A STUDY ON THE FORCED VIBRATION OF A TIMOSHENKO BEAM

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SUMMARY

By using Galerkin's variational method we build up an approximate solution for a system of two differential equations with linear partial derivatives of the second order. This system of differential equations corresponds to the physical model, known in the literature as the Timoshenko Beam. The results obtained can be finally applied to two particular cases representing respectively:
the case of a beam with a rectangular section, with a constant height and a basis with a linear variation:
the case of a beam with a constant basis and a height with cubic variation.

INTRODUCTION

We are taking into consideration a heterogenous elastic straight beam possessing variable geometrical and mechanical characteristics all along the beam.

We are considering the small, cross-cut non-damping forced oscillations. The mathematical model chosen to be subjected to analysis consists in a system of two linear equations with partial derivatives of second order, corresponding to the physical model known in the literature under the name of Timoshenko Beam. This model is more exact than the classical one usually employed in the engineering calculations, that is the Euler-Bernoulli model. The difference between them consists in the fact that while for the Euler-Bernoulli model only the deformations given by the bending moment or by the translation inertia are taken into account, in the Timoshenko model the transverse shear and the rotational inertia are also taken into consideration. As a result the Timoshenko model reflects more exactly the physical reality. It is well-known that (ref. 1) the differences between the two theories become significant in the case of (relatively) short beams and this cannot be neglected any longer.

Although the literature referring to the dynamics of the Timoshenko Beam is abundant enough, the matter of the non-damping beam has been insufficiently treated.

In the present paper we try to determine the approximate solutions of the phenomenon by means of the Galerkin variational method. We are of the opinion that the above mentioned method is most suitable in solving the subject considered. The choosing of the system of coordinates required by the Galerkin method assures the convergence of the obtained solutions.

SYMBOLS

$\delta(x-\zeta)$	Dirac function
$\beta(p,g)$	Euler's Beta function: $\beta(p,g) = \int_0^1 x^{p-1}(1-x)^{g-1}dx$
K	coefficient of the form of the section
G	cross-cut modulus of elasticity
ρ	density of material
E	longitudinal modulus of elasticity (Young)
A(x)	area of cross-cut section

$I(x)$	moment of inertia of cross-cut section
$W(x,t)$	cross-cut displacement
$\psi(x,t)$	rotation angle
$(f(x),g(x))$	scalar product: $(f,g) = \int_0^1 f(x)g(x)dx$
l	length of beam
$V(x)$	time-independent cross-cut displacement
$U(x)$	time-independent rotation angle
α,λ	cross-sectional area parameters
β	moment of inertia parameter
$C_0^\infty[0,1]$	class of functions defined on 0 to 1

THE DIFFERENTIAL EQUATIONS OF THE PHENOMENON

The differential equations for the phenomenon are as follows: (ref. 2)

$$\begin{aligned} &KGA(x)\frac{\partial^2 W}{\partial x^2} - KGA(x)\frac{\partial \psi}{\partial x} = \rho A(x)\frac{\partial^2 W}{\partial t^2} - f(x,t) \\ &EI(x)\frac{\partial^2 \psi}{\partial x^2} + KGA(x)\left[\frac{\partial W}{\partial x} - \psi\right] = \rho I(x)\frac{\partial^2 \psi}{\partial t^2} \end{aligned} \tag{1}$$

Solutions for the differential equations are determined as follows:

$$W(x,t) = V(x)e^{i\omega t}; \quad \psi(x,t) = U(x)e^{i\omega t} \tag{2}$$

for boundary conditions

$$W(0,t) = W(l,t) = 0 \; ; \; \psi(0,t) = \psi(l,t) = 0 \tag{3}$$

and f(x,t) is a perturbance force, a mobile, but concentrated force for a unit magnitude:

$$f(x,t) = \delta(x-\zeta)e^{i\omega t}$$

By considering equation (2) the system of equation (1) becomes two differential equations of the fourth order for V(x) and U(x) as follows:

$$a_0(x)U^{IV} + a_1(x)U^{III} + a_2(x)U^{II} + a_3(x)U^{I} + a_4(x)U = a_5(x) \tag{4}$$

$$b_0(x)V^{IV} + b_1(x)V^{III} + b_2(x)V^{II} + b_3(x)V^{I} + b_4(x)V = b_5(x) \tag{5}$$

The differential equations (4) and (5) for the following two cases are as follows:

Case 1: $A(x) = \alpha(1+\lambda x) \; ; \; J(x) = \beta(1+\lambda x)$

$$\begin{aligned}
&(a_{30}x^3 + a_{31}x^2 + a_{32}x + a_{33})U^{IV} + (a_{10}x + a_{11})U^{III} + \\
&(b_{30}x^3 + b_{31}x^2 + b_{32}x + b_{33})U^{II} + (b_{10}x + b_{11})U^{I} + \\
&(c_{30}x^3 + c_{31}x^2 + c_{32}x + c_{33})U = \\
&\qquad = (a_{20}x^2 + a_{21}x + a_{22})\delta^{I}(x-\zeta) - (c_{10}x + c_{11})\delta(x-\zeta)
\end{aligned} \tag{6}$$

$$
\begin{aligned}
&(a_{50}x^5+a_{51}x^4+a_{52}x^3+a_{53}x^2+a_{54}x+a_{55})V^{IV}+\\
&+(a_{30}x^3+a_{31}x^2+a_{32}x+a_{33})V'''+\\
&+(b_{50}x^5+b_{51}x^4+b_{52}x^3+b_{53}x^2+b_{54}x+b_{55})V''+\\
&+(b_{30}x^3+b_{31}x^2+b_{32}x+b_{33})V'+\\
&+(c_{50}x^5+c_{51}x^4+c_{52}x^3+c_{53}x^2+c_{54}x+c_{55})V=\\
&=(a_{40}x^4+a_{41}x^3+a_{42}x^2+a_{43}x+a_{44})\delta(x-\zeta)+\\
&+(c_{30}x^3+c_{31}x^2+c_{32}x+c_{33})\delta'(x-\zeta)-(b_{40}x^4+b_{41}x^3+b_{42}x^2+b_{43}x+b_{44})\delta''(x-\zeta)
\end{aligned}
\tag{7}
$$

Case 2: $A(x)=\alpha(1+\lambda x)$; $I(x)=\beta(1+\lambda x)^3$

$$
\begin{aligned}
&(a_{50}x^5+a_{51}x^4+a_{52}x^3+a_{53}x^2+a_{54}x+a_{55})U^{IV}+(a_{40}x^4+a_{41}x^3+a_{42}x^2+a_{43}x+a_{44})U'''+\\
&(b_{50}x^5+b_{51}x^4+b_{52}x^3+b_{53}x^2+b_{54}x+b_{55})U''+(b_{40}x^4+b_{41}x^3+b_{42}x^2+b_{43}x+b_{44})U'+\\
&(c_{50}x^5+c_{51}x^4+c_{52}x^3+c_{53}x^2+c_{54}x+c_{55})U=\\
&\qquad=(a_{20}x^2+a_{21}x+a_{22})\delta'(x-\zeta)-(a_{10}x+a_{11})\delta(x-\zeta)
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&(a_{90}x^9+a_{91}x^8+a_{92}x^7+a_{93}x^6+a_{94}x^5+a_{95}x^4+a_{96}x^3+a_{97}x^2+a_{98}x+a_{99})V^{IV}+\\
&\qquad+(a_{60}x^6+a_{61}x^5+a_{62}x^4+a_{63}x^3+a_{64}x^2+a_{65}x+a_{66})V'''+\\
&+(b_{90}x^9+b_{91}x^8+b_{92}x^7+b_{93}x^6+b_{94}x^5+b_{95}x^4+b_{96}x^3+b_{97}x^2+b_{98}x+b_{99})V''+\\
&\qquad+(b_{60}x^6+b_{61}x^5+b_{62}x^4+b_{63}x^3+b_{64}x^2+b_{65}x+b_{66})V'+
\end{aligned}
$$

$$+\left(c_{90}x^9+c_{91}x^8+c_{92}x^7+c_{93}x^6+c_{94}x^5+c_{95}x^4+c_{96}x^3+c_{97}x^2+c_{98}x+c_{99}\right)V =$$

$$= \left(a_{80}x^8+a_{81}x^7+a_{82}x^6+a_{83}x^5+a_{84}x^4+a_{85}x^3+a_{86}x^2+a_{87}x+a_{88}\right)\delta(x-\xi) + \tag{9}$$

$$+\left(a_{70}x^7+a_{71}x^6+a_{72}x^5+a_{73}x^4+a_{74}x^3+a_{75}x^2+a_{76}x+a_{77}\right)\delta'(x-\xi) -$$

$$-\left(b_{80}x^8+b_{81}x^7+b_{82}x^6+b_{83}x^5+b_{84}x^4+b_{85}x^3+b_{86}x^2+b_{87}x+b_{88}\right)\delta''(x-\xi)$$

THE APPROXIMATE SOLUTION

We shall integrate the differential equations (6), (7), (8), and (9) by means of the Galerkin method.

In the case of boundary conditions of equation (3) we shall consider l=unit which is always possible by

$$\frac{x}{l} = X \; : \; 0 \leq x \leq l \; \Rightarrow \; 0 \leq X \leq 1$$

Using the Galerkin method, we shall determine an approximate solution for equation (6) as follows:

$$U_n(x) = \sum_{k=1}^{n} \alpha_k \varphi_k(x) \tag{10}$$

We choose $\phi_k(x)$ of the form (ref. 3)

$$\varphi_k(x) = x^k (1-x)^{m-k} \; ; \; m = n+1$$

The system of coordinate functions $\phi_k(x)$ has to satisfy the boundary conditions of equation (3) which become equivalent with the following conditions:

$$U(0) = U(1) = 0$$

The approximate solution (10) becomes:

$$U_n(x) = \sum_{k=1}^{n} \alpha_k x^k (1-x)^{m-k} \tag{11}$$

The α_k constants are determined out of the following algebraic system:

$$\sum_{k=1}^{n} \alpha_k (L\varphi_k, \varphi_j) = (g, \varphi_j) \; ; \; j = 1, 2, .., n. \tag{12}$$

where L is the left part of equation (6), and g is the right part of the same equation, that is:

$$L = (a_{30}x^3 + a_{31}x^2 + a_{32}x + a_{33})\frac{d^4}{dx^4} + (a_{10}x + a_{11})\frac{d^3}{dx^3} + $$
$$+ (b_{30}x^3 + b_{31}x^2 + b_{32}x + b_{33})\frac{d^2}{dx^2} + (b_{10}x + b_{11})\frac{d}{dx} + $$
$$+ (c_{30}x^3 + c_{31}x^2 + c_{32}x + c_{33}) \; ;$$

$$L\varphi_k = (a_{30}x^3 + a_{31}x^2 + a_{32}x + a_{33})\frac{d^4\varphi_k}{dx^4} + (a_{10}x + a_{11})\frac{d^3\varphi_k}{dx^3} + $$
$$+ (b_{30}x^3 + b_{31}x^2 + b_{32}x + b_{33})\frac{d^2\varphi_k}{dx^2} + (b_{10}x + b_{11})\frac{d\varphi_k}{dx} + $$
$$+ (c_{30}x^3 + c_{31}x^2 + c_{32}x + c_{33})\varphi_k \; ;$$

$$g=\left(a_{20}x^2+a_{21}x+a_{22}\right)\delta'(x-\zeta)-\left(c_{10}x+c_{11}\right)\delta(x-\zeta)$$

$$(L\varphi_k,\varphi_j)=\int_0^1\varphi_j L\varphi_k\,dx\;;\;(g,\varphi_j)=\int_0^1 g\varphi_j\,dx$$

The system of equation (11) is a non-damped algebraic system of n equations with n indeterminates. This system is compatible because the determinant formed with the coefficients of the undeterminants is a Gramm determinant of a linear independent system of functions. For the calculation of the scalar product $(L\phi_k,\phi_j)$ and (g,ϕ_j), we have kept in view the following points:

We have used the Euler's Beta function

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$$

We have used the following formula (ref. 4) in calculating the scalar product:

if $\varphi\in C_0^\infty[0,1]$ then

$$\int_0^1\varphi(x)\delta^{(n)}(x-\zeta)dx=\left(\delta^{(n)}(x-\zeta),\varphi(x)\right)=(-1)^n\varphi^{(n)}(\zeta)$$

For equation (7) with the boundary conditions of equation (3) which mean $V(o)=V(1)=o$ we are going to give an approximate solution of the following form:

$$V_n(x)=\sum_{k=1}^{n}\beta_k x^k(1-x)^{m-k}\;;\;m=n+1\qquad(13)$$

where the constant β_k is drawn from the following algebraic system:

$$\sum_{k=1}^{n} \beta_k (L\varphi_k, \varphi_j) = (g, \varphi_j) \ ; \ j = 1, 2, \ldots, n.$$

where L is the left side of equation (7) and g is the right side of the same equation.

Analogous to equation (8) we build up an approximate solution of the following form:

$$U_n(x) = \sum_{k=1}^{n} \gamma_k x^k (1-x)^{m-k} \ ; \ m = n+1$$

where the γ_k constants are determined from the following algebraic system:

$$\sum_{k=1}^{n} \gamma_k (L\varphi_k, \varphi_j) = (g, \varphi_j) \ ; \ j = 1, 2, \ldots, n.$$

where L is the left side of equation (8) and g the right side of the same equation.

Finally, for equation (9) we build up a solution of the following form:

$$V_n(x) = \sum_{k=1}^{n} \delta_k x^k (1-x)^{m-k} \ ; \ m = n+1$$

where the δ_k constants are determined from the following algebraic system:

$$\sum_{k=1}^{n} \delta_k (L\varphi_k, \varphi_j) = (g, \varphi_j) \ ; \ j = 1, 2, \ldots, n.$$

where L and g are the left side and right side of equation (9).

As a conclusion to case 1 the approximate solutions built up by the Galerkin method are the following:

$$\psi_n(x,t) = e^{i\omega t}\sum_{k=1}^{n}\alpha_k x^k(1-x)^{m-k}; \quad W_n(x,t) = e^{i\omega t}\sum_{k=1}^{n}\beta_k x^k(1-x)^{m-k}$$

and, for case 2 the approximate solutions are the following

$$\psi_n(x,t) = e^{i\omega t}\sum_{k=1}^{n}\gamma_k x^k(1-x)^{m-k}; \quad W_n(x,t) = e^{i\omega t}\sum_{k=1}^{n}\delta_k x^k(1-x)^{m-k}$$

PECULIAR CASES

In the following lines we shall use the obtained solution for two particular cases, which will be also an indirect checking of the accuracy of the obtained results.

We build up the first two approximations ψ_1; ψ_2 and respectively W_1; W_2 for the following situations:

$$A(x) = A_0(1+\lambda x); \quad J(x) = J_0(1+\lambda x) \tag{14}$$

$$A(x) = A_0(1+\lambda x); \quad J(x) = J_0(1+\lambda x)^3 \tag{15}$$

They represent respectively the case of a beam with a rectangular section, having a constant height and a base with a linear variation, and the case of a beam with a constant base and a height with a cubic variation and this because, from an applicative point of view the beam sections are in many cases considered rectangular. Case (a) The equation (6), if we consider (14) is reduced to the following equation

$$(1+\lambda x)^2\left[EI_0U^{IV}+\rho\omega^2\left(1+\frac{E}{KG}\right)I_0U''+\rho\omega^2\left(\frac{\rho\omega^2}{KG}I_0-A_0\right)U\right]=$$
$$=(1+\lambda x)\delta'(x-\zeta)-\lambda\delta(x-\zeta)$$

The first and second approximations are respectively:

$$\psi_1=\alpha x(1-x)e^{i\omega t}\ ;\ \psi_2=\left[\alpha_1 x(1-x)^2+\alpha_2 x^2(1-x)\right]e^{i\omega t}$$

If we compare ψ_1 with ψ_2 for a rectangular beam made of steel we come to the conclusion that the two approximations are comparable: $\psi_1 = \psi_2$ for certain λ values and for certain x values

λ	0,1	0,2	0,3	0,4	0,5
x	0,252	0,541	0,528	0,573	0,525

This conclusion results from the following calculation:

$$\alpha=\frac{(g,\varphi)}{(L\varphi,\varphi)}=\frac{4\lambda\zeta^2-3\lambda\zeta+2\zeta-1}{\rho\omega^2\left(\frac{\rho\omega^2}{KG}I_0-A_0\right)\left(\frac{1}{105}\lambda^2+\frac{1}{30}\lambda+\frac{1}{30}\right)-\rho\omega^2\left(1+\frac{E}{KG}\right)I_0\left(\frac{1}{10}\lambda^2+\frac{1}{3}\lambda+\frac{1}{3}\right)}$$

The α_1,α_2 constants are determined from the following algebraic system:

$$\alpha_1(L\varphi_1,\varphi_1)+\alpha_2(L\varphi_2,\varphi_1)=(g,\varphi_1)$$
$$\alpha_1(L\varphi_1,\varphi_2)+\alpha_2(L\varphi_2,\varphi_2)=(g,\varphi_2)$$

where

$$(g,\varphi_1) = -5\lambda\zeta^3 + (8\lambda-3)\zeta^2 + (4-3\lambda)\zeta - 1\,; \quad (g,\varphi_2) = 5\lambda\zeta^3 + (3-4\lambda)\zeta^2 - 2\zeta$$

$$(L\varphi_1,\varphi_1) = \rho\omega^2\left(\frac{\rho\omega^2}{KG}I_o - A_o\right)\left(\frac{1}{630}\lambda^2 + \frac{1}{140}\lambda + \frac{1}{105}\right) - \rho\omega^2\left(1+\frac{E}{KG}\right)I_o\left(\frac{1}{105}\lambda^2 + \frac{1}{15}\lambda + \frac{2}{15}\right)$$

$$(L\varphi_1,\varphi_2) = \rho\omega^2\left(\frac{\rho\omega^2}{KG}I_o - A_o\right)\left(\frac{1}{504}\lambda^2 + \frac{1}{140}\lambda + \frac{1}{140}\right) + \rho\omega^2\left(1+\frac{E}{KG}\right)I_o\left(\frac{1}{105}\lambda^2 - \frac{1}{30}\right)$$

$$(L\varphi_2,\varphi_1) = \rho\omega^2\left(\frac{\rho\omega^2}{KG}I_o - A_o\right)\left(\frac{1}{504}\lambda^2 + \frac{1}{140}\lambda + \frac{1}{140}\right) - \rho\omega^2\left(1+\frac{E}{KG}\right)I_o\left(\frac{1}{42}\lambda^2 + \frac{1}{15}\lambda + \frac{1}{30}\right)$$

$$(L\varphi_2,\varphi_2) = \rho\omega^2\left(\frac{\rho\omega^2}{KG}I_o - A_o\right)\left(\frac{1}{252}\lambda^2 + \frac{1}{84}\lambda + \frac{1}{105}\right) - \rho\omega^2\left(1+\frac{E}{KG}\right)I_o\left(\frac{8}{105}\lambda^2 + \frac{1}{5}\lambda + \frac{2}{15}\right)$$

and for the steel in S.I. units

$$K = \frac{5}{6}\,; \quad G = 8\times 9{,}8\times 10^9\,\frac{N}{m^2}\,; \quad E = 2{,}1\times 9{,}8\times 10^{10}\,\frac{N}{m^2}\,; \quad l = 1\,m.$$
$$h = 10^{-1}\,m.\,; \quad \rho = 8\times 10^3\,\frac{Kg}{m^3}\,; \quad A_o = a_o h\,; \quad I_o = a_o\frac{h^3}{12}$$

Equation (7) then becomes:

$$(1+\lambda x)^4\left(aV^{IV} + bV^{II} + cV\right) = \left[d_1(1+\lambda x)^3 - 2\lambda^2 d(1+\lambda x)\right]\delta(x-\zeta) +$$
$$+\, 2\lambda d(1+\lambda x)\delta^{I}(x-\zeta) - d(1+\lambda x)^3\delta^{II}(x-\zeta)$$

where $\quad a = EGKA_o^4\,; \quad b = \rho\omega^2(E+KG)I_oA_o^3\,; \quad d = EI_oA_o^2$

$$c = \rho\omega^2(\rho\omega^2 I_o - KGA_o)A_o^3\,; \quad d_1 = (KGA_o - \rho\omega^2 I_o)A_o^2$$

The first and second approximations are

$$W_1 = \beta x(1-x)e^{i\omega t} \text{ ; and } \quad W_2 = \left[\beta_1 x(1-x)^2 + \beta_2 x^2(1-x)\right]e^{i\omega t}$$

here

$$\beta = \frac{A_1 d_1 + B_1 d}{Ac + Bb}$$

$$A = \frac{1}{252}\lambda^4 + \frac{1}{42}\lambda^3 + \frac{2}{35}\lambda^2 + \frac{1}{15}\lambda + \frac{1}{30}$$

$$B = -\left(\frac{1}{21}\lambda^4 + \frac{4}{15}\lambda^3 + \frac{3}{5}\lambda^2 + \frac{2}{3}\lambda + \frac{1}{3}\right)$$

$$A_1 = \frac{1}{4}\left(1+\frac{\lambda}{2}\right)^3$$

$$B_1 = -\frac{3}{4}\lambda^3 - \lambda^2 + 3\lambda + 2$$

$$\beta_1 = \frac{1}{\Delta}\left[(A_2 d_1 + B_2 d)(P_6 c + P_7 b) - (A_3 d_1 + B_3 d)(P_3 c + P_5 b)\right]$$

$$\beta_2 = \frac{1}{\Delta}\left[(A_3 d_1 + B_3 d)(P_1 c + P_2 b) - (A_2 d_1 + B_2 d)(P_3 c + P_4 b)\right]$$

$$\Delta = (P_1 c + P_2 b)(P_6 c + P_7 b) - (P_3 c + P_4 b)(P_3 c + P_5 b)$$

$$A_2 = \frac{1}{8}\left(1+\frac{\lambda}{2}\right)^3$$

$$B_2 = 1 + \frac{7}{2}\lambda + \frac{5}{4}\lambda^2$$

$$A_3 = \frac{1}{8}\left(1+\frac{\lambda}{2}\right)^3$$

$$B_3 = 1 - \frac{1}{2}\lambda - \frac{9}{4}\lambda^2 - \frac{3}{4}\lambda^3$$

$$P_1 = \frac{1}{2310}\lambda^4 + \frac{1}{315}\lambda^3 + \frac{1}{105}\lambda^2 + \frac{1}{70}\lambda + \frac{1}{105}$$
$$P_2 = -\left(\frac{1}{105}\lambda^3 + \frac{2}{35}\lambda^2 + \frac{2}{15}\lambda + \frac{2}{15}\right)$$
$$P_3 = \frac{1}{1320}\lambda^4 + \frac{1}{210}\lambda^3 + \frac{1}{84}\lambda^2 + \frac{1}{70}\lambda + \frac{1}{140}$$
$$P_4 = \frac{1}{84}\lambda^4 + \frac{1}{21}\lambda^3 + \frac{2}{35}\lambda^2 - \frac{1}{30}$$
$$P_5 = -\left(\frac{1}{84}\lambda^4 + \frac{1}{15}\lambda^3 + \frac{1}{7}\lambda^2 + \frac{2}{15}\lambda + \frac{1}{30}\right)$$
$$P_6 = \frac{1}{495}\lambda^4 + \frac{1}{90}\lambda^3 + \frac{1}{42}\lambda^2 + \frac{1}{42}\lambda + \frac{1}{105}$$
$$P_7 = -\left(\frac{1}{24}\lambda^4 + \frac{5}{24}\lambda^3 + \frac{16}{35}\lambda^2 + \frac{2}{5}\lambda + \frac{2}{15}\right)$$

Because $W_2 = W_1\left[1 + \frac{(\beta_1 - \beta) + (\beta_2 - \beta_1)x}{\beta}\right]$ the conclusion is made that the two approximations are comparable for certain λ values and for certain x values such as

λ	0,1	0,2	0,3	0,4	0,5
x	0,675	0,515	0,585	0,525	0,567

CONCLUSIONS TO THESE PECULIAR CASES

For equations (8) and (9) we come to the same result, that is: the first two approximate solutions are equal for the given values of λ for the same value of x :0,5: $0 \le x \le 1$ that is, the approximate solutions are comparable among themselves in the vicinity of where the concentrated perturbance force is applied: when $x = \zeta = \frac{1}{2}$.

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