ON THE STABILITY OF A CLASS OF IMPLICIT ALGORITHMS FOR NONLINEAR STRUCTURAL DYNAMICS

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SUMMARY

Stability in energy for the Newmark β-family of time integration operators for nonlinear material problems is examined. It is shown that the necessary and sufficient conditions for unconditional stability are equivalent to those predicted by Fourier methods for linear problems.

INTRODUCTION

In this paper, stability in energy for the Newmark family (ref. 1) of time integration operators is examined. Stability for these operators was considered in the original paper of Newmark, who used essentially Fourier techniques which are strictly applicable only to linear problems. Belytschko and Schoeberle (ref. 2) have shown the unconditional stability of the particular form of the Newmark β-operator that corresponds to the trapezoidal rule ($\gamma=\frac{1}{2}$, $\beta=\frac{1}{4}$) for nonlinear material problems by energy methods. Hughes (ref. 3) extended this proof to the range of parameters ($\gamma=\frac{1}{2}$, $\beta\geq\frac{1}{4}$). In this paper, it is shown by generalizing the definition of discrete energy, sufficient conditions for unconditional stability in energy on both γ and β can be obtained. These conditions are equivalent to the necessary conditions for the unconditional stability of the Newmark operators in linear problems, so the conditions obtained herein are necessary and sufficient for the unconditional stability for nonlinear material problems.

PRELIMINARY EQUATIONS

The equations will here be presented in the formalism of the finite element method. As indicated in Belytschko, et al (ref. 4), the spatial discretization in finite difference methods is basically identical, so the choice of finite element notation is only a matter of convenience, not a restriction on the proof. The equations will only be outlined; details may be found in Zienkiewicz (ref. 5).

The fundamental step in any spatial discretization, which is often called the semidiscretization, is a separation of variables

in the form

$$\underline{u}(\underline{x},t) = \underline{\phi}(\underline{x})\ \underline{d}^{(e)}(t) \tag{1}$$

where $\underline{x}$ is the Cartesian coordinate, t the time, $\underline{u}$ the displacement field, $\underline{\phi}$ the shape functions, and $\underline{d}^{(e)}$ the nodal displacement of element e. The strains can then be related to the nodal displacements by

$$\underline{\varepsilon} = \underline{B}\,\underline{d}^{(e)} = \underline{B}\ \underline{L}^{(e)}\ \underline{d} \tag{2}$$

where $\underline{B}$ consists of derivatives of the shape functions and $\underline{L}^{(e)}$ is the connectivity matrix. The discrete equations of motion are then

$$\underline{M}\ \underline{a} + \underline{f} = \underline{p} \tag{3}$$

where $\underline{M}$ is the mass matrix, $\underline{a}$ the nodal accelerations (second derivatives of $\underline{d}$ with respect to time), $\underline{p}$ the external nodal forces and $\underline{f}$ the internal nodal forces, which are given by

$$\underline{f} = \sum_e \underline{L}^{(e)T}\underline{f}^{(e)} = \sum_e \underline{L}^{(e)T} \int_{V^{(e)}} \underline{B}^T\underline{\sigma}\ dV \tag{4}$$

Equations (3) and (4) can be derived from the principle of virtual work with the inertial forces included in a d'Alembert sense; see for example Belytschko, et al (ref. 6).

We define a discrete internal energy by

$$U_1 = 0$$

$$U_{I+1} = U_I + \tfrac{1}{2} \sum_e \int_{V^{(e)}} \Delta\underline{\varepsilon}_I^T \left[(1-\mu)\underline{\sigma}_I + \mu\underline{\sigma}_{I+1}\right] dV \tag{5}$$

where upper case subscripts denote the time step and Δ denotes a forward difference

$$\Delta\underline{\varepsilon}_I = \underline{\varepsilon}_{I+1} - \underline{\varepsilon}_I \tag{6a}$$

and

$$0 \le \mu \le 1 \tag{6b}$$

When $\mu=\frac{1}{2}$, eq. (5) represents a trapezoidal integration of the nonlinear stress strain curve, while $\mu=0$ corresponds to Euler integration.

By means of eqs. (2) and (4), eq. (5) can also be written as

$$U_{I+1} = U_I + \tfrac{1}{2}\,\Delta\underline{d}_I^T\left[(1-\mu)\,\underline{f}_I + \mu\underline{f}_{I+1}\right] \tag{7}$$

We require that the discrete internal energy be positive definite, so that

$$\sum_{J=1}^{I} \Delta\underline{\varepsilon}_J^T\left[(1-\mu)\,\underline{\sigma}_J + \mu\underline{\sigma}_{J+1}\right] \geq 0 \tag{8}$$

The kinetic energy T is given by

$$T_I = \tfrac{1}{2}\,\underline{v}_I^T\,\underline{M}\,\underline{v}_I \tag{9}$$

where $\underline{v}$ are the nodal velocities (first derivative of $\underline{d}$ with respect to time).

The Newmark difference formulas are

$$\underline{v}_{I+1} = \underline{v}_I + \Delta t\left[(1-\gamma)\,\underline{a}_I + \gamma\underline{a}_{I+1}\right] \tag{10}$$

$$\underline{d}_{I+1} = \underline{d}_I + \Delta t\,\underline{v}_I + \Delta t^2\left[(\tfrac{1}{2}-\beta)\,\underline{a}_I + \beta\underline{a}_{I+1}\right] \tag{11}$$

When $\beta > 0$, these equations are implicit, and hence for nonlinear materials, the solution of a nonlinear system of equation is necessary. The exact solution of the nonlinear system of equations at each time step is not possible; at each time step there will be an error $\underline{f}_I^{err}$ given by

$$\underline{f}_I^{err} = \underline{p}_I - \underline{f}_I - \underline{M}\underline{a}_I \tag{12}$$

We define an energy error criterion

$$\left|\Delta\underline{d}_I^T\left[(1-\mu)\,\underline{f}_I^{err} + \mu\underline{f}_{I+1}^{err}\right]\right| < \varepsilon\,(U_I + T_I) \tag{13}$$

where ε is a small constant and require that the solution of the nonlinear equations at each time step satisfy this criterion.

PROOF OF UNCONDITIONAL STABILITY

We will now show that the error criterion, eq. (13), is a sufficient condition for the unconditional stability in energy of the Newmark integration formulas for $\gamma \geq \frac{1}{2}$, $\beta \geq \gamma/2$. Stability in energy is described in Richtmyer and Morton (ref. 7) and has previously been used for the derivation of stability conditions for the solution of linear problems by the Newmark β-method by Fujii (ref. 8).

The demonstration of unconditional stability in energy requires that it be shown that a positive definite norm of the solution is bounded regardless of the size of the time step. As pointed out in reference 7, the norm need not be the physical energy, though in many cases it is. For the purposes of this proof, we define the norm by

$$S_I = T_I + U_I + (2\beta-\gamma)\underline{a}_I^T \underline{M}\, \underline{a}_I \tag{14}$$

Because of the requirement of eq. (8), U_I is positive definite, whereas the positive definiteness of the mass matrix $\underline{M}$ assures the positiveness definiteness of the remaining two quantities if

$$2\beta \geq \gamma \tag{15}$$

Stability in energy is then assured if we can show that S_I is always bounded, i.e. that

$$S_{I+1} \leq (1+\varepsilon^*) S_I \tag{16}$$

where ε^* is an arbitrarily small quantity. The interpretation of the condition of eqs. (14) and (16) is as follows. Provided that the discrete internal energy is a monotonically increasing function of the displacements, the boundedness of S_I implies that the velocities and displacements are bounded, which corresponds to the notion of stability.

The proof of stability then consists of deducing eq. (16) from eqs. (7) to (11) and the homogeneous form of eq. (3). From eqs. (9) and (10), it follows that

$$\begin{aligned} T_{I+1} = T_I &+ \Delta t\, \underline{v}_I^T \underline{M}\, [(1-\gamma)\underline{a}_I + \gamma \underline{a}_{I+1}] \\ &+ \frac{\Delta t^2}{2} [(1-\gamma)\underline{a}_I^T + \gamma \underline{a}_{I+1}^T]\, \underline{M}\, [(1-\gamma)\underline{a}_I + \gamma \underline{a}_{I+1}] \end{aligned} \tag{17}$$

so

$$T_{I+1} = T_I + \left[\Delta \underline{d}_I^T + \Delta t^2(\beta-\tfrac{1}{2}\gamma)\underline{a}_I^T + \Delta t^2(\tfrac{1}{2}\gamma-\beta)\underline{a}_{I+1}^T\right] \underline{M}\left[(1-\gamma)\underline{a}_I + \gamma\underline{a}_{I+1}\right] \tag{18}$$

Thus if we let

$$\gamma = \mu \tag{19}$$

then eqs. (7) and (18) yield

$$T_{I+1} + U_{I+1} = T_I + U_I + \Delta \underline{d}_I^T\left[(1-\gamma)(\underline{M}\,\underline{a}_I + \underline{f}_I) + \gamma(\underline{M}\,\underline{a}_{I+1} + \underline{f}_{I+1})\right] + \Delta t^2\left[(\beta - \tfrac{\gamma}{2})\underline{a}_I^T + (\tfrac{\gamma}{2}-\beta)\underline{a}_{I+1}^T\right]\underline{M}\left[(1-\gamma)\underline{a}_I + \gamma\underline{a}_{I+1}\right] \tag{20}$$

The second term on the right hand side of eq. (20) corresponds to the error in energy as defined by eq. (13), and the last term can be rearranged so that we obtain

$$T_{I+1} + U_{I+1} \leq (1+\varepsilon)(T_I+U_I) + \Delta t^2(\gamma-2\beta)(\underline{a}_{I+1}^T\,\underline{M}\,\underline{a}_{I+1} - \underline{a}_I\,\underline{M}\,\underline{a}_I) + \Delta t^2(\gamma-\tfrac{1}{2})(\gamma-2\beta)(\underline{a}_{I+1}^T - \underline{a}_I^T)\,\underline{M}\,(\underline{a}_{I+1} - \underline{a}_I) \tag{21}$$

The last term on the right hand side of eq. (21) is negative semi-definite if

$$\gamma \geq \tfrac{1}{2} \qquad \text{and} \qquad 2\beta \geq \gamma \tag{22}$$

or if both of the above inequalities are reversed. However, if the inequalities are reversed, as can be seen from eqs. (14) and (15), the norm S_I is not positive definite. Hence, only the conditions given by eq. (22) are pertinent. Under these conditions, the inequality of eq. (21) applies even if the last term is dropped. The remaining terms then yield eq. (16).

DISCUSSION AND CONCLUSIONS

Several remarks should be noted in applying these results to computations. First, the stability hinges on the achievement of a solution in each time step that satisfies eq. (13). The convergence of solution schemes, such as the modified Newton Raphson method, cannot be assured, and is therefore the primary obstacle in obtaining stable solutions. The difficulties are particularly severe in elastic-plastic problems if the tangential stiffness method is used whenever unloading takes place over a large part of the mesh.

It is also not clear whether the form of the error criterion, eq. (13), is suitable for very fine meshes. Numerical experiments indicate that it becomes increasingly difficult to satisy eq. (13) for finer meshes, for although the criterion appears to be mesh-independent in that the right hand side increases with the size of the physical problem, the right hand side does not vary as a mesh is refined. Furthermore, in very large meshes there is a possibility of cancellation of errors, i.e. positive error energy transfer in one portion, with negative error energy transfer in another portion. This can be avoided by placing the absolute value within the summation.

Results have been reported for a special case of this operator ($\gamma=\frac{1}{2}$, $\beta=\frac{1}{4}$) in reference 2. Both material and geometric nonlinearities were included in those problems. However, the proofs given here and in reference 2 require the absence of geometric nonlinearities; if geometric nonlinearities are included, eq. (5) does not imply eq. (7), for in geometrically nonlinear problems $\Delta\underline{B}$ does not vanish. Hence, as shown in reference 9, in geometrically nonlinear problems, energy transfer is associated with the rotation of a stressed member: this effect results in the generation of energy if the stress is tensile and is hence destabilizing under those conditions. In many structural dynamics problems, the total rigid body rotation that takes place is insufficient for this energy generation to be significant. However, test problems have been devised where the energy error is so large that for practical purposes the computation can be considered unstable.

Finally, we comment on some experience with the requirement of eq. (8). This condition requires that the numerical integration of the internal work always yield a positive quantity. In elastic-plastic materials and other strongly dissipative materials, this condition poses no problems. However, when the stress is a single-valued function of the strain, eq. (8) can easily be violated in cyclic load paths. However, numerical experiments do not indicate that violation of eq. (8) results in any catastrophic failure of the computation.

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