A RAYLEIGH-RITZ APPROACH TO THE SYNTHESIS OF LARGE STRUCTURES WITH ROTATING FLEXIBLE COMPONENTS*

L. Meirovitch** and A. L. Hale***
Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University

SUMMARY

The equations of motion for large structures with rotating flexible components are derived by regarding the structure as an assemblage of substructures. Based on a stationarity principle for rotating structures, it is shown that each continuous or discrete substructure can be simulated by a suitable set of admissible functions or admissible vectors. This substructure synthesis approach provides a rational basis for truncating the number of degrees of freedom both of each substructure and of the assembled structure.

INTRODUCTION

The methodology for analyzing large complex structures has developed along different lines. One approach represents a natural extension of methods developed originally for civil and aircraft structures, culminating in the finite-element method (ref. 1) and the component-mode synthesis (refs. 2,3). Although rotation of the structure could be accounted for through rigid-body modes, work using the approach of references 1-3 has been concerned mainly with nonspinning structures. On the other hand, an entirely different approach was developed in conjunction with spinning and nonspinning spacecraft structures. This approach was dominated by the fact that early spacecraft could be treated as entirely rigid. Hence, in the early stages of development, structures were assumed to consist of point-connected rigid bodies arranged in "topological trees" (refs. 4,5). With time, the rigidity assumption was relaxed gradually by first allowing for flexible "terminal bodies" (refs. 6,7) and then finally for all flexible bodies (ref. 8). A third approach to the problem of spinning flexible spacecraft was concerned with spacecraft consisting of a rigid body with flexible appendages (ref. 9,10). This latter approach can be regarded as an early application of the component-mode synthesis to spinning structures.

Most papers concerned with structures simulated by point-connected rigid bodies, such as references 4, 5, proposed to derive the equations of motion by

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**Professor. ***Graduate Research Assistant.

the Newtonian approach, on the assumption that such a derivation was more suitable for digital computation. Of course, an early difficulty became immediately apparent in the form of the handling of interbody constraints, a major criticism of the Newtonian approach in most circumstances. Another difficulty was the relatively large number of degrees of freedom involved, a difficulty only compounded by permitting various bodies to be flexible. As a result, there are no meaningful ways of truncating the problem.

This paper is concerned with the mathematical simulation of large structures, where the structure is regarded as an assemblage of substructures. Indeed, the mathematical model is assumed to consist of a central substructure with a number of appended substructures, where some of the latter can rotate relative to the central substructure. To ensure that the various substructures act as parts of a whole structure, an orderly kinematical procedure is used which takes into account automatically the superposition of motion of the central substructure on the motion of the interconnected substructures. The system equations of motion are derived by means of the Lagrangian approach, which, when used in conjunction with the kinematical procedure just described, does away with the question of constraints. The equations of motion are derived from scalar functions, namely, the kinetic and potential energy, where the first requires the calculation of velocities only. In addition, discretization of the kinetic and potential energy in conjunction with linearization ensures proper symmetry and skew symmetry of the coefficient matrices in the final equations of motion. Using a Rayleigh-Ritz approach, the motion of each continuous (discrete) substructure can be represented by a linear combination of admissible functions (vectors) rather than substructure natural modes. This approach is based on a stationarity principle for rotating structures developed recently by the first author (ref. 11). Finally, the truncation problem can be handled much more efficiently by the substructure synthesis approach, as the possibility of truncating the number of degrees of freedom both of the individual substructures and of the assembled structure provides a much more rational basis for an overall truncation decision.

KINEMATICAL CONSIDERATIONS

Let us consider a general structure consisting of a central substructure C and a given number of appended substructures (see fig. 1), where the latter are of three types: rigid and rotating relative to the central substructure (type R), elastic and nonrotating relative to the central substructure (type E), and elastic and rotating relative to the central substructure (type A). Clearly, there can be more than one appendage of a given type, but we shall confine our discussion to a representative one of each type, with summation implied over the entire number of substructures. Although we consider here only peripheral substructures, the formulation can be easily extended to chains of substructures, as discussed later.

Let us introduce the inertial system of axes XYZ with the origin at O and identify a system of axes $x_C y_C z_C$ with the origin at an arbitrary point C of the central substructure. Then, denoting by $\underset{\sim}{w}_{OC}$ the radius vector from O to C, by $\underset{\sim}{r}_C$ the position vector of any mass point in the substructure, and by $\underset{\sim}{u}_C$ the

elastic displacement of that point measured relative to $x_C y_C z_C$, and recognizing that w_{OC} is in terms of components along XYZ and r_C and u_C are in terms of components along $x_C y_C z_C$, the absolute position of the mass point in question in terms of components along $x_C y_C z_C$ is $w_C = T_{OC} w_{OC} + r_C + u_C$, where T_{OC} is the matrix of direction cosines between XYZ and $x_C y_C z_C$. Moreover, if Ω_C is the angular velocity of the frame $x_C y_C z_C$ relative to XYZ, the absolute velocity of the mass point is

$$\dot{w}_C = T_{OC}\dot{w}_{OC} - (\tilde{r}_C + \tilde{u}_C)\Omega_C + \dot{u}_C \tag{1}$$

where $\tilde{r}_C + \tilde{u}_C$ is a skew symmetric matrix associated with $r_C + u_C$ and $\dot{u}_C$ is the elastic velocity of the point relative to axes $x_C y_C z_C$.

To calculate the absolute velocity of a point in the substructure R, we must first obtain the velocity of point R as well as the angular velocity of a reference frame $x_{CR} y_{CR} z_{CR}$ attached to the central substructure at R and with axes parallel to the rotor axes $x_R y_R z_R$ when at rest and when the central substructure is undeformed. Due to geometry alone the orientation of axes $x_{CR} y_{CR} z_{CR}$ relative to $x_C y_C z_C$ is given by the constant matrix of direction cosines L_{GR}. Denoting by u_{CR} the elastic deformation vector at point R of the central substructure and assuming that the components u_{CRx}, u_{CRy}, u_{CRz}, of u_{CR} are small, the rotation vector of axes $x_{CR} y_{CR} z_{CR}$ due to elastic deformation can be written in the form

$$\tilde{\nabla}_{CR}(L_{GR} u_{CR}) = \left[\frac{\partial u_{CRz}}{\partial y_{CR}} - \frac{\partial u_{CRy}}{\partial z_{CR}} \quad \frac{\partial u_{CRx}}{\partial z_{CR}} - \frac{\partial u_{CRz}}{\partial x_{CR}} \quad \frac{\partial u_{CRy}}{\partial x_{CR}} - \frac{\partial u_{CRx}}{\partial y_{CR}}\right]^T \tag{2}$$

where $\tilde{\nabla}_{CR}$ is a skew symmetric differential operator matrix corresponding to the curl operator. Hence, the matrix of direction cosines between axes $x_{CR} y_{CR} z_{CR}$ before and after deformation is

$$L_{CR} = \begin{bmatrix} 1 & \frac{\partial u_{CRy}}{\partial x_{CR}} - \frac{\partial u_{CRx}}{\partial y_{CR}} & -\left(\frac{\partial u_{CRx}}{\partial z_{CR}} - \frac{\partial u_{CRz}}{\partial x_{CR}}\right) \\ -\left(\frac{\partial u_{CRy}}{\partial x_{CR}} - \frac{\partial u_{CRx}}{\partial y_{CR}}\right) & 1 & \frac{\partial u_{CRz}}{\partial y_{CR}} - \frac{\partial u_{CRy}}{\partial z_{CR}} \\ \frac{\partial u_{CRx}}{\partial z_{CR}} - \frac{\partial u_{CRz}}{\partial x_{CR}} & -\left(\frac{\partial u_{CRz}}{\partial y_{CR}} - \frac{\partial u_{CRy}}{\partial z_{CR}}\right) & 1 \end{bmatrix} \tag{3}$$

Moreover, letting L_R be the matrix of direction cosines between axes $x_R y_R z_R$ and $x_{CR} y_{CR} z_{CR}$, the transformation matrix between axes $x_R y_R z_R$ and $x_C y_C z_C$ is simply $T_{CR} = L_R L_{CR} L_{GR}$.

Denoting by ω_R the angular velocity of the rotor relative to axes $x_{CR} y_{CR} z_{CR}$, the absolute angular velocity of $x_R y_R z_R$ in terms of components along $x_R y_R z_R$ is

$$\Omega_R = T_{CR}\,\Omega_C + L_R\,\tilde{\nabla}_{CR}(L_{GR}\,\dot{u}_{CR}) + \omega_R \tag{4}$$

where the second term in equation (4) is the angular velocity of axes $x_{CR}y_{CR}z_{CR}$ due to the elastic motion of the central substructure. Because the rotor is rigid, the position of a mass point relative to R is simply $\underset{\sim}{r}_R$. Hence, the absolute velocity of the point in question is simply

$$\dot{\underset{\sim}{w}}_R = T_{CR}\,\dot{\underset{\sim}{w}}_{CR} - \tilde{\underset{\sim}{r}}_R\,\underset{\sim}{\Omega}_R \tag{5}$$

where $\dot{\underset{\sim}{w}}_{CR}$ is the velocity of point R obtained from $\dot{\underset{\sim}{w}}_C$ by substituting the coordinates of the point R for those of an arbitrary point.

Next, let us turn our attention to the substructure E and denote by $x_Ey_Ez_E$ any convenient set of axes with the origin at E and attached to the substructure. Using the analogy with equation (4), the angular velocity of $x_Ey_Ez_E$ is

$$\underset{\sim}{\Omega}_E = T_{CE}\,\underset{\sim}{\Omega}_C + \tilde{\nabla}_{CE}(L_{GE}\,\dot{\underset{\sim}{u}}_{CE}) \tag{6}$$

where $T_{CE} = L_{CE}L_{GE}$. Moreover, by analogy with equation (5), the absolute velocity of a mass point in the substructure is

$$\dot{\underset{\sim}{w}}_E = T_{CE}\,\dot{\underset{\sim}{w}}_{CE} - (\tilde{\underset{\sim}{r}}_E + \tilde{\underset{\sim}{u}}_E)\underset{\sim}{\Omega}_E + \dot{\underset{\sim}{u}}_E \tag{7}$$

where $\dot{\underset{\sim}{u}}_E$ is the elastic displacement relative to axes $x_Ey_Ez_E$.

The extension to elastic substructures rotating relative to the central substructure is quite obvious. Letting $\underset{\sim}{\omega}_A$ be the angular velocity of the substructure A relative to a set of axes $x_{CA}y_{CA}z_{CA}$ attached to the central body at point A, the absolute angular velocity of $x_Ay_Az_A$ is simply

$$\underset{\sim}{\Omega}_A = T_{CA}\,\underset{\sim}{\Omega}_C + L_A\,\tilde{\nabla}_{CA}(L_{GA}\,\dot{\underset{\sim}{u}}_{CA}) + \underset{\sim}{\omega}_A \tag{8}$$

where $T_{CA} = L_AL_{CA}L_{GA}$, and the absolute velocity of an arbitrary point in A is

$$\dot{\underset{\sim}{w}}_A = T_{CA}\,\dot{\underset{\sim}{w}}_{CA} - (\tilde{\underset{\sim}{r}}_A + \tilde{\underset{\sim}{u}}_A)\underset{\sim}{\Omega}_A + \dot{\underset{\sim}{u}}_A \tag{9}$$

Finally, let us consider chains of substructures. First, we note that the angular velocity of a peripheral substructure and the absolute velocity of an arbitrary point in a peripheral substructure are written in terms of the angular velocity of a set of axes attached to the central substructure and with origin at the interconnecting point and the translational velocity of the interconnecting point. As an example, see equations (4) and (5). To write the angular velocity and absolute velocity of an arbitrary point of a substructure in a chain, we simply replace quantities pertaining to the central substructures, such as T_{CR}, $\underset{\sim}{\Omega}_C$, $\nabla_{CR}(L_{GR}\,\dot{\underset{\sim}{u}}_{CR})$, and $\dot{\underset{\sim}{w}}_{CR}$ in equations (4) and (5) by analogous quantities pertaining to the immediately preceding substructure in the chain.

SYSTEM DISCRETIZATION AND/OR TRUNCATION

In general, each elastic substructure possesses a large number of degrees of freedom. In fact, if the substructure is continuous, then its number of degrees of freedom is infinite. For practical reasons, we must limit the formulation not only to a finite number of degrees of freedom but also to as small a number as possible consistent with a good simulation of the system dynamic characteristics. In this regard, we wish to use a Rayleigh-Ritz approach and represent the elastic displacements of a continuous substructure by a linear combination of space-dependent admissible functions multiplied by time-dependent generalized coordinates of the substructure. If the substructure is discrete, then instead of admissible functions we must use admissible vectors. Note that it is common practice to use as admissible functions and admissible vectors the eigenfunctions and eigenvectors of the substructure. In view of the stationarity principle for gyroscopic systems developed in reference 11, however, this is not really necessary, and a reasonable set of admissible functions or admissible vectors should suffice. Hence, we shall use the discretization and/or truncation scheme

$$\underset{\sim}{u}_C = \Phi_C \underset{\sim}{\eta}_C \quad , \quad \underset{\sim}{u}_E = \Phi_E \underset{\sim}{\eta}_E \quad , \quad \underset{\sim}{u}_A = \Phi_A \underset{\sim}{\eta}_A \tag{10}$$

where $\underset{\sim}{\eta}_C$, $\underset{\sim}{\eta}_E$, and $\underset{\sim}{\eta}_A$ are time-dependent vectors of generalized displacements with dimensions n_C, n_E, and n_A, respectively, and Φ_C, Φ_E, and Φ_A are $3 \times n_C$, $3 \times n_E$, and $3 \times n_A$ space-dependent matrices of admissible functions or admissible vectors, as the case may be. Note that for a continuous substructure $\underset{\sim}{u}$ depends on continuous space variables and for a discrete substructure it depends on discrete space variables. In the latter case, the partial derivatives involved in the quantity $\nabla \underset{\sim}{u}$ are to be replaced by corresponding slopes.

Although we have mentioned both continuous and discrete substructures in the above, we have made no attempt to make clear distinction between the two types of mathematical models. Neither have we elaborated on the various types of discrete models, such as lumped models, finite-element models, etc. Of course, the mathematical model used depends on the substructure mass and stiffness distributions, but this is of no particular concern here. The reason for this is that, independently of the mathematical model postulated for the substructure, the general idea is the same, namely, to eliminate the spatial dependence by the use of admissible functions or admissible vectors and to truncate the problem by limiting the number of these functions or vectors.

LAGRANGE'S EQUATIONS OF MOTION

To derive Lagrange's equations of motion it is necessary to produce first expressions for the kinetic energy, potential energy, and nonconservative virtual work. Assuming that in equilibrium the central substructure C, substructure R, and substructure A rotate with the uniform angular velocities $\underset{\sim}{\Omega}_C$ about z_C, $\underset{\sim}{\Omega}_R$ about z_R, and $\underset{\sim}{\Omega}_A$ about z_A, respectively, while any other motion is zero, we can write $\underset{\sim}{\Omega}_C = \Omega_C \underset{\sim}{\ell}_C + \Theta_C \dot{\underset{\sim}{\theta}}_C$, $\underset{\sim}{\omega}_R = \Omega_R \underset{\sim}{\ell}_R + \Theta_R \dot{\underset{\sim}{\theta}}_R$, $\underset{\sim}{\omega}_A = \Omega_A \underset{\sim}{\ell}_A + \Theta_A \dot{\underset{\sim}{\theta}}_A$, where $\underset{\sim}{\ell}_A$ is the vector of direction cosines between z_C and $\tilde{XYZ}$, $\underset{\sim}{\ell}_R$ is the vector of

direction cosines between z_R and $x_{CR}y_{CR}z_{CR}$, and ℓ_A is the vector of direction cosines between z_A and $x_{CA}y_{CA}z_{CA}$. Moreover, Θ_C, Θ_R, and Θ_A are 3 x 3 matrices depending on oscillation of the axes $x_Cy_Cz_C$ relative to XYZ, etc. Using equations (10) and retaining only linear terms, the absolute velocities of typical points in the various substructures become

$$\dot{w}_C = C_1 \dot{q}_C + C_2 q_C \quad , \quad \dot{w}_R = R_1 \dot{q}_R + R_2 q_R$$
$$\dot{w}_E = E_1 \dot{q}_E + E_2 q_E \quad , \quad \dot{w}_A = A_1 \dot{q}_A + A_2 q_A \tag{11}$$

where $q_C = [w_{OC}^T \vdots \theta_C^T \vdots \eta_C^T]^T$, $q_R = [w_{OC}^T \vdots \theta_C^T \vdots \eta_C^T \vdots \theta_R^T]^T$, $q_E = [w_{OC}^T \vdots \theta_C^T \vdots \eta_C^T \vdots \eta_E^T]^T$, and $q_A = [w_{OC}^T \vdots \theta_C^T \vdots \eta_C^T \vdots \theta_A^T \vdots \eta_A^T]^T$ are configuration vectors for the substructures

The system kinetic energy can be written in the form

$$T = T_C + T_R + T_E + T_A \tag{12}$$

where

$$T_C = \frac{1}{2}\int_{m_C} \dot{w}_C^T \dot{w}_C \, dm_C = \frac{1}{2}\dot{q}_C^T \bar{C}_{11} \dot{q}_C + q_C^T \bar{C}_{12} \dot{q}_C + \frac{1}{2} q_C^T \bar{C}_{22} q_C$$
$$T_R = \frac{1}{2}\int_{m_R} \dot{w}_R^T \dot{w}_R \, dm_R = \frac{1}{2}\dot{q}_R^T \bar{R}_{11} \dot{q}_R + q_R^T \bar{R}_{12} \dot{q}_R + \frac{1}{2} q_R^T \bar{R}_{22} q_R$$
$$T_E = \frac{1}{2}\int_{m_E} \dot{w}_E^T \dot{w}_E \, dm_E = \frac{1}{2}\dot{q}_E^T \bar{E}_{11} \dot{q}_E + q_E^T \bar{E}_{12} \dot{q}_E + \frac{1}{2} q_E^T \bar{E}_{22} q_E \tag{13}$$
$$T_A = \frac{1}{2}\int_{m_A} \dot{w}_A^T \dot{w}_A \, dm_A = \frac{1}{2}\dot{q}_A^T \bar{A}_{11} \dot{q}_A + q_A^T \bar{A}_{12} \dot{q}_A + \frac{1}{2} q_A \bar{A}_{22} q_A$$

in which

$$\bar{C}_{ij} = \int_{m_C} C_j^T C_i \, dm_C \quad , \quad \bar{R}_{ij} = \int_{m_R} R_j^T R_i \, dm_R$$
$$\bar{E}_{ij} = \int_{m_E} E_j^T E_i \, dm_E \quad , \quad \bar{A}_{ij} = \int_{m_A} A_j^T A_i \, dm_A \tag{14}$$

Note that the square matrices $\bar{C}_{ij}$, $\bar{R}_{ij}$, $\bar{E}_{ij}$, and $\bar{A}_{ij}$ have partitioned forms, with many of the off-diagonal submatrices equal to zero. Introducing the n-dimensional configuration vector for the entire system in the form $g = [w_{OC}^T \vdots \theta_C^T \vdots \theta_R^T \vdots \theta_A^T \vdots \eta_C^T \vdots \eta_E^T \vdots \eta_A^T]^T$, where n is the number of degrees of freedom of the system, the kinetic energy can be written in the general form

$$T = \frac{1}{2}\dot{\underset{\sim}{q}}^T M \dot{\underset{\sim}{q}} + \underset{\sim}{q}^T F \dot{\underset{\sim}{q}} + \frac{1}{2}\underset{\sim}{q}^T K_T \underset{\sim}{q} \tag{15}$$

where M and K_T are symmetric matrices. Similarly, the potential energy for the entire system is

$$V = \frac{1}{2}\underset{\sim}{q}^T K_V \underset{\sim}{q} \tag{16}$$

where K_V is a symmetric matrix, and the nonconservative virtual work has the form

$$\delta W = \underset{\sim}{Q}^T \delta\underset{\sim}{q} \tag{17}$$

where $\underset{\sim}{Q}$ is the nonconservative generalized force vector.

In general, the matrices M and F depend explicitly on time. However, under certain circumstances, such as when the substructures R and A are symmetric, the time dependence disappears. A helicopter with a symmetric rotor rotating relative to an airframe while in hover is an example, where the entire rotor is considered as a substructure. Another possibility is to consider each rotor blade as a separate substructure. In this case, a combination of substructures forms a symmetric rotor and M and F will once again be constant matrices.

Lagrange's equations can be written in the symbolic form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial L}{\partial \underset{\sim}{q}} = \underset{\sim}{Q} \tag{18}$$

where L = T - V is the system Lagrangian. Assuming that M and F are constant, introducing equations (15) and (16) into the Lagrangian L, and using equation (18), we obtain the Lagrange's equations of motion

$$M\ddot{\underset{\sim}{q}} + (F^T - F)\dot{\underset{\sim}{q}} + (K_V - K_T)\underset{\sim}{q} = \underset{\sim}{Q} \tag{19}$$

where $F^T - F$ is a skew symmetric matrix. Hence, equation (19) represents a typical gyroscopic system. The natural frequencies and natural modes of the complete structure and the closed-form solution of equation (19) can be obtained by the methods developed in references 12 and 13. The interest here is not so much in the response as in the dynamic characteristics of the system, and in particular, the truncation effect on these characteristics.

THE EIGENVALUE PROBLEM AND TRUNCATION IMPLICATIONS

Introducing the 2n-dimensional state vector $\underset{\sim}{x}(t)$ and the associated 2n-dimensional force vector $\underset{\sim}{X}(t)$ in the form

$$\underset{\sim}{x}(t) = [\underset{\sim}{q}^T(t) \vdots \dot{\underset{\sim}{q}}^T(t)]^T \quad , \quad \underset{\sim}{X}(t) = [\underset{\sim}{Q}^T(t) \vdots \underset{\sim}{0}^T]^T \tag{20}$$

where $\underset{\sim}{0}$ is the n-dimensional null vector, as well as the 2n x 2n matrices

$$I = \begin{bmatrix} M & 0 \\ 0 & K_V - K_T \end{bmatrix} \qquad G = \begin{bmatrix} F^T - F & K_V - K_T \\ K_T - K_V & 0 \end{bmatrix} \tag{21}$$

where 0 is the null matrix of order n, the n second-order differential equations of motion, equation (19), can be replaced by the 2n first-order differential equations in the state space $\underset{\sim}{x}(t)$, where the equations have the eigenvalue problem

$$\lambda I \underset{\sim}{x} + G \underset{\sim}{x} = \underset{\sim}{0} \tag{22}$$

It is shown in reference 12 that the eigenvalue problem (22) can be reduced to the real symmetric form

$$\omega^2 I \underset{\sim}{y} = K \underset{\sim}{y} \quad , \quad \omega^2 I \underset{\sim}{z} = K \underset{\sim}{z} \tag{23}$$

where $K = G^T I^{-1} G$ is a real symmetric matrix. The eigenvalue problem (23) is in terms of two real symmetric matrices and is known to possess real eigenvalues. Assuming that I is positive definite, it follows that K is positive definite, so that the eigenvalues are not only real but also positive. Moreover, the eigenvalues ω_r^2 (r = 1,2,...,n) have multiplicity two, so that to each ω_r^2 belong the eigenvectors $\underset{\sim}{y}_r$ and $\underset{\sim}{z}_r$. Because I and K are positive definite all the eigenvectors are independent. In fact, they are orthogonal with respect to the matrix I.

Next, let us use the Cholesky decomposition and write I in the form $I = LL^T$, where L is a lower triangular matrix. Introducing the notation $\underset{\sim}{y}'_r = L^T \underset{\sim}{y}_r$, $\underset{\sim}{z}'_r = L^T \underset{\sim}{z}_r$, (r = 1,2,...,n), the eigenvalue problem (23) becomes

$$\omega_r^2 \underset{\sim}{y}'_r = K' \underset{\sim}{y}'_r \quad , \quad \omega_r^2 \underset{\sim}{z}'_r = K' \underset{\sim}{z}'_r \tag{24}$$

where $K' = L^{-1} K L^{-T}$ is a real symmetric positive definite matrix, in which $L^{-T} = (L^{-1})^T$.

Denoting by $\underset{\sim}{v}$ an arbitrary 2n-vector, Rayleigh's quotient associated with the eigenvalue problem (24) can be written in the form (ref. 11)

$$R(v) = \frac{\underset{\sim}{v}^T K' \underset{\sim}{v}}{\underset{\sim}{v}^T \underset{\sim}{v}} \tag{25}$$

Because K' is real and symmetric, it is well known that Rayleigh's quotient has a stationary value in the neighborhood of an eigenvalue. Note that the symmetric formulation (24) permits us to conclude that a stationarity principle exists also for gyroscopic systems.

Next, we wish to examine the truncation effect on the system characteristics. To this end, let us examine the eigenvalue problem $A\underset{\sim}{v} = \lambda\underset{\sim}{v}$, where A is a real symmetric matrix of order N, and assume that the eigenvalues of A are ordered so that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. Now, let us form the matrix B by deleting the last row and column of A and write the eigenvalue problem $B\underset{\sim}{u} = \gamma\underset{\sim}{u}$, where the eigenvalues γ_j $(j = 1,2,\ldots,N-1)$ are ordered so that $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{N-1}$. The question arises as to how the eigenvalues γ_j relate to the eigenvalues λ_i. To this end, one can use the Courant's maximum-minimum theorem (ref. 14) and prove that

$$\lambda_1 \leq \gamma_1 \leq \lambda_2 \leq \gamma_2 \leq \cdots \leq \lambda_{N-1} \leq \gamma_{N-1} \leq \lambda_N \tag{26}$$

We shall refer to inequalities (26) as the inclusion principle.

Now, let us return to the truncation problem. The 2n x 2n matrix K' was obtained as the result of representing the spinning structure by an n-degree-of-freedom system. Note that the rotational coordinates are also included in these degrees of freedom. This representation is tantamount to the imposition of a given number of constraints on the original structure. For example, the first of equations (10) can be written in the form

$$\underset{\sim}{u}_C = \sum_{i=1}^{n_C} \underset{\sim}{\Phi}_{Ci}\, \eta_{Ci} \tag{27}$$

so that the constraints imposed on the system are $\eta_{C,n_{C+1}} = \eta_{C,n_{C+2}} = \ldots = 0$. Truncating the series (27) by assuming that $\eta_{C,n_C} = 0$, we obtain a matrix K" obtained from K' by deleting two rows and the corresponding two columns. If the eigenvalues ω_r^2 of K' are such that $\omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_n^2$ and the eigenvalues β_r^2 of K" are such that $\beta_1^2 \leq \beta_2^2 \leq \ldots \leq \beta_{n-1}^2$, then we have

$$\omega_1^2 \leq \beta_1^2 \leq \omega_2^2 \leq \beta_2^2 \leq \cdots \leq \omega_{n-1}^2 \leq \beta_{n-1}^2 \leq \omega_n^2 \tag{28}$$

Note that the fact that the eigenvalues of K' and K" have multiplicity two is automatically taken into account in inequalities (28). On the other hand, by relaxing one constraint, i.e., by adding one term to the series (27), we obtain a (2n + 2) x (2n + 2) matrix K"' which is obtained by adding two rows and columns to K'. The eigenvalues α^2 of K"' are such that

$$\alpha_1^2 \leq \omega_1^2 \leq \alpha_2^2 \leq \omega_2^2 \leq \cdots \leq \alpha_n^2 \leq \omega_n^2 \leq \alpha_{n+1}^2 \tag{29}$$

The above developments permit us to conclude that the system estimated natural frequencies tend to decrease monotonically with each additional degree of freedom. At the same time there is a new frequency added which is higher than any of the previous ones.

The question remains as to how to select the admissible functions or admissible vectors. The first thing that comes to mind is to take them as the

eigenfunctions and eigenvectors of the various substructures. In many cases, the solution of the eigenvalue problem for a substructure can be quite a task in itself, so that in such cases one may wish to use deformation patterns only approximating the actual modes. This can be regarded as imposing additional constraints on the system, which tends to raise the natural frequencies of the system, but this may be considered as a viable alternative, particularly when the validity of the solution of the eigenvalue problem is questionable. Experience with the Rayleigh-Ritz approach shows that the system natural frequencies are not very sensitive to the admissible functions used, which can be traced to the stationarity principle. But a stationarity principle exists also for discrete systems, so that the same conclusion can be extended to admissible vectors.

The truncation by substructures has a clear advantage over truncation of the structure as a whole. The reason is that it permits a more rational judgement based on the substructure properties, such as the mass and stiffness distributions. Generally one is interested in only a limited number of lower modes of the complete structure. Hence, a very stiff and light substructure is likely to have less effect on the modes of the complete structure than a flexible heavy substructure. Hence, one can truncate the first more severely than the second. Some ideas for truncation can be obtained by estimating the natural frequencies of the substructures. This by no means implies that one need solve the eigenvalue problem for the substructures exactly. Indeed, using a Rayleigh-Ritz procedure for continuous or discrete systems, in conjunction with a preselected set of admissible functions or admissible vectors, it is possible to obtain a reasonable estimate of the lower frequencies of each substructure. Note that the Rayleigh-Ritz method can be used to produce and solve an eigenvalue problem of considerably lower dimension than that of the full eigenvalue problem for the substructure. The estimated lower natural frequencies of the substructure, when compared to those of other substructures, can be used merely as a guide for truncation purposes. In fact, the eigenvectors serve no useful purpose and need not be calculated, as the same admissible functions or vectors can be used to represent the substructure in the generation of the eigenvalue problem for the complete assembled structure. This conclusion is based on results shown in reference 11.

If the dimension of the eigenvalue problem for the complete assembled structure is still too large, and the higher modes are not really necessary, then one can solve only for a given number of lower modes by using such techniques as subspace iteration.

CONCLUDING REMARKS

A procedure has been shown whereby the equations of motion for large structures with rotating flexible components can be derived by the Lagrangian approach. A fundamental consideration in the derivation of Lagrange's equations is the superposition of substructure motions by means of an orderly kinematical procedure, which automatically eliminates the problem of constraints. Using a Rayleigh-Ritz approach, it is shown that each continuous or discrete flexible substructure can be simulated by a finite number of ad-

missible functions or admissible vectors and exact substructure modes are not really necessary. This conclusion is based on a stationarity principle for rotating structures developed recently by the first author (ref. 11). Finally, the substructure synthesis approach provides a rational basis for truncating the number of degrees of freedom both of each individual substructure and of the assembled substructure.

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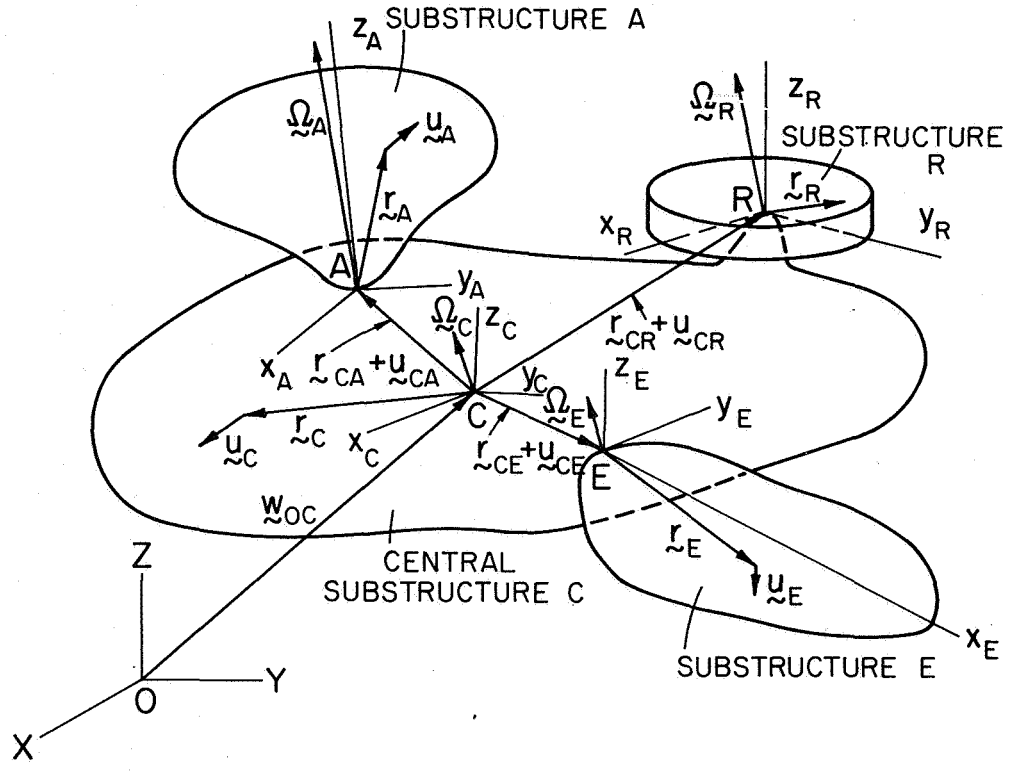

Figure 1.- The mathematical model.