General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)

Final Report

National Aeronautics and Space Administration, Research Center,

Hampton, Langley, Va., Grant NSG 1140, Computer Automation and

Feedback System Design, June 30, 1976.

SUMMARY

Previous reports described a completely automatic computer program for the

design of single input-output, single-loop feedback systems with parameter un-

certainty, to satisfy time-domain bounds on the system response to step commands

and disturbances. The inputs to the program are basically the specified time-

domain response bounds, the form of the constrained plant transfer function and

the ranges of the uncertain parameters of the plant. The program output consists

of the transfer functions of the two free compensation networks, in the form of

the coefficients of the numerator and denominator polynomials, and the data

on the prescribed bounds and the extremes actually obtained for the system

response to commands and disturbances. This program was delivered and demon-

strated to NASA personnel with reports describing the computer program. This

report is therefore devoted to additional work since done.

1. Explicit solution for optimum $L(j\omega)$.

At some point in the program, there are prepared bounds $\psi(\omega)$ in the complex

plane on the frequency response of the loop transmission function $L(j\omega)$ -- see

Fig. 1, i.e. $L(j\omega)$ must lie outside the region described by $\psi(\omega)$, in order to

satisfy the response specifications over the range of plant uncertainty. The

computer then seeks to find a rational function $L(j\omega)$ which satisfies these bounds, using as a vital aid that the optimum $L(j\omega)$ lies on its bound $\psi(\omega)$ at each ω. The technique used has been described in detail in previous reports, but basically a primitive form of $L(j\omega)$ is first used which satisfies the $\psi(\omega)$ bounds and then local corrections are made in order to push $L(j\omega)$ towards the $\psi(\omega)$, with iterations continued until the global 'error' (sum of distances from the $\psi(\omega)$), is less than a prescribed amount. This is one of the most difficult parts of the program.

A different approach was attempted, wherein equations have been developed which actually give $L(j\omega)$ explicitly as functions of the bounds $\psi(\omega)$. As expected, the equations are integral equations, so in the general case an iteration process is necessary to solve them. In the special case when the bounds $\psi(\omega)$ are straight lines, no iteration is necessary. Hence, one approach is to first use present techniques to obtain a reasonable first approximation to $L_{opt}(j\omega)$, and then represent $\psi(\omega)$ as straight lines in the intervals where the approximate $L_{opt}(j\omega)$ crosses the $\psi(\omega)$. As of this date, there has not been acquired sufficient experience to justify incorporation of either of these techniques into the automatic computer program. The analytic background and development is given in Appendix 1.

2. Nonminimum-phase (nmp) systems.

The computer program delivered is confined to minimum-phase (mp) systems, i.e. for all possible plant parameter combinations over their range of uncertainty, the resulting plant transfer function $P(s)$ has no right half-plane zeros (rhpz). Then almost any set of performance specifications over almost any bounded parameter

uncertainty set (see (1) for limitations), are achievable. However, if $P(s)$ has a rhpz, say at b then $L(j\omega) = P(j\omega) G(j\omega)$ cannot achieve a crossover frequency ω_c (defined by $|L(j\omega_c)| = 1$) significantly more than $b/2$ (see (1)). Sensitivity reduction is achieved only for $\omega < \omega_c/2$ approximately, so the adaptive capabilities of nmp systems are severely limited. Given an uncertain plant which is nmp for one or more parameter combinations and specific performance specifications, it is generally impossible (at least at the present state of the subject) to know apriori whether the problem is solvable, i.e. whether the performance specifications are achievable. One may therefore have to proceed with the synthesis details and be stopped (if the problem is not solvable), when he tries to find $L_{opt}(j\omega)$ -- it does not exist. It is then necessary to relax the performance specifications in order to make the problem solvable.

The objective was to incorporate nmp systems into the automatic design program. However, it was first necessary to solve the problem analytically and thoroughly understand the detailed steps in solving it, if done by hand or interactively with the computer -- especially the reasoning involved in relaxing the performance specifications the minimum amount, should the attempted ones prove not solvable. Also, the proof that was used to derive the properties of the optimum $L(j\omega)$ in mp systems, was not valid for nmp systems. It was necessary to first establish on a rigorous basis the theoretical foundations of nmp systems. This proved to be a first class research problem in itself, as can be seen from the results achieved, described in Appendix 2.

3. Translation of time-domain bounds into equivalent frequency-domain bounds

The synthesis technique operates in the frequency (ω) domain, whereas performance bounds are often given in the time domain. The present program, therefore, has a routine for translating the latter into ω-domain bounds, but which does not work satisfactorily for higher order systems, where the effective 'time-delay' (say the time $t_{.1}$, for the unit step response to reach 0.1) is relatively large. The problem of translation from time to frequency is also more difficult in non-minimum-phase systems. Considerable effort was invested in obtaining a good understanding of the time-frequency relations, in order to make significant improvements in this translation problem. The results are given in Appendix 3.

APPENDIX 1

Explicit Solution for Optimum $L(j\omega)$

There is given a boundary $\psi(\omega)$ in the complex plane at each ω, such that the loop transmission $L(j\omega)$ must lie on or outside $\psi(\omega)$. A typical family of $\psi(\omega)$ in the logarithmic complex plane (Nichols chart NC) where the abscissa B represents $\text{Arg } L(j\omega)$ and A the ordinate represents $|L(j\omega)|$ in logarithmic units, is shown in Fig. 1. The reasons for the portions labelled B_h, Γ in Fig. 1, have been given in detail in Reference 2. A certain excess e of poles over zeros is $\underline{a\ priori}$ assigned to $L(s)$, so that as $s \to \infty$, $L(s) \to \dfrac{k}{s^e}$. The objective is to find $L(j\omega)$ which satisfies the boundary constraints with a minimum value of k. Reference 2 details the reasons for this definition and the $\underline{a\ priori}$ assumption of a value for e.

The $\psi(\omega)$ are of two types. The one at $\omega = .5$ in Fig. 1 is an example of the first which is single-valued for all B and exists for B from -360° to 0°, whereas ψ_h is an example of the second which is multiple-valued and generally exists for only part of B. It is assumed here that each ψ has a continuous derivative at each point. Any sharp corners of ψ can always be rounded with infinitessimal effect on ψ. Consider a segment of ψ in Fig. 2. At each point on ψ a tangent line T_o can be drawn as shown at point P for example, whose equation is

$$A \cos \theta - B \sin \theta = C_o \tag{1}$$

θ and C_o will, in general, be functions of the point on ψ, i.e. of the value of B being considered. The intercepts of this line are: on the B axis, $B = -C_o/\sin \theta$ and on the A axis, $A = C_o/\cos \theta$. Suppose that the

acceptable ln $L(j\omega)$ at $B = B_p$ must be on or above B so that point P_1 is acceptable. Draw a line T_1 parallel to T_0 , through P_1 . The equation of this line is

$$A \cos \theta - B \sin \theta = C \tag{2}$$

with $C \geq C_o$. Let ln $L(j\omega) = A(\omega) + jB(\omega)$, and suppose the boundary segment shown in Fig. 2 is for $\omega = \omega_1$. The condition that at $B = B_p$, $A(\omega_1)$ must lie above P , is expressed by writing

$$A(B_p,\omega_1) \cos \theta(B_p,\omega_1) - B_p(\omega_1) \sin \theta(B_p,\omega_1) = C > C_o(B_p,\omega_1) \tag{3}$$

Note that C_o is a function both of B and of ω as is θ . The permissible A is consequently also a function of these two, explaining (3).

Consider the situation at point Q on $\psi(\omega_1)$ in Fig. 2. We want θ to be a continuous function of B , so clearly $-\frac{\pi}{2} < \theta(B_Q) < 0$ and $C_o(B_Q) > 0$, in order that the A intercept is positive. The condition that $L(j\omega_1)$ lies on or above Q if $B = $ Im ln $L(j\omega_1) = B_Q$, is the same as (3) if P is replaced by Q , i.e. it is $C \geq C_o(B_Q,\omega_1)$. It is found that this is the condition at all points on $\psi(\omega_1)$; also, it makes no difference if all or some of the tangent lines have A intercepts which are negative. Thus, it is noted that for ψ of the first type, (i.e., $|\theta| < \pi/2$, $\cos \theta > 0$), for all B values the acceptable condition is $C \geq C_o$.

Consider a boundary of type 2 with the acceptable region for $L(j\omega)$ outside ψ , as in Fig. 3. If we start at point P_1 , define its $\theta_1 > 0$ and move clockwise then the previous paragraph shows that from P_1 to P_2 , the condition $C \geq C_o$ applies. At P_2 , $\theta_2 = -\pi/2$ so Eq. (1) becomes $B = C_o$. The acceptable condition is hence again $C \geq C_o$. At P_3 , continuity of θ gives, $-\pi < \theta_3 <$

$-\pi/2$, $\cos \theta_3 < 0$ and the intercept of the tangent line being $C_0/\cos \theta_3$, the condition that L lie on or below P_3 again requires $C \geq C_0$ because $C/\cos \theta_3$ $< C_0/\cos \theta_3$, if $C > C_0$ and $\cos \theta_3 < 0$. At P_4 , continuity of θ gives $-1.5\pi < \theta_4 < -\pi$, $\cos \theta_4 < 0$, so again $C \geq C_0$ is needed so that L lies on or below P_4 . Thus, if θ and C_0 are properly defined as functions of B (and, of course, of ω) the condition that L lies on or outside a closed boundary is given by $C \geq C_0$.

The problem of choosing an optimum $L(j\omega)$ can therefore be stated as follows. There is given an infinite set of equations, one for each ω in $[0,\infty)$

$$A \cos \theta (\omega, B) - B \sin \theta (\omega, B) = C_0 (\omega, B) \quad . \tag{4}$$

θ, C_0 are continuous functions of ω, B . Find $\ln L(j\omega) \triangleq A(\omega) + jB(\omega)$, regular in the right half-plane and the boundary (rrhp), such that when the latter A,B are substituted into (4), they satisfy the equation with $C \geq C_0$. Given that $L(s)$ has a specified excess e of poles over zeros, so that $L(s) \to k_\ell/s^e$ as $s \to \infty$, the optimum L is that with minimum k_ℓ .

The requirement that $\ln L(j\omega) = A(\omega) + jB(\omega)$ satisfy (4) with C_0 replaced by $C \geq C_0$, may be put in the form

$$\mathcal{R}_{Le} \; e^{j\theta}(A + jB) \; = C \tag{5}$$

The general complex equation then has the form $A + jB = e^{-j\theta}(C + jD)$. One difficulty is that $\ln e^{-j\theta(\omega)}$ is not rrhp, because $e^{-j\theta}$ is all-pass. We therefore deliberately introduce $e^{\sigma(\omega)}$ such that $e^{\sigma+j\theta}$ and its ln are rrhp. This is done by choosing $\sigma(\omega)$ as the Hilbert image of $\theta(\omega)$, using the Bode integral [1]

$$\sigma(\omega) - \sigma(0) = \frac{2\omega^2}{\pi} \int_0^\infty \frac{\theta(y)dy}{y(\omega^2 - y^2)} \tag{6}$$

The desired equation is now

$$e^{\sigma+j\theta} (A + jB) = (C + jD) e^\sigma \triangleq \hat{C} + j\hat{D} \tag{7}$$

whose real part is Eq. (5) multiplied by e^σ. In order to ensure that $A + jB$ is rrhp as required, $\hat{D}$ must be the Hilbert image of $\hat{C} = C e^\sigma$, i.e. using the Bode integral [1],

$$D(\omega) \; e^{\sigma(\omega)} = \frac{-2\omega}{\pi} \int_0^\infty \frac{C(y) \; e^{\sigma(y)}}{\omega^2 - y^2} dy \tag{8}$$

Then (7) can be written as:

$$A(\omega) = C(\omega) \cos \theta(\omega) - \frac{2\omega e^{-\sigma(\omega)} \sin \theta(\omega)}{\pi} \int_0^\infty \frac{e^{\sigma(y)} C(y)dy}{\omega^2 - y^2} \tag{9a}$$

$$B(\omega) = -C(\omega) \sin \theta(\omega) - \frac{2\omega e^{-\sigma(\omega)} \cos \theta(\omega)}{\pi} \int_0^\infty \frac{e^{\sigma(y)} C(y)dy}{\omega^2 - y^2} \tag{9b}$$

It is understood that in the general case A, C, θ, etc. are functions of B as well. Recall that the objective is to minimize k_ℓ where $\lim_{s \to \infty} L(s) = k_\ell/s^e$.

Hence, $\lim_{\omega \to \infty} \ln|L(j\omega)| = \lim_{\omega \to \infty} A(\omega) = \ln k_\ell - e \ln \omega$ and $\ln k_\ell = \lim_{\omega \to \infty}[e \ln \omega + A(\omega)]$.

Minimization of k_ℓ means minimization of $\lim_{\omega \to \infty} A(\omega)$. In Fig. 1, by recalling the evaluation of θ by means of the continuity of θ, it is seen that $\lim_{\omega \to \infty} \theta(\omega) = -\pi/2$, so the first term on the right side of (9a) disappears and the

second term becomes

$$2\omega e^{-\sigma(\omega)} \int_0^\infty \frac{e^{\sigma(y)} \ C(y) \ dy}{\omega^2 - y^2} \tag{10}$$

whose limit as $\omega \to \infty$ is to be minimized. Since every factor in (10) is positive (recall $\sigma(y)$ is real) except possibly for the free $C(y)$, it is clear that a necessary condition for minimization is to make $C(y)$ as small as possible. But $C(y) \geq C_0(y)$. Hence, choose $C(y) = C_0(y)$. This proves that the optimum $L(j\omega)$ (if it exists) lies on its boundary $\psi(\omega)$ at each ω. It has previously been proven [2] that if any solution exists at all to the problem, then an optimum must also exist. If the plant transfer function set is minimum-phase over its entire range of uncertainty then it is known [3] that a solution exists.

Application to Numerical Work

The simplest application of the above under consideration for numerical work, is to first obtain an approximation to L_{opt} by means of the computer program previously delivered to NASA. This approximation gives one a good idea of the region on each $\psi(\omega)$, where L_{opt} crosses. A straight line approximation is then made of this region. Then $\theta(\omega)$ and $C(\omega) = C_0(\omega)$ in (9a,b) are known as functions of ω, and (6) can be used to evaluate $\sigma(\omega) - \sigma(0)$. Note in (9a,b) that the constant $\sigma(0)$ cancels out and need not be known, so it can be set at zero. On the vertical line of the 'universal high-frequency boundary' ψ_h in Fig. 1, $C = B$ a constant, corresponding to the value of the constant phase. It is clear from Eqs. (9a,b) that the straight line approximation is

the simplest to use because no iteration is necessary as C_0, θ are then not functions of B . Thus, A and B can be calculated independently.

Experimentation with this approach has not as yet been completed. In numerical examples where the $\psi(\omega)$ are well approximated over a significant interval near the point where $L_{APPROX}(j\omega)$ intersects $\psi(\omega)$, the results have been quite good, permitting quick convergence to the optimum L . However, it must be noted that the final result gives $\ln L(j\omega) = A(\omega) + jB(\omega)$ in the form of numerical data, not as a rational function. It is necessary to use the RFA (rational function approximation) subroutine of the computer program package previously delivered to NASA, in order to secure L(s) as a rational function approximation of the numerical data. In numerical examples where the straight line approximation is valid only over a narrow interval in the region where $L_{APPROX}(j\omega)$ intersects $\psi(\omega)$, the results have been poorer, as could be expected. Application of (9a,b) still gives a clear-cut answer, but on some $\psi(\omega)$ the results put the point on a part of the straight line where it is no longer a reasonably accurate representation of $\psi(\omega)$. One must then take a compromise straight line in between, i.e. iterate. If so, one might as well incorporate the iteration directly into (9a,b) by using a second-order approximation of $\psi(\omega)$ in the region of interest, at least at those ω values where it appears necessary. Then θ and $C = C_0$ in (9a,b) must be written as functions of B as well as of ω . There has not been sufficient time, as yet, to thoroughly evaluate these alternatives and incorporate them into the computer program. However, we are confident that this approach will significantly improve the program for finding the optimum L , and intend to continue working on this phase. The improved program will be made available to NASA when it is ready.

REFERENCES

1. I. Horowitz, Synthesis of Feedback Systems, Academic Press, New York, 1963.

2. I. Horowitz, Optimum loop transfer function in single-loop minimum-phase feedback systems, Int. J. Control, 18, 1973, pp. 97-113.

3. I. Horowitz, Synthesis of feedback systems with nonlinear time-varying uncertain plants to satisfy quantitative performance specifications, IEEE Proceedings, Jan. 1976, pp. 123-130.
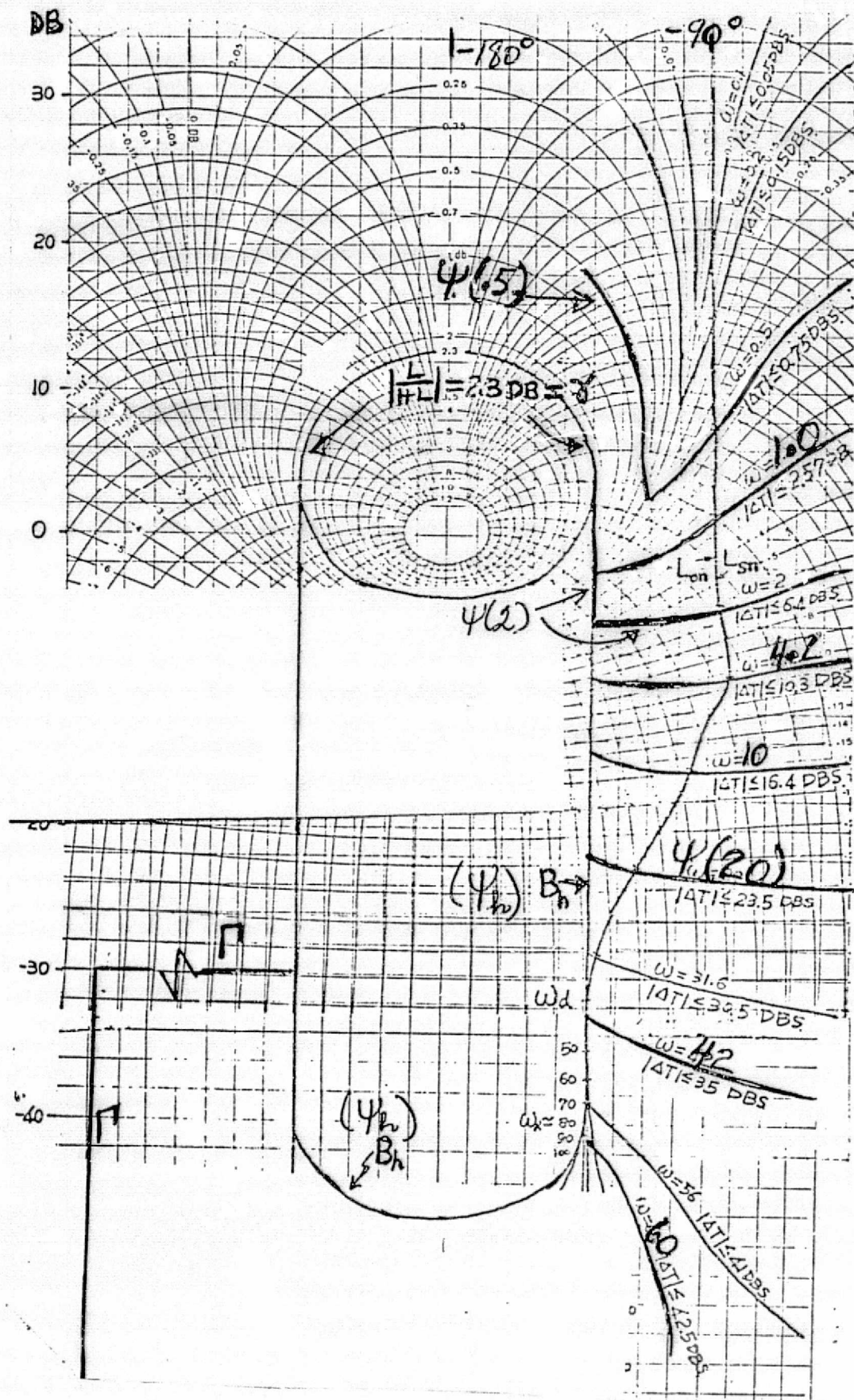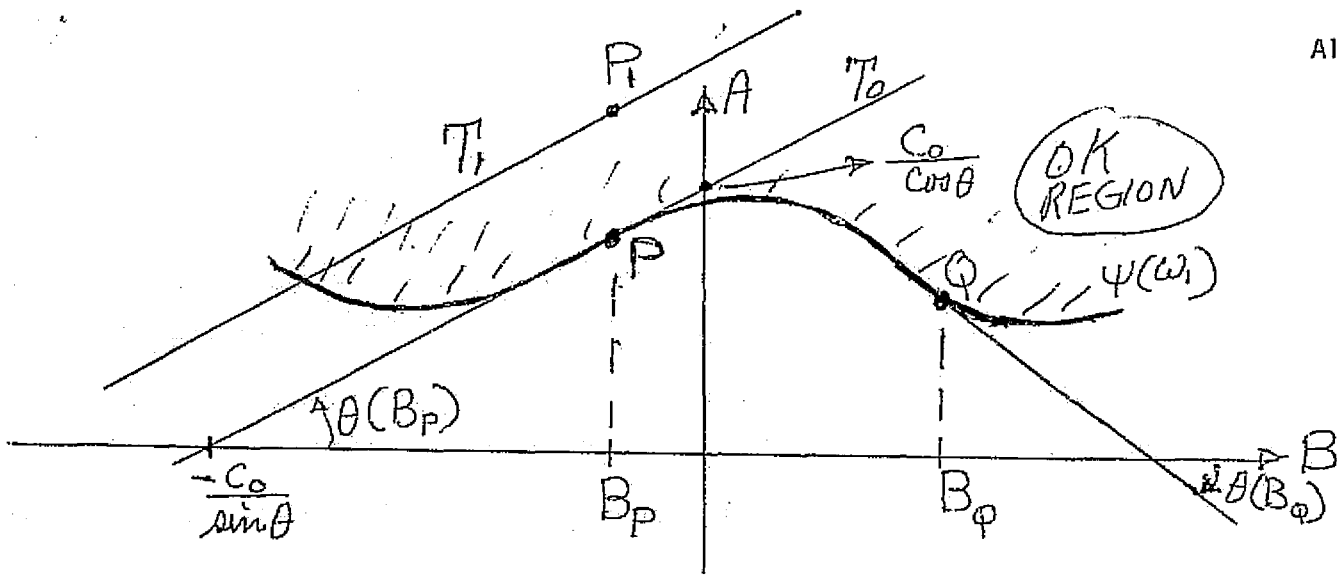
Fig. A1-1. Typical boundaries $\psi(\omega)$ on Nichols chart

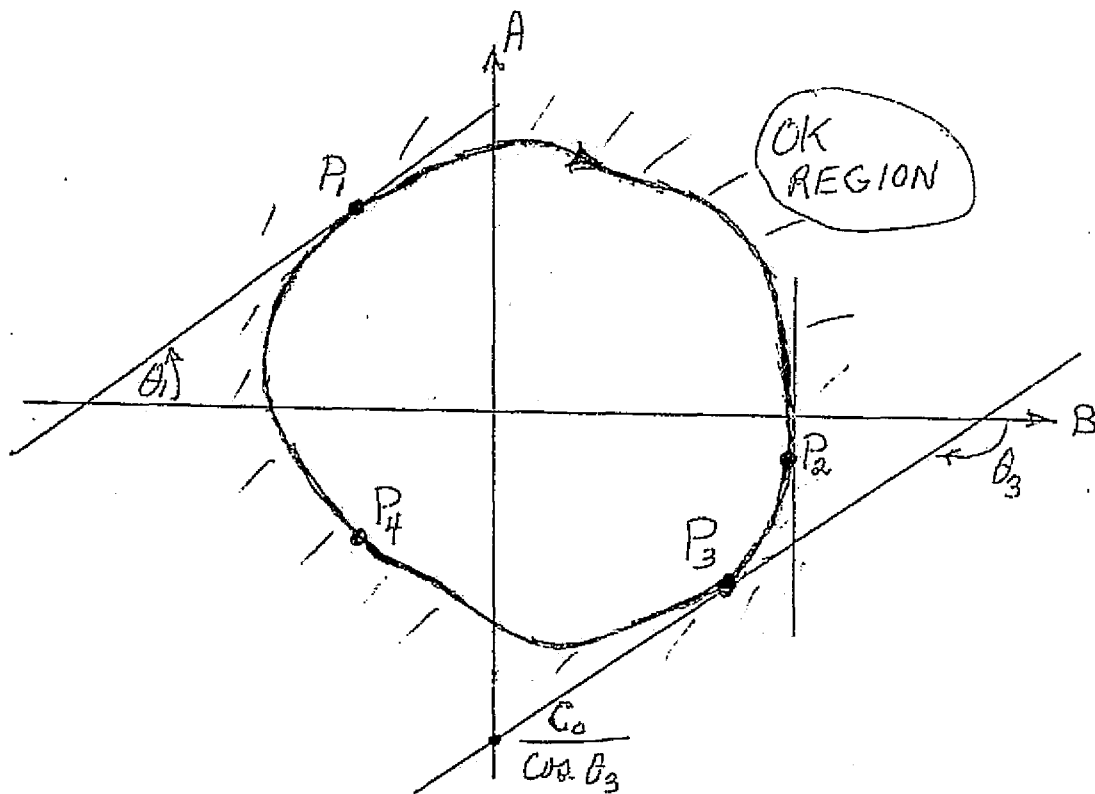Al-2. Definitions of $\psi(\omega)$ parameters -- type 1

Fig. Al-3. Definitions of $\psi(\omega)$ parameters -- type 2

APPENDIX 2

OPTIMUM SYNTHESIS OF NONMINIMUM-PHASE

FEEDBACK SYSTEMS WITH PLANT UNCERTAINTY

Isaac Horowitz* and Marcel Sidi*

ABSTRACT

Linear, time invariant feedback systems in which the constrained plant transfer function has right half-plane zeros, are perforce nonminimum-phase, and their attainable benefits of feedback are inherently restricted. This paper presents criteria for determining whether a given set of performance specifications are achievable, and if so, a synthesis procedure is included for deriving the optimum design, defined as that with effectively minimum loop transmission bandwidth. The properties of the optimum design are derived and its uniqueness proven, for both the minimum and nonminimum-phase feedback systems.

OPTIMUM SYNTHESIS OF NONMINIMUM-PHASE FEEDBACK SYSTEMS WITH PARAMETER UNCERTAINTY

I. INTRODUCTION

There is given a linear time-invariant constrained plant with minimum-phase (mp) transfer function $P(s)$ (see Definitions Il.). There is uncertainty in the parameters of $P(s)$, but the ranges of the uncertain parameters are known. The uncertainty may include the excess (e_p) of poles over zeros of the mp plant, up to three [1], e.g. $4 \leqslant e_p \leqslant 7$, and parameter ranges in which the plant is unstable [2]. It is desired that in response to command inputs $r(t)$ and disturbance inputs $d(t)$ in Figure 1, the system output should satisfy, over the entire range of parameter uncertainty, specifications of the form

$$A_T(\omega) \leqslant |T(j\omega)| \leqslant B_T(\omega) \quad , \quad T(s) \triangleq \frac{C(s)}{R(s)}$$

$$A_d(\omega) \leqslant |T_d(j\omega)| \leqslant B_d(\omega) \quad , \quad T_d(s) \triangleq \frac{C(s)}{D(s)}$$

(1a,b)

for given $A_i(\omega) \leqslant B_i(\omega)$.

It is possible to demand $A_i(\omega) = B_i(\omega)$ at a finite number of ω values, at which the loop transmission must then be infinite. Although these performance tolerances are in the frequency domain, it has been shown [3] that they suffice to satisfy time-domain specifications of the form $a_{im}(t) \leqslant c_i^{(m)}(t) \leqslant b_{im}(t)$ $i = r,d$

where $c_i^{(m)}(t)$ is the mth derivative of $c_i(t)$ for $m = 0,1,\ldots,n$ finite, and $a_i(t) = b_i(t)$ allowed, at most, only at zero and infinite t .

A detailed synthesis procedure has been presented [2] for the above problem, and an existence theorem [1] guaranteeing its validity, providing $P(s)$ is mp over its entire range of uncertainty. In addition, under a certain definition of optimality and under simplifying restrictions reasonable only for a mp plant with a relatively large range of uncertainty and/or narrow performance tolerances, the properties of the optimum design were derived and its existence and uniqueness proven [4]. This paper extends the synthesis procedure and the optimization results to both mp and nonminimum-phase (nmp) plants in which most of the simplifying constraints are eliminated. It is necessary in any case to remove the simplifying restrictions in order to cope with the nmp system. Additional reasons for removing them are:

1. They do not necessarily hold in problems with moderate parameter uncertainty and/or large performance tolerances.

2. It has been shown [1,3] that the above linear, time-invariant (lti) synthesis technique based on frequency response, may be rigorously used in linear and nonlinear time-varying systems with uncertainty. The key ingredient is the replacement of the non-linear (or linear) time-varying plant set by an equivalent lti plant set. The restrictions which reasonably apply to a true lti plant set (i.e. from a lti model) may not at all apply to this equivalent lti plant set. Also, this equivalent lti plant set, obtained from a nonlinear or linear time-varying plant set, is much more likely

to be nmp.   This is one of the principal motivations for solving
the nmp problem - to greatly extend the ℓtv and nonlinear problems
solvable by the 'equivalent ℓti' method.

3.   Optimum ℓti multiloop and multivariable synthesis based on
frequency response, is to a large extent a matter of the optimum
shaping of analytic functions (the loop transmissions) in the
complex plane. While some progress has been made [8,9], the
optimization problems are in general unsolved. The techniques and
theorems used in this paper for extending the single-loop results,
are useful and suggestive for the more difficult multiloop problems.

It is noted that the nonminimum-phase concept was first
introduced by Bode [7]. The problem in control systems has been
treated for deterministic and stochastic inputs, notably by
Chang [14], Horowitz [5] and very thoroughly by Youla et al [15].
However, these treatments have been confined to systems with
no parameter uncertainty.

I 1. Definitions.

Throughout this paper, the left half-plane (ℓhp) includes the imaginary axis, while the right half-plane (rhp) excludes it. If F(s) is a rational function in the complex variable s then it is here called minimum-phase (mp) if it has no zeros in the finite rhp. Otherwise, it is nonminimum-phase (nmp). A simple way to handle a nonrational function is to approximate it by a rational function over the frequency range of interest and deal with the approximation. In this way it is seen that any reasonable approximation of e^{-sT} is nmp. Note that rhp poles are admissible in a mp function.

This paper is concerned only with problems in which the relevant functions can be approximated as closely as desired by rational functions, so the above definition suffices.

I 2. Constraints on L(s) , T(s) due to nmp P(s)

Figure 1 presents a two-degree-of-freedom [5] structure in which F(s), G(s) are to be chosen to satisfy the system specifications (1). If P(s) has one or more rhp *zeros* at $\tau_1,\ldots,\tau_q$, let them be explicitly shown by writing

$$P(s) = P_1(s)(1-\tau_1 s) \ldots (1-\tau_q s)$$

$$= \{P_1(s) \prod_1^q (1+\tau_i s)\}[\frac{\Pi(1-\tau_i s)}{\Pi(1+\tau_i s)}] \overset{\Delta}{=} \{P_m(s)\}[A(s)] , \quad (2a,b)$$

$$L(s) \overset{\Delta}{=} G(s) P(s) = (GP_m)A(s) \overset{\Delta}{=} L_m(s) A(s) .$$

$P_m(s)$, $L_m(s)$ are mp while A(s) is an 'all-pass' function, because $|A(j\omega)| = 1$ for all ω . If the system is to be stable then it is impossible to cancel the rhp zeros of P by means of rhp poles of G because cancellation, being imperfect, results in rhp dipoles for the loop transmission L = GP and therefore for T = FL/(1+L) . Hence, both L and T are constrained to have the rhp zeros of P as zeros. While this is actually also true for the lhp zeros of P , these can at least be cancelled at a nominal set of parameter values, so the resulting nominal L_o , T_o can be free of these zeros. In the nmp case they must appear even in the nominal L_o , T_o .

Consider the effect of a single all-pass section $A(s) = (1-\tau s)/(1+\tau s)$ in $L(s) = L_m(s) A(s)$. The phase of $A(j\omega)$ decreasess monotonically from zero at $\omega = 0$, to -90° at $\omega = 1/\tau$ and tends to -180° as $\omega \to \infty$. In a realistic feedback design,

$|L(j\omega)| > 1$ by a reasonable margin over some ω range, in order to obtain the desired benefits of feedback. Beyond this range it is important to reduce $|L(j\omega)|$ vs ω, as rapidly as possible [2,8]. One pays with phase lag for the rate of reduction of $|L(j\omega)|$, with the stability requirement limiting the fund of phase lag available. $A(j\omega)$ contributes phase lag, thereby diminishing the fund, without contributing anything, being all-pass, to the reduction of $|L(j\omega)|$. At $\omega = 0.5/\tau$, the amount of phase-lag lost is 53.2° which when added to say 35° for phase margin, leaves only about 90° for magnitude reduction. It is therefore impossible to obtain a crossover frequency ω_c (defined by $|L(j\omega_c)| \overset{\Delta}{=} 1$), much larger than $0.5/\tau$. The above discussion is qualitative. The quantitative limitations will be forthcoming from the quantitative synthesis procedure in this paper.

II. SYNTHESIS PROCEDURE FOR THE NMP SYSTEM

The synthesis procedure parallels that for the mp system [2], with certain modifications. The frequency response specifications are assumed to be given by (1a,b). The problem of translating time-domain into ω-domain tolerances in mp system has been discussed [2,6]. While no rigorous translation exists, there is in practice little difficulty in effecting one satisfactory for engineering purposes. Appendix 5 discusses the additional complexities that arise in the translation, due to the system being nmp. In presenting the technique, primary

attention will be paid to (1a) (see Figures 3, 4 for a typical example of (1a)). It will then be obvious what to do about (1b). Also, to simplify the presentation, the order of the plant is assumed known. The modification needed for a plant of uncertain order (up to a factor of three) will then be evident.

Let all the rhp zeros of the plant and only these, be contained in the polynomial $N(-s)$, $-s$ being used to emphasize its nmp character. Let $N_o(-s)$ be the nominal value of $N(-s)$ obtained at a set of plant parameter values arbitrarily defined as nominal. There is therefore no uncertainty in $N_o(-s)$. $P(s)$ may also contain poles whose range of uncertainty includes portions of both the ℓhp and rhp. Denote these by the polynomial $d(s)$, with nominal value $d_o(s)$ deliberately chosen with all its zeros in the ℓhp.

The above rhp zeros and poles are explicitly displayed by writing $P = N(-s) P_1(s)/d(s)$, and

$$L = PG = \frac{N(-s)P_1 G}{d(s)} = \left[\frac{N_o(-s)}{N_o(s)}\right] \left[\frac{P_{10}(s)N_o(s)G}{d_o(s)}\right] \left[\frac{N(-s)P_1(s)\ d_o(s)}{N_o(-s)P_{10}(s)\ d(s)}\right]$$

$$\overset{\Delta}{=} [A_o(s)][L_{mo}(s)][V(s)] \overset{\Delta}{=} L_{no}V(s)$$

(3)

where $P_{10}(s)$ is the nominal (and hence not uncertain) represent-ation of $P_1(s)$ when the plant parameter have their nominal values. $V(s)$ is the only unceratin part of L and is unity at the nominal values of the plant parameters. $L_{mo}(s)$ is mp and stable of course, and L_{no} is the nominal nmp loop transmission. Since

$$T = F \frac{L}{1+L} \quad , \quad \Delta \ln T = \Delta \ln \frac{L}{1+L} \quad , \quad \text{with}$$

$$\Delta \ln L = \Delta \ln V \qquad\qquad (4a,b,c)$$

due to uncertainty. As the plant parameters range over their regions of uncertainty, $\Delta \ln L(j\omega) = \Delta \ln V(j\omega)$ occupies a known region in the complex plane at any fixed ω . It is a straightforward matter to use (4b) to find those values of $L_{no}(j\omega) = A_o(j\omega) L_{mo}(j\omega)$ which satisfy (1a) over the entire range of $\Delta \ln V$. The boundary of the acceptable L_{no} is denoted by B_ω^n , and of the corresponding $L_{mo} = A_o^{-1} L_{no}$ by B_ω . The logarithmic complex plane (Nichols chart) is useful for hand calculations and for seeing the design 'tradeoffs' [2, 8], but of course the procedure easily lends itself to computer calcualtions. The procedure is illustrated by the following simple example.

Example. $P(s) = \frac{k(1-\tau s)}{s(1+\beta s)}$ with parameter uncertainty: $k \in [1,3]$, $\beta \in [0.3,1]$, $\tau \in [.05,.1]$. The arbitrarily chosen nominal values are $k_o = 1$, $\beta_o = 0.3$, $\tau_o = .05$ giving

$$V(s) = \frac{k(1-\tau s)(1+.3s)}{(1-.05s)(1+\beta s)} \quad .$$

The range of $V(j4)$ on the Nichols chart shown in Figure 2. Note that it includes the point $1 = 0$ db $\underline{/0^\circ}$. Suppose that $|T(j4)|$ is allowed to be between 0.5 and -3.5 db i.e. $\Delta \ln |L(j4)| \leq 4$db . Using (4b) it is quite easy to find the permissible values of

$L_{mo}(j4)$ of (3), such that $\Delta \ln|T(j4)| \leq 4db$ for all $V(j4)$,

a portion of which labelled as B_4^n is shown in Figure 2, i.e.

$\Delta \ln|T(j4)| \leq 4db$ so long as $L_{no}(j4)$ lies on or above B_4^n .

Such bounds on the nmp L_{no} exist for each $\omega \in [0,\infty)$. How-

ever, in actual numerical design it is more convenient to work

with a mp function (the Bode integrals [5, 7] can then be used),

so the corresponding boundary of $L_{mo}(j\omega)$ of (3) is easily

derived as that of $L_{no}(j4)$ multiplied by $A_o^{-1}(j4) = \underline{/22.6}^o$,

in this case, giving the boundary B_4 in Figure 2. Similar

boundaries exist at each ω , with the amount of shift in angle

due to $A_o^{-1}(j\omega)$, tending to 180° as ω becomes large.

II 1   Disturbances and the Single High-Frequency Boundary

In the above, only (1a) was considered. The disturbance

specifications (1b) may require all or part of the boundary to

be changed, so as to satisfy both (1a,b). It is important to note

however, that even in the absence of significant disturbances

(so assumed in the balance of this paper), the latter should

not be entirely ignored. If it is, then design economy (as will

be seen) will result in very large peak values of $|L/(1+L)|$ at

higher ω . Such peaking does not violate (1a) because in (4a),

the prefilter F is abailable to drastically attenuate the

higher frequency components of the command input. It is however,

assumed impossible to prefilter the disturbance D in Figure 1.

Hence, one should always add a specification of the form

$$\left|\frac{L}{1+L}\right| \leqslant \gamma \qquad\qquad (5)$$

with γ chosen by the designer. This leads eventually, in the lti mp problem, to a single high frequency boundary B_h^n, effective for all ω greater than some ω_h, for the following reason.

At large ω, $P \rightarrow k/s^{e_p}$ so $V(s) \rightarrow k/k_o$ whose range (·template·) is a vertical line in the Nichols chart, of magnitude $20 \log(k_{max}/k_{min})$db. Assuming γ in (5) is a constant in this range, the resulting boundary of $L_{no}(j\omega)$ is the same for all ω for which the above is applicable, e.g. B_h^n in Figure 2 for $\gamma = 2.3$db, $k_{max}/k_{min} = 10$. In nmp systems this 'universal' high-frequency boundary of L_{no} is translated to the right by $|\text{Arg } A_o(j\omega)|$ in ordr to obtain the boundary of the mp $L_{mo}(j\omega)$. The resulting boundaries are denoted by $B_{h\omega}$ to emphasize their dependence on ω. Figure 2 displays some $B_{h\omega}$ for $A_o(s) = (1-.05s)/(1+.05s)$. Note how the corner point V_ω moves to the right as ω increases. In general, $\text{Arg } A_o(j\omega)$ tends at laest to 180° (if $A_o(s)$ is of order 1, $180n^o$ for order n), so that sooner or later the right side of $B_{h\omega}$ lies in the positive angle region. It is this excursion of $B_{h\omega}$ into the positive region which may render it impossible to satisfy a given set of specifications. It is proven in Appendix 4 that if $L_{mo}(j\omega)$ cannot reach the corner V_ω before V_ω is on the zero angle line (i.e. while $\text{Arg } L_{mo}$ is still negative), then it is impossible for L_{mo} to satisfy the specifications and also $\rightarrow 0$ as $\omega \rightarrow \infty$.

II 2 Design Optimization

In mp systems, any realistic specifications of the (1a,b) type (see [1] for precise conditions), may be satisfied for a large class of plant parameter uncertainty problems, by an infinitude of designs. A nmp problem if it is solvable at all, will also generally have an infinitude of acceptable designs. Uniqueness is achieved by defining an optimal design, here chosen as follows. Let e be an apriori specified excess of poles over zeros for the nominal loop transmission $L_o(s)$, so that as $\omega \to \infty$, $L_o(s) \to k_o/s^e$.

Definition: The optimal design is defined as that which satisfies the system specifications with minimum value of k_o .

This definition is motivated by the practical importance of decreasing $|L(j\omega)|$ vs ω as fast as possible, in order to reduce the effect of high-frequency plant parasites and of sensor noise [2, 5, 8]. It is noted that essentially this criterion was used by Bode [5, 7] for the derivation of his 'approximately ideal characteristic.' It is important to note that by working with the mp $L_{mo}(s)$ of (3) instead of the nmp $L_{no}(s)$, the problem of finding L_{opt} becomes one of finding $(L_{mo})_{opt}$. Henceforth, L_{opt} denotes $(L_{mo})_{opt}$, unless otherwise specified.

The following property of L_{opt} is worth noting, derived from the Bode integral [5,7]

$$\ell n|L(0)| \; - \; \ell n|L(\infty)| \;\; = \; - \frac{\pi}{2} \int_{-\infty}^{\infty} Arg \; L(j\omega) d\ell n \; \omega \;\; . \qquad (6)$$

This relation may be applied to our problem by modifying L at

very large ω , to be nonzero as $\omega \to \infty$ and, if necessary, at very small ω to be noninfinite as $\omega \to 0$. It shows that the optimum design strives to maximize its phase lag, subject to the boundaries B_ω .

It is proven in Appendices 1-3 that if any design exists which satisfies the specifications, then L_{opt} is unique and exists in a limiting sense, i.e. may be approximated as clearly as desired. Also, L_{opt} lies on its boundary B_ω at every ω from zero to infinity. It is crucial however, to carefully include certain conditions in order that the problem be well defined. Figure 2 may be used for this purpose. Let $B_h = B_h^h$ be the applicable 'universal' high-frequency boundary in a purely mp problem, i.e. $A_O(s) = 1$. As $\omega \to \infty$, L_{opt} must at least asymptotically approach the vertical line $W''\ YZ$ (in Figure 2) corresponding to Arg $L = -90\ e^o$. L_{opt} must therefore make the transition from B_h to this vertical line and as soon as it 'turns the corner' at QV there is nothing to stop it from very rapidly increasing its phase lag. However, a rapid increase in phase lag is accompanied by a sharp increase in magnitude [5, 7] so L_{opt} would follow QVW and then instantaneously increase its phase lag to $-90\ e$ degrees, where-in $|L_{opt}|$ abruptly goes to ∞ along $QVWW'$ and returns on $W''YZ$. To prevent such a discontinuity, and for obvious practical reasons, the supplementary 'transition' boundary W_1V_2YZ is added [4], i.e. that for all ω more than some ω_x , L must lie in the closed region bounded by $..QVW\ W_1W_2YZ$.

The above suffices for mp &ti problems with large parameter

uncertainty and/or narrow performance tolerances. In these,

$L_{opt}(j\omega)$ must be so large over a large ω range, that it can

reach the NQV region in Figure 2 only at ω values so large

that the plant function $P(j\omega)$ is very close to its asymptotic

high frequency value ks^{-e_p} . The range (template) of $V(j\omega)$

of (3, 4) is then a line so B_h is the single effective universal

boundary for all higher ω . This is not necessarily so for the

problems listed in the Introduction. In these L_{no} may cross the

-180° line before $P(j\omega)$ has degenerated into ks^{-e_p} . The

corresponding B_ω^n need not form a nested set in this frequency

range, but instead cross each other, expand and contract in size

and shape etc. A more careful definition must be made of the

transition from the boundaries B_ω^n obtained from Equation (4),

to the high-frequency asymptote $\text{Arg } L_{no}(j\omega) = -90e^o$, in order

that a continuous $L_{opt}(j\omega)$ may result. To do this note that there

must exist a ω^* , $\ni$, $\forall \omega > \omega^*$, the boundaries of L_{no} do

not enclose the origin of the arithmetic complex plane, and lie

entirely in the second and third quadrants, $-270^o < \text{Arg} < -180^o$.

Otherwise, it is impossible for $L_{no}(j\omega)$ to approach zero as

$\omega \to \infty$. Sooner or later $L_{no}(j\omega)$ must cross the -180° line

between the origin and the boundaries, and some interval after

it has done so, its phase may be sharply decreased. L_{opt} seeks

to 'escape' from the boundaries to its final asymptote as quickly

as possible. The transition boundary consists of three parts.

The last is a segment of the line $\text{Arg } L_{no} = -90e^o$ (e.g. ODE

in Figure A5a for the case $e = 4$) , corresponding to YZ of Figure 2. The second is one or more portions of an arc of constant magnitude $|L_{no}| = CM$ corresponding to $W_2 W_1$ of Figure 2 (see Figure A4 for $CM > 1$ and Figure A5 for $CM < 1$) .

THe first part (ρ_ω) of the transition boundary must join CM to the B_ω^n and is defined as follows. For each B_ω^n , $\omega > \omega^*$, choose ρ_ω a straight line tangent to B_ω^n and terminating on CM (e.g. ρ_{ω_1} in Figure A4a where $B_\omega^n = B_\omega$) . If several such points of tangency exist, choose one for which ρ does not intersect the interior of B_ω^n . If CM intersects $B_{\omega_1}^n$, ρ_{ω_1} may, of course, not exist. Henceforth B_ω^n refers to the boundary modified by ρ_ω and it is therefore also necessary that the ρ_ω be chosen so that this set $\{B_\omega^n\}$ of modified boundaries continue to satisfy (A1). It is conceivable for $L_{no}(j\omega)$ to be on CM for some ω range, be forced to return to B_ω^n , but after one or more such excursions eventually return to CM while in the second quadrant, and leave it then only for $\text{Arg } L_{no} = 90e^\circ$.

Given the above problem definition, it is proven in Appendix 1 that L_{opt} must lie on its associated B_ω^n at every ω , in Appendix 2 that if any L exists which satisfies the B_ω^n then L_{opt} exists

and in Appendix 3 that if L_{opt} exists, it is unique. The
above apply to both mp and nmp systems. It is well known and
formally proven in [2] that in the mp case a solution does
exist. It is possible, however, that no solution exists for
a specific nmp problem. Appendix 4 presents some criteria
for determining whether this is so.

III. DESIGN EXAMPLES

Example 1

The plant transfer function

$$P(s) = \frac{k(1-ds)}{s(1+bs)} \quad \text{with} \quad k \in [1,3] \quad , \quad b \in [0.3,1] \quad d \in [0.05,0.1].$$

The step response bounds were originally those shown as I in
Figure 3a, resulting in the equivalent frequency-response
bounds on the magnitude of the minimum-phase $T_m(j\omega)$ (see
Appendix 5) I in Figure 3b. The disturbance response parameter
γ of Equation (5) is 6 db.

Design

The range (templates) of V of Equation (3) were obtained
and used to find first, the bounds on L_{no} and then as
discussed in Section II (Figure 2), the resulting bounds B_ω
on $L_{mo}(j\omega)$, shown in Figure 4a. Using the technique of
Appendix 4, it is found that L_{mo} cannot be reduced sufficiently
by the time Arg L_{mo} is zero degrees. Therefore the specifications
on the system step response must be relaxed. In Figure 4a

it is seen that primarily the bounds for $\omega \in [2,10]$ must be relaxed as B_2 compels $|L_{mo}(j2)|$ to be $>$ 0 db and B_{10} compels Arg $L_{mo}(j10)$ to be rather large. A new set (II) of frequency domain bounds and their corresponding time domain bounds are shown in Figures 3b, a respectively. The resulting new bounds on $L_{mo}(j\omega)$ are shown in Figure 4b together with the $L_{mo}(j\omega)$ used. Experimental design results are shown in Figure 3a.

Example 2.

In this example $P = (1-\tau s)ka/s(s+a)$ with uncertainties $k \in [1,10]$, $a \in [1,10]$, and two separate values of τ were considered: 1) $\tau = 1/4$ 2). $\tau = 1/60$. The bounds on the nmp $L_{no}(j\omega)$, shown in Figures 5a, b, are therefore the same in both cases. But the bounds on the mp $L_{mo}(j\omega)$ for $\tau = 1/4$ in Figure 5a, are shifted to the right much more than those in Figure 5b, where $\tau = 1/60$. A computer program was prepared which derives an $L(j\omega)$ function which lies on the boundaries [9]. In Figure 5a this program perforce gives an L function which crosses into the positive phase region, as shown. Such a L function cannot go to zero at large ω. In Figure 5b, L_{mo} is attainable, and the resulting nmp $L_{no}(j\omega)$ is also sketched. With such a computer program it is a simple matter to check whether any specific nmp problem is solvable.

IV. CONCLUSIONS

Optimum single-loop ℓti synthesis for plants with parameter uncertainty, previously developed for a special class of mp problems [4], has been extended to a much larger class of mp systems and to nmp systems as well. This extension permits the 'equivalent ℓti' technique developed for linear time-varying uncertain systems which are effectively mp [1], to be also applied to ℓtv uncertain systems which are effectively nmp, the resutls to be made available in the near furture. The results of this paper are also obviously applicable to ℓti synthesis techniques previously developed for more complex mp structure such as the cascaded multiple-loop [8] and the multivariable [9]. In all of these, the availability of a computer program for finding the L(jω) function which satisfies the bounds, makes it an easy matter to determine whether the original nmp specifications can be satisfied or whether they must be modified. The synthesis technique provides insight into the minimum required modification.

APPENDIX 1.

$L_{opt}(j\omega)$ LIES ON ITS BOUNDARY

It was shown in Section II that the problem of optimization of the nmp $L_{no}(s)$ of (3), may be transformed into that of the mp $L_{mo}(s)$. Until specified otherwise L , L_{opt} , L_a , B_ω etc refer to the mp items. The following restrictions on the boundaries B_ω are highly realistic and obvious in any real problem, and are listed for the sake of a clear mathematical problem statement:

> B_ω bound simply connected regions in the complex plane. They are continuous on $[0,\infty)$ in the sense that given any point z on B_ω , then for $\Delta\omega$ small enough, there is a point $z+\Delta z$ on $B_{\omega+\Delta\omega}$ which is as close as desired to z , in the Euclidean metric. Also each B_ω is a continuously differentiable curve in the complex plane.

(A1)

The continuity requirement does not require similar continuity in ω of the range (template) of $P(j\omega)$. Even if the original B_ω is not continuously differentiable it can easily be deformed to be so with infinitesimal changes in its actual shape. The region bounded by B_ω which is forbidden to $L(j\omega)$ is denoted by B_ω^o .

Theorem 1. $L_{opt}(j\omega)$ if it exists lies on B_ω .

<u>Proof.</u> Let $L_a(j\omega)$ be submitted as candidate for optimum and suppose there is at least one ω interval (not a point because of (A1)), denoted by $I = (A,B)$, in which $L_a \notin B_\omega$ itself. Suppose that it is always possible to find an $H(j\omega)$ such that $L_a H(j\omega) \notin B_\omega^O$ for any ω, with $\ell n|H(\infty)| < 0$, and $1+L_a H$ having no rhp zeros. Note that both L_a and $L_a H$, as candidate functions, must have the assigned excess e of poles over zeros, so $H(\infty)$ must be finite. Then $L_a H$ is better than L_a in the sense of the definition of optimality. The theorem is therefore proven if such an $H(j\omega)$ can be found whenever such an interval $I = (A,B)$ exists.

The main problem in choosing H is to ensure that $L_a H(j\omega) \notin B_\omega^O$, $\forall \omega$. The problem does not exist for $\omega \in I = (A,B)$ or in any other intervals in which $L_a \notin B_\omega$, because $L(j\omega)$ may there be shifted in <u>any direction</u> by a finite amount without entering $B^O(\omega)$. Thus, the worst case to consider is when $L_a(j\omega) \in B_\omega$, $\omega \in \bar{I}$ (complement of I). Consider in Figure A1, a very general form for B_{ω_1} in the arithmetic complex plane, $\omega_1 \in \bar{I}$. In Figure A1, suppose $L_a(j\omega_1)$ has the value $A \in B_{\omega_1}$. If $\text{Arg } H(j\omega_1) = 0$ and $\ell n |H(j\omega_1)| \leqslant 0$, then clearly $L_a H(j\omega_1) \notin B_{\omega_1}^O$. If $L_a(j\omega_1) = E$ in Figure A1, then $\text{Arg } H(j\omega_1) = 0$, $\ell n|H(j\omega_1)| \geqslant 0$ is satisfactory. This strategy ($\text{Arg } H = 0$, $\ell n|H|$ positive or negative) fails only at points such as C, i.e. if $L_a(j\omega_1) = C$. Such points are temporarily ignored as if nonexistant. Let

$$\ell n\, H(j\omega) = \ell n\, |H| + j\, \text{Arg } H \overset{\triangle}{=} H_1(\omega) + j\, H_2(\omega) \qquad (A2)$$

and choose $\text{Arg } H_1(\omega) = 0$, $\forall\, \omega \in \bar{I}$. It is required that

$H_1 \leqslant 0$ in those intervals in $\bar{I}$ in which $L_a(j\omega)$ has the property

of A in Figure A1, and that $H_1 \geqslant 0$ in those intervals in which

$L_a(j\omega)$ has the property of D or E . These requirements dictate

say, n zero crossings of $H_1(\omega)$ at $\omega_1, \omega_2, \ldots$ in $\bar{I} = (0,A) \cup (B,\infty)$

(see Figure A2), giving the n+2 intervals $R_1 = (0,\omega_1), \ldots, R_{k+1} =$

(ω_k, A) , $R_{k+2} = (B, \omega_{k+1}), \ldots, R_{n+2} = (\omega_n, \infty)$. Only the values of

$H_2(\omega)$ in I are available to secure these required properties for

$H_1(\omega)$, inasmuch as $H_2(\omega)$ has been chosen zero for $\omega \in \bar{I}$. The

candidate L_a can be such that a combination of the following may

be necessary for H :

(Aa)　$H_1(\omega) \geqslant 0$ in $R_1, R_3, \ldots$ in (0,A), with $H_1(\omega) \leqslant 0$ in

$R_2, R_4, \ldots$ in (0,A)

(Ab)　the opposite of (Aa), i.e. $H_1(\omega) \leqslant 0$ for R_1, R_3 , etc.

(this case is shown in Figure A2).

(Ba)　If the number of 'intervals' in (B,∞) is even then the

firs, third, etc. of them must be $\geqslant 0$ and the others $\leqslant 0$,

in order that $H_1(\infty) < 0$.

(Bb)　If the numbers of 'intervals' in (B,∞) is odd then the

first, third, ... must be $\leqslant 0$ etc.

The following combinations are possible: (1) Aa, Ba (2) Aa,

Bb (3) Ab, Ba (4) Ab, Bb, and the objective is to prove that H_2

can be constructed to give any desired one of these four combinations.

Obviously $H_1(0) = 0$ may be assigned for all cases.

Consier the Hilber transform relating the real (H_1) and imaginary (H_2) parts of the analytic function $\ln H(s)$, $s = j\omega$[5,p. 309],

$$E_1^*(\omega) \overset{\Delta}{=} \frac{H_1(\omega)-H_1(0)}{\omega^2} = \frac{2}{\pi}\int_0^\infty \frac{H_2(\zeta)d\zeta}{\zeta(\omega^2-\zeta^2)} \overset{\Delta}{=} \int_A^B \frac{\theta(\zeta)d\zeta}{\omega^2-\zeta^2} \qquad (A3)$$

with $H_2(\zeta) = 0$ deliberately picked $\forall \omega \in \overline{I}$, giving the limits A, B instead of $(0,\infty)$. Equation (A3) is used to prove that the desired properties of H_1 in $\overline{I}$ are achievable by a suitable choice of H_2 in I. The technique is borrowed from those used in Total Positivity [10, 11]. Pick a continuous $H_2(\zeta)$ in I with n zero crossings $A \leqslant \eta_1 < \eta_2 \ldots < \eta_n \leqslant B$ and in addition $H_2(A) = H_2(B) = 0$.

Step 1. Define $K\begin{pmatrix} \omega_1, \ldots\ldots, \omega_n \\ \zeta_1, \ldots\ldots, \omega_n \end{pmatrix}$ as the $n \times n$ determinant whose ij th element is $k(\omega_i,\zeta_j) = (\omega_i^2-\zeta_j^2)^{-1}$, $\eta_{j-1} \leqslant \zeta_j \leqslant \eta_j$ and $\omega_i \in \overline{I}$ ordered as in Figure A2. It is now proven that $K(\ldots)$ has constant sign for any ζ_i, ω_j pair satisfying the above ordering conditions i.e. $k(\omega,\zeta)$ is totally positive [10]. Since each ζ_i, ω_j can be continuously varied (so long as the ordering is kept), a sign change of $K(\ldots)$ requires $K(\ldots) = 0$ at some ζ_i, ω_j pair. Let the i th column of $K(\ldots)$ be the vector α_i. If $K(\ldots) = 0$ at some ζ_i, ω_j pair, then $\exists$ scalars $a_1, \ldots a_n$ not all zero, $\ni \sum_1^n a_i \alpha_i = 0$. Hence $\sum_{i=1}^n a_i(\omega_j^2-\zeta_i^2)^{-1} = 0$ for $j = 1,2,\ldots,n$, which means that $\sum_i a_i(\omega^2-\zeta_i^2)^{-1} \overset{\wedge}{=} N(\omega^2)/\pi(\omega^2-\zeta_i^2)$ has n zeros at ω_j, $j = 1,2,\ldots,n$. This is impossible because $N(\omega^2)$ is of degree $n-1$ in ω^2 and $(\omega^2-\zeta_i^2) \neq 0$

for $\omega \in \overline{I}$, $\zeta_i \in I$, The only other possibility that $K \equiv 0$ is impossible, as follows.

The determinant $K(..)$ is expanded by its first column,

giving $K(..) = \sum_1^n \dfrac{J_i}{\omega_i^2 - \zeta_1^2} = \psi(\zeta_1^2) \stackrel{\Delta}{=} \psi(z_1)$ letting $z_1 = \zeta_1^2$ with J_i the cofactor of the $(i,1)$ element of the corresponding matrix, and ζ_1 does not appear in any J_i . Thus $K(..)$ can be regarded as a function of $\zeta_1^2 = z_1$. If $K(..) \equiv 0$ then the derivatives $\psi^{(m)}(z_1) \equiv 0$, $m = 1,2,..$. $\psi^{(1)}(z_1)$ can be obtained by differentiation of the first column of $K(..)$ with respect to z_1 , giving a determinant whose first column is $1/(\omega_i^2 - z_1)^2$; $i = 1,2,.. n$, while all other columns are the same as in $K(..)$. Similarly $\psi^{(m)}(z_1)$ consists of $K(..)$ with the first column only replaced by $m!/(\omega_i^2 - z_1)^{m+1}$. As m becomes sufficiently large, one term will dominate in the expansion of $\psi^m(z_1)$ by the first column – the term with the smallest $|\omega_i^2 - z_1|$. Hence only its cofactor need be considered. This process is repeated, with the cofactor considered as a function of ζ_2^2 , etc until the final cofactor to be considered has only one term $1/(\omega_j^2 - \zeta_n^2) \neq 0$ because $\omega_j \in \overline{I}$, $\zeta_n \in I$. Hence $K(..)$ has the same sign $\forall$ $\omega_i \in \overline{I}$, $\zeta_j \in I$, ordered as previously indicated.

Step 2. From A3 and the definition of $k(\omega_i, \zeta_j)$ in $K(..)$,
$H_1^*(\omega) = \displaystyle\int_A^B k(\omega, \zeta)\theta(\zeta)d\zeta$, in which from (A4) below, $\theta(\zeta)$ has

$n+2$ zeros θt $\eta_0 = A < \eta_1 < \eta_2 \cdots \,_n < B = \eta_{n+1}$, and has

the same sign in between any two of these consecutive zeros
and the opposite sign between the next two. It is next shown
that $H_1^*(\omega)$ of (A3) cannot have more than n internal zeros
in $\overline{I}$. Assume this is not so and that it has $n+1$ internal
zeros at $0 < \omega_1 < \omega_2 \ldots < \omega_{n+1}$ in $\overline{I}$. Let ω_{n+2} be <u>any</u>
internal point in $\overline{I}$, and consider the $n+2$ square matrix M
whose ij th element is

$$\int_{\eta_{j-1}}^{\eta_j} k(\omega_i,\zeta)|\theta(\zeta)|d\zeta \ , \quad \text{for} \ \ i = 1,2,\ldots,n+2 \ ; \ j=1,2,\ldots n+1 \ ,$$

while the $(n+2)$ column consist of $H_1^*(\omega_i)$ of (A2), $i = 1,\ldots,n+2$.
(Det M) is zero because from (A2,3), its last column is a
linear combination of the others. Since by assumption, $H_1^*(\omega_i) = 0$
for $i = 1,2,\ldots,n+1$, expansion of det M by its last column,
gives

$$|\det M| = | \ H_1^*(\omega_{n+2})| \int_{A}^{\eta_1} \cdots \int_{\eta_n}^{B} K\begin{pmatrix} \omega_1,\ldots,\omega_{n+1} \\ \zeta_1,\ldots,\zeta_{n+1} \end{pmatrix} \prod_{i=1}^{n+1} |\theta(\zeta_i)|d\zeta_1\ldots d\zeta_{n+1} \ ,$$

with $\zeta_i \in (\eta_{i-1},\eta_i)$, $\eta_o = A$, $\eta_{n+1} = B$. From step 1, $K(..)$
has a fixed sign. Hence det M can't be zero, which contradicts
the previous conclusion that det M = 0 . This is true for
any $\omega_{n+2} \in \overline{I}$. Hence, either $H_1^*(\omega) = 0$, $\forall \omega \in \overline{I}$ or $H_1^*(\omega)$
has at most n zeros (excluding the origin) in $\overline{I}$. The former
is impossible, because both H_1^* and its argument identically
zero in $\overline{I}$, would require them to be identically zero $\forall \omega$.

Step 3. It is finally shown that $\theta(\zeta)$ can be chosen so that $H_1^*(\omega)$ has n zeros (exluding the origin) in $\bar{I}$ so that in view of Step 2, it has precisely n. For $\zeta \in I$,

$$\theta(\xi) = \prod_1^n k(\omega_i^2 - \zeta^2)\psi(\zeta)(\zeta^2 - A^2)(\zeta^2 - B^2), \quad \psi(\zeta) = \sum_0^n B_i \zeta^{2(n-i)}$$
$$B_0 = 1 . \tag{A4}$$

From (A3), $H_1^*(\omega_j) = \int_A^B k \prod_{j \neq i} (\omega_i^2 - \zeta^2)\psi(\zeta)(\zeta^2 - A^2)(\zeta^2 - B^2)d\zeta$. It is possible to choose $\psi(\zeta)$ so that $\int_A^B (\zeta^2 - A^2)(\zeta^2 - B^2)\zeta^{2x}\psi(\zeta)d\zeta = 0$ for $x = 0,\ldots,n-1$. For each x , this definite integral gives a linear equation in the coefficients B_i of $\psi(\zeta)$, resulting in n linear simultaneous equations in the unknown B_i . It has been shown that this set has a unique solution [12, p.236], with n simple zeros of $\psi(\zeta)$ in (A,B) .

For the present purpose, it is imperative to know the relation between the sign of $H_1(\omega)$ in R_{k+1} and its sign in R_{k+2} (see Figure A2), because in view of the previous four combinations shown possible for H_1, this relation must be flexible. By making use of the known result for the Tchebycheff system on $[A,B]$ [11, p. 11] it can be shown that $\text{Sign}(H_1^*$ in $R_{k+1}) = (-1)^{n+1} \text{Sign}(H_1^*$ in $R_{k+2})$. Hence, to secure the desired flexibility it may be necessary to insert an extra zero say at the origin, or at A^- or B^+ , where it does no harm. Of course, any desired sign of H_1 in R_1 is possible by choosing the sign of k in (A3). Also the scale factor of H is controlled by the magnitude of k , in order that $L_a H$ does not cross into $B^0(\omega)$, $\forall \omega$.

It is necessary to prove that $1+L_aH$ has no rhp zeros. Obviously, $1+L_a$ has no rhp zeros, or L_a would not be an acceptable candidate for L_{opt} . The transition from L_a to L_aH can easily be made continuous with $L_a H \notin B^O(\omega)$ over the entire transition range, simply by means of the gain factor k in (A4). The zeros of $1+L_aH$ are continuous functions of k in H . Hence if these zeros are in the ℓhp at $k = 0$ and do not lie on the imaginary axis during the transition (by having $L_a H \notin B^O(\omega)$, $\forall\omega$) , then $1+L_a H$ also has no rhp zeros. The theorem is therefore proven, providing $L_a(j\omega)$ has no values such as C in Figure A1, $\forall \omega \in [0,\infty)$.

If $L_a(j\omega) = C$ (Figure A1) at one or more ω values, one may instead specify $H_1(\omega)$ of (A1) to be zero in $\overline{I}(A,B)$ and choose it in I so that $H_2(\omega)$ may have the required zero crossings in $\overline{I}$. THe Hilbert transform giving H_2 in terms of H_1 (corresponding to (A3)), has the same total positivity property as (A3). This is satisfactory, providing $L_a(j\omega)$ has no values such as D in Figure A_1 , which is, in this respect, the dual of point C . In the most general case where $L_a(j\omega)$ has both kinds of values over one or more intervals, proof of the theorem requires a more general Hilbert transform – one in which H_2 is specified in alternate intervals—say in $I_A = (0, \omega_a)$, $I_C = (\omega_b,\omega_c)$ etc, H_1 in $I_B = (\omega_a,\omega_b)$, $I_D = (\omega_c,\omega_d)$ etc. H_1 is made zero in intervals of C type in Figure A1, $H_2 = 0$ in the D type and either in the A, E, F type.

Again, there is only one segment $I = (A,B)$ in one of these intervals in which nonzero $H_2(\zeta)$ (or $H_1(\zeta)$) may be assigned. A certain pattern of zero crossings is required in $\bar{I}$. Figure A3 shows two examples, with the desired zero crossings and sign of H_1 in I_A, part of I_C, and I_E the same for both examples. The zero crossings for both the H_2 shown are really the same, but the desired sign of H_2 is the opposite in the second in the interval I_B. It will be seen later, that it is therefore necessary in the second example to introduce two more zeros, one at $\omega_a{}^+$, the second at $\omega_b{}^-$, i.e. $n = 11,13$ respectively for the first and second examples.

It has been shown how to derive the Hilbert transform for such problems in general and the solution has been given for the case of two intervals [7; 5, p.319]. For the special case here, where the specified function is nonzero only over $I = (A,B)$, the same technique gives

$$H_1^*(\omega) = \nu(\omega) \int_A^B \frac{\zeta H_2(\zeta)d\zeta}{(\omega^2-\zeta^2)\lambda(\zeta)} \overset{\Delta}{=} \nu(\omega) \int_A^B \frac{\theta(\zeta)d\zeta}{\omega^2-\zeta^2}$$

$$= \begin{cases} (-1)^i 0.5\pi H_1(\omega), & \omega \in I_A \cdot I_C, \ldots; \quad i = 0 \text{ for } \omega \in I_a; \ 1 \text{ for } \omega \in I_C; \ldots \\ (-1)^j 0.5 \, H_2(\omega), & \omega \in I_B, I_D, \ldots, \quad j = 0 \text{ for } \omega \in I_B; \\ & \hspace{4cm} 1 \text{ for } \omega \in I_d; \ldots \end{cases}$$

$$\lambda(\zeta) = |\pi(\omega_x^2-\zeta^2)|^{1/2}, \quad \nu(\omega) = |\pi(\omega_x^2-\omega^2)|^{1/2}$$

$$x = a, b, \ldots .$$

(A5)

Note that $\lambda(\zeta) > 0$, $\forall \zeta \in I$; and $\nu(\omega) > 0$, $\omega \in I_A$, $I_B, \ldots$ with $\nu(\omega) \to 0$ as $\omega \to \omega_a, \omega_b, \ldots$. The problem is to choose $\theta(\zeta)$ of (A5) so as to obtain the desired zero crossings

and signs of $H_1(\omega)$ in I_A, I_C, ... and of $H_2(\omega)$ in I_B, I_D, ...

The proof for this general case is amost identical to that for Figure A2 and equation (A3). The ordering of the ω_i, η_j is the same. Step 1 is exactly the same. Step 2 is also identical but note that the zeros referred to are the internal ones $\omega_1, \omega_2, \ldots$ in Figure A3, not to the zeros at ω_a, ω_b, $\omega_c, \ldots$ the latter being obtained by means of $\nu(\omega)$ in (A5). Step 3 is also the same, as are also the remarks re R_{k+1}, R_{k+2} which now refer to the relative signs of $H_1(\omega)$ at $\omega = A-$, $B+$. (In Figure A3, n+1 is even and the signs are the same, so no additional zeros are needed for this purpose). However, one new complication concernes the relative signs of $H_1(\omega_a-)$, $H_2(\omega_a+)$; of $H_2(\omega_b-)$, $H_1(\omega_b+)$, etc. It is seen that because of the factors $(-1)^i$, $(-1)^j$ on the right side of (A5), no extra zeros are needed if $Sg\, H_1(\omega_a-) = Sg\, H_2(\omega_a+)$, $S_g\, H_2(\omega_b-) = -Sg\, H_1(\omega_b+)$, $Sg\, H_1(\omega_c-) = Sg\, H_2(\omega_c+)$. $Sg\, H_2(\omega_d-) = -Sg\, H_1(\omega_d+)$, etc. An extra zero is needed for each violation of the above. Thus, in Example 2 of Figure A3, there are violations at ω_a, ω_b requiring a zero at ω_a+ (or ω_a-) and at ω_b (or ω_b+), giving $n = 13$.

This completes the proof of Theorem 1. It has been shown that if a loop function exists at all which satisfies B_ω, then the optimum, if it exists, lies on B_ω at each ω. It is implied, of course, that the B_ω are bounded which explains the need for the supplementary $\wedge$ *transition* boundary discussed in II2.

APPENDIX 2

EXISTENCE OF L_{opt}

To prove existence, use is made of the analog of the Weierstrass theorem which states [13] that if $J(L)$ is a continuous functional defined on the normal and compact family $\{L(s)\}$, then $J(L) = min$ has a solution within the family. Here $J(L)$ is equivalent to $\lim_{\omega \to \infty} |(j\omega)^e L(j\omega)|$. If the nominal L has no rhp poles then the family $\{L(s)\}$ consists of functions regular in the right half-plane, including the $j\omega$ axis which satisfy the bounds on $L(j\omega)$ at each ω , and with $\lim_{\omega \to \infty} arg\, L(j\omega) = -0.5\pi e$. If all $P(s)$ over the range of uncertainty have rhp poles then the nominal L must also have rhp poles. If at the nominal parameter set, these rhp poles are at $1/\tau_1, \dots 1/\tau_m$ then let $L = L_1(s)/(1-\tau_1 s) \dots (1-\tau_m s)$ and consider optimization on $L_1(s)$ with the family $\{L_1(s)\}$ regular in the rhp etc, and $\lim\, arg$ $\lim_{\omega \to \infty} arg\, L_1(j\omega) \to 0.5\pi(m-e)$ [5, Section 7.14]. If L has no poles at the origin then it is obviously possible to bound $|L(j\omega)|$, (or $|L_1(j\omega)|$) by some very large number M , if necessary . If L has poles at the origin these poles can be removed and optimization considered on $\{sL\}$ (or $\{s\, L_1\}$). If in (1), $A(\omega) = B(\omega)$ is wanted at a finite number of values ω_a , ω_b then any candidate L would have factors $(s^2 + \omega_a^2)(s^2 + \omega_b^2) \dots$ in the denominator. Such poles would be removed in the same manner. In Appendices 1-2, L is therefore

assumed mp and finite on the jω axis. By the principle

of the maximum modulus $|L(s)|$ (or $|L_1(s)|$, or $|sL(s)|$ etc) < M

over the entire rhp including the boundary. By Montels'

theorem [13], the family is therefore normal and compact. The

functional J(L) is clearly continuous, so the conditions of

the analogue of the Weirstrass theorem are satisfied and an

optimum L(s) (or L_1 or s L etc) exists, providing the family

{L} is not empty. {L} is certainly not empty in the mp

problem [1] but it might be empty in the nmp problem. However,

if the nmp problem can be solved at all, then the optimum

solution exists and lies on B_ω at all ω . From the nature

of B_ω in the transition range (Figure 2), L_{opt} is clearly

not realizable precisely as a rational function but may be

approximated as closely as desired by a rational function.

APPENDIX 3

L_{opt} IS UNIQUE

Uniqueness of L_{opt} is first proven for the mp problem

under the assumption that the $B_\omega(\theta')$ are single-valued functions

of θ' , the angle of the line to B_ω from the -1 point.

Conditions (A1) on B_ω and the definition of the transition

boundary (ρ, CM, -90e) at the end of Section II are important

and should be noted. Suppose there exist two functions L_a , L_b

which satisfy all the necessary conditions for optimality.

Let ω_a be the frequency at which L_a 'reaches' CM and similarly

define ω_b . Let Ω = smaller of the sets $[0,\omega_a]$, $[0,\omega_b]$,

and define

$$N(j\omega) = \frac{L_a(j\omega)-L_b(j\omega)}{1+L_a(j\omega)} \quad , \quad \theta_N = \text{Arg } N \;, \; \alpha_i = \text{Arg }(1+L_i)$$

$$\psi(\omega) = \text{Arg}[L_a(j\omega)-L_b(j\omega)] \; . \tag{A6}$$

All candidates for L_{opt} are constrained to have the same excess (e) of poles over zeros and the same number (n) of poles at the origin. In the following development $n = 0$ is used, but if $n \neq 0$ replace L_i everywhere by $s^n L_i$. Hence $\alpha_a(0)=\alpha_b(0)$ giving $L_a(0) = L_b(0)$ because both are on the unique $B_o(0)$.

Uniqueness is first proven on the condition L_a , L_b have no PM (phase-magnitude) crossing, defined as follows:

$$L_a \;, \; L_b \;\; \text{have a PM crossing at } \omega_x \;\; \text{iff}$$
$$L_a(j\omega_x) = L_b(j\omega_x) \;\; \text{and Sgn Arg}\left(\frac{L_a}{L_b}\right)_{\omega_x-} = -\text{Sgn Arg}\left(\frac{L_a}{L_b}\right)_{\omega_x+}$$

$$\tag{A7}$$

i.e. there is a zero crossing of $\text{Arg }(L_a/L_b)$ at ω_x .

To simplify the proof, $L_a = L_b$ tangentially, i.e. with no zero crossing of $\text{Arg }(L_a/L_b)$ also does not exist at any $\omega \in \Omega$, although they can be arbitrarily close. Another assumption, later dropped, is that after L reaches the CM portion of the transition boundary (Figures A4,5), it remains on it until the final portion $\text{Arg } L = -90e°$ (EDO in Figure A5a). Suppose $\alpha_a(0+) > \alpha_b(0+)$, then it is seen that $\alpha_a(\omega) > \alpha_b(\omega)$ and $\theta_N(\omega) > 0 \;,\forall \; \omega \in \Omega$, as follows. The first is true because

$B_\omega(\theta')$ is single-valued, so $\alpha_a = \alpha_b$ at ω_1 would require $L_a = L_b$ at ω_1, disallowed. To prove the latter, take $L_a(j\omega_o)$ as fixed with $\alpha_a(\omega_o) > \alpha_b(\omega_o)$. Inf $\theta_N(\omega_o)$, as $|1+L_b(j\omega_o)|$ ranges from zero to ∞ = Inf $\{Arg[(1+L_a)-(1+L_b)] - \alpha_a\} = 0$, at $|1+L_a| = 0$ which is an impossible value (the loop is then unstable). Hence $\theta_N > 0$, $\forall \omega \in \Omega$.

Consider (A) the case $CM > 1$ and $\omega_b < \omega_a$ (Case Al) at which, from the previous, $\alpha_a(\omega_b) > \alpha_b(\omega_b)$ with $\psi(\omega_b)$ of (A6) $> \pi$ (Figure A4). Suppose in Figure A4a, L_a reaches CM 'behind' L_b (Case Ala) meaning by this that Arg $L_b(j\omega_a) <$ Arg $L_a(j\omega_a)$. Then L_a is 'behind' L_b in this sense $\forall \omega > \omega_a$, because PM crossings and departures from CM are not allowed. It is readily seen that $\theta_N \to 0$ as $\omega \to \infty$. If L_a reaches CM 'ahead' of L_b (Case Alb), i.e. Arg $L_b(j\omega_a) >$ Arg $L_a(j\omega_a)$, then it is seen in Figure A4b that $\psi(\omega_a) > 2\pi$, L_a stays ahead for all $\omega > \omega_a$, and $\theta_N \to \pi$ as $\omega \to \infty$. The third possibility is (Case A2a) $\omega_b > \omega_a$ with L_a reaching CM ahead of L_b, giving $\theta_N \to -\pi$ as $\omega \to \infty$. The fourth (A2b) is $\omega_b > \omega_a$ and L_b ahead of L_a on CM, giving $\theta_N \to 0$ as $\omega \to \infty$. There are similarly four possibilities withing the alternative $CM < 1$, of which $\theta_N \to -\pi$ or zero, as $\omega \to \infty$, in three of them. But in only one case ($\omega_b < \omega_a$ and L_b ahead of L_a on $CM < 1$) does $\theta_N \to -0.5\pi$ e as $\omega \to \infty$. This is the only case in which the behavior of θ_N at infinity is consistent with the fact that from (A6), $|N(j\omega)| \doteq 0\omega^{-e}$, inasmuch as L_i and L_j is

each Os^{-e} , and therefore if $N(s)$ is mp and has no right half-plane poles, $\theta_N(\infty) = -0.5\pi e$. $N(s)$ is obviously stable from its definition in (A3). It is conceivable that N is nmp , but if so $\theta_N(\infty)$ must be even less than $-0.5\pi e$, because right half-plane zeros effectively contribute [5, p. 332] phase lag at ∞ . It is therefore necessary to prove that this case is impossible, as follows.

In Figure A5a at $\omega_b < \omega_a$, $L_b(j\omega_b)$ is say, at point B on CM and $L_a(j\omega_b)$ at A must be to the left of $0'X$, because $\omega_b \in \Omega$ and $\alpha_a > \alpha_b$ in Ω. Both A and B must be on $B_{\omega_b}(\theta')$ which is single-valued, and hence must be as shown. Since L_b is 'ahead' of L_a on CM , $\text{Arg } L_b(j\omega_a) < \text{Arg } L_a(j\omega_a)$, both on CM . The transition to the latter from the situation in Figure A5a with $L_a(j\omega)$ in the interior of CM for ω $[\omega_b,\omega_a)$, necessitates $\alpha_a(\omega_o) = \alpha_b(\omega_o)$ for some $\omega_o \in (\omega_b,\omega_a)$ as shown in Figure A5b . But B_{ω_o} must cross CM below B' (otherwise $L_b(j\omega_o)$ would be in a forbidden region) and pass through A' . Note that for all $\omega \in (\omega_b,\omega_a)$, B_ω must cross CM below $L_b(j\omega)$, and in addition the B_ω must be continuous (from Al). These two conditions, together with B_{ω_b} of Figure A5a, compel B_{ω_a} to have the shape shown in Figure A5b, violating its single-valuedness. Thus, this case is impossible. The conclusion is that if no PM crosssings are allowed of L_a , L_b and each stays on CM once it reaches CM then there cannot exist two such functions satisfying the optimality requirements.

Suppose that departures from CM are now permitted (but obviously only in the second quadrant). It is clear from the preceding that for CM < 1 the starting point is at the smaller of ω_a, ω_b with $\alpha_a > \alpha_b$ and $\psi > \alpha_a > 0$ at this point. In order that $\theta_N(\infty)$ be consistent with $|N(\infty)|$, it is essential that $(L_a - L_b)$ execute a clockwise rotation while satisfying all the constraints. The closest to such a rotation previously was in Figures A5a, b, invalid because point A' had to be on B_{ω_o} in the interior of CM (otherwise, since no departures from CM or PM crossings were allowed, L_a always remained 'ahead' of L_b, giving $\psi(\infty) = -\pi$ for $e = 4$ and in general $-0.5\pi e + \pi$). But now A' can be on CM, but the resulting B_{ω_o} must be as shown in Figure A6 (it would be possible but more difficult to draw B_{ω_o} if B' was on CM). The desired rotation of $(L_a - L_b)$ requires that at some $\ddot{\omega}_d$, $L_a(j\omega_d)$ depart from CM so that B_{ω_d} must pass through $L_a(j\omega_d)$, as shown in Figure A6. Also, in order that the achievment at ω_o not be lost, $L_b(j\omega_d)$ at B" must be in the shaded region and must of course not be in B_{ω_d}, necessitating B_{ω_d} as shown (it is easier to draw the latter if B" is CM). It is clearly impossible to continuously distort B_{ω_o} into B_{ω_d} without crossing any continuous curve drawn between B' and B", thus forcing $L_b(j\omega) \in B_\omega$ for some $\omega \in (\omega_o, \omega_d)$. Hence, the desired rotation of $(L_a - L_b)$ is not achievable, even if CM departure are permitted.

The same result is obtained in the case $CM > 1$, where the best chance to achieve the desired rotation of $L_a - L_b$ is when $\omega_a < \omega_b$ giving $\pi > \psi > 0.5\pi$ as in Figure A7. The zero crossing of $\alpha_a - \alpha_b$ from positive to negative is achievable if L_a is on CM - at say ω_x. But the desired rotation requires that L_a depart from CM while $\alpha_a - \alpha_b$ remains negative. B_ω must therefore advance clockwise, overtaking L_a on CM and forcing it off. However, in the process $\alpha_a - \alpha_b$ is forced to become positive again, in view of the definition of ρ_ω in II2. The desired rotation of $L_a - L_b$ is not achievable. Hence, L_{opt} is unique if no PM crossings are allowed, even if CM departures are permitted.

Let PM crossings of L_a, L_b now be permitted. If there is a single one at $\omega_x \in \Omega$, the the roles of L_a, L_b are simply interchanged in the proof above. This is so for any odd number of PM crossings in Ω, while the proof is unaltered if there are an even number of PM crossings in Ω. In the interval between ω_a, ω_b (or ω_b, ω_a) there can be no PM crossings because, by definition, one L is on CM while the second is in the interior of CM. There can, however, be PM crossings in the interval (ω_c, ∞) where ω_c is the larger of ω_a, ω_b. In view of the previous argument, PM crossings in Ω need not be considered, only PM crossings in (ω_c, ∞). At any such crossing $|N| = 0$, $\ln|N| = -\infty$, which is accompanied by an abrupt positive change [5, p. 336] in θ_N. Thus PM crossings in (ω_c, ∞) increase the value of θ_N at infinity, increasing even further the inconsistency between $|N|$ and θ_N as $\omega \to \infty$.

Proof of uniqueness is essentially the same in the nmp system. From Equation (3), the nominal loop transmission L_{no} $L_{no} = A_o(s) L_{mo}(s)$ is nmp because L_{mo}^* is mp while A_o is all-pass nmp. The boundaries B_ω^n of L_{no} have the same properties as B_ω of L_{mo} in the purely mp problem previously studied, i.e. $\exists \omega^*$, $\exists$, $\forall \omega > \omega^*$ they do not enclose the origin and lie entirely in the second and third quadrants. Also they satisfy (A1). The constraint that $B_\omega^n(\theta')$ are single-valued in θ' is also meanwhile retained. $N(j\omega)$ of (A6) is defined exactly as before with nmp L_a, L_b two candidates for the optimum L_{no}, so that each must be of the form $L_m A_o(s)$, L_m mp. Thus $N(s)$ has $A_o(s)$ as a factor. The only difference in the proof is in the relation between $|N(j\omega)|$ and $\theta_N(\omega)$ as $\omega \to \infty$. Since $|N|$ is $0 s^{-e}$ $\theta_N(\infty) = -0.5\pi e - m\pi$, where m is the number of zeros of $A(s)$. Hence the inconsistency between $|N(\infty)|$ and $\theta_N(\infty)$ which was used to prove uniqueness, is magnified in the nmp system. Therefore in the nmp case, too, there is only one L function which satisfies the optimality conditions.

Multivalued Boundaries $B_\omega(\theta')$

The final step is to remove the restriction that $B_\omega(\theta')$ is single-valued in θ'. There is no difference between the mp and nmp cases here so the mp case and notation is used in the following. It is worth noting that this restriction of single-valuedness is actually a very reasonable one in practice. It is readily seen that to obtain multivalued $B_\omega(\theta')$ it is necessary to have plant templates (ranges of V in Equation (3))

which can have at least four values for a given θ', and
system specifications $\Delta\,\ln|T| > \Delta\,\ln|L|$ and $|\frac{L}{1+L}| \gg 1$,
thereby permitting B to closely follow the boundary of the
plant template. The latter might be in special cases permitted
at relatively large ω, at which however, the plant templates
tend to be vertical lines in the Nichols chart. Hence, the
combination required for multivalued $B_\omega(\theta')$ is highly unusual.
Nevertheless this possibility is considered, for the sake of
generality.

Suppose that some B_ω are multivalued in θ', as in Figure
A8 in the sector XY. Let a single-valued approximation B_{a1}
be made of B, differing from B only for some part of this
sector, e.g. B_{a1} may be taken as ...$QEFGH_1JA_1$... in Figure A8.
Let such closed portions of B_{a1} at any ω be denoted by Δ_ω
i.e. in Figure A8, Δ consists of GH_1J. Let Δ_ω be chosen
so that the resulting $\{B_{a1\omega}\}$ satisfy (A1). This is certainly
possible if the multivalued B_ω satisfy (A1). It has been
shown that the optimum solution L_{opt1} to the $\{B_{a1\omega}\}$ problem
exists and is unique, if any solution at all exists. If this
L_{opt1} does not lie on Δ_ω at any ω (e.g. it is at A_1 in
Figure A8), then it is also the unique optimum solution to the
original multivalued problem. This is seen as follows. Suppose
that B_{a2} consisting of ...$E'F'IJ$... had been used, instead of
B_{a1} (and similarly for all other ω in such a manner that
conditions (A1) are preserved). Clearly the resulting L_{opt1}
would be precisely the same as before, as otherwise the
optimum for the single-valued case would not be unique. In this
manner every part of the multivalued boundaries in the $XO'Y$

sector can be given an opportunity to be part of a B_{ai} ,
without affecting the original unique L_{optl} .

On the other hand, suppose that on some $B_{al\omega}$, L_{optl}
lies in the interior of EFG-type segment e.g. at A_2 in
Figure A8, but there is no ω for which L_{optl} lies on the
GH_1J-type segment of $B_{al\omega}$. Then L_{optl} is again the unique
optimum for the original multivalued problem, seen as follows.
Let $...QE'EG'HGH_1JA_1..$ (and corresponding ones at other ω ,
in such a manner that conditions (A1) are satisfied), be used in
B_{a2} . Then as before, L_{optl} would still be the unique
optimum, the essential point being that a part of the boundary
containing A_2 as an interior point, is common to B_{al} and B_{a2} .
In this manner, all parts of EFGHIJ (except for the isolated
points A_2' , A_2'') have their chance to be parts of the boundary,
without affecting the result. An infinitesimal modification
of B_{al} at A_2 would result in an infinitesimal change of L_{optl}
and permit A_2'' (or A_2') to have its chance etc.

Clearly, one need reconsider the problem only if for some ω
values L_{optl} lies on Δ_ω , e.g. the closed GH_1J -type portion
of $B_{al\omega}$. As $L(j\omega)$ and the Δ_ω are continuous in ω , and
B_o , B_∞ are single-valued, L_{optl} must be in Δ_ω over some
finite interval $[\omega_x , \omega_y]$. It is necessary to modify the $B_{a\omega}$
until there is no ω for which L_{opt} lied on Δ_ω . It will be
clear from L_{optl} how to proceed. For example, suppose L_{optl}
is in the A_1J-type protion in Figure A8, for ω immediately
less than ω_x and more than ω_y , then clearly the true L_{opt}
wants to be in the JIH-type portion and return to the JA_1-type.

The $B_{a\omega}$ should be modified accordingly (while preserving conditions (A1) etc) to permit it to do so. If L_{opt1} traverses Δ , e.g. is in a JA_1-type segment for $\omega < \omega_x$ and in a GF-type segment for $\omega > \omega_y$, then L_{opt} wants to go right through the multivalued portion, and the $B_{a\omega}$ must be modified to permit it to do so. In this manner an L_{opt} can be conceptually obtained which lies on no Δ_ω except possibly at its boundary points. From the previous discussion, such an L_{opt} is unique.

APPENDIX 4

ON THE EXISTENCE OF A SOLUTION TO THE NMP PROBLEM

As noted previously, there may be no solution to the nmp problem. It is convenient here to work with the mp L_{mo} of Equation (3) and the shifted B_ω (see Example in II and Figure 2). Following the discussion there, it is shown here, that if an L exists which lies on the B_ω , it must have turned the corner at V , before Arg L is zero.

Form Equation (A3), it is easily found that

$$\frac{\pi}{2}\,[R(\omega_2)-R(\omega_1)] = \frac{\pi}{2}\{\,[R(\omega_2)-R(0)]-[R(\omega_1)-R(0)]\}$$

$$= \int_0^\infty X(\zeta)\lambda(\zeta)d\zeta \; , \quad \lambda(\zeta) = \frac{\zeta\,(\omega_1^2-\omega_2^2)}{(\omega_1^2-\zeta^2)(\omega_2^2-\zeta^2)} \qquad . \qquad (A8)$$

where $R = \ell n\,|L|$, $X = \mathrm{Arg}\,L$. It is assumed that the high-frequency boundaries $B_{h\omega}$ are as shown in Figure 2, i.e. have the same shape but are shifted by Arg $A(j\omega)$. Suppose that at

ω_1 , $\underline{/L(j\omega_1)} = 0^\circ$, but $|L(j\omega_1)|$ has not managed to decrease to the corner point V which would have permitted it to turn the corner and become negative in phase. Hence, for $\omega > \omega_1$, $\underline{/L}$ must therefore be positive. If a solution L exists, then at some $\omega_2 > \omega_1$, $\underline{/L}$ must cross from positive to negative values and remain negative, eventually reaching $-0.5\,\pi e$. Hence, in (A8), . $X(\xi) < 0$, $\zeta \in (\omega_2,\infty) = I4$, > 0 in $(\omega_1,\omega_2) = I3$, and may have both positive and negative values from zero to scme ω_a (I1) and must be negative from ω_a to ω_1 (I2) . In (A8), $\lambda(\zeta) < 0$ in I1, I2 ; > 0 in I3, < 0 in I4. Hence the net effect in each of I2, I3, I4 is positive, while in I1 there may be a net negative effect, which it will be soon shown, is overwhelmed by the others in any realistic problem. Hence $|L(j\omega_2)| > |L(j\omega_1)|$. However, $|L(j\omega_1)| > V$, and since $|L(j\omega_2)| > |L(j\omega_1)|$ there is no possibility whatsoever that the corner may be turned at $\omega_2 > \omega_1$, unless the B_{ha} change in shape and move upwards in Figure 2, as ω increases from ω_1 to ω_2 .

To prove that the effect of I1 even if negative, is overwhelmed by the others, consider $R(\omega_1)-R(\omega_a)$ using Equation A8 with ω_1 , ω_a replacing ω_2 , ω_1 respectively, with the same $X(\zeta)$ but λ replaced by $\lambda_1(\zeta) = \dfrac{\zeta(\omega_a^2-\omega_1^2)}{(\omega^2-\zeta^2)\,(\omega_1^2-\zeta^2)}$, whose . sign is opposite to that of $\lambda(\zeta)$ in (ω_a,ω_2) , and the same in $(0,\omega_a)$, (ω_2,∞) . In (ω_2,∞) $|\lambda_1(\zeta)| < |\lambda(\zeta)|$, while in $(0,\omega_a)$, $|\lambda_1(\zeta)| > |\lambda(\zeta)|$, which is the only interval in which

$X(\zeta)\lambda(\zeta) < 0$. Hence if this interval suffices to make

$$\int_0^\infty X(\zeta)\lambda(\zeta)d\zeta < 0 \ , \ \ \text{it certainly makes} \ \int_0^\infty X(\zeta)\lambda_1(\zeta)d\zeta < 0 \ ,$$

giving $|L(j\omega_a)| < |L(j\omega_1)|$. This is inconceivable in any realistic feedback problem because it involves a region of positive phase angle in the low ω range (unusual but not inconceivable) up to ω_a , but in addition requires that at ω significantly $>$ ω_a , the bounds on L require $|L(j\omega_1)|$ significantly $>$ $|L(j\omega_a)|$. This is therefore rejected.

In practice in a specific problem, how may one determine apriori whether L can reach V before arg L is zero? A necessary condition is given here. Let $L = L_1 M$ with $\text{Arg } L_1 = \text{Arg } L_1(j\omega_1) = 0$ for $\omega > \omega_1$, $\theta_M = \text{Arg } M = 0$ $\omega \leqslant \omega_1$, $\theta_M \leqslant 0$ for $\omega > \omega_1$, $|M(0)| = 1$. From equation A3, $\text{Sgn}\{\ln|M(j\omega_1)|-\ln|M(0)|\} =$ $\text{Sgn} \int_0^\infty \theta_M(\zeta)\mu(\zeta)d\lambda$ with $\mu(\zeta) = [\zeta(\omega_1^2-\zeta^2)]^{-1} > 0$ in $(0,\omega_1)$, <0 in (ω_1,∞) , so that $|M(j\omega_1)| > |M(0)| = 1$. Hence $|L(j\omega_1)| > |L_1(j\omega_1)|$. Therefore $L(j\omega_1)$ certainly cannot reach the the level V if $L_1(j\omega_1)$ cannot do so. One can therefore search for such a L_1 function. Define the optimum L_1 as that which satisfies B_ω for $\omega \in [0,\omega_1]$ with minimum value of $|L_1(j\omega_1)|$, and the constraint $\underline{/L_1} = \underline{/L_1}(j\omega_1) = 0$, $\omega \geqslant \omega_1$. Then it can be proven by the same methods as in Appendix 1 that L_{opt} lies on B_ω for each $\omega \in [0,\omega_1]$. Therefore in the appropriate ω range, specify $\text{Arg } L_1$ to have the values dictated by the right hand vertical parts of $B_{h\omega}$, and use a reasonable estimate of $\text{Arg } L_1$ in the lower ω range.

It is fairly easy to answer the above question if a flexible computer program is available for determining $L(j\omega)$ which lies on the B_ω. In a nmp problem for which no L exists which lies on the B_ω and goes to zero as $\omega \to \infty$, such a program gives an $L(j\omega)$ whose phase becomes positive before L has decreased to V. The program must break down at some point thereafter, if its angle at large ω must be $-90e^{o}$.

APPENDIX 5

SOME TIME DOMAIN PROPERTIES OF NMP SYSTEMS

This appendix presents some properties of nmp systems which are helpful in assigning realistic time-domain bounds and to their frequency-domain equivalent. An important effect of the all-pass factor $A(s) = (1-\tau s)/(1+\tau s)$ in $T_n(s) = T_m(s) A(s)$, T_m mp , is that of 'time-delay'. $A(s)$ is a first approximation to the pure time delay $e^{-2s\tau}$, reasonably accurate up to $\omega \approx 2/\tau$. This suggests and it is experimentally confirmed that if the bandwidth of the mp $T_m(j\omega)$ is $< 2/\tau$, then the nmp time response is closely that of the mp , delayed by 2τ . The accuracy of this approximation increases with time and it is of course, not valid for $t < 2/\tau$, the step response being negative over approximately this interval. Similar time-delay estimates may be made for higher order all-pass functions, by finding the frequency range for which $\text{Arg } A(j\omega) \approx \omega\tau_d$, τ_d the equivalent time-delay.

In the nmp system the step-response $c_m(t)$ may begin with undershoot i.e. $c_n(0+) < 0$. This is obvious from the initial-value theorem, if $A(s)$ has an odd number of rhp zeros. In fact, if the mp system has the typical continuously differentiable unit step response with $c_m(0) = 0$, $c_m(0+) > 0$, $c_m(\infty)$ a positive constant, then the nmp system with step response $C_n(s) = (1-\tau_1 s) \ldots (1-\tau_x s) C_m(s)$ with all τ_i positive real, has at least x zero crossings and $c_n(0+) > 0$ if x is even, < 0 if x is odd. This is proven by induction.

Let $x = 1$, $c_m(0+) > 0$ and then from the initial value theorem $c_n(0+) < 0$. For large enough t , $\dot{c}_m(t)$ is negligibly small while $c_m(t) > 0$. Hence, at large enough t, $c_n(t) > 0$ and so must have at least one zero crossing. In the general case, write $C_n(S) = (1-\tau_1 s) C_y(s)$, so $c_n(t) = 1-\tau_1 \dot{c}_y$. Suppose the proposition is true for $x-1$, so that $c_y(t)$ has at least $n-1$ zero crossings, while $\dot{c}_y(t)$ must have at least one zero crossing between any two of $c_y(t)$. Also $\text{Sgn } c_n(0+) = - \text{Sgn } c_y(0+)$ by the initial value theorem. Hence $c_n(t)$ must have at least x zero crossings. The proposition was proven true for $x = 1$, so is true for all positive integer x .

The maximum magnitude of the negative undershoot in the nmp unit step response is of interest. For the case of a single rhp zero at $1/\tau$, it was found experimentally that the following approximate relation gave fair results, $M = 0.1\tau\omega_{-6}$, where ω_{-6} is the -6db point of $|T_n(j\omega)|$. Thus, to reduce M for a given τ , one must decrease the system bandwith and thereby its speed of response. This empirical result can be theoretically justified by noting that in the small t interval where the peak undershoot occurs, the step response of the mp system $T_n =$ $T_n = (1-s\tau) T_m$, can be well approxiamted by $h_m = At^r$, so $h_n = h_m - \tau\dot{h}_m$ $c_n = \int_0^t h_m \, d\xi - \tau h_m$. At the instant t_u of peak undershoot $\dot{c}_n(t_u) = h_n(t_u) = 0$, giving $t_u = n\gamma$ and $M = -c_n(t_u) = \dfrac{An^n \tau^{n+1}}{n+1}$. In considering alternatives for $h_m = At^n$ it is reasonable to demand that at some t_o larger than t_u ,

the mp step response $\int_0^{t_o} h_m \, d\varsigma$ have the same value K, i.e.

$\dfrac{At_o^{n+1}}{n+1} = K$. Susbstituting in the above gives $M = K\left(\dfrac{\tau}{t_o}\right)^{n+1} n^n$,

with n the only free variable, and M a minimum at $n < 1$.
This value of n is not feasible because $n+1$ is the excess
of poles over zeros of $T_m(s)$ and this excess must at least equal
that of the plant.

Acknowledgement

REFERENCES

1. I. Horowitz, "A synthesis theory for linear time-varying feedback systems with plant uncertainty", IEEE Trans. Automat. Control AC-20, pp. 454-464, Aug. 1975.

2. I. Horowitz and M. Sidi, "Synthesis of feedback systems with large plant ignorance for prescribed time-domain tolerences", Int. J. Cont. 16, No. 2, pp. 287-309, 1972.

3. I. Horowitz, "Synthesis of feedback systems with nonlinear time-varying uncertain plants to satisfy quantitative performance specifications", Proc. IEEE 64, No. 1, pp. 123-130, Jan. 1976.

4. I. Horowitz, "Optimum loop transfer function in single-loop minimum-phase feedback systems", Int, J. Contr., vol. 18, no. 1, pp. 97-113, 1973.

5. I. Horowitz, Synthesis of Feedback Systems. New York: Academic, 1963.

6. M. Sidi, "Synthesis of feedback systems with large plant ingnorance for prescribed time-domain tolerances", Ph.D. dissertation, Dept. Appl. Math., Weizmann Institute of Science, Rehovot, Israel, 1974.

7. H. W. Bode, Network Analysis and Feddback Amplifier Design, Van Nostrand, N.Y. 1945.

8. I. Horowitz and M. Sidi, "Synthesis of cascaded multiple-loop feedback systems with large plant parameter ignorance, Automatica 9, pp. 589-600, 1973.

9. I. Horowitz, U. Shaked, S. Golde, "Synthesis of multivariable, basically non-interacting systems with significant plant uncertainty" Automatica 12, pp. 61-72, 1976.

10. S. Karlin, Total Positivity, Stanford University Press, 1968.

11. S. Karlin and W. Studden, Tchebycheff Systems with Applications in Analysis and Statistics, Interscience, N.Y. 1966.

12. P.J. Davis, Interpolation and Approximation. Blaisdell, N.Y.
     1963.

13. Z. Nehari, COnformal Mapping, McGraw-Hill, N.Y., 1952.

14. S.S.L. Chang, Synthesis of Optimum Control Systems, McGraw-Hill,
     N.Y. 1961.

15. D. Youla, J. Bongiorno, H. Jabr, Modern Wiener-Hopf design of
     optimal controllers, IEEE Trans. Automat. Control, AC-21, pp.3-31,
     1976.

Abbreviations

ℓhp   left half-plane

ℓti   linear time-invariant

ℓtv   linear time-varying

mp   minimum-phase

nmp   nonminimum-phase

rhp   right half-plane

d(t)
D(s)

R(s)
r(t)

G(s)

P(s)

c(t)
C(s)

−1
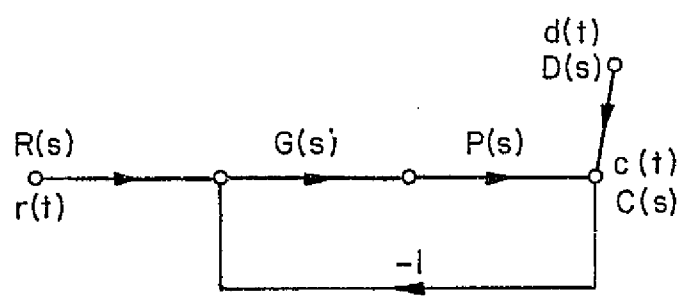
1.   Two-degree-of-freedom feedback structure.

(DB)

20

W'

10

$\left|\frac{L}{j+L}\right| = 2.3\,DB$

W"

-90e

-180

B_h^n

-10

W₂

W₁

W

N

$B_{h_{10}}$

-20

Q

V

V_{10}

Y

Z

B_4^n B_4

V(j4)

-90

0   (DEGREES)

$B_{h_{20}}$

$B_{h_{30}}$

V_{30}
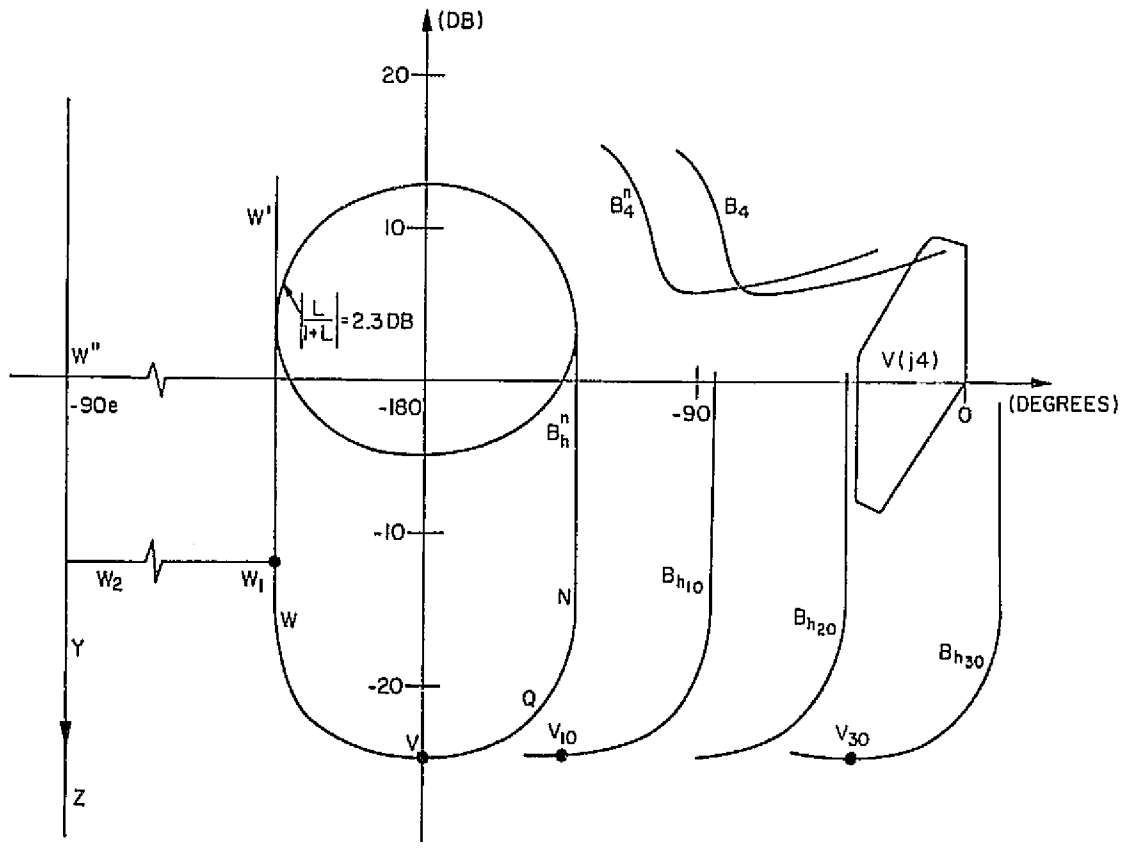
2.   Boundaries of mp and nmp L which satisfy system specifications
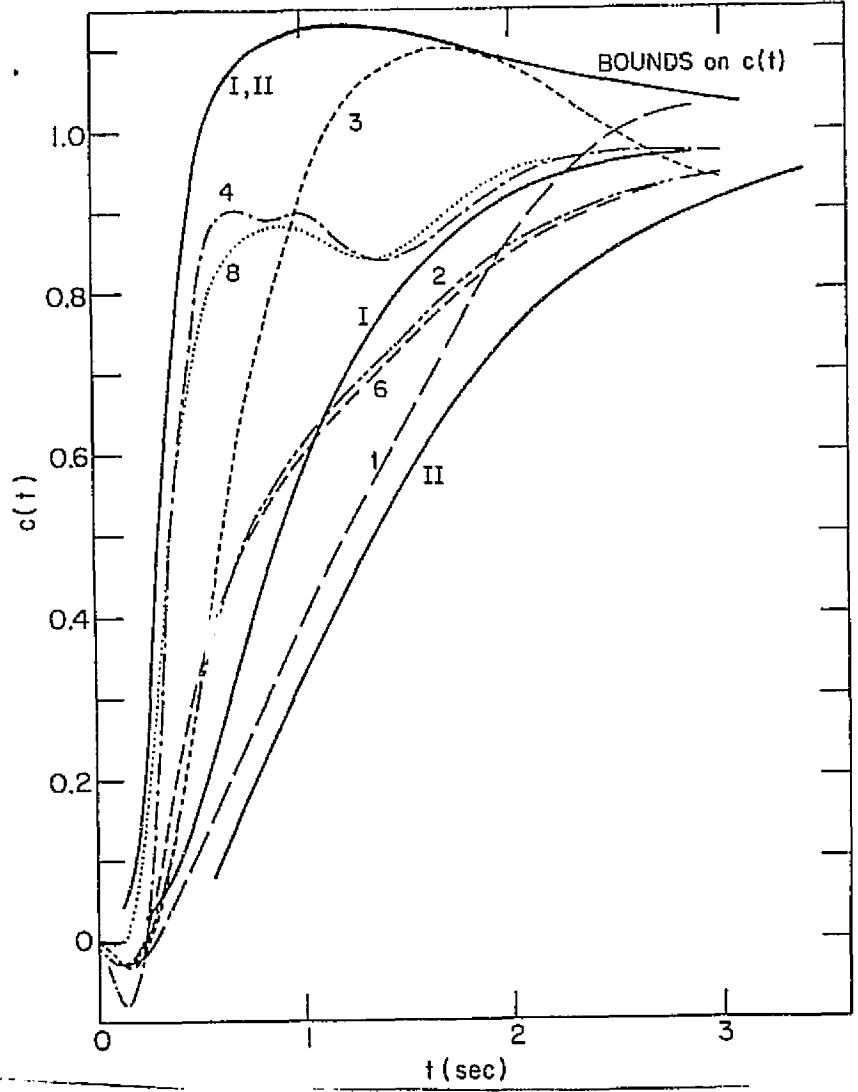
3. Example 1: Bounds on step response and the equivalent bounds on mp |T(jω)|

case	k	d	b
1	1.	0.1	1.
2	1.	0.1	0.3
3	3.	0.1	1.
4	3.	0.1	0.3
5	1.	0.05	1.
6	1.	0.05	0.3
7	3.	0.05	1.
8	3.	0.05	0.3
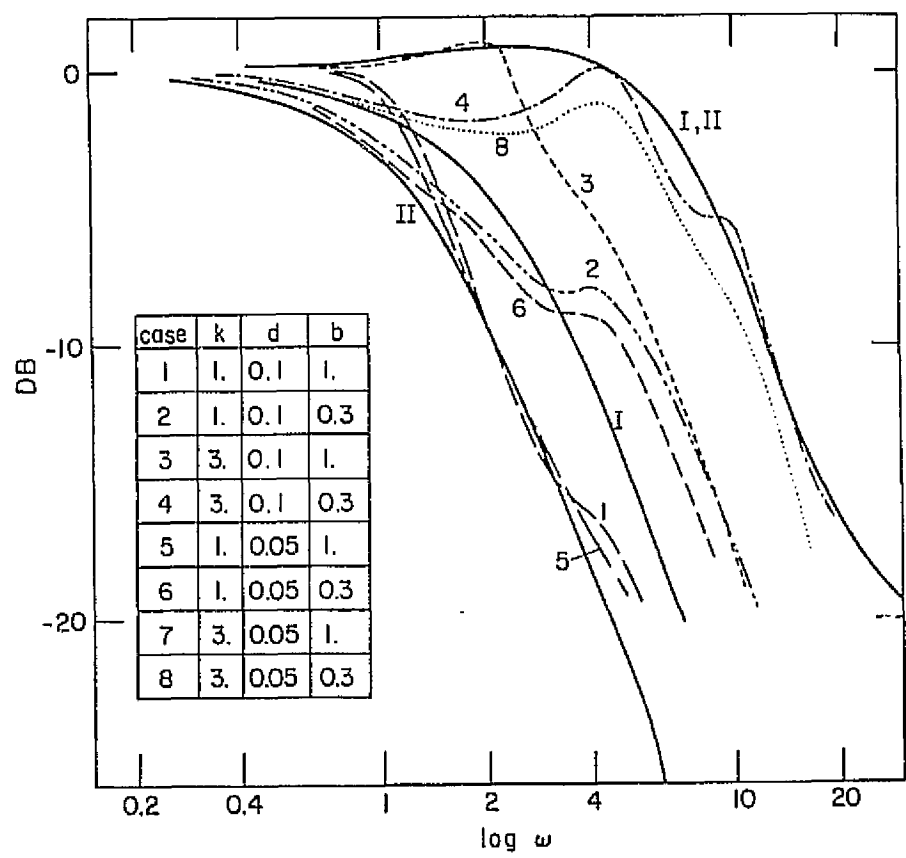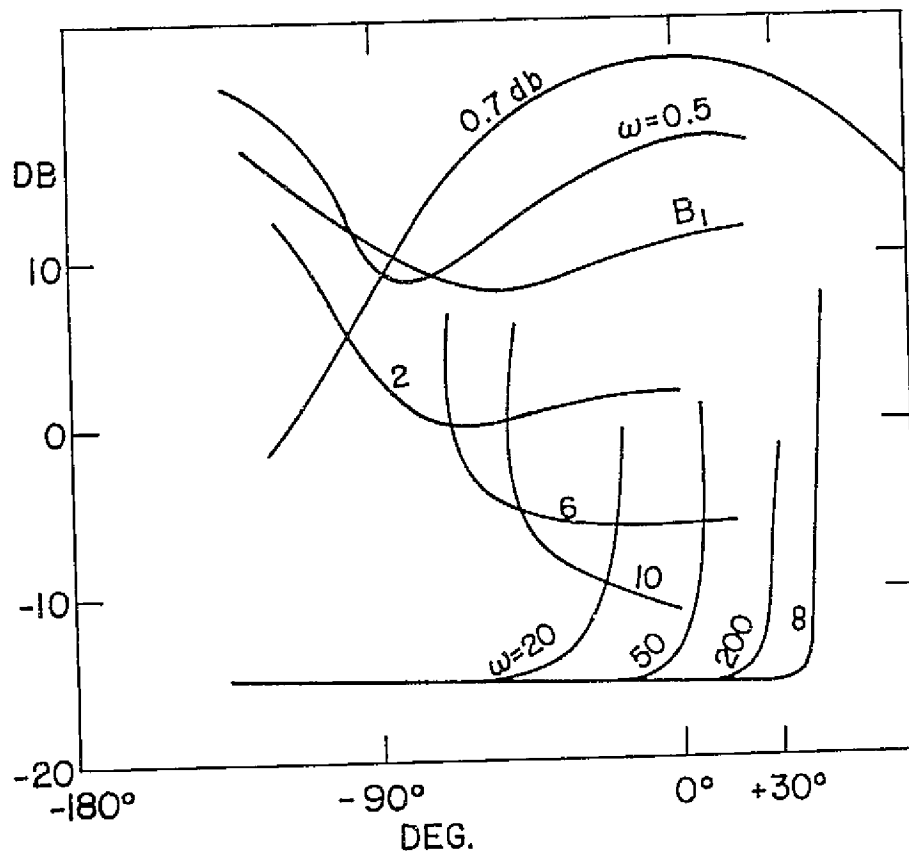
4a.   Example 1:   Boundaries of $L_{mo}(j\omega)$ – Specifications I .
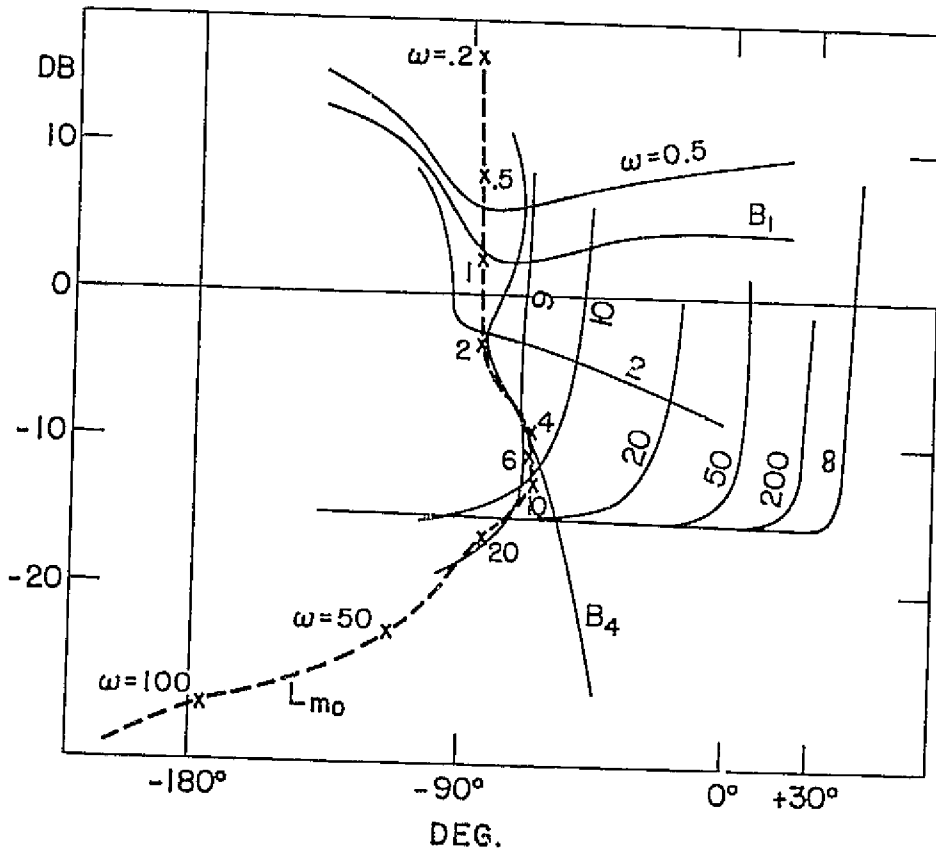
4b. Boundaries of $L_{mo}(j\omega)$ - II
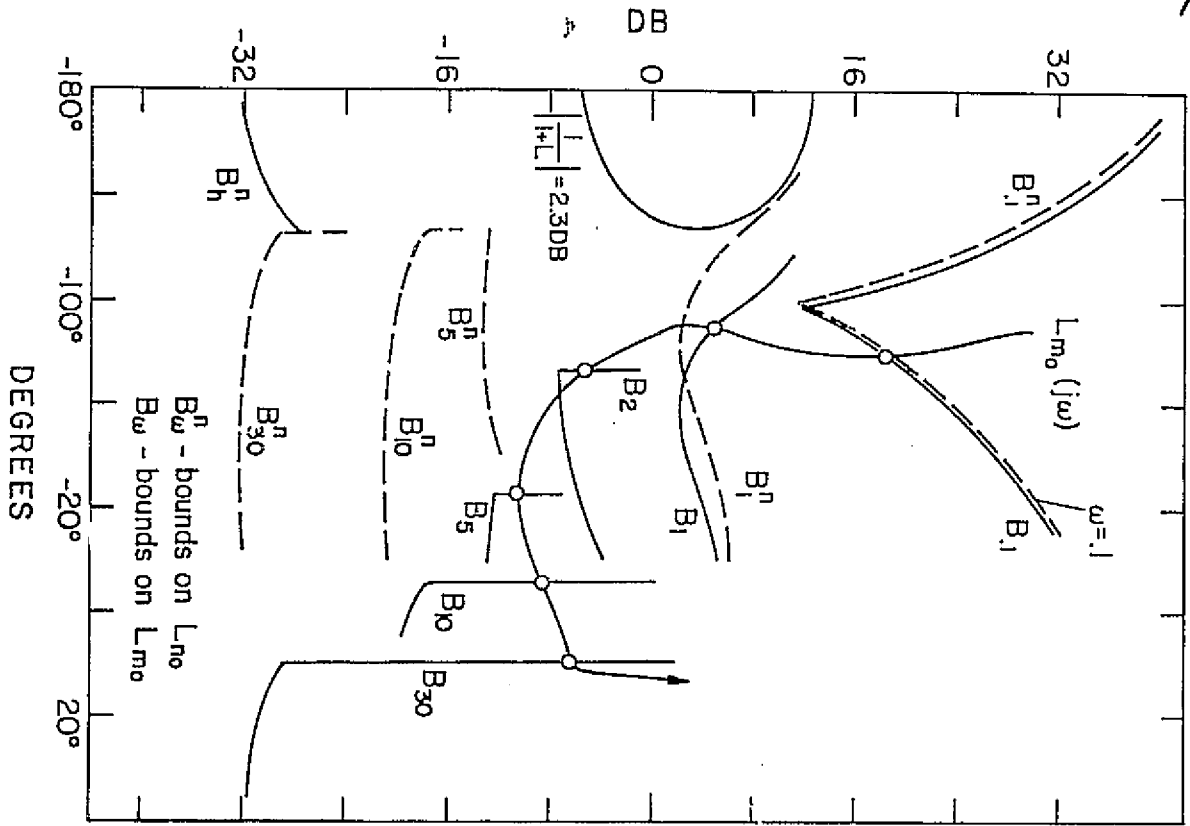
5a. Example 2: Unachievable design bounds, $\tau = 1/4$.
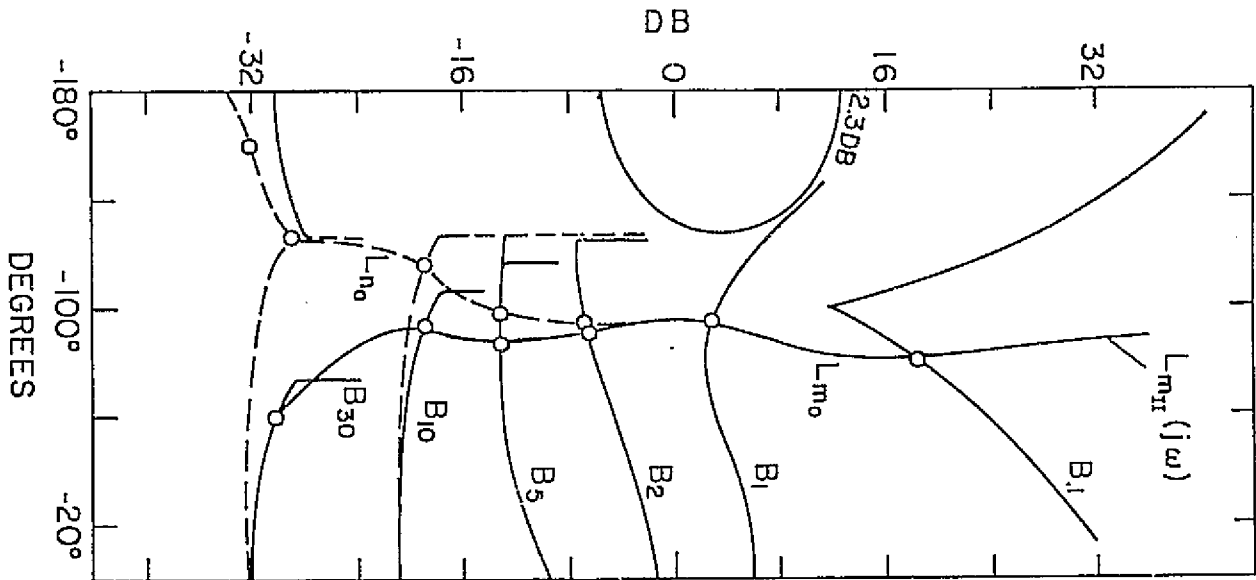
5b. Achievable design bounds, $\tau = 1/60$.
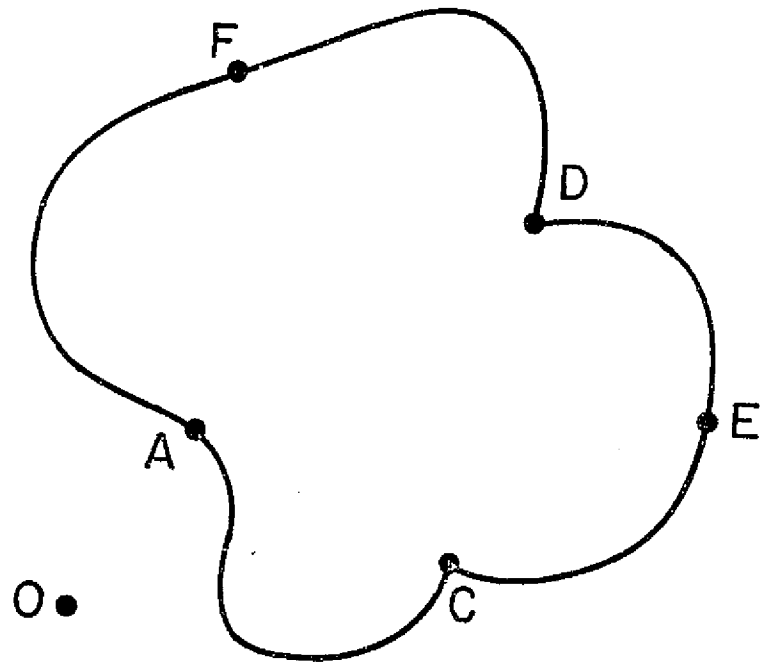
Fig. A1

A1.   Hypothetical boundary B_{ω_1} on $L(j\omega)$.
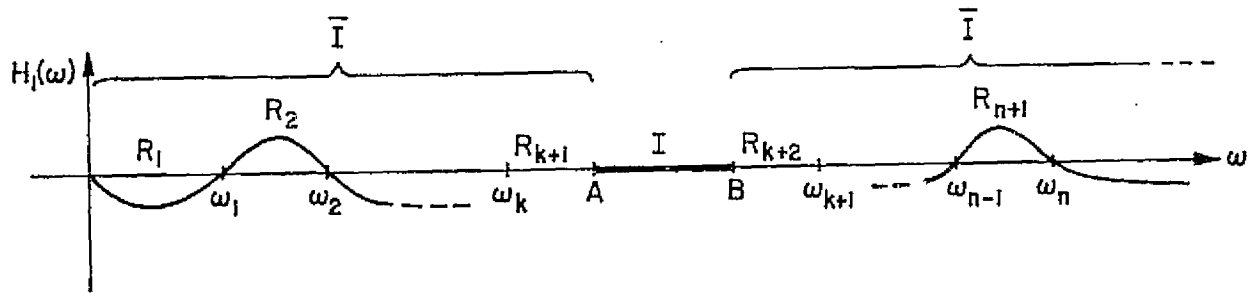
Fig. A2

A2. Zero crossings of a simple building block $\ell n H$.
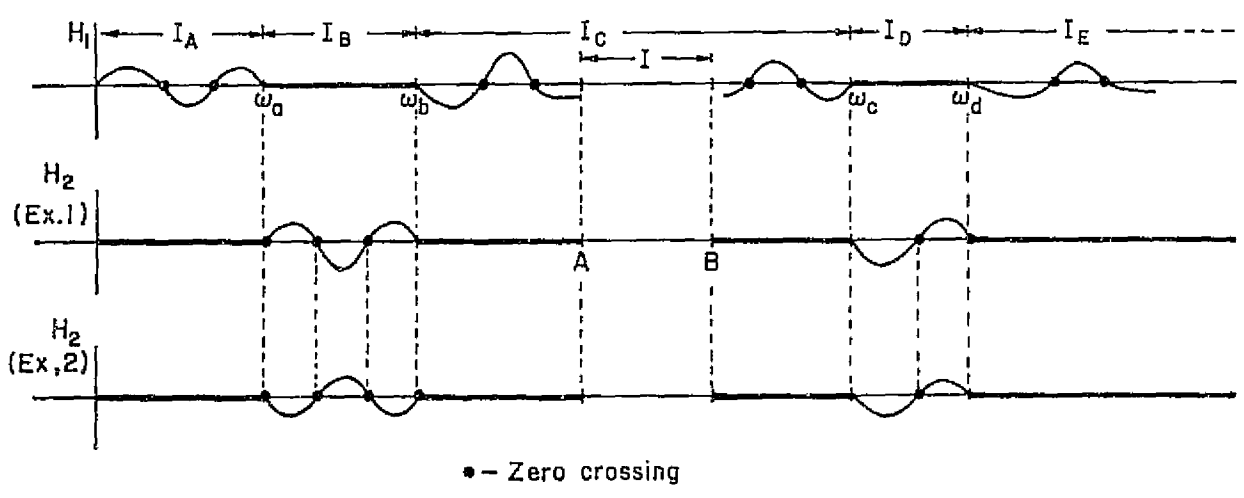
●— Zero crossing

Fig. A3

A3. Zero crossings of two complex building blocks.

Figure A4a. Case A1a.

Fig. A4b

Figure A4b. Case A1b.

Proof of unique L_{opt} , CM > 1 .
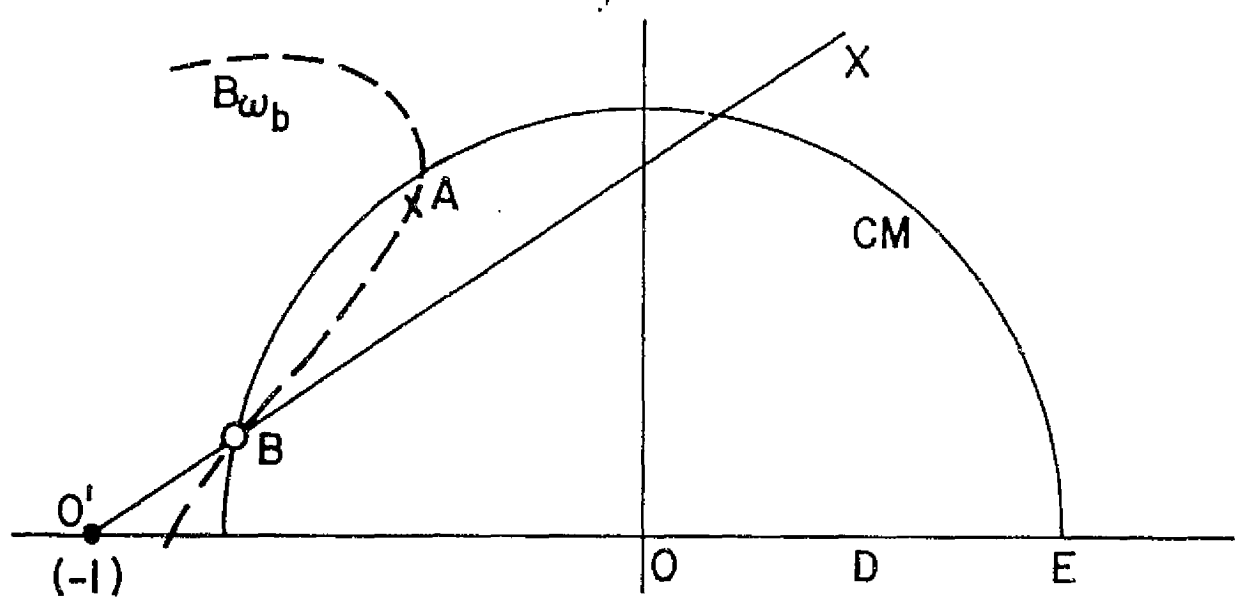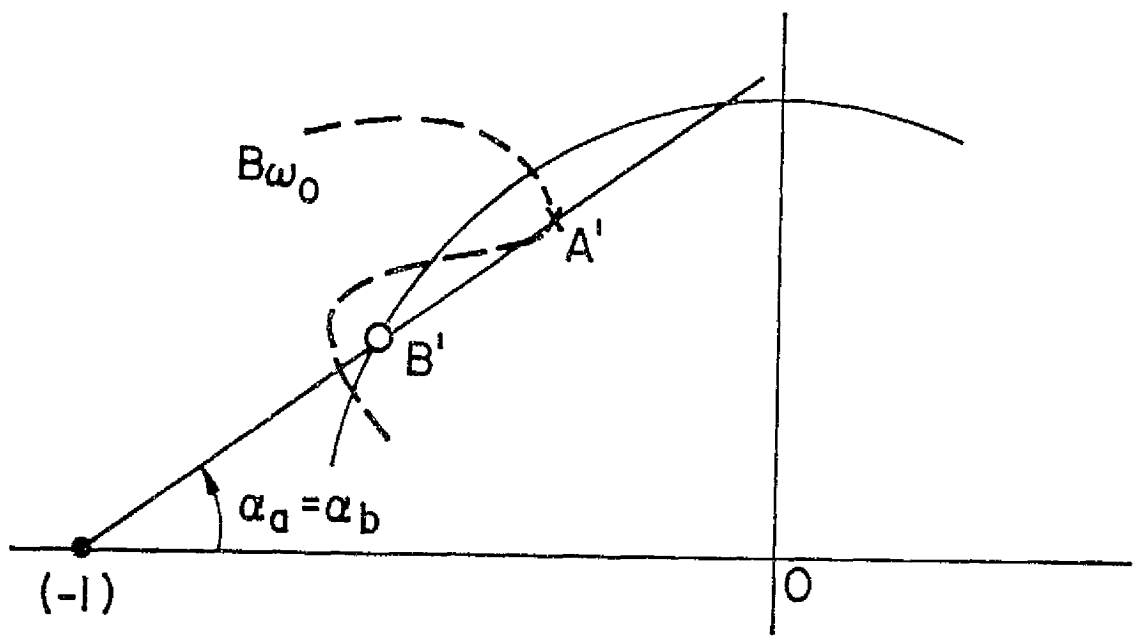
Fig. A5a

Fig. A5b

Figure A5.   Proof of unique L_{opt} , CM < 1 .
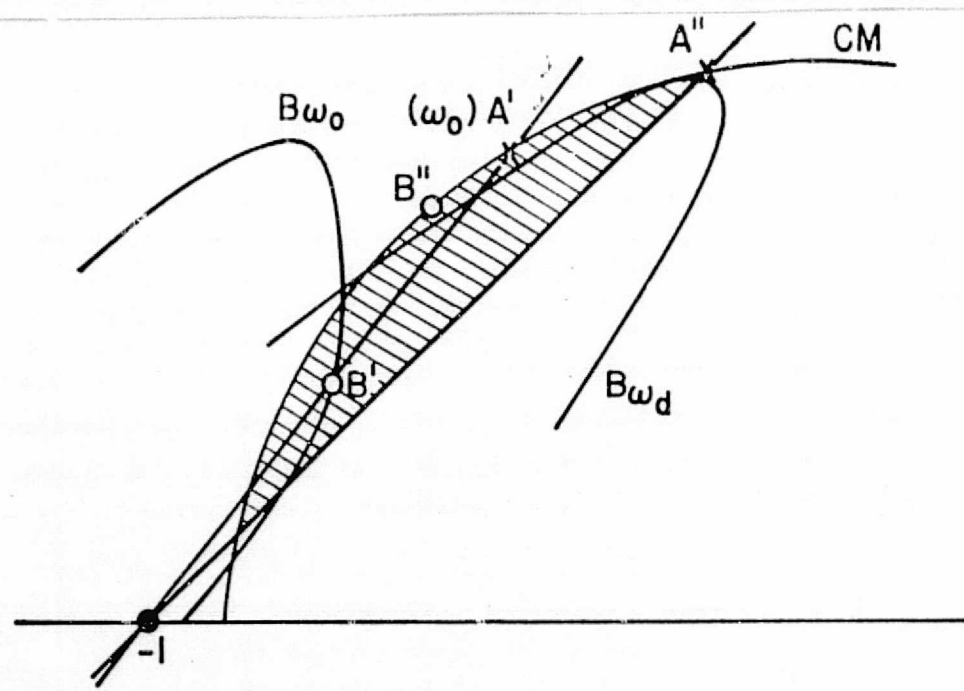
Figure A6. Proof of unique L_{opt} , CM departure permitted, CM < 1 .
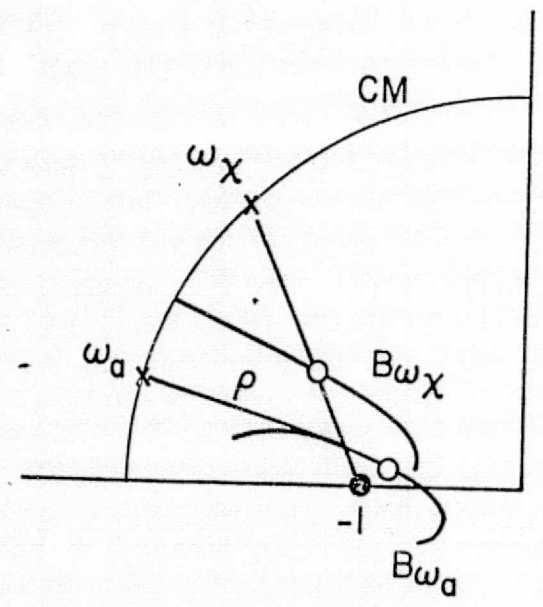
Fig. A7

Figure A7. Proof of unique L_{opt} , CM departure permitted CM > 1.
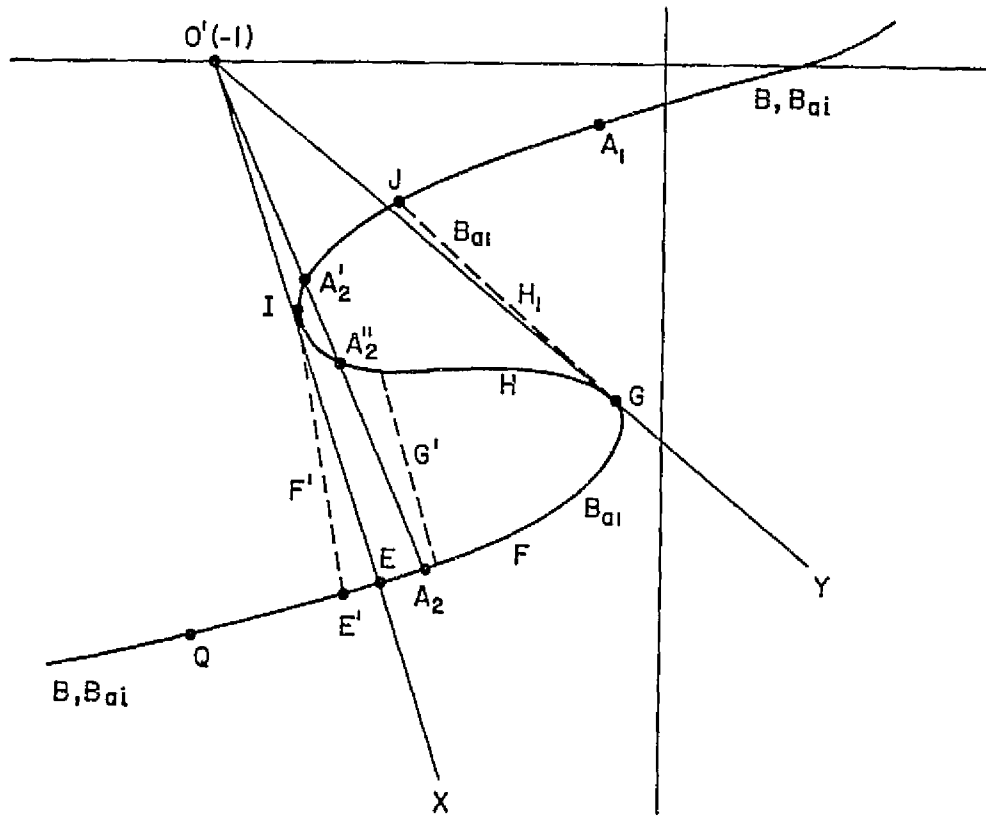
Figure A8. Proof of unique L_{opt} , multivalued boundaries.

APPENDIX 3

Translation of Time-Domain Bounds on Unit Step Response

Into Equivalent Frequency-Domain Bounds

The results given here are largely empirical and were obtained by examining
the works of many authors dealing with the problem and by considerable experi-
mental work with many additional sample responses. Definition of the parameters
of ω and t-domain parameters are given in Figs. A3-1a,b.

Time delay = $t_{.1}$ (Time for unit step response to reach 0.1)

Our empirical relation is

$$\omega_{.1} \, t_{.1} \approx A_{.1} \tag{A3.1}$$

where $A_{.1}$ is a function of the slope of $|T(j\omega)|$ for $\omega > \omega_{.1}$, the latter
being the frequency at which the normalized frequency response is 0.1 (-20 db)
in magnitude. The relation is shown in Fig. A3-2. This result is important in
our design problem because it is essential that at high-frequencies the tolerance
in $|T(j\omega)|$ be greater than the high-frequency plant uncertainty. This requires
'widening' the nigh-frequency $|T(j\omega)|$ bounds, and one should know that the
price paid is in the time delay.

Rise time, $t_r = t_{.9} - t_{.1}$ ($t_{.9}$ = time for step response to reach 0.9)

Let n_1 be the average slope factor of $|T(j\omega)|$ for $\omega_{-10} > \omega > \omega_{-3}$
(defined by $|T(j\omega_{-3})| = -3$ db, and ω_{-10} similarly defined). The results are
given in Fig. A3-3. Among the samples used to obtain these results are those

shown in Figs. A3-4 to 6 which vary very widely in shape. The agreement is within ±10%.

Relative Overshoot V

A useful result is given by Papoulis [A1] for $|T(j\omega)|$ having a single peak at ω_p and steady state response of unity,

$$1 + V \leq 1.18 \frac{|T(j\omega_p)|}{|T(0)|} \tag{A3.2}$$

Here we are also interested in the instant of peak overshoot denoted by t_p. The following empirical relation has been found by examining a large number of examples, 'or the case $|T(j\omega_p)| \leq 12$ db, with accuracy of ±20%,

$$(t_p - t_{.9})(\omega_p + \omega_o) \approx K_p \tag{A3.3}$$

with K_p a function of $|T(j\omega_p)|$ given in Fig. A3-7. In Eq. (A3.3), ω_o is the value at which $|T(j\omega)| = 1$, normally $\omega_o = 0$ in servo systems. The above is valid for cases in which there is overshoot but no 'oscillation' (see below). Note that Eq. (A3.3) is used to find t_p, since $t_{.9}$ is known from the previous. Fig. A3-8 gives the peak response $(1 + V)$ as a function of $\Delta\omega|T(j\omega_p)|/\omega_p$, where $\Delta\omega = \omega_2 - \omega_1$ has been defined in Fig. A3-1a. The accuracy of Fig. A3-8 is within ±15%.

Oscillations in step response

A criterion for existence of 'oscillations' is difficult to define. For example, we do not here regard (a) in Fig. A3-9 as oscillatory but do so for (b). Roughly, there should be two discernible cycles with almost the same period. But of course we are here concerned with oscillations like (b) in Fig. A3-9, and not

like (c). The latter is associated with the peak at ω_{p2} in Fig. A3-10, the former with the peak at ω_{p1}. A great number of samples were taken with the result shown in Fig. A3-11, where the 'peaking criterion'

$$\text{P.C.} \triangleq \frac{\Delta\omega_i}{\omega_p} \; |T(j\omega_{p1})| \tag{A3.4}$$

$\Delta\omega_i$ = smaller of $\Delta\omega_1$, $\Delta\omega_2$, $\Delta\omega_i = \omega_p - \omega_i$ defined in Fig. A3-1a. From Fig. A3-11, oscillations exist only if P.C. $\tilde{<}$ .07.

Nonminimum-phase systems

Appendix 5 of Appendix 2 presents some results for one or more right half-plane zeros in the system transfer function $T(s)$. Here we only present an empirical relation for the peak negative undershoot which occurs in the initial part of the unit (positive) step response, when the system has a single right half-plane zero at 1/a. The results are presented in Figs. A3-12a,b. Here ω_{-k} refers to the -k db frequency for $T(s)$ of the form $T(s) = T_M(s)\frac{(1 - as)}{(1 + as)}$, T_M minimum-phase. The error Δ/MUS in the figures refers to the respective error obtained if the following relations are used

$$\begin{aligned} \text{MUS} &= (0.1)a\omega_{-6} \\ \text{MUS} &= (.14)a\omega_{-3} \end{aligned} \tag{A3.5a,b}$$

MUS is the maximum undershoot.

References

A1. A. Papoulis, The Fourier Integral and its Application, McGraw-Hill, New York, 1962.
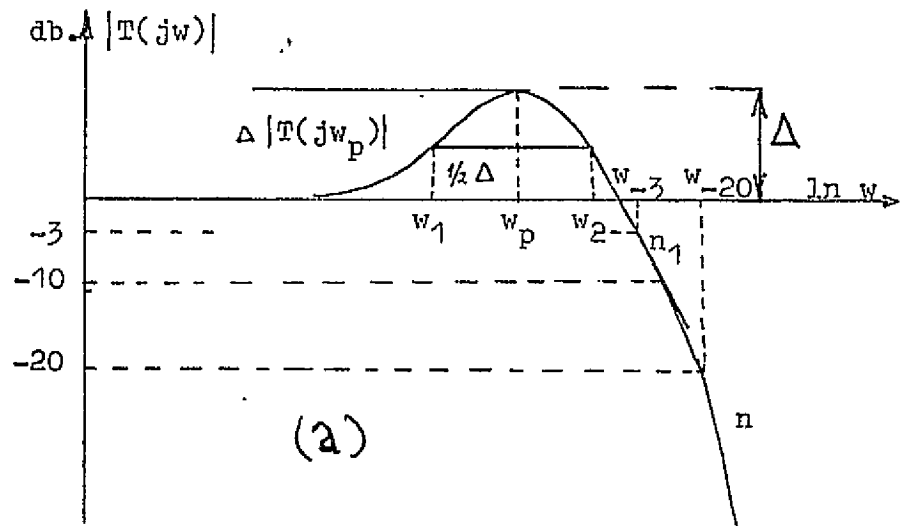
Fig. A3-1. Definition of
time and frequency domain
parameters
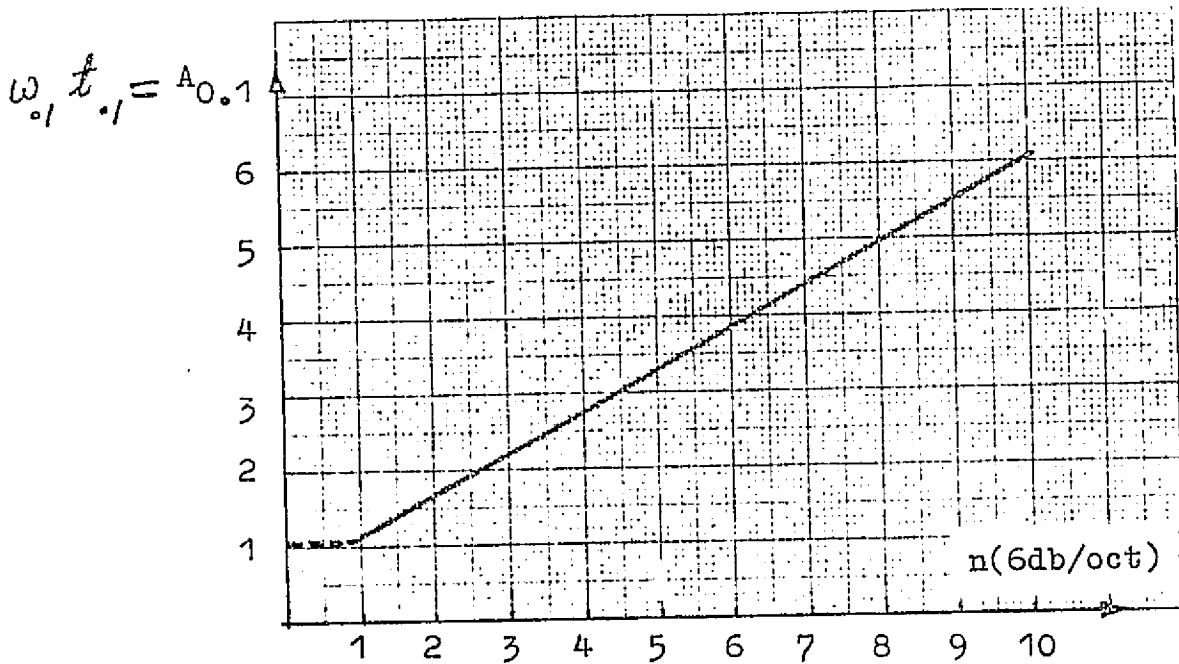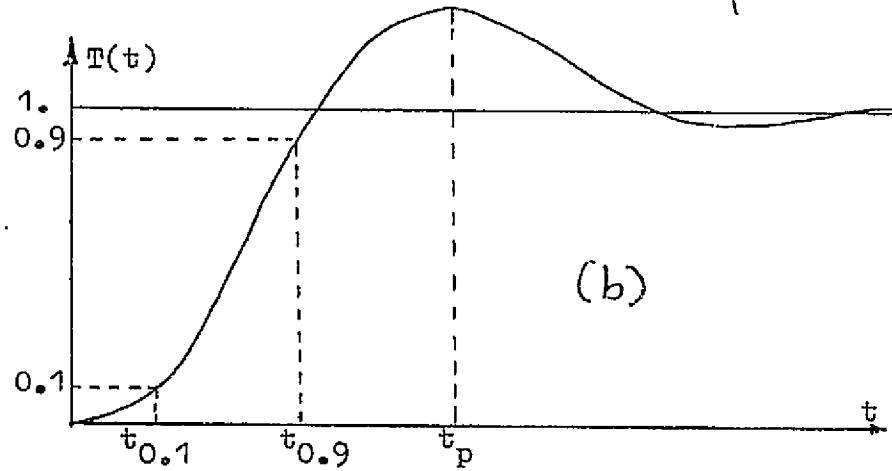
Fig. A3-2. $A_{.1}$ as a function of n

Fig. A3-3. K
vs. $|T(j\omega)|_{max}$
with n_1 as
parameter
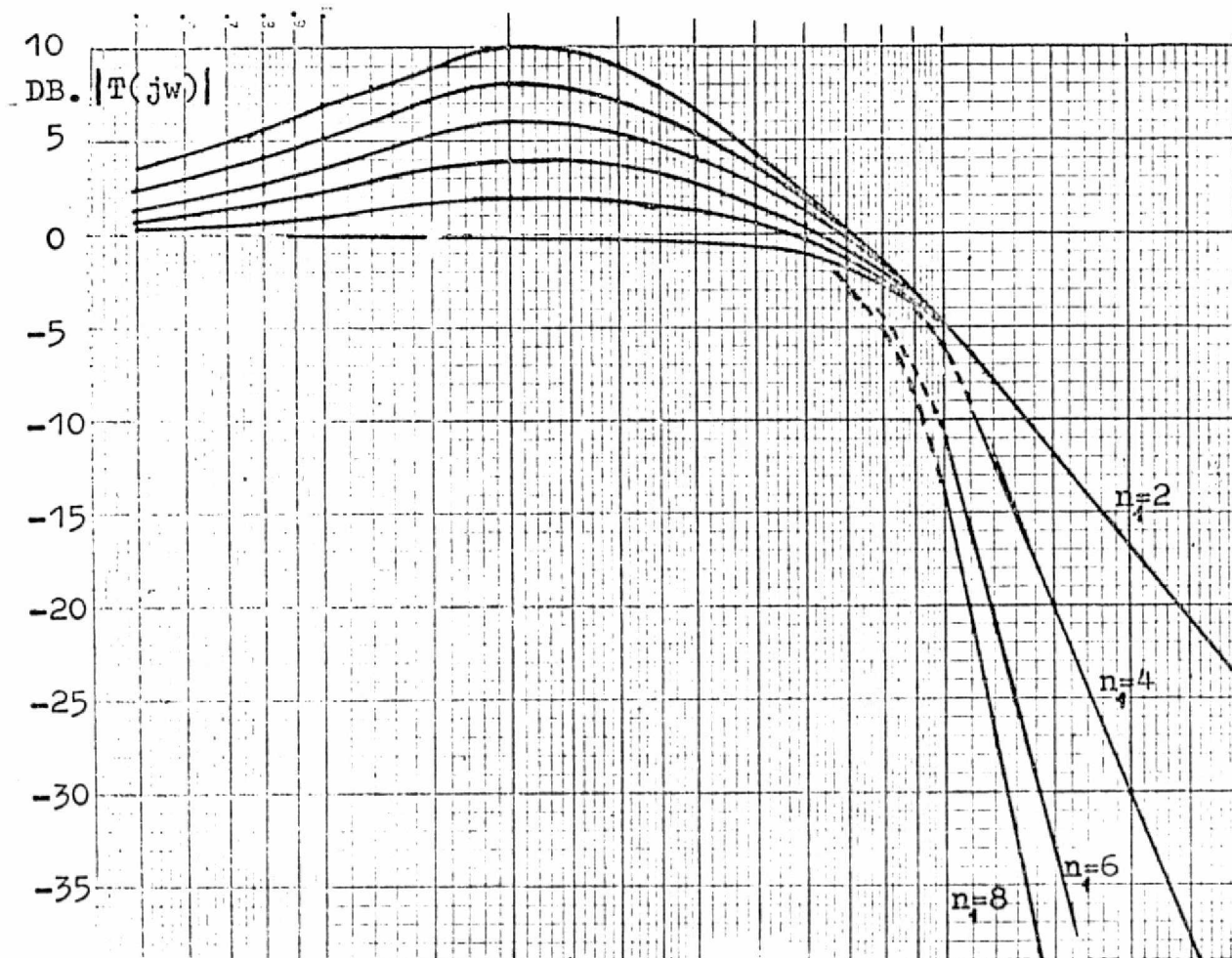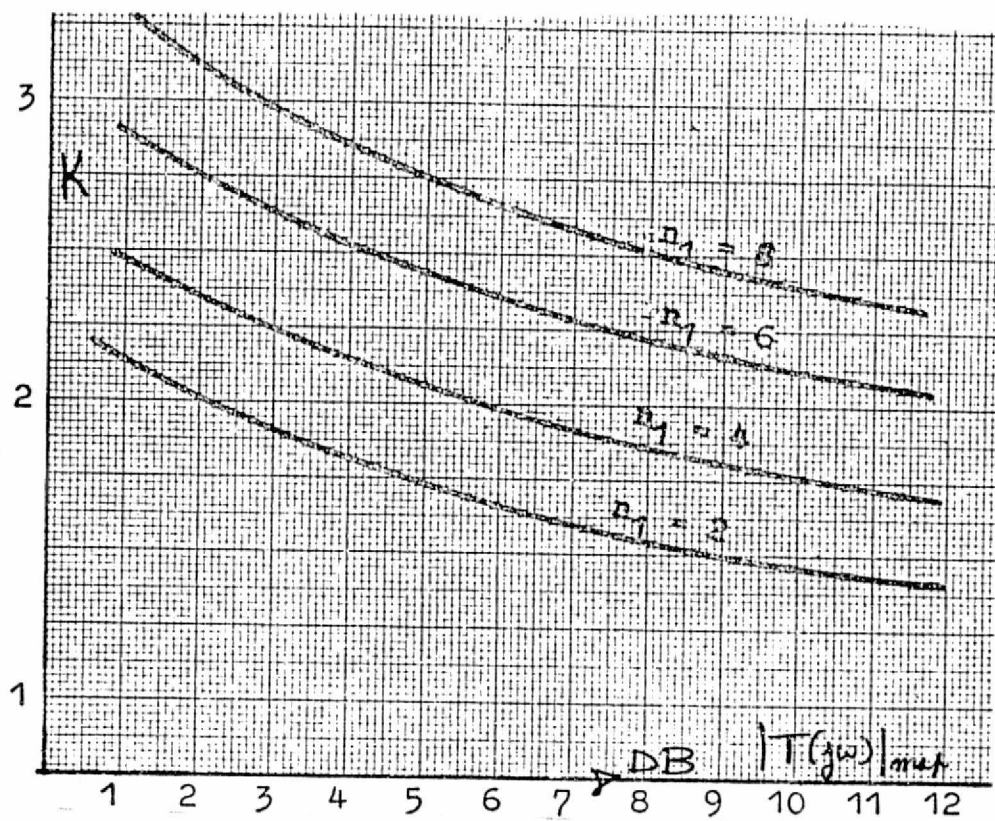
Fig. A3-4. Sample runs
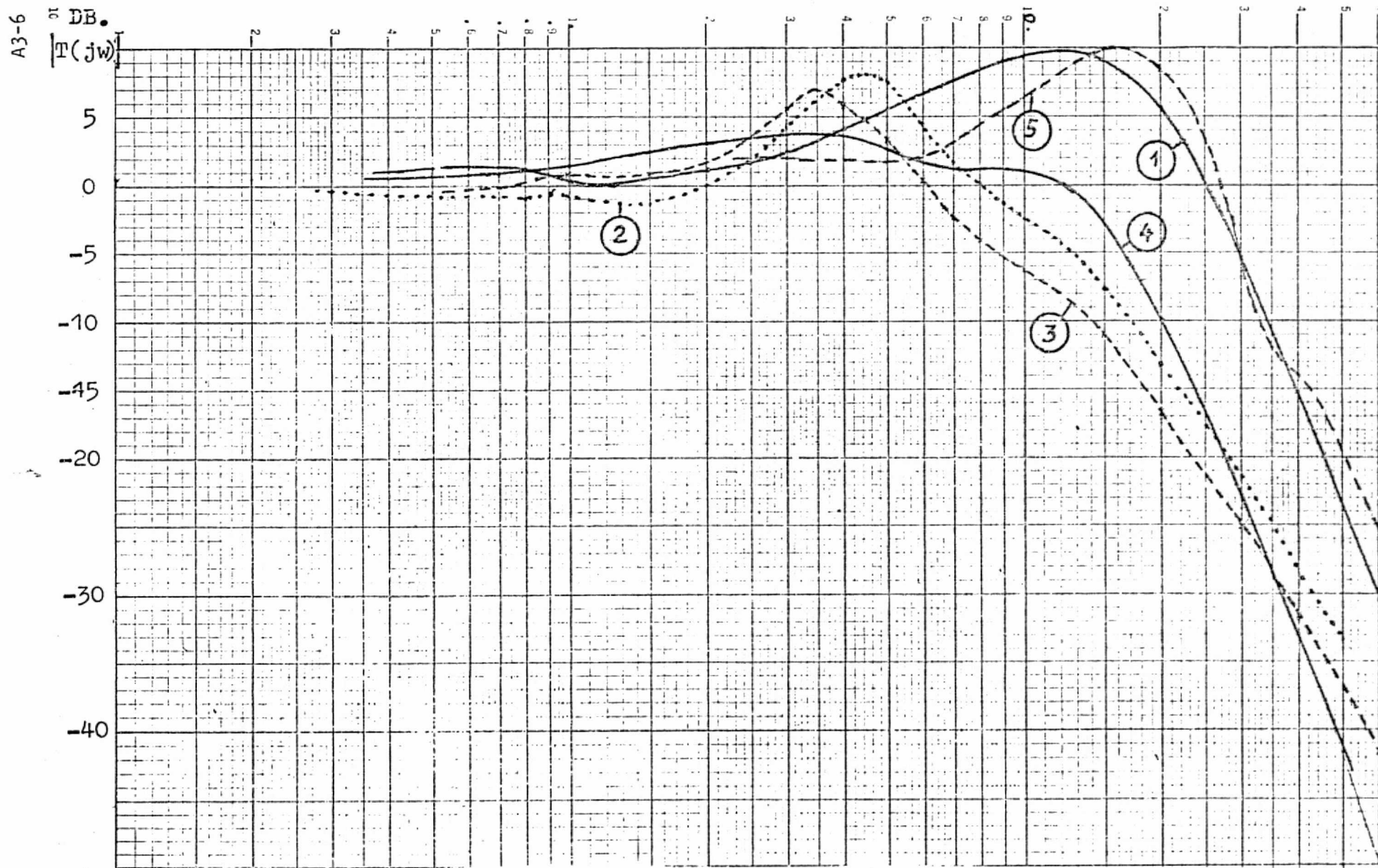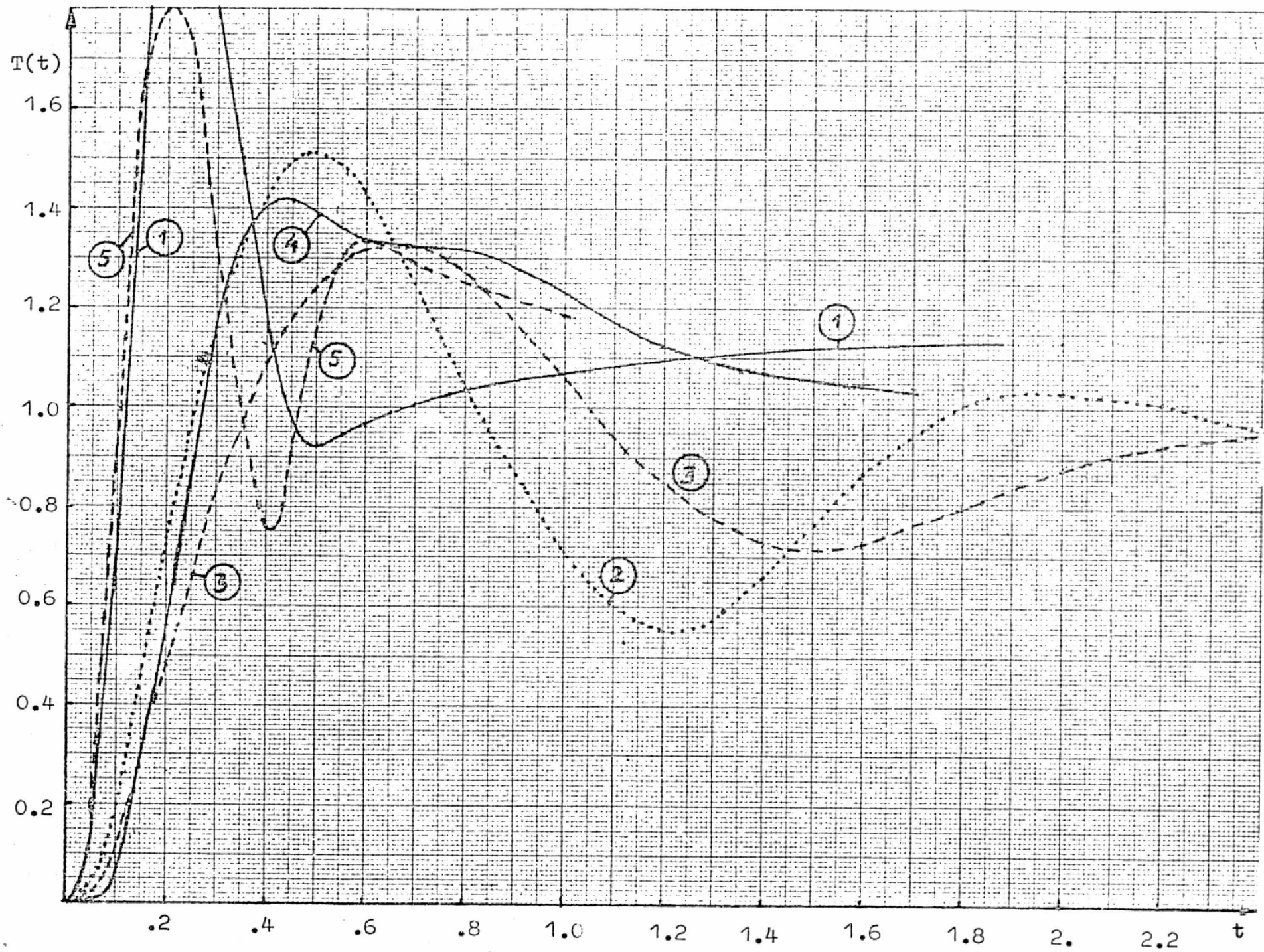
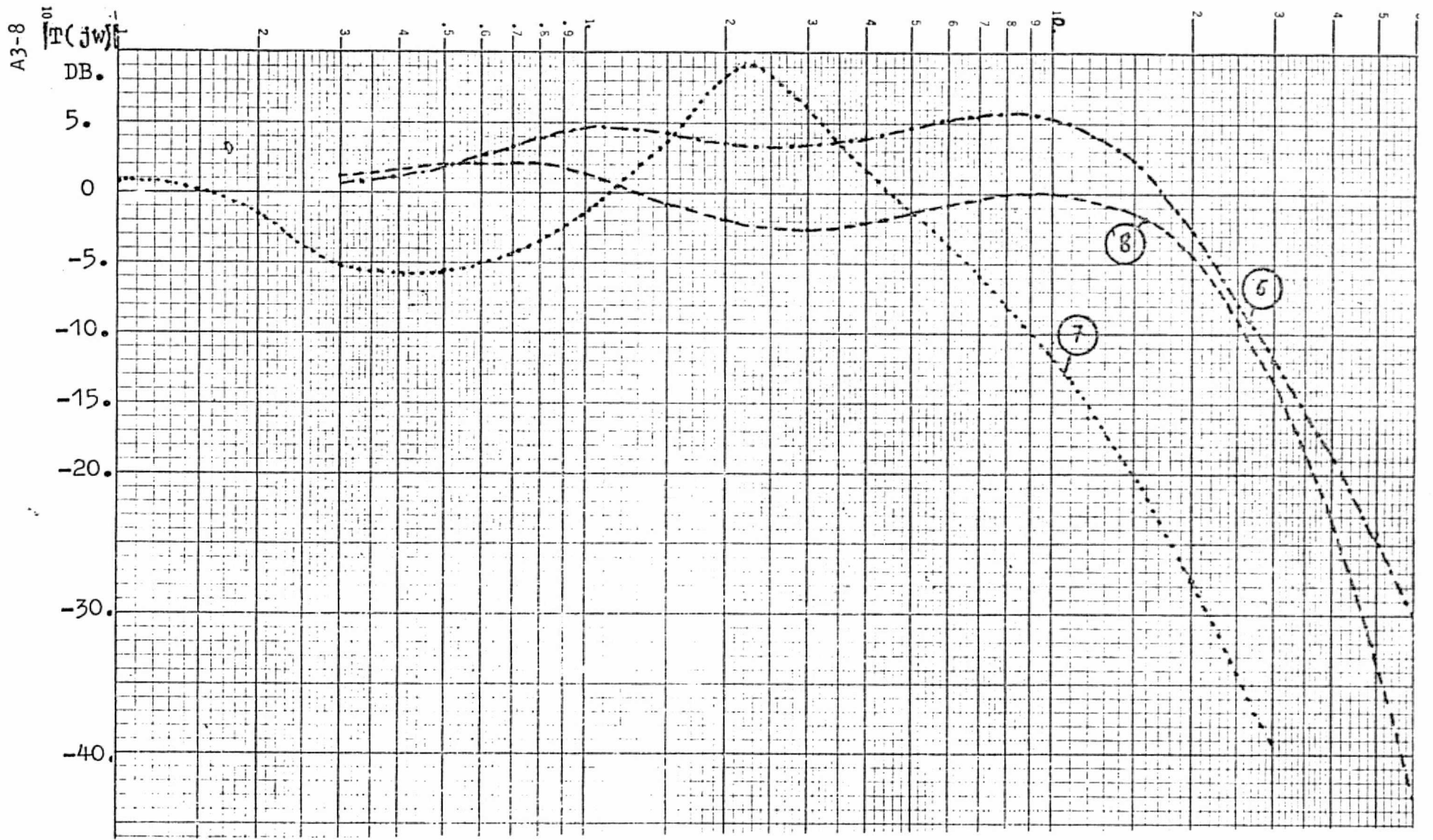Fig. A3-5a.   Sample runs -- frequency response

Fig. A3-5b. Sample runs -- step response
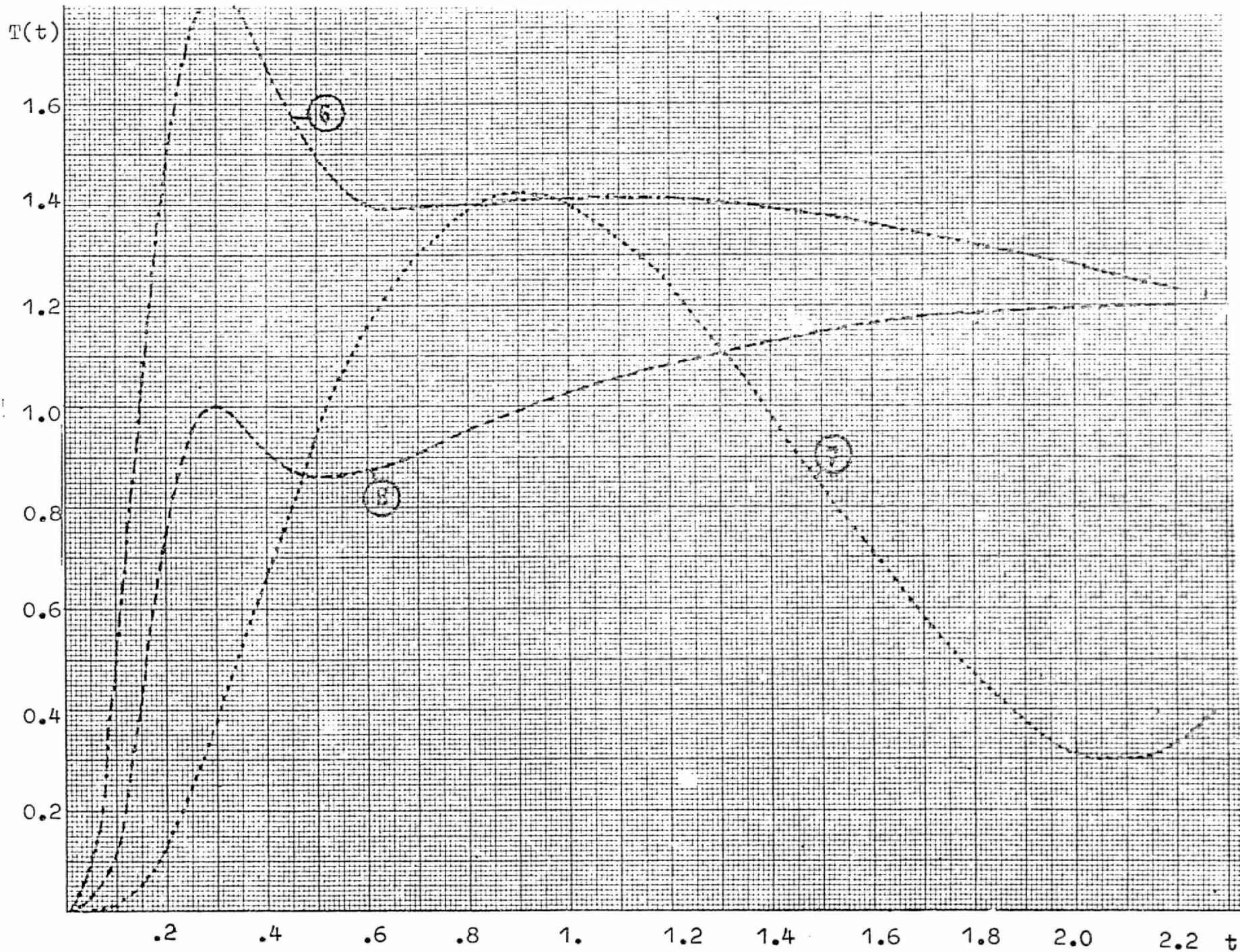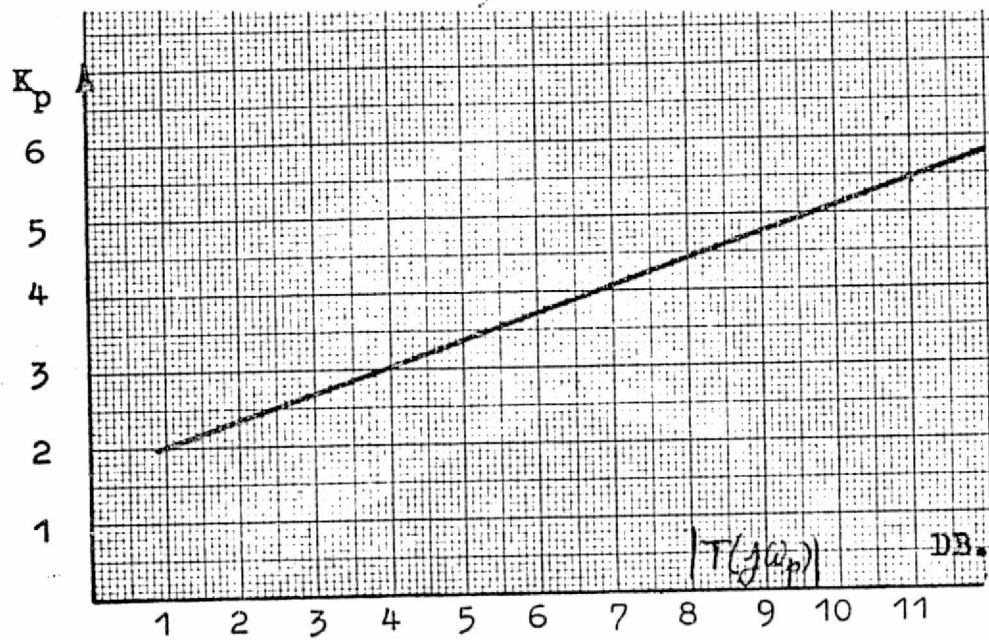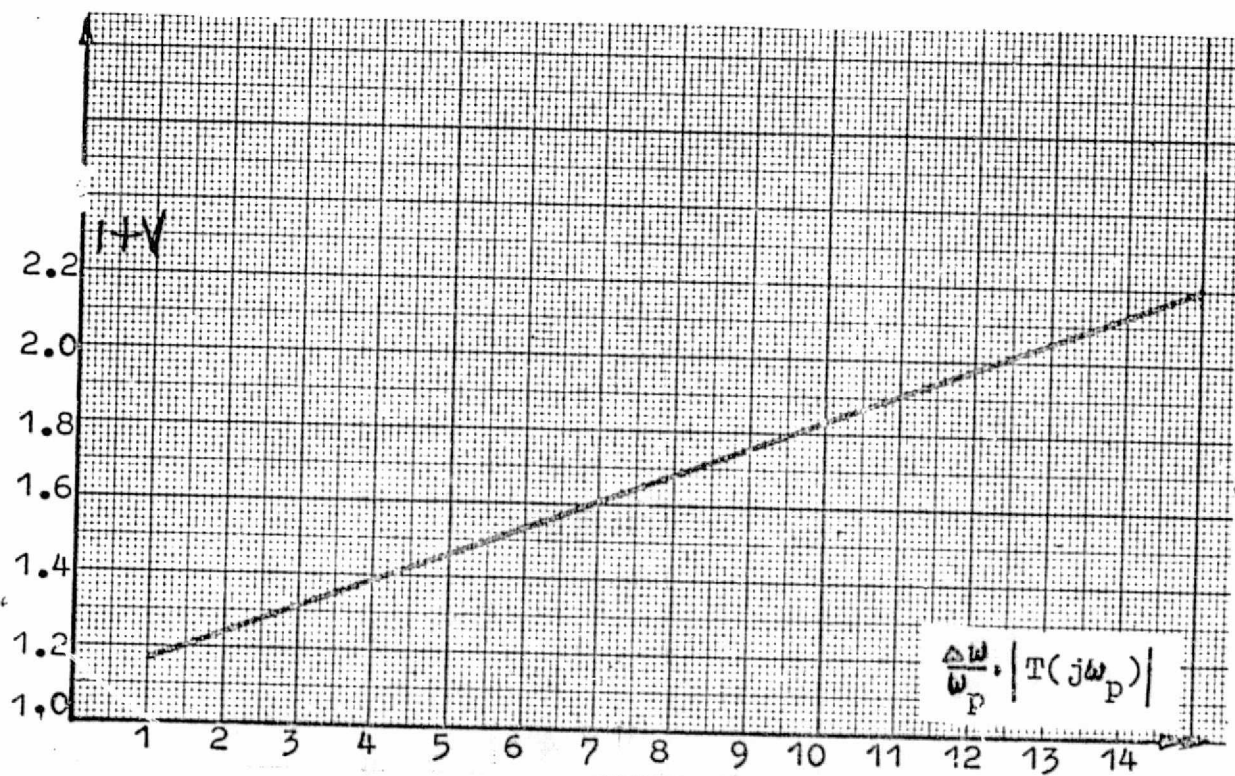
Fig. A3-6a. Sample runs -- frequency response

Fig. A3-6b. Sample runs -- step response

Fig. A3-7. K_p vs. $|T(j\omega_p)|$

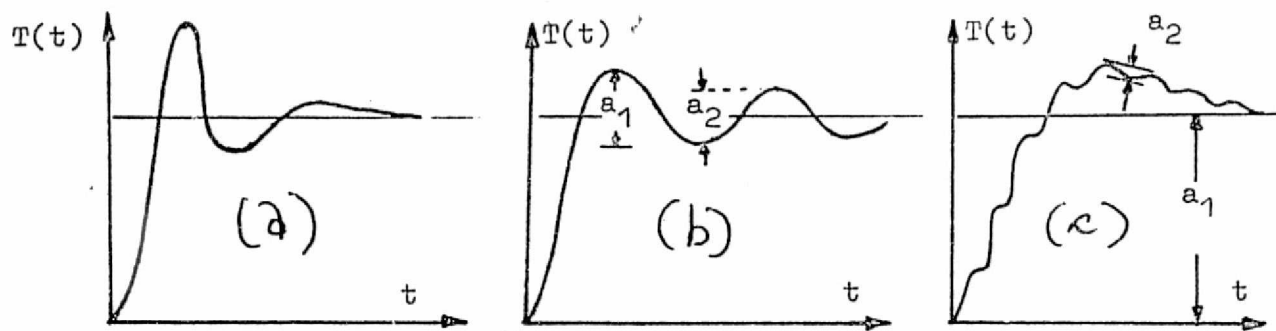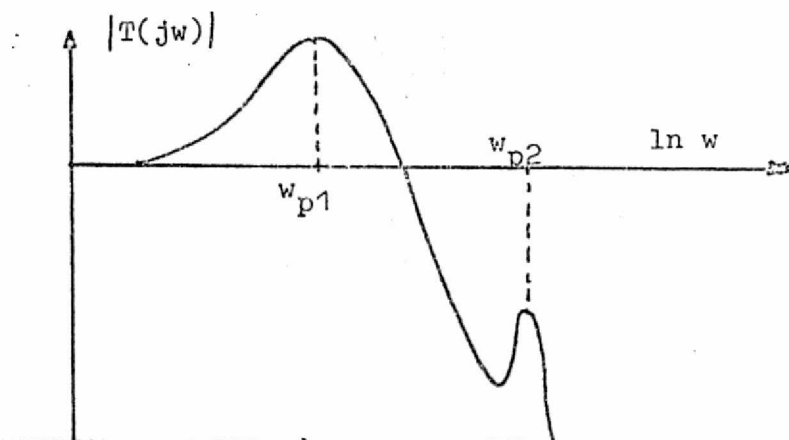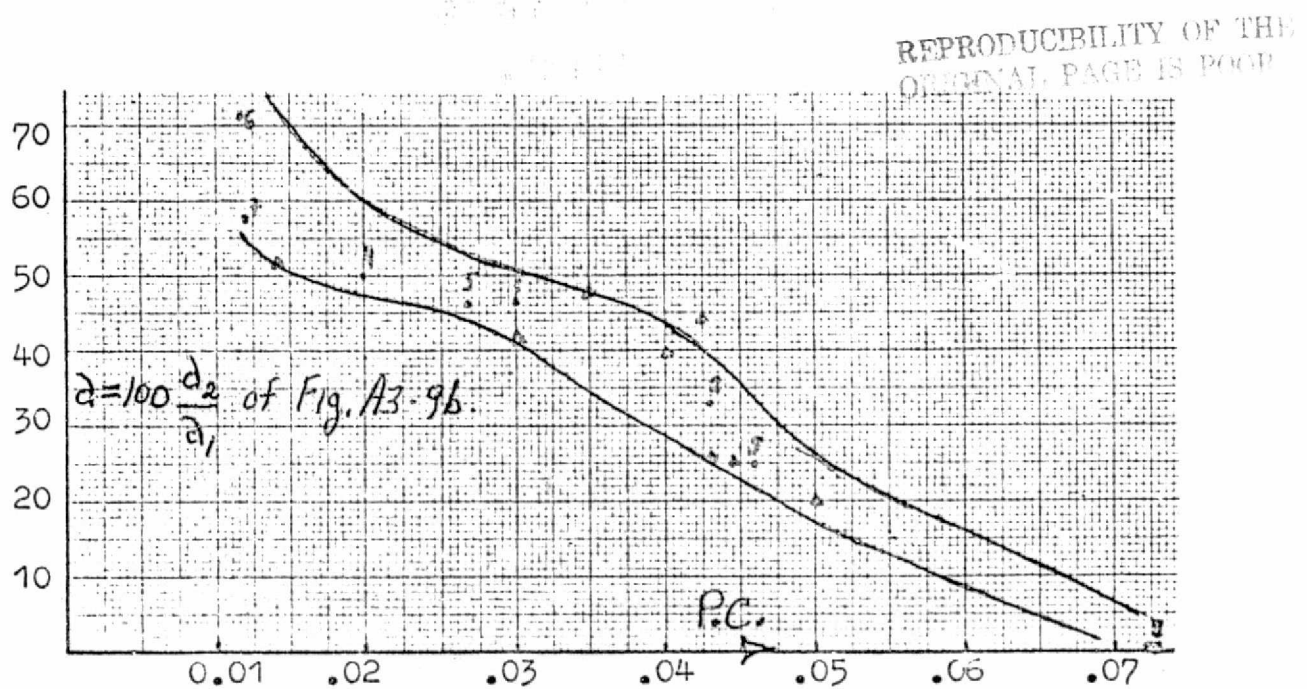Fig. A3-8. $(1 + V)$ vs. $\frac{\Delta\omega}{\omega_p} |T(j\omega_p)|$

Fig. A3-9. Oscillating features in step response

Fig. A3-10. Two kinds of peaking in $|T(j\omega)|$

$a = 100 \frac{a_2}{a_1}$ of Fig. A3-9b.

P.C.

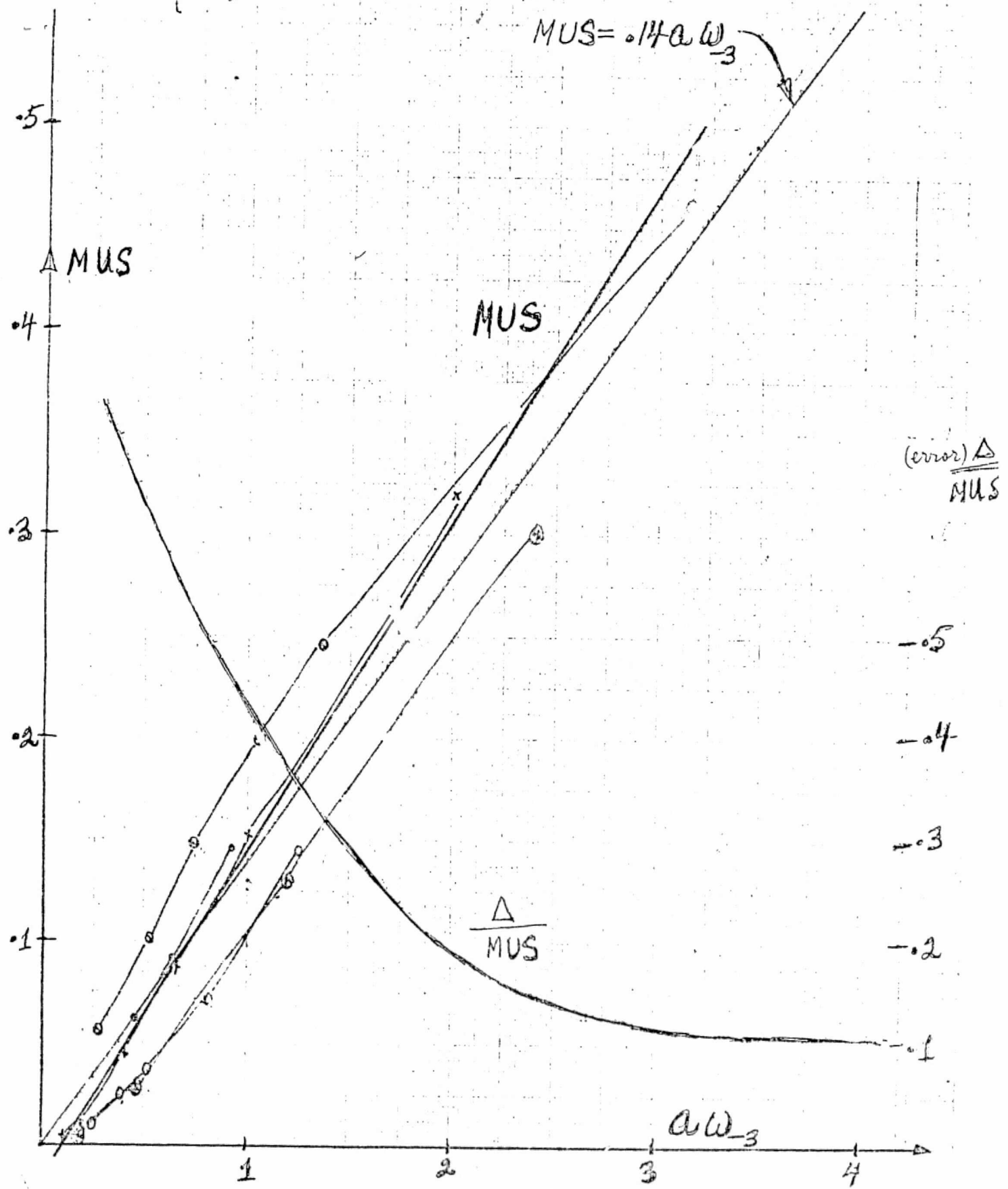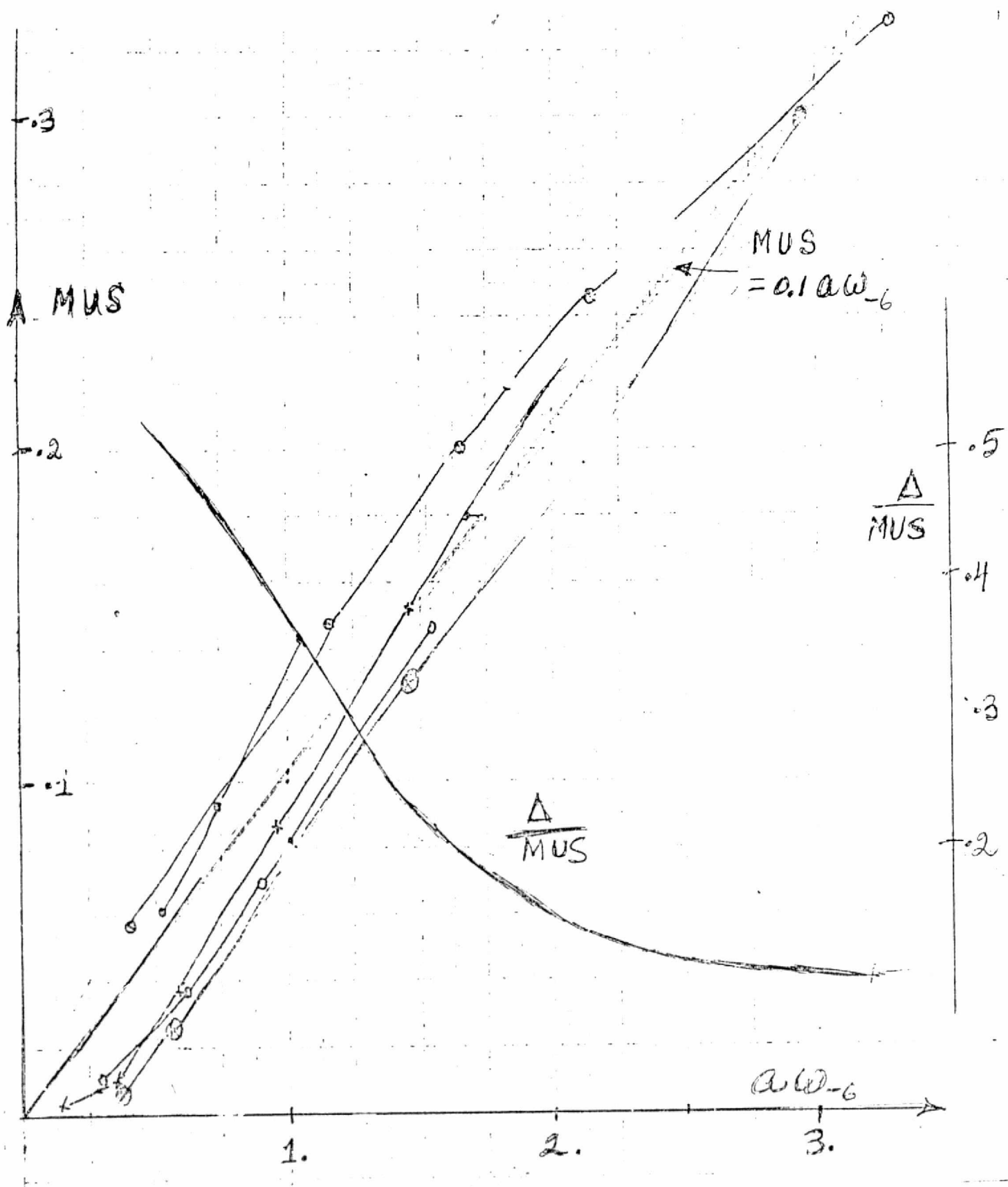Fig. A3-11. Approximate relation between a and P.C.

Fig. A3-12a. Results on maximum undershoot MUS

Fig. A3-12b.   Results on maximum undershoot MUS