CORIOLIS EFFECTS ON NONLINEAR OSCILLATIONS OF ROTATING CYLINDERS AND RINGS

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SUMMARY

The effects which moderately large deflections have on the frequency spectrum of rotating rings and cylinders are considered. To develop the requisite solution, a variationally constrained version of the Lindstedt-Poincare procedure is employed. Based on the solution developed, in addition to considering the effects of displacement induced nonlinearity, the role of Coriolis forces is also given special consideration.

INTRODUCTION

Numerous engineering applications (tires, turbines, satellites, etc.) contain rotor systems which are essentially rings or shells of revolution rotating about their axes. Obviously, in order to properly influence their design, a thorough dynamic analysis is necessary. In this regard, numerous papers have been published which deal with the free vibration properties of such systems. Most such work has centered on stationary configurations, as can be seen from the excellent surveys by references 1 and 2. The effects of rotation, in particular Coriolis forces, have been discussed by references 3 to 7. With the exception of references 6 and 7 which treated small dynamic deformations superposed on large static deformations, the previous investigations incorporating Coriolis acceleration forces have been limited to linear shell theories. This is a shortcoming since numerous rotor systems, tires, satellites, and turbines are flexible enough to undergo significant deflections in the form of moderately large rotations.

It is the purpose of this paper to consider the effects which such moderately large rotations have on the frequency spectrum of rotating structures. In particular, the analysis presented will consider the free vibration characteristics of rotating rings and cylinders wherein the deflections involve moderately large rotations. Since the analytical model used to characterize the stated problem involves nonlinear partial differential equations, a modified version of the renormalized perturbation procedure is employed to evaluate the overall solution. This modification was undertaken since the usual renormalized procedure is unwieldy for systems of equations involving a multitude of frequency eigenvalue branches and secondly yields steady state results which are irregular for the linearized case. The modification employed involves prescribing the system energy in advance; hence, a hierarchy of energy states is obtained from which the strained parameter can be evaluated. The resulting solution employing this procedure is regular, and thus, the proper limiting behavior is obtained for the linearized

case. Based on the solution, in addition to considering the global effects of nonlinearity, special emphasis is centered on determining the effects of Coriolis forces in the range of deformations marked by moderately large rotations. Hence the effects on the backward and forward traveling waves will be evaluated.

GOVERNING EQUATIONS

Since the nonlinear oscillations of rotating, elastically supported rings and infinite cylinders undergoing deflections involving moderately large rotations are considered herein, the governing displacement equations of motion employed to model the stated problem are defined by (refs. 2,4,6, and 7)

$$A_1 W_{,\theta\theta\theta}+A_2(V_{,\theta}+W)+(\kappa+\frac{P}{R})W-\varepsilon A_2(\frac{1}{2}W_{,\theta}^2+V_{,\theta\theta}W_{,\theta}+V_{,\theta}W_{,\theta\theta}+WW_{,\theta\theta}) -$$

$$\frac{3}{2}\varepsilon^2 A_2 W_{,\theta}^2 W_{,\theta\theta}+f\cos(m\theta)\cos(\omega t)+\rho h(W_{,tt}-2\Omega V_{,t}-\Omega^2 W) = 0 \qquad (1)$$

$$A_2(V_{,\theta\theta}+W_{,\theta}+\varepsilon W_{,\theta}W_{,\theta\theta})-\rho h(V_{,tt}+2\Omega W_{,t}-\Omega^2 V) = 0 \qquad (2)$$

where

$$A_1 = \frac{EI}{R^4} \quad , \quad A_2 = \frac{Eh}{R^2} \qquad (3)$$

such that $\varepsilon = W_m/R$ and θ, t, $(\ )_{,\theta}$, $(\ )_{,t}$, W, V, W_m, E, I, h, R, ρ, P, κ, ω, and Ω respectively represent circumferential space, time, space and time differentiation, radial and circumferential shell displacements, maximum radial displacement, Young's modulus, moment of inertia, shell thickness, radius and density, internal pressure, foundation elasticity, exciting frequency, and lastly, the rotational speed of the shell. Due to the inherent nature of the circumferential coordinate space and the fact that the steady state response is being sought, it follows that W and V are periodic in both space and time. To round out the requisite field equations, the following potential energy functional is associated with equations (1) and (2), namely

$$\gamma = \int_0^T \int_0^{2\pi} \{A_1 W_{,\theta\theta}^2+A_2(V_{,\theta}^2+2V_{,\theta}W+W^2)+\varepsilon A_2(V_{,\theta}W_{,\theta}^2+WW_{,\theta}^2) +$$

$$\frac{1}{4}\varepsilon^2 A_2 W_{,\theta}^4+(\kappa+\frac{P}{R})W^2+2f\cos(m\theta)\cos(\omega t)W-\rho h[\Omega^2(R^*+W)^2+W_{,t}^2 +$$

$$\Omega^2 V^2+V_{,t}^2+2\Omega(R^*+W)V_{,t}-2\Omega W_{,t}V]\}d\theta dt \qquad (4)$$

where $T = \frac{1}{2\pi\omega}$ and $R^* = R/W_m$.

SOLUTION

As noted earlier, the standard renormalized perturbation procedure has the twofold difficulty of yielding irregular results as $\varepsilon \to 0$ and secondly, is unwieldy when more than one equation of motion involving several frequency eigenvalue branches is considered. This difficulty is circumvented by prescribing the systems potential energy in advance such that $(W; V) = (W(\theta,t,f,m,\gamma); V(\theta,t,f,m,\gamma))$. Once the solution is obtained, the role of γ and W are reversed to that employed in the traditional version of the renormalized procedure. To initiate the solution, ω is treated as the strained parameter; hence W, V, and ω are expanded in the following perturbation series

$$\langle W; V; \omega \rangle = \sum_{i=0}^{\infty} \langle W_i; V_i; \omega_i \rangle \varepsilon^i \tag{5}$$

such that time is stretched so that $\tau = \omega t$.

In order to obtain the zeroth order equations, ε is set to zero; this yields

$$A_1 W_{o,\theta\theta\theta\theta} + (A_2 + \kappa + \frac{P}{R}) W_o + A_2 V_{o,\theta} + \rho h(\omega_o^2 W_{o,tt} -$$

$$2\omega_o \Omega V_{o,\tau} - \Omega^2 W_o) + f \cos(m\theta) \cos(\tau) = 0 \tag{6}$$

$$A_2 (V_{o,\theta\theta} + W_{o,\theta}) = \rho h(\omega_o^2 V_{o,\tau\tau} + 2\omega_o \Omega W_{o,\tau} - \Omega^2 V_o) \tag{7}$$

$$\Gamma = \int_0^{2\pi} \int_0^{2\pi} \{A_1 W_{o,\theta\theta}^2 + A_2 (V_{o,\theta}^2 + 2V_{o,\theta} W_o + W_o^2) + (\kappa + \frac{P}{R}) W_o^2 +$$

$$2f \cos(m\theta) \cos(\tau) W_o - \rho h[\Omega^2 (R^* + W_o)^2 + \omega_o^2 W_{o,\tau}^2 + \Omega^2 V_o^2 + \omega_o^2 V_{o,\tau}^2 +$$

$$2\omega_o \Omega (R^* + W_o) V_{o,\tau} - 2\omega_o \Omega W_{o,\tau} V_o]\} d\theta d\tau \tag{8}$$

whereas with time, the potential energy space is stretched so that $\Gamma = \gamma/\Omega$. Since the steady state solution is sought,

$$(W_o; V_o) = (W_{co}; V_{co}) \cos(m\theta) + (W_{so}; V_{so}) \sin(m\theta) \tag{9}$$

where $W_{co}, \ldots$ are time dependent. Employing equations (9), (6), and (7) reduce to the following matrix set of ordinary differential equations, namely

$$\omega_o^2 [B_{1m}] \underset{\sim}{Y}_{mo,\tau\tau} + \omega_o [B_{2m}] \underset{\sim}{Y}_{mo,\tau} + [B_{3m}] \underset{\sim}{Y}_{mo} + \underset{\sim}{f} \cos(\tau) = \underset{\sim}{0} \tag{10}$$

such that

$$\underset{\sim}{Y}_{mo} = (W_{co},\ V_{so},\ W_{so},\ V_{co})^T \tag{11}$$

Noting that $[B_{2m}]$ is skew symmetric while $[B_{1m}]$ and $[B_{3m}]$ are purely symmetric, the steady state form of $\underset{\sim}{Y}_{mo}$ is given by

$$\underset{\sim}{Y}_{mo} = \underset{\sim}{Z}_{mc} \cos(\tau) + \underset{\sim}{Z}_{ms} \sin(\tau) \tag{12}$$

where $\underset{\sim}{Z}_{mc}$ and $\underset{\sim}{Z}_{ms}$ satisfy the matrix equation

$$\begin{Bmatrix} \underset{\sim}{f} \\ \underset{\sim}{0} \end{Bmatrix} = \begin{bmatrix} \omega_o^2[B_{1m}]-[B_{3m}] & \omega_o[B_{2m}] \\ \omega_o[B_{2m}] & \omega_o[B_{1m}]-[B_{3m}] \end{bmatrix} \begin{Bmatrix} \underset{\sim}{Z}_{mc} \\ \underset{\sim}{Z}_{ms} \end{Bmatrix} \tag{13}$$

Noting that the pencil of equation (13) yields the characteristic equation of equation (10), equation (12) becomes unbounded for ω_o equal to the natural frequency eigenvalues of the linear case. The properties of such eigenvalues can be ascertained by developing the appropriate Rayleigh quotient. This is possible by inserting $\gamma_m \alpha \underset{\sim}{Z}_m e^{j\tau}$ into equation (10) to yield a complex second order regular polynomial matrix problem. The inner product of this expression and $\underset{\sim}{Z}_m$ yields a bilinear form from which the following modified version of Rayleigh's quotient is obtained, namely

$$\omega_o = j\,\frac{\bar{\underset{\sim}{Z}}_m^T[B_{2m}]\underset{\sim}{Z}_m}{2\rho h \bar{\underset{\sim}{Z}}_m^T \underset{\sim}{Z}_m} + \left[\frac{\bar{\underset{\sim}{Z}}_m^T[B_{3m}]\underset{\sim}{Z}_m}{\rho h \bar{\underset{\sim}{Z}}_m^T \underset{\sim}{Z}_m} - \left(\frac{\bar{\underset{\sim}{Z}}_m^T[B_{2m}]\underset{\sim}{Z}_m}{4\rho h \bar{\underset{\sim}{Z}}_m^T \underset{\sim}{Z}_m}\right)^2\right]^{1/2} \tag{14}$$

As can be seen from equation (14), Coriolis forces cause a twofold bifurcation in the number of eigenvalue branches. Following the previous comments, the relationship between Γ and ω_o, W_o, and V_o must be evaluated by inserting equations (9) and (12) into equation (8); this yields

$$\begin{aligned}
&[A_1 m^4 + A_2 + \kappa + \frac{P}{R} - \rho h(\Omega^2 + \omega_o^2 m^2)](W_{cco}^2 + W_{cso}^2 + W_{sco}^2 + W_{sso}^2) + \\
&[A_2 m^2 - \rho h(\Omega^2 + m^2\omega_o^2)](V_{cco}^2 + V_{cso}^2 + V_{sco}^2 + V_{sso}^2) + 2mA_2(V_{sco}W_{cco} + \\
&V_{sso}W_{cso} - V_{cco}W_{sco} - V_{cso}W_{sso}) - 2\rho h\omega_o \Omega m(W_{cco}V_{cso} - W_{cso}V_{cco} + \\
&W_{sco}V_{sso} - W_{sso}V_{sco} - V_{cco}W_{cso} + V_{cso}W_{cco} - V_{sco}W_{sso} + \\
&V_{sso}W_{sco})] = \Gamma/\pi^2
\end{aligned} \tag{15}$$

where $W_{cco}, \ldots, V_{sso}$ denote coefficients of the W_o and V_o solution, namely

$$(W_o;V_o) = (W_{cco};V_{cco})\cos(m\theta)\cos(\tau)+\ldots+(W_{sso};V_{sso})\sin(m\theta)\sin(\tau)$$

As can be seen from equations (13) and (15), four potential energy resonances are initiated for $\omega_o \sim O(\omega_{mf})$ wherein ω_{mf} are the frequency eigenvalues of the linear problem. Hence equation (5) is regular for $\varepsilon \to 0$ (the linear case).

The first order set of field equations can be obtained by taking the first derivative of equations (1), (2), and (4) with respect to ε and then setting ε to zero. This yields

$$A_1 W_{1,\theta\theta\theta}+A_2(V_{1,\theta}+W_1)+(\kappa+\frac{P}{R})W_1+\rho h(\omega_o^2 W_{1,\tau\tau}-2\Omega\omega_o V_{1,\tau}-\Omega^2 W_1) =$$

$$A_2(\frac{1}{2}W_{o,\theta}^2-V_{o,\theta\theta}W_{o,\theta}+V_{o,\theta}W_{o,\theta\theta}+W_o W_{o,\theta\theta}) - 2\rho h(\omega_1\omega_o W_{o,\tau\tau}-\Omega\omega_1 V_{o,\tau}) \quad (16)$$

$$A_2(V_{1,\theta\theta}+W_{1,\theta})-\rho h(\omega_o^2 V_{1,\theta\theta}+2\Omega\omega_o W_{1,\tau}-\Omega^2 V_1) =$$

$$-A_2 W_{o,\theta}W_{o,\theta\theta}+2\rho h(\omega_1\omega_o V_{o,\tau\tau}+\Omega\omega_1 W_{o,\tau}) \quad (17)$$

$$\int_0^{2\pi}\int_0^{2\pi} \{2A_1 W_{o,\theta\theta}W_{1,\theta\theta}+2A_2(V_{o,\theta}+W_o)(V_{1,\theta}+W_1)+A_2(V_{o,\theta}+W_o)W_{o,\theta}^2+$$

$$2(\kappa+\frac{P}{R})W_o W_1+2f\cos(m\theta)\cos(\tau)W_1-2\rho h[\Omega^2(R^*+W_o)W_1+$$

$$\omega_o\omega_1 W_{o,\tau}^2+\omega_o^2 W_{o,\tau}W_{1,\tau}+\Omega^2 V_o V_1+\omega_o\omega_1 V_{o,\tau}^2 +$$

$$\omega_o^2 V_{o,\tau}V_{1,\tau}+\Omega\omega_1 R^* V_{o,\tau}+\Omega\omega_o R^* V_{1,\tau}+\Omega\omega_o W_1 V_{o,\tau} +$$

$$\Omega\omega_o W_o V_{1,\tau}+\Omega\omega_1 W_o V_{o,\tau}-\Omega\omega_o W_{1,\tau}V_o-\Omega\omega_o W_{1,\tau}V_1-$$

$$\Omega\omega_1 W_{1,\tau}V_o]\}d\theta d\tau = 0 \quad (18)$$

Noting the form of the inhomogeneities appearing in equations (16) and (17), it follows that W_1 and V_1 can be taken in the form

$$(W_1;V_1) = (W_{c1};V_{c1})\cos(2\tau)+(W_{s1};V_{s1})\sin(2\tau)+$$

$$\ldots+(W_{ss1};V_{ss1})\sin(m\theta)\sin(\tau) \quad (19)$$

where the coefficients $W_{c1},\ldots$ are directly obtained upon inserting equation (19) into equations (16) and (17). Furthermore, employing equation (19) in conjunction with the first order potential energy constraint, equation (18), the following functional relationship is obtained for ω_1, namely

$$\omega_1 = \omega_1\ (1/D(2\omega_0,0),\ 1/D(0,2m),\ 1/D(2\omega_0,2m)) \tag{20}$$

where D is the determinant of the pencil of equation (13). Hence for $\omega_0 \sim 0(\omega_{mf})$, ω_1 is bounded and positive definite. This follows since $1/D(2\omega_0,0)$,...etc. remain bounded for $\forall\omega_0\varepsilon(0,\infty)$. Therefore, unlike the zeroth order set, W_1 and V_1 remain bounded for $\forall\omega_0$.

In order to obtain the second order field equations, equations (1), (2), and (4) are differentiated twice and then ε is set to zero. This operation yields

$$A_1W_{2,\theta\theta\theta\theta}+A_2(V_{2,\theta}+W_2)+(\kappa+\frac{P}{R})W_2+\rho h[\omega_0^2W_{2,\tau\tau}-2\Omega\omega_0V_{2,\tau}-$$

$$\Omega^2W_2] = A_2[W_{1,\theta}W_{0,\theta}+V_{1,\theta\theta}W_{0,\theta}+V_{0,\theta\theta}W_{1,\theta}+V_{1,\theta}W_{0,\theta\theta}+V_{0,\theta}W_{1,\theta\theta}+$$

$$W_1W_{0,\theta}+W_0W_{1,\theta\theta}]+\frac{1}{2}A_2W_{0,\theta}^2W_{0,\theta\theta}-\rho h[2\omega_2\omega_0W_{0,\tau\tau}+$$

$$\omega_1^2W_{0,\tau\tau}+2\omega_1\omega_0W_{1,\tau\tau}-2\Omega\omega_2V_{0,\tau}-2\Omega\omega_1V_{1,\tau}] \tag{21}$$

$$A_2(V_{2,\theta\theta}+W_{2,\theta})-\rho h(\omega_0^2V_{2,\tau\tau}+2\Omega\omega_0W_{2,\tau\tau}-\Omega^2V_2) =$$

$$-A_2(W_{1,\theta}W_{0,\theta\theta}+W_{0,\theta}W_{1,\theta\theta})+\rho h(\omega_1^2V_{0,\tau\tau}+2\omega_0\omega_2V_{0,\tau\tau}+$$

$$2\omega_2\Omega W_{0,\tau}+2\omega_1\Omega W_{1,\tau}) \tag{22}$$

$$0 = \int_0^{2\pi}\int_0^{2\pi}\{A_1W_{1,\theta\theta}^2+2A_1W_{0,\theta\theta}W_{2,\theta\theta}+A_1(V_{1,\theta}+W_1)^2+$$

$$2A_2(V_{0,\theta}+W_0)(V_{2,\theta}+W_2)+A_2(2(V_{0,\theta}+W_0)W_{0,\theta}W_{1,\theta}+$$

$$(V_{1,\theta}+W_1)W_{0,\theta}^2)+\frac{1}{4}A_2W_{0,\theta}^4+(\kappa+\frac{P}{R})(W_1^2+2W_0W_2)+$$

$$2f\cos(m\theta)\cos(\tau)\,W_2-\rho h[2\Omega^2(R^*+W_0)W_2+\Omega^2W_1^2+$$

$$\omega_0^2(2W_{2,\tau}W_{0,\tau}+W_{1,\tau}^2)+4\omega_0\omega_1W_{0,\tau}W_{1,\tau}+2\omega_2\omega_0W_{0,\tau}^2+$$

$$\Omega^2(2V_0V_2+V_1^2)+\omega_0^2(2V_{2,\tau}V_{0,\tau}+V_{1,\tau}^2)+4\omega_1\omega_0V_{1,\tau}V_{0,\tau}+$$

$$2\omega_0\omega_2V_{0,\tau}^2+2\Omega\omega_0(R^*+W_0)V_{2,\tau}+2\Omega\omega_1(R^*+W_0)V_{1,\tau}+$$

$$2\Omega\omega_2(R^*+W_0)V_{0,\tau}+2\Omega\omega_0W_1V_{1,\tau}+2\Omega\omega_1W_1V_{0,\tau}+2\Omega\omega_0W_{2,\tau}V_0-$$

(continued)

$$2\Omega\omega_0 W_{0,\tau} V_2 - 2\Omega\omega_0 W_{2,\tau} V_0 - 2\Omega\omega_1 W_{0,\tau} V_1 - 2\Omega\omega_1 W_{1,\tau} V_0 -$$

$$2\Omega\omega_2 W_{0,\tau} V_0]\}d\theta d\tau \qquad (23)$$

As in the zeroth and first order cases, noting the inhomogeneities of equations (21) and (22), W_2 and V_2 take the form, namely

$$(W_2;\ V_2) = (W_{c2};\ V_{c2}) \cos(2\tau) + \ldots$$

$$+ (W_{ss2};\ V_{ss2}) \sin(3m\theta) \cos(3\tau) \qquad (24)$$

Employing equations (24), (21), and (22), it can be shown that the following proportionalities exist, that is

$$(W_2;\ V_2) \propto (W_2(1/D^3(\omega_0,m));\ V_2(1/D^3(\omega_0,m))) \qquad (25)$$

Hence W_2 and V_2 become unbounded for $\omega_0 \sim O(\omega_{mf})$. The requisite form of ω_2 can be obtained by inserting equation (24) into the second order potential energy functional, namely equation (23). After extensive manipulations, this operation yields the following proportionalities for ω_2, that is

$$\omega_2 = \omega_{2NUM}(1/D^4(\omega_0,m))/\omega_{2DEN}(1/D^2(\omega_0,m)) \qquad (26)$$

Thus for $\omega_0 \sim O(\omega_{mf})$, $\omega_2 \sim O(1/D^2(\omega_0,m))$ where, since $D^2(\omega_{mf},m)$ is singular, ω_2 is itself unbounded and negative definite. Additionally W_2 and V_2 are themselves unbounded at such values of ω_0.

DISCUSSION

Stopping the solution at this point, W, V, and ω are given by

$$(W;\ V;\ \omega) \sim (W_0;\ V_0;\ \omega_0) + (W_1;\ V_1;\ \omega_1)\varepsilon +$$

$$(W_2;\ V_2;\ \omega_2)\varepsilon^2 + O(\varepsilon^3) \qquad (27)$$

Due to the procedure employed, it follows that W and V are regular in ε, including $\varepsilon \equiv 0$. This result is, in contrast to standard renormalized perturbation procedures which do not yield zeroth order solutions exhibiting the proper unbounded behavior for ω on the order of the linear system frequencies.

The softening behavior of the ring or infinite cylinder can be directly obtained by considering the fundamental relationship between ω and Γ. Before doing this, the nature of the ω_0 dependency of ω must be ascertained. In particular, for $\omega_0 \sim O(\omega_{mf})$,

$$\omega \sim \omega_0 + \varepsilon 0(1) - \varepsilon^2 0 \left(\frac{1}{D^2(\omega_0, m)}\right) + 0(\varepsilon^3) \tag{28}$$

where since ω_2 is negative definite and unbounded, ω is itself negative definite and unbounded. Such unboundedness occurs at each of the eigenvalues of the pencil of equation (13). Note as Ω is set to zero, the two pairs of eigenvalue branches merge back to the two frequency branches of the stationary state, and hence, the traditional frequencies are obtained.

Eliminating ω_0 from equations (28) and (15), it follows that since ω is unbounded and negative definite for $\omega_0 \sim 0(\omega_{mf})$, the overall steady state harmonic behavior of the ring or infinite cylinder is of the softening type. Hence, as ω is raised or lowered, the usual softening type jump phenomenon is encountered.

In the context of the foregoing, the results can be summarized by the following remarks:

(1) Coriolis forces induce bifurcations in the frequency spectrum;

(2) Such bifurcations extend into the range of deflections marked by moderately large rotations;

(3) All branches exhibit a softening type behavior; this applies to the branches associated with forward as well as backward traveling waves;

(4) Driving frequencies in the neighborhood of the linear system frequency may induce jump phenomena;

(5) Setting $\Omega \to 0$ yields the results for stationary rings and cylinders.

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