and without controls and with m, g, and k as given by Eqs. (103) follows the pattern established in Secs. 6-10.

The above formulation was used to determine the response of a spacecraft with the following parameters:

$$A = 1{,}000 \text{ kg m}^2, \quad B = 6{,}000 \text{ kg m}^2$$
$$C = 8{,}000 \text{ kg m}^2, \quad \Omega = 0.6 \text{ rad s}^{-1}$$

The panels were modelled by the finite element method. The first natural frequencies for in-plane, out-of-plane, and torsional vibration are

$$\Lambda_1 = 0.0647 \text{ rad s}^{-1}, \ \Lambda_2 = 0.1743 \text{ rad s}^{-1},$$
$$\Lambda_3 = 0.0227 \text{ rad s}^{-1}$$

Simulations were made of the system response for: 1) uncontrolled spacecraft, 2) proportional control, 3) on-off control with deadband. Figures 4a, 4b, 5a, 5b, and 6a, 6b show typical computer plots for the nutation angle θ_1 and the out-of-plane mode ζ_2 for the three cases, respectively. The initial conditions and the various control parameters were:

$$\theta_1(0) = \theta_2(0) = 10^{-4} \text{ rad}, \ \zeta_1(0) = \zeta_2(0) = \zeta_3(0) = 10^{-4} \text{ m}, \quad c = 0.1 \text{ s}^{-1}$$

$$d_1 = d_2 = 10^{-3}, \ d_3 = d_4 = d_5 = 10^{-4}$$
$$k_1 = 2 \times 10^{-3}\ \omega_1^2 \text{ s}^{-2}, \ k_2 = 2 \times 10^{-3}\ \omega_2^2 \text{ s}^{-2}$$
$$k_3 = 10^{-4}\ \omega_3^2 \text{ s}^{-2}, \ k_4 = 10^{-4}\ \omega_4^2 \text{ s}^{-2},$$
$$k_5 = 10^{-4}\ \omega_5^2 \text{ s}^{-2}$$

The inclusion of Figs. 4, 5, 6 is merely to show typical results. The results are not meant to represent optimal control. Indeed, the on-off control results could be greatly improved by a reduction of the dead band constants.

12. Conclusions

This paper develops a modal procedure for the control of a flexible spacecraft exhibiting gyroscopic behavior. Control via decoupling has distinct computational advantages over control of the coupled system, particularly for large order systems, as it permits the use of methods of solution generally associated with second-order systems. Design procedures are demonstrated for two types of control algorithms, proportional control and on-off control with dead band.

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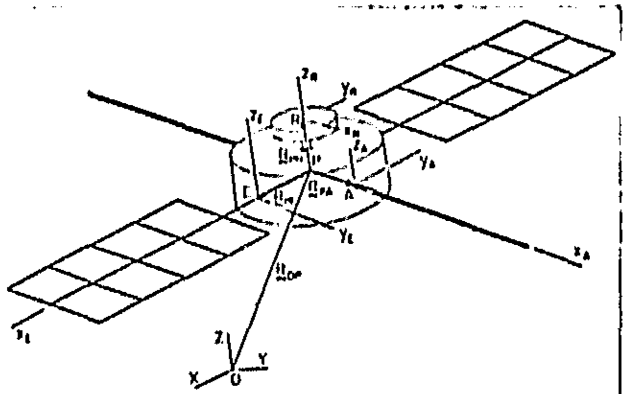

Fig. 1 The Flexible Spacecraft

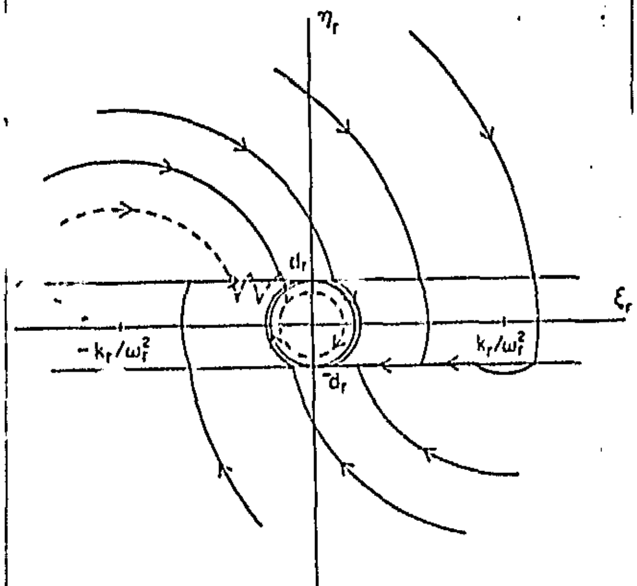

Fig. 2 Typical Response for an Uncoupled Single-Mode Nonlinear Control

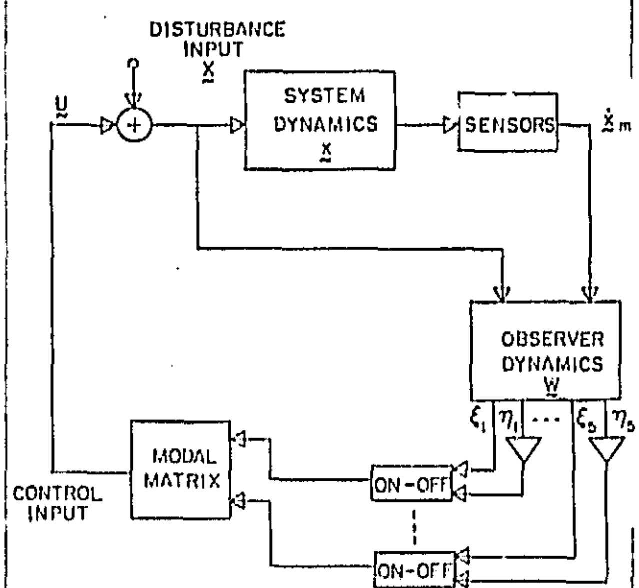

Fig. 3 General Control System Structure

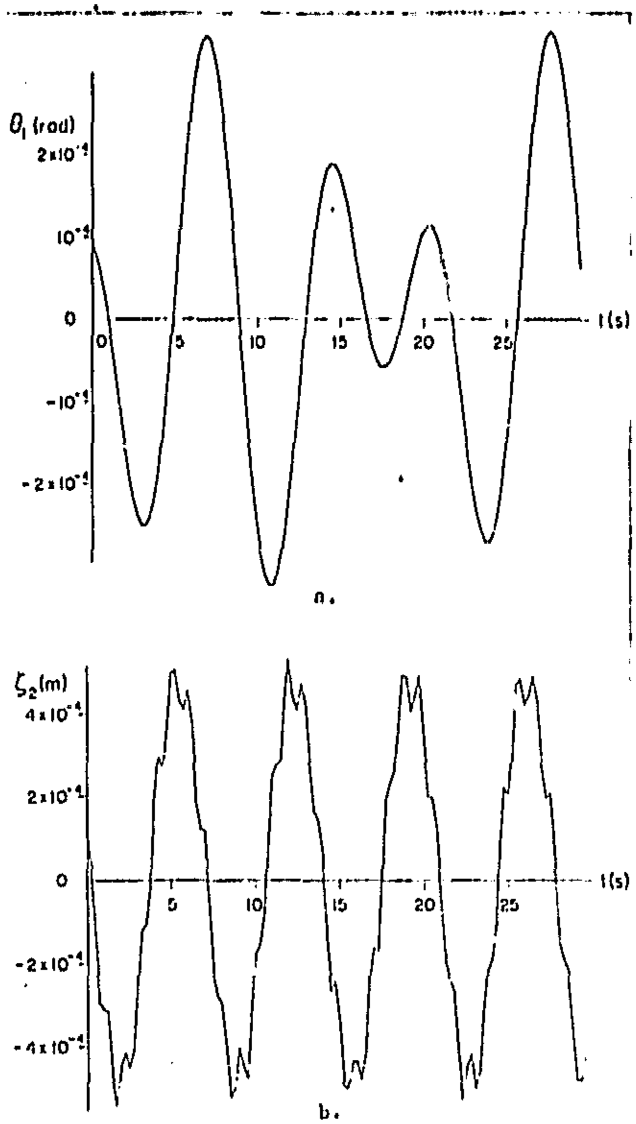

Fig. 4 Uncontrolled System Response

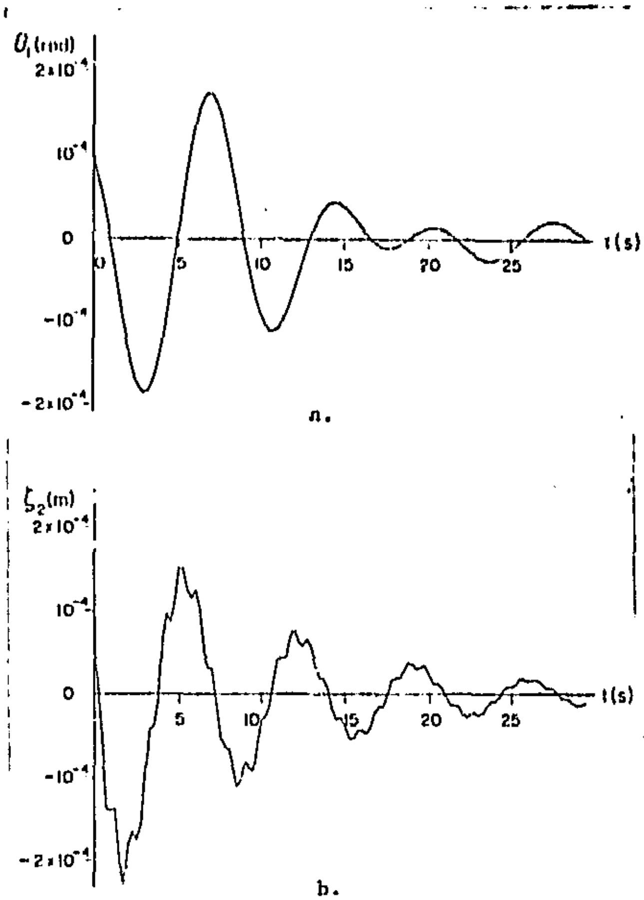

Fig. 5 System Response with Proportional Control

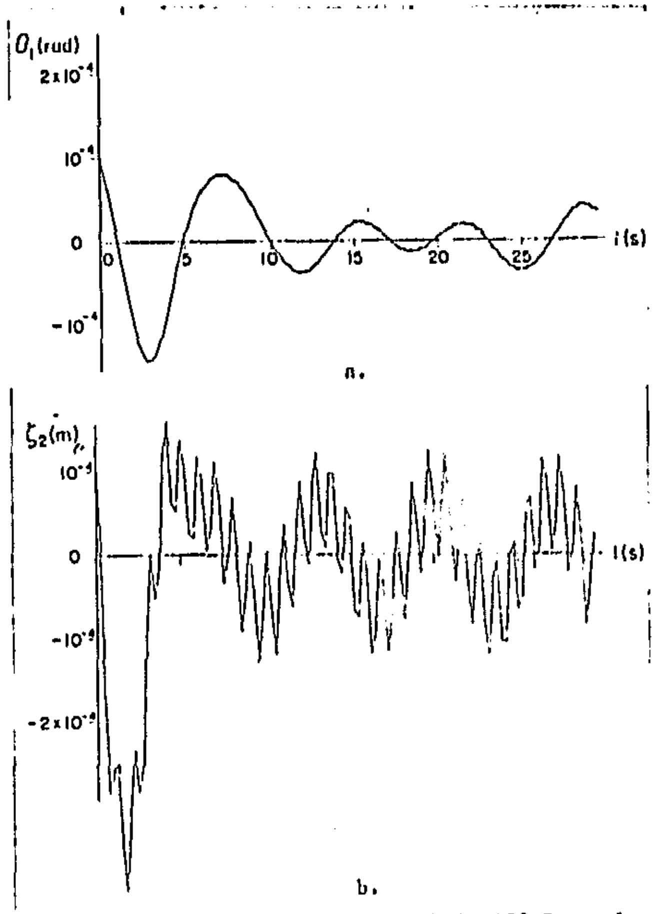

Fig. 6 System Response with On-Off Control