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NASA TECHNICAL MEMORANDUM

NASA TM X-73323

SOLUTION TO THE DIFFERENTIAL EQUATION FOR COMBINED RADIATIVE AND CONVECTIVE COOLING FOR A HEATED SPHERE

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June 1, 1976

NASA

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama

(NASA-TM-X-73323) SOLUTION TO THE DIFFERENTIAL EQUATION FOR COMBINED RADIATIVE AND CONVECTIVE COOLING FOR A HEATED SPHERE (NASA) 11 p HC $3.50 CSCL 20M
N76-28515
Unclas
G3/34 47624

TECHNICAL REPORT STANDARD TITLE PAGE

1. REPORT NO.	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
NASA TM X-73323		

4. TITLE AND SUBTITLE

Solution to the Differential Equation for Combined Radiative and Convective Cooling for a Heated Sphere

5. REPORT DATE

June 1, 1976

6. PERFORMING ORGANIZATION CODE

7. AUTHOR(S)

Fred D. Wills and Lester Katz

8. PERFORMING ORGANIZATION REPORT #

9. PERFORMING ORGANIZATION NAME AND ADDRESS

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama 35812

10. WORK UNIT NO.

11. CONTRACT OR GRANT NO.

12. SPONSORING AGENCY NAME AND ADDRESS

National Aeronautics and Space Administration
Washington, D.C. 20546

13. TYPE OF REPORT & PERIOD COVERED

Technical Memorandum

14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Prepared by Space Sciences Laboratory, Science and Engineering

16. ABSTRACT

A solution is presented for the differential equation relating the combined effects of radiative and forced convective cooling for a heated sphere. The equation has the form

$$\frac{dT}{dt} = -K_o(T^4 - T_o^4) - H(T - T_o) \quad ,$$

where T and t are the variables temperature and time, respectively, and K_o, T_o, and H are constants. The solution can be used as a guideline for the design and understanding of space processing phenomena.

17. KEY WORDS

18. DISTRIBUTION STATEMENT

Unclassified — Unlimited

Fred D Wills

19. SECURITY CLASSIF. (of this report)	20. SECURITY CLASSIF. (of this page)	21. NO. OF PAGES	22. PRICE
Unclassified	Unclassified	12	NTIS

LIST OF SYMBOLS

Symbol	Definition
A	surface area of spherical drop
C_1, C_2	quantities defined by equation (5)
c	specific heat coefficient
H	hA/mc
H_o	H/K_o
h	total convective heat transfer coefficient
i	$\sqrt{-1}$
K_o	$\epsilon A\sigma/mc$
k_1, k_2	quantities defined by equation (6)
m	mass
r_1, r_2, r_3	roots of equation (3)
T	temperature
t	time
ϵ	emissivity constant
σ	Boltzmann constant

Subscripts

i	initial
o	ambient, referring to temperature

TECHNICAL MEMORANDUM X-73323

SOLUTION TO THE DIFFERENTIAL EQUATION FOR COMBINED RADIATIVE AND CONVECTIVE COOLING FOR A HEATED SPHERE

INTRODUCTION

The simulation of space processing (i.e., low gravity) cooling of metal droplets in free fall leads to a differential equation of heat transfer that has apparently not been solved in closed form [1]. Consider a small heated sphere or droplet falling vertically through a large diameter tube in which it is subjected to simultaneous cooling by radiation to the tube walls and forced convection to a gaseous medium. It is required that the time to reach a specific (cooler) temperature be explicitly determined. This will be done by solving the differential equation for time as a function of temperature.

SOLUTION

Let T denote the temperature of the sphere at any time t. The energy balance relating the heat transfer from the droplet by radiation and convection is expressed by

$$mc\ \frac{dT}{dt} = -\epsilon A\sigma(T^4 - T_o^{\ 4}) - hA(T - T_o) \tag{1a}$$

or

$$\frac{dT}{dt} = -K_o(T^4 - T_o^{\ 4}) - H(T - T_o) \quad . \tag{1b}$$

The first term on the right expresses the Stefan-Boltzmann law of thermal radiation; the second is the forced convection term. It is assumed that the coefficients do not vary during the short time of fall involved in the drop tube, which, in effect, constrains the problem to the class: slow cooling. Thus, $K_o = \epsilon A\sigma / mc$ and $H = hA/mc$. In addition, the droplet is considered so small that any thermal lag between its center and surface may be neglected.

The differential equation may be written in integral form as

$$t = -\frac{1}{K_o} \int \frac{dT}{(T - T_o)(T^3 + T_o T^2 + T_o{}^2 T + T_o{}^3 + H_o)} + \text{const.} \qquad (2)$$

where $H_o = H/K_o$. Consider the cubic equation which may be formed from the cubic polynomial in the denominator

$$T^3 + T_o T^2 + T_o{}^2 T + T_o{}^3 + H_o = 0 \quad . \qquad (3)$$

The three roots of this equation are:

$$r_1 = -\frac{T_o}{3}(1 + C_1)$$

$$r_2 = \frac{T_o}{3}\left[\left(\frac{C_1}{2} - 1\right) + i\sqrt{3}\left(\frac{C_2}{2}\right)\right] \qquad (4)$$

$$r_3 = \frac{T_o}{3}\left[\left(\frac{C_1}{2} - 1\right) - i\sqrt{3}\left(\frac{C_2}{2}\right)\right] \quad ,$$

with

$$C_1 = \sqrt[3]{\frac{k_1 + k_2}{2}} + \sqrt[3]{\frac{k_1 - k_2}{2}}$$

$$C_2 = \sqrt[3]{\frac{k_1 + k_2}{2}} - \sqrt[3]{\frac{k_1 - k_2}{2}} \qquad (5)$$

and

$$k_1 = 20 + 27 \frac{H_o}{T_o^3}$$

$$k_2 = \sqrt{(k_1)^2 + 32} \quad . \tag{6}$$

The single fraction of equation (2) may now be broken into four fractions using the quantities r_1, r_2, and r_3:

$$t = -\frac{1}{K_o}\left(\int \frac{\mathcal{A}\, dT}{T - T_o} + \int \frac{B\, dT}{T - r_1} + \int \frac{C\, dT}{T - r_2} + \int \frac{D\, dT}{T - r_3}\right) + \text{const.} \tag{7}$$

where $\mathcal{A}$, B, C, and D are constants that are functions of T_o, r_1, r_2, and r_3. Integration of equation (7) yields

$$t = -\frac{1}{K_o}\left[\mathcal{A}\, \ell n\,(T - T_o) + B\, \ell n\,(T - r_1) + C\, \ell n\,(T - r_2) + D\, \ell n\,(T - r_3)\right] + \text{const.} \tag{8}$$

Using standard techniques from the theory of rational fractions, the constants $\mathcal{A}$, B, C, and D are evaluated:

$$\mathcal{A} = \frac{1}{(T_o - r_1)(T_o - r_2)(T_o - r_3)}$$

$$B = \frac{1}{(r_1 - T_o)(r_1 - r_2)(r_1 - r_3)}$$

$$C = \frac{1}{(r_2 - T_o)(r_2 - r_1)(r_2 - r_3)}$$

$$D = \frac{1}{(r_3 - T_o)(r_3 - r_1)(r_3 - r_2)} \quad . \tag{9}$$

Substituting the system, equation (9), into equation (8) yields

$$t = \frac{1}{K_o(T_o - r_1)(T_o - r_2)(T_o - r_3)} \left[-\ln (T - T_o) + \frac{(T_o - r_2)(T_o - r_3)}{(r_1 - r_2)(r_1 - r_3)} \ln (T - r_1) + \frac{(T_o - r_1)(T_o - r_3)}{(r_2 - r_1)(r_2 - r_3)} \ln (T - r_2) + \frac{(T_o - r_1)(T_o - r_2)}{(r_3 - r_1)(r_3 - r_2)} \ln (T - r_3) \right] + \text{const.} \quad . \tag{10}$$

Substituting the roots, equation (4), for the coefficients in equation (10) results in

$$(T_o - r_1)(T_o - r_2)(T_o - r_3) = 4(T_o)^3 + H_o \tag{11}$$

$$\frac{(T_o - r_2)(T_o - r_3)}{(r_1 - r_2)(r_1 - r_3)} = \frac{(C_1)^2 - 4C_1 + 22}{3[(C_1)^2 + 2]} \tag{12}$$

$$\frac{(T_o - r_1)(T_o - r_3)}{(r_2 - r_1)(r_2 - r_3)} = \frac{(C_1 + 4)[C_2(C_1 - 2) - 2\sqrt{3}\, i(C_1 + 1)]}{3C_2[(C_1)^2 + 2]} \tag{13}$$

$$\frac{(T_o - r_1)(T_o - r_2)}{(r_3 - r_1)(r_3 - r_2)} = \frac{(C_1 + 4)[C_2(C_1 - 2) + 2\sqrt{3}\,i(C_1 + 1)]}{3C_2[(C_1)^2 + 2]} \quad . \qquad (14)$$

The roots r_2 and r_3 are complex conjugates, as are the coefficents of equations (13) and (14). From complex number theory, the following formula will be used to evaluate the last two terms of equation (10) which have conjugate coefficients and arguments, respectively:

$$(u + iv)\,\ell n\,(z + iy) + (u - iv)\,\ell n\,(z - iy) = u\,\ell n\,(z^2 + y^2) - 2v\,\tan^{-1}\frac{y}{z} \quad . \qquad (15)$$

The conjugate roots are used to form

$$T - r_2 = \left[T - \frac{T_o}{3}\left(\frac{C_1}{2} - 1\right)\right] - i\,\frac{T_o C_2}{2\sqrt{3}} \qquad (16)$$

$$T - r_3 = \left[T - \frac{T_o}{3}\left(\frac{C_1}{2} - 1\right)\right] + i\,\frac{T_o C_2}{2\sqrt{3}} \quad . \qquad (17)$$

Using equations (13) through (17), the following identifications of real and imaginary components are made:

$$u = \frac{(C_1 + 4)(C_1 - 2)}{3[(C_1)^2 + 2]}$$

$$v = \frac{-2(C_1 + 1)(C_1 + 4)}{\sqrt{3}\,C_2[(C_1)^2 + 2]}$$

$$z = T - \frac{T_o}{3}\left(\frac{C_1}{2} - 1\right) \qquad (18)$$

$$y = \frac{-T_o C_2}{2\sqrt{3}} \quad .$$

Since the identifications, equation (18), and equation (15) fit the form of the last two terms of equation (10), proper substitution of the results of equation (11) through (18) yields for equation (10):

$$t = \frac{1}{K_o[4(T_o)^3 + H_o]} \Bigg[- \ln (T - T_o) + \left(\frac{[(C_1)^2 - 4C_1 + 22]}{3[(C_1)^2 + 2]}\right) \ln \left[T + \frac{T_o}{3}(C_1 + 1)\right] + \left(\frac{(C_1 + 4)(C_1 - 2)}{3[(C_1)^2 + 2]}\right) \ln \left\{ \left[T - \frac{T_o}{3}\left(\frac{C_1}{2} - 1\right)\right]^2 + \frac{(T_o)^2(C_2)^2}{12} \right\} + \left(\frac{4(C_1 + 1)(C_1 + 4)}{\sqrt{3}\, C_2[(C_1)^2 + 2]}\right) \tan^{-1}\left(\frac{T - \frac{T_o}{3}\left(\frac{C_1}{2} - 1\right)}{\frac{T_o C_2}{2\sqrt{3}}} \right) \Bigg] + \text{const.} \qquad (19)$$

This is the desired solution for the cooling time required to reach a specific temperature, T. The constant is evaluated in the usual way by using the initial temperature T_i at $t = 0$, where, in general, $T_i \geq T > T_o$.

For the case of radiative cooling only, the forced convection constant $H \equiv 0$, so that $C_1 = 2$ and $C_2 = \pm 2\sqrt{3}$, and equation (19) yields

$$t = \frac{1}{4K_o T_o^3} \left(\ln \frac{T + T_o}{T - T_o} + 2 \tan^{-1} \frac{T}{T_o} \right) + \text{const.} \qquad (20)$$

This is the same solution that the integration of equation (1b) gives with $H = 0$, thus affording a check of the solution, equation (19), for a simpler well-known case. However, if in the solution, equation (19), we allow $T_o \rightarrow 0$, then the time for cooling is given by a limiting process as

$$t = \frac{1}{3H} \ln \left(K_o + \frac{H}{T^3} \right) + \text{const.} \qquad (21)$$

This same result is obtained if at the outset in equation (1b) we set $T_o \equiv 0$, forming the Bernoulli [2] differential equation for this case whose solution also yields equation (21). This affords another check of the solution.

CONCLUSIONS

The solution to this differential equation will be applied as a guideline in the physics and engineering analysis of the cooling of small metal droplets after solidification from molten metal as they fall freely down a tube. Such a free fall will simulate for a few seconds the process of forming such droplets in a space laboratory. There are perhaps other physical processes in which the solution to such a differential equation could be useful.

REFERENCES

1. Vinzens, K.: Cooling of Geometrically Simple Bodies / Flat Plate, Cylinder, Sphere / by Convection and Radiation. Dr.-Ing. Dissertation, Technische Universität, Berlin, 1971, p. 26.

2. Kamke, E.: Differentialgleichungen, Lösungsmethoden und Lösungen. Akademische Verlagsgesellschaft, Geest & Portig, K. G., Leipzig, 1959, p. 303.

APPROVAL

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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