VIBRATIONS AND STRESSES IN LAYERED ANISOTROPIC CYLINDERS

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SUMMARY

An equation describing the radial displacement in a k layered anisotropic cylinder has been obtained. The cylinders are initially unstressed but are subjected to either a time-dependent normal stress or a displacement at the external boundaries of the laminate. The solution is obtained by utilizing the Vodicka orthogonalization technique. Numerical examples are given to illustrate the procedure.

INTRODUCTION

The problems associated with the vibrations of plates and shells have been of concern to many investigators over the years. Most of these works for a single layered homogeneous material are summarized in two monographs by Leissa (ref. 1,2) and the reader is referred there for further references. Since composite materials have become popular due to their mechanical and thermal properties, it has become necessary to study their behavior to determine their unique characteristics before they can be used effectively. Recently Cobble (ref. 3) and Dong and Nelson (ref. 4) considered the vibration problem in laminated plates and the references contained in these papers summarize the work in this area quite well. For works concerned with anisotropic and layered cylinders, the book of Ambartsumyan (ref. 5) and Hearmon (ref. 6) and the papers of Gulati and Essenburg (ref. 7), Stavsky and Smolash (ref. 8), Cheung and Wu (ref. 9), and Nelson et al. (ref. 10) are representative.

In this paper, the radial vibrations of a layered anisotropic cylinder are considered. The cylinders are solidly joined at their interfaces, are initially unstressed, and can be subjected to either arbitrary time-dependent normal stresses or displacements at the external boundaries of the system. The solution is obtained by using a dependent variable transformation in the displacement equation thereby obtaining a new partial differential equation with homogeneous external boundary conditions; the Vodicka orthogonality conditions are then applied to this new system to obtain the final solution. The plane strain situation is considered for this analysis.

To illustrate the efficient and straight-forward manner in which solutions can be obtained with this method, numerical examples are given for a two-layered

composite. Results are presented for the displacement, normal stress, tangential stress, and axial stress components at two interior positions.

SYMBOLS

B_i, D_i	constants, (eq. (1))
C_{i11}, C_{i12}	constants, (eq. (2))
E_{i1}, E_{i2}, E_{i3}	Young's modulus for r, θ, and z directions, dyne/cm^2
$F_j(t)$	function of time, (eq. (5))
$H_{ij}(r,t)$	function of displacement and time, (eq. (10))
$J_{Di}(r)$	Bessel function of first kind of order D_i
$L_{ij}(r)$	function of r, (eq. (4))
P_m, q_m	constants, (eq. (23))
r	radial coordinate, cm
t	time, seconds
$u_i(r,t)$	radial displacement, cm
$u_m(t)$	function of time, (eq. (11))
W_i	weighting function, (eq. (17))
$X_{im}(r)$	eigenfunction
Y_{Di}	Bessel function of the second kind of order D_i
α_m	eigenvalues, 1/sec
Δ	constant
ν	Poisson's ratio
σ_{ir}, $\sigma_{i\theta}$, σ_{iz}	normal stress in r, θ, and z directions, dyne/cm^2
$\phi_1(t)$, $\phi_2(t)$	functions of time, (eq. (2))
$\psi_1(r)$, $\psi_2(r)$	functions of r, (eq. (9))

The partial differential equation describing the displacement u_i for the ith layer of a multilayered cylindrical composite whose material properties are constant for each layer is given by

$$\frac{\partial^2 u_i}{\partial r^2}(r,t) + \frac{1}{r}\frac{\partial u_i}{\partial r}(r,t) - \frac{D_i^2}{r^2} u_i(r,t) = \frac{1}{B_i^2}\frac{\partial^2 u_i}{\partial t^2}(r,t) \tag{1}$$

where

$$D_i^2 = \frac{E_{2i}}{E_{1i}}\frac{1-\nu_{31i}\nu_{13i}}{1-\nu_{32i}\nu_{23i}}$$

$$B_i^2 = \frac{E_{1i}}{\rho_i}\frac{1-\nu_{32i}\nu_{23i}}{\Delta_i}$$

$$\Delta = (1-\nu_{31i}\nu_{13i})(1-\nu_{32i}\nu_{231}) - (\nu_{21i} + \nu_{31i}\nu_{23i})(\nu_{12i} + \nu_{32i}\nu_{13i})$$

The boundary and initial conditions associated with equation (1) are:

a) $\quad \sigma_r(r_1,t) = C_{111}\frac{\partial u_1}{\partial r}(r_1,t) + C_{112}\frac{u_1(r_1,t)}{r_1} = \phi_1(t)$

b) $\quad \sigma_r(r_{k+1},t) = C_{k11}\frac{\partial u_k}{\partial r}(r_{k11},t) + C_{k12}\frac{u_k}{r_{k+1}}(r_{k+1},t) = \phi_2(t)$

c) $\quad u_i(r_{k+1},t) = u_{i+1}(r_{i+1},t)$

d) $\quad C_{i11}\frac{\partial u_i}{\partial r}(r_{i+1},t) + C_{i12}\frac{u_i}{r_{i+1}}(r_{i+1},t) = C_{i+1,11}\frac{\partial u_{i+1}}{\partial r}(r_{i+1},t) \tag{2}$

$$+ C_{i+1,12}\frac{u_{i+1}(r_{i+1},t)}{r_{i+1}}$$

e) $\quad u_i(r,o) = 0$

f) $\quad \frac{\partial u_i}{\partial t}(r,o) = 0$

where

$$C_{i11} = \frac{E_{1i}}{\Delta_i} (1-\nu_{32i}\nu_{23i})$$

$$C_{i12} = \frac{E_{1i}}{\Delta_i} (\nu_{21i} + \nu_{31i}\nu_{23i})$$

The boundary and initial conditions given by equation (2) assume that either the radial stresses or displacements are known at the external boundaries and that the radial stresses and displacements are continuous at the interfaces.

To obtain homogeneous external boundary conditions, let

$$u_i(r,t) = U_i(r,t) + \sum_{j=1}^{2} L_{ij}(r)F_j(t) \qquad (3)$$

where

$$L_{ij}(r) = A_{ij}r^{Di} + B_{ij}/r^{Di}, \quad j=1,2 \qquad (4)$$

$$F_j(t) = \phi_j(t), \quad j=1,2 \qquad (5)$$

and

$$\nabla^2 L_{ij}(r) - \frac{D_i^2}{r^2} L_{ij}(r) = 0 \qquad (6)$$

For a cylinder with $r_1 = 0$ (solid cylinder) and $D_1 \leq 1$, Eq. (4) and (6) take the following form for $i = 1$:

$$L_{1j}(r) = A_{1j}$$

and

$$\nabla^2 L_{1j} - \frac{D_1^2}{r^2} L_{1j}(r) = \frac{-A_{1j}D_1^2}{r^2}$$

The functions $L_{ij}(r)$ satisfy the following boundary conditions:

a) $C_{111} \dfrac{dL_{ij}}{dr} (r_1) + C_{112} \dfrac{L_{ij}(r_1)}{r_1} = 1, \; j=1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0, \; j=2$

b) $C_{k11} \dfrac{dL_{kj}}{dr} (r_{k+1}) + C_{k12} \dfrac{L_{kj}}{r_{k+1}} (r_{k+1}) = 0, \; j=1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1, \; j=2$ $\qquad\qquad$ (7)

c) $L_{ij}(r_{i+1}) = L_{i+1,j}(r_{i+1})$

d) $C_{i11} \dfrac{dL_{ij}}{dr} (r_{i+1}) + C_{i12} \dfrac{L_{ij}(r_{i+1})}{r_{i+1}} = C_{i+1,11} \dfrac{dL_{ij}}{dr} (r_{i+1})$

$\qquad\qquad + C_{i+1,12} \dfrac{L_{i+1,j}(r_{i+1})}{r_{i+1}}$

Substitution of equation (3) into equations (1) and (2) yields the following partial differential equation with homogeneous external boundary conditions:

$$\dfrac{\partial^2 U_i}{\partial r^2} (r,t) + \dfrac{1}{r} \dfrac{\partial U_i}{\partial r} (r,t) - \dfrac{D_i^2}{r^2} U_i(r,t) = \dfrac{1}{B_i^2} \dfrac{\partial^2 U_i}{\partial t^2} (r,t) + H_{ij}(r,t) \qquad (8)$$

with

a) $C_{111} \dfrac{\partial U_1}{\partial r} (r_1,t) + C_{112} \dfrac{U_1(r_1,t)}{r_1} = 0$

b) $C_{k11} \dfrac{\partial U_k}{\partial r} (r_{k+1},t) + C_{k12} \dfrac{U_k(r_{k+1},t)}{r_{k+1}} = 0 \qquad\qquad\qquad (9)$

c) $U_i(r_{i+1},t) = U_{i+1}(r_{i+1},t)$

d) $C_{i11} \dfrac{\partial U_i}{\partial r} (r_{i+1},t) + C_{i12} \dfrac{U_i(r_{i+1},t)}{r_{i+1}} = C_{i+1,11} \dfrac{\partial U_{i+1}}{\partial r} (r_{i+1},t)$

$\qquad\qquad + C_{i+1,12} \dfrac{U_{i+1}(r_{i+1},t)}{r_{i+1}}$

e) $U_i(r,o) = -\displaystyle\sum_{j=1}^{2} L_{ij}(r) \, F_j(0) = \psi_1(r)$

f) $\dfrac{\partial U_i}{\partial t}(r,o) = -\sum\limits_{j=1}^{2} L_{ij}(r)\, F_j'(0) = \psi_2(r)$

and where

$$H_{ij}(r,t) = \dfrac{1}{B_i^{\,2}} \sum\limits_{j=1}^{2} L_{ij}(r) F_j''(t) \qquad (10)$$

SOLUTION: $U_i(r,t)$

The problem has now been sufficiently simplified so that a series solution for $U_i(r,t)$ can be assumed where the orthogonality conditions developed by Vodicka (ref. 11) can be utilized. Let

$$U_i(r,t) = \sum\limits_{m=1}^{\infty} u_m(t) X_{im}(r) \qquad (11)$$

$$r_i \le r \le r_{i+1}, \quad i = 1,2,3,\ldots,k, t \ge 0$$

where the function $u_m(t)$ is to be determined from the initial conditions and the functions $X_{im}(r)$ are eigenfunctions of the eigenvalue problem

$$\dfrac{B_i^{\,2}}{r}\dfrac{d}{dr}\left[r \dfrac{dX_{im}}{dr}(r) \right] - \dfrac{B_i^{\,2} D_i^{\,2}}{r^2} X_{im}(r) + \alpha_m^{\,2} X_{im}(r) = 0 \qquad (12)$$

with

a) $C_{111}\dfrac{dX_{1m}}{dr}(r_1) + C_{112}\dfrac{X_{1m}(r_1)}{r_1} = 0$

b) $C_{k11}\dfrac{dX_{km}}{dr}(r_{k+1}) + C_{k12}\dfrac{X_{km}(r_{k+1})}{r_{k+1}} = 0 \qquad (13)$

c) $X_{im}(r_{k+1}) = X_{i+1,m}(r_{k+1})$

d) $C_{i11}\dfrac{dX_{im}}{dr}(r_{i+1}) + C_{i12}\dfrac{X_{im}(r_{i+1})}{r_{i+1}} = C_{i+1,11}\dfrac{dX_{i+1,m}}{dr}(r_{i+1})$

$\qquad + C_{i+1,12}\dfrac{X_{i+1,m}(r_{i+1})}{r_{i+1}}$

The solution of equation (12) is

$$X_{im}(r) = A_{im}J_{Di}\left(\frac{\alpha_m}{B_i}r\right) + B_{im}J_{-Di}\left(\frac{\alpha_m}{B_i}r\right) \quad , \quad D_i = \text{non-integer} \tag{14}$$

$$X_{im}(r) = A_{im}J_{Di}\left(\frac{\alpha_m}{F_i}r\right) + B_{im}Y_{Di}\left(\frac{\alpha_m}{B_i}r\right) \quad , \quad D_i = \text{integer} \tag{15}$$

The eigenvalues, α_m, are found by substituting equations (14) or (15) into the boundary conditions, (eq. (13)). The 2k linear homogeneous equations that result from this substitution are then solved for the constants A_{im} and B_{im} (ref. 12).

The orthogonality condition for the eigenfunctions is

$$\sum_{i=1}^{k} \int_{r_1}^{r_{i+1}} W_1^2\, r\, X_{im}(r)\, X_{in}(r)dr = \begin{matrix}(\text{const. } m = n)\\ 0,\ m \neq n\end{matrix} \tag{16}$$

where

$$W_i^2 = C_{i11}/B_i^2 = \rho_i \tag{17}$$

The functions $L_{ij}(r)$ and $H_{ij}(r,t)$ will satisfy Dirichlet's conditions so they can be expanded in an infinite series of the eigenfunctions

$$L_{ij}(r) = \sum_{m=1}^{\infty} \ell_{mj}X_{im}(r), \quad j = 1,2 \tag{18}$$

and

$$B_i^2 H_{ij}(r,t) = \sum_{m=1}^{\infty} [\ell_{mj}F_j''(t)]\, X_{im}(r), \quad j = 1,2 \tag{19}$$

where

$$\ell_{mj} = \frac{1}{N_m}\sum_{i=1}^{k}\rho_i \int_{r_1}^{r_{i+1}} rL_{ij}(r)\, X_{im}(r)\, dr, \quad j = 1,2 \tag{20}$$

and

$$N_m = \sum_{i=1}^{k} \rho_i \int_{r_1}^{r_{i+1}} r \, [X_{im}^2 (r)] dr \qquad (21)$$

Substituting equations (11), (18), and (19) into equation (8), we obtain the following relationship:

$$\sum_{m=1}^{\infty} \left[\frac{d^2 u_m}{dt^2} (t) + \alpha_m^2 u_m(t) + \sum_{j=1}^{2} \ell_{mj} F_j''(t) \right] X_{im}(r) = 0 \qquad (22)$$

The initial conditions asssociated with equation (28) are obtained in the following manner:

$$U_i(r,o) = \sum_{m=1}^{\infty} u_m(o) X_{im}(r) = \psi_1(r) = -\sum_{m=1}^{\infty} \ell_{mj} F_j(o) X_{im}(r)$$

and

$$\frac{\partial U_i}{\partial t} (r,o) = \sum_{m=1}^{\infty} u_m'(o) X_{im}(r) = \psi_2(r) = -\sum_{m=1}^{\infty} \ell_{mj} F_j'(o) X_{im}(r)$$

Thus

a) $\quad u_m(o) = -\sum_{j=1}^{2} \ell_{mj} F_j(o) = p_m$

$$\qquad (23)$$

b) $\quad u_m'(o) = -\sum_{j=1}^{2} \ell_{mj} F_j'(o) = q_m$

The solution of equation (22) subject to the initial conditions (eq. (23)) is

$$u_m(t) = \frac{q_m}{\alpha_m} \sin \alpha_m t + p_m \cos \alpha_m t - \sum_{j=1}^{2} \frac{\ell_{mj}}{\alpha_m} F_j''(t) * \sin (\alpha_m t) \qquad (24)$$

where the symbol * denotes convolution. Substitution of equation (24) into equation (11) and that result into equation (3) gives the desired relationship for the radial displacement of the composite cylinders:

$$u_i(r,t) = \sum_{j=1}^{2} L_{ij}(r) F_j(t) + \sum_{m=1}^{2} u_m(t) X_{im}(r) \qquad (3)$$

where the functions $L_{ij}(r)$, $F_j(t)$, $X_{im}(r)$ and $u_m(t)$ are given by equations (4), (5), (14) or (15), and (24), respectively.

The stress in the ith section of the composite is given by

$$\sigma_{ir} = \frac{E_1}{\Delta_i} \left((1 - \nu_{32}\nu_{23}) \frac{\partial u_i}{\partial r} (r,t) + (\nu_{21} + \nu_{31}\nu_{23}) \frac{u_i(r,t)}{r} \right) \tag{25}$$

$$\sigma_{i\theta} = \frac{E_2}{\Delta_i} \left((\nu_{12} + \nu_{32}\nu_{13}) \frac{\partial u_i}{\partial r} (r,t) + (1 - \nu_{31}\nu_{13}) \frac{u_i(r,t)}{r} \right) \tag{26}$$

$$\sigma_{iz} = \frac{\nu_{13}E_3}{E_1} \left(\sigma_{ir} + \frac{\nu_{23}E_3}{E_2} \nu_{i\theta} \right) \tag{27}$$

EXAMPLE

Consider a two-layered composite with the following properties:

Layer 1

$\rho_1 = 1.73$ gm/cm^3

$\nu_{121} = \nu_{131} = 0.11$

$\nu_{211} = \nu_{311} = 0.16$

$\nu_{231} = \nu_{321} = 0.1$

$E_{11} = 7.93 \times 10^5$ newton/cm^2

$E_{21} = E_{31} = 1.14 \times 10^6$ newton/cm^2

Layer 2

$\rho_2 = 1.75$ gm/cm^3

$\nu_{122} = \nu_{132} = 0.14$

$\nu_{212} = \nu_{312} = 0.18$

$\nu_{232} = \nu_{322} = 0.22$

$E_{12} = 6.6 \times 10^5$ newton/cm^2

$E_{22} = E_{32} = 8.76 \times 10^5$ newton/cm^2

The above properties are typical of some of the more common graphites (ATJ and CHQ) (ref. 13). Assume further that there is a normal stress applied at the outer boundary of the cylinder.

$$\phi_2(t) = 6895 \sin(10t) \text{ N/cm}^2$$

and the physical dimensions are

$$r_1 = 0; \quad r_2 = 2.54 \text{ cm}; \quad r_3 = 5.08 \text{ cm}$$

Following the procedures outlined in the text, the radial displacement and the radial and tangential stresses within the composite are obtained; values at two positions are shown in Figures 1 through 3.

SUMMARY

A closed-form solution for the radial displacement in layered orthotropic cylinders has been obtained. The solution can be programmed on a modern computer which enables one to calculate natural frequencies, displacements and stresses quite easily. The functions ℓ_{mj} and N_m can either be integrated directly by hand or a numerical integration subroutine can be written to perform the calculations.

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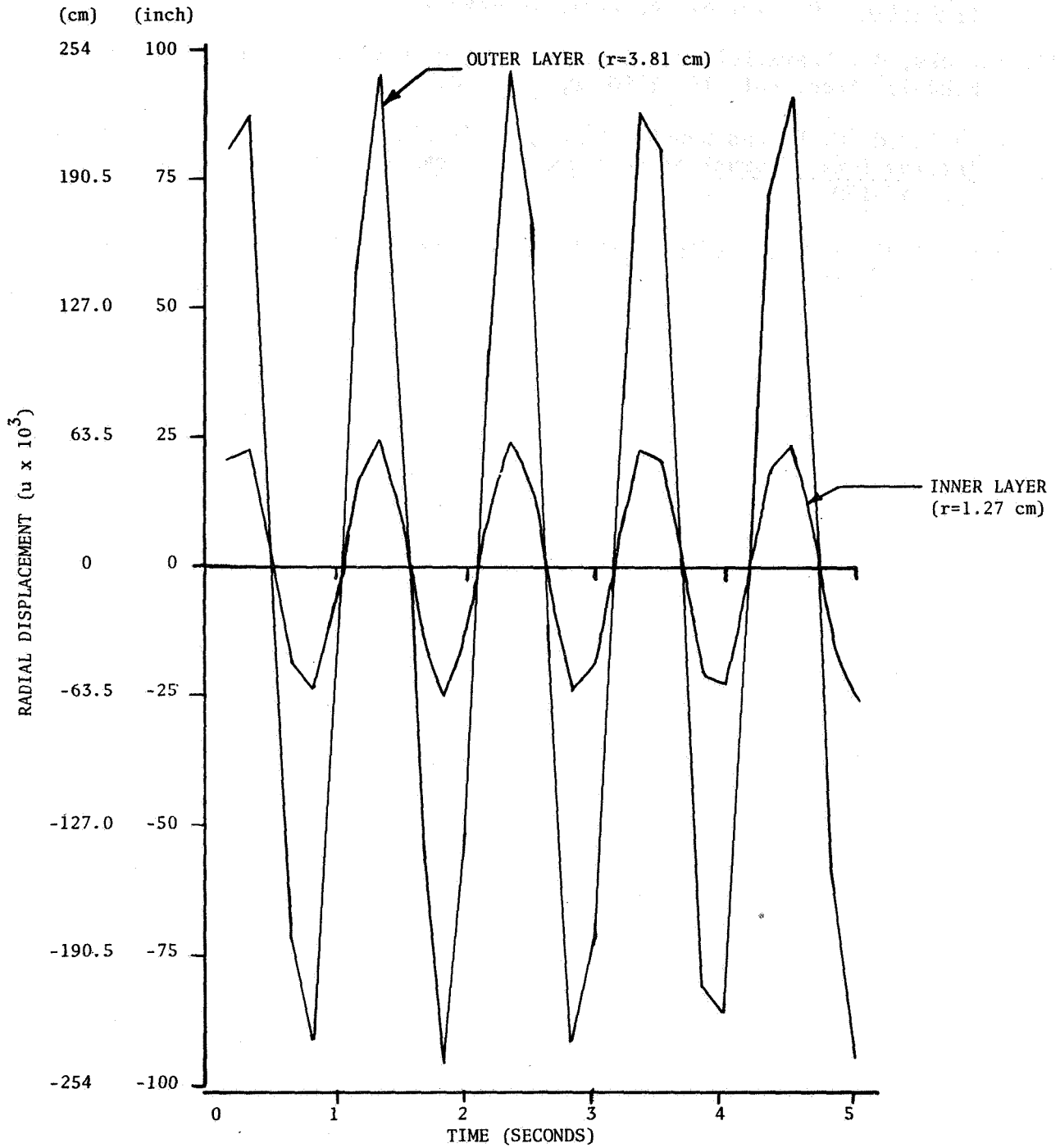13. Industrial Graphite Engineering Handbook, Union Carbide Corporation, New York, 1970.

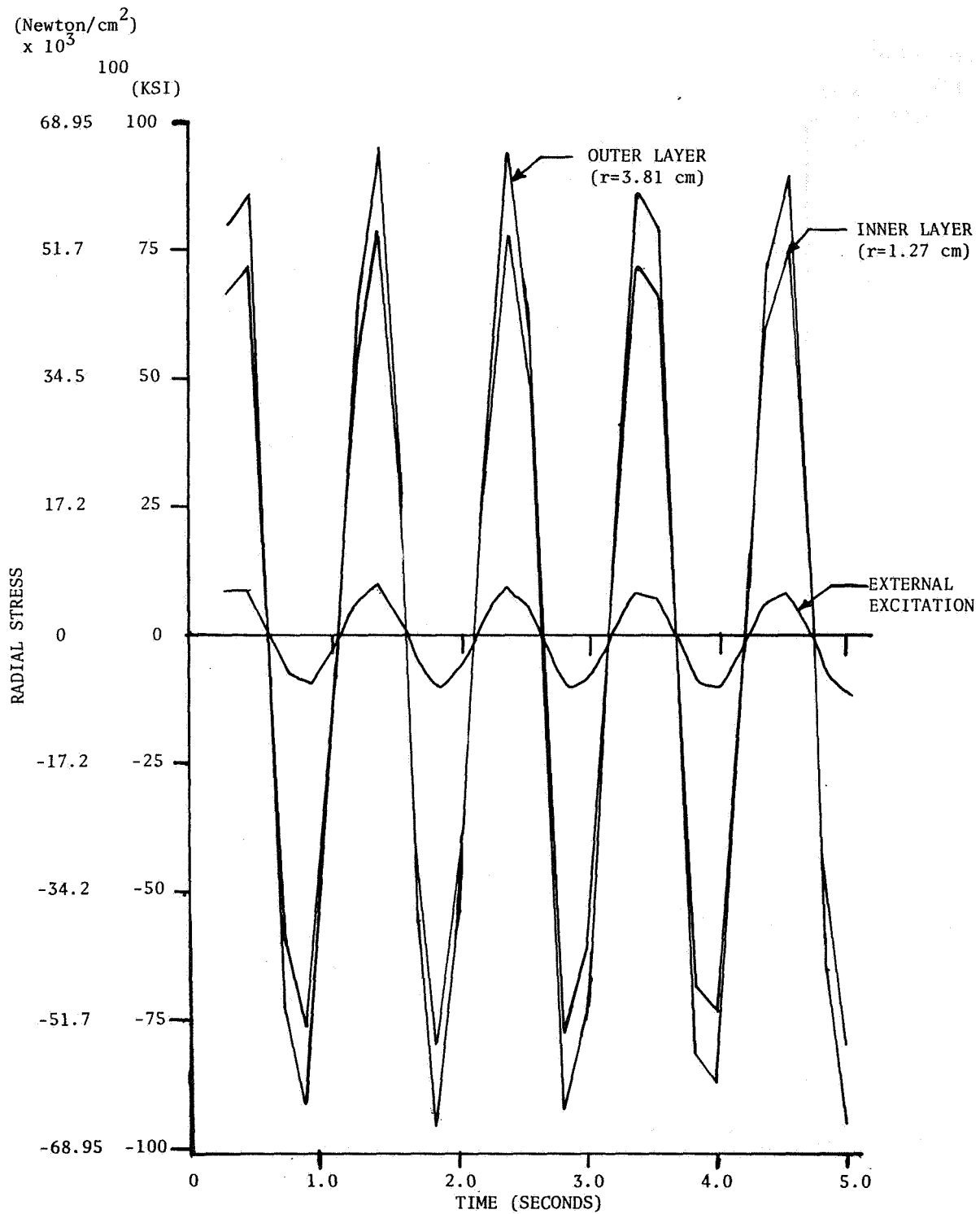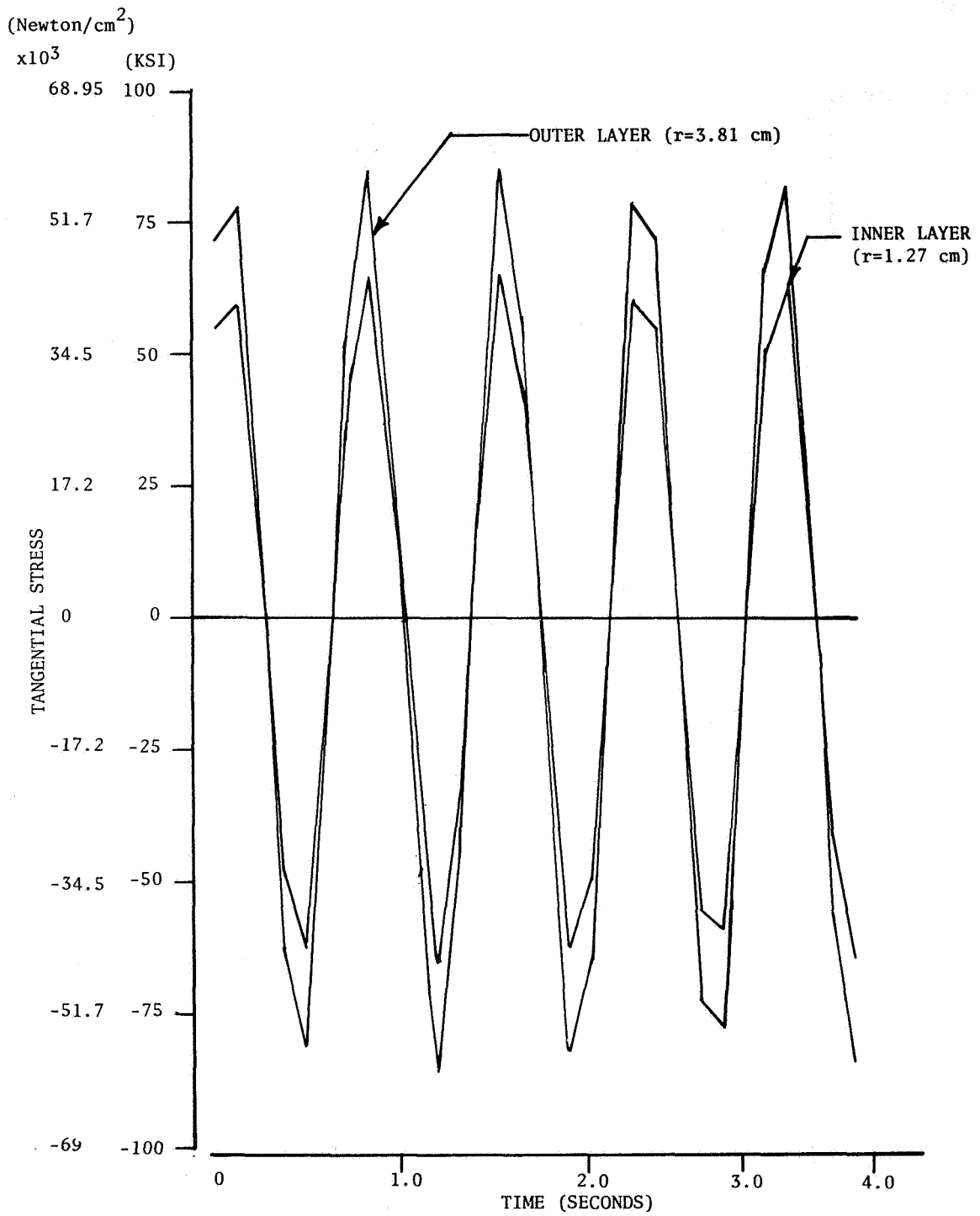Figure 1.- Radial displacement of composite.

Figure 2.- Radial stress compared to external excitation.

Figure 3.- Tangential stresses within composite.