TRANSFER MATRIX APPROACH TO LAYERED SYSTEMS WITH AXIAL SYMMETRY

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SUMMARY

The stress and displacement distribution in a layered medium is found by means of transfer matrices. The surface loading exhibits axial symmetry, and each layer is of infinite extent in the horizontal direction, of constant depth, and is considered to be linearly elastic, homogeneous, and isotropic. The method developed has the built-in advantage of enforcing interface continuity conditions automatically. Its application to layered composites shows the flexibility with which it predicts the local as well as the global response of the medium.

INTRODUCTION

Recently, this writer developed a transfer matrix approach to various problems in mechanics by combining the method of initial functions due to Vlasov (ref. 1), with the integral transform method developed by Sneddon (ref. 2).

The method employed by this writer consists in applying the state space approach, which has been used extensively to analyze linear systems in various areas of systems engineering, such as modern control theory (ref. 3), to the field of elastomechanics.

The topics so far analyzed through this approach cover two-dimensional elastostatics (ref. 4), one-dimensional elastodynamics (ref. 5), application to a typical elasticity problem (ref. 6), examination of the basic foundation of the theory (ref. 7), application to numerical integration of equations of motion to predict dynamic response (ref. 8), heat conduction (ref. 9), boundary value problems (ref. 10) and earthquake engineering with emphasis on soil-structure interaction (ref. 11). Additional references pertaining to each topic considered will be found in the references cited above and will not be repeated here.

This paper extends the work described in (ref. 4) which was restricted to a plane stress (or plane strain) problem, to a three-dimensional one with axially symmetric loading. The motivation for considering the present approach is to develop a flexible method for the analysis of layered media subjected for instance to concentrated loads, ranging from classical problems in soil mechanics, to the prediction of impulsive response of laminated composites. In the latter case inertial effects must be included.

The main advantage of the method is due to the fact that continuity of stresses and displacements at interfaces is automatically satisfied. Therefore, upon determination of the missing initial displacements from boundary conditions, the field quantities can be determined upon multiplication of the initial state vector by the chain of layer transfer matrices by the field matrix of the layer of interest. A Hankel inversion gives the actual field quantities.

In contrast, the classical formulation requires the construction of a transformed Airy stress function that contains four arbitrary parameters per layer, thus producing a total of 4n equations in 4n unknowns for a medium of n layers. These are determined by enforcing the continuity of stresses and displacements across each interface, which yields 4(n-1) conditions to which the four boundary conditions are added.

DERIVATION OF THE TRANSFER MATRIX

The equations governing the state of stress of an axially symmetric, homogeneous, isotropic, linearly elastic solid, are given by the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1a}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \tag{1b}$$

in the absence of body forces and inertial effects. These equations must be adjoined by the constitutive relations

$$\sigma_r = (\lambda + 2\mu) \frac{\partial u}{\partial r} + \lambda\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right) \tag{2a}$$

$$\sigma_\theta = \lambda\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right) + (\lambda + 2\mu)\frac{u}{r} \tag{2b}$$

$$\sigma_z = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + (\lambda + 2\mu) \frac{\partial w}{\partial z} \tag{2c}$$

$$\tau_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{2d}$$

The four stresses given by equations (2) are functions of the partial derivatives of two displacements only; it follows that two of these stresses can be eliminated.

For reasons of convenience, σ_θ and σ_r are chosen for this purpose. Upon substitution of equations (2a) and (2b) into equation (1a), the latter can be rewritten as

$$(\lambda + 2\mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + \frac{\partial \tau_{rz}}{\partial z} + \lambda \frac{\partial^2 w}{\partial z \partial r} = 0 \tag{3}$$

Differentiation of equation (2c) with respect to r yields

$$\frac{\partial \sigma_z}{\partial r} = \lambda \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + (\lambda + 2\mu)\frac{\partial^2 w}{\partial z \partial r} \tag{4}$$

Elimination of the mixed derivative between equations (3) and (4) results in the relation

$$\lambda \frac{\partial \sigma_z}{\partial r} + 4\mu(\lambda + \mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + (\lambda + 2\mu)\frac{\partial \tau_{rz}}{\partial z} = 0 \tag{5}$$

Consider a semi-infinite elastic medium which extends to infinity in the r-direction as shown in figure 1. The medium is loaded by an axially symmetric load as shown. Under the circumstances, taking Hankel Transforms of order one of equations (5) and (2d), and of order zero of equations (2c) and (1b), results in the system of equations cast in matrix form as follows:

$$\frac{d}{dz}\begin{Bmatrix} \mu\bar{u}_1 \\ \mu\bar{w}_0 \\ \bar{\sigma}_0 \\ \bar{\tau}_1 \end{Bmatrix} = \begin{bmatrix} 0 & \xi & 0 & 1 \\ -\lambda\xi/(\lambda+2\mu) & 0 & \mu/(\lambda+2\mu) & 0 \\ 0 & 0 & 0 & -\xi \\ 4(\lambda+\mu)\xi^2/(\lambda+2\mu) & 0 & \lambda\xi/(\lambda+2\mu) & 0 \end{bmatrix}\begin{Bmatrix} \mu\bar{u}_1 \\ \mu\bar{w}_0 \\ \bar{\sigma}_0 \\ \bar{\tau}_1 \end{Bmatrix} \tag{6}$$

where the subscripts indicate the order of the Hankel transform. Equation (6) can be integrated by considering the column vector of transformed stresses and displacements as the state vector $\bar{X}(\xi,z)$, and rewriting it as

$$\frac{d}{dz}\{\bar{X}(\xi,z)\} = [A(\xi)]\{\bar{X}(\xi,z)\} \tag{7}$$

As shown in (ref. 4), equation (7) can be integrated to yield

$$\{\bar{X}(\xi,z)\} = \exp[zA(\xi)]\{\bar{X}(0)\} \tag{8}$$

where the matrix exponential has to be evaluated explicitly. The characteristic roots of the determinant associated with the matrix $A(\xi)$ are the double roots $\pm\xi$, identical to the result obtained in ref. 4 . Therefore, the results are analogous to those obtained in that paper, in which it is shown that

$$\exp(zA) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 \tag{9}$$

where

$$\begin{aligned}
a_0 &= \cosh \xi z - (\xi z/2) \sinh \xi z \\
a_1 &= [3 \sinh \xi z - \xi z \cosh \xi z]/2\xi \\
a_2 &= [z \sinh \xi z]/2\xi \\
a_3 &= [\xi z \cosh \xi z - \sinh \xi z]/2\xi^3
\end{aligned} \tag{10}$$

Upon substitution of these values into equation (9), the transfer matrix is obtained, and equation (8) gives, in turn, the state vector which consists of the transformed stresses and displacements at an arbitrary depth in the field. The details pertaining to the evaluation of the transfer matrix are given in the Appendix. The results can be summarized in matrix form as

$$\begin{Bmatrix} \mu\bar{u}_1(\xi,z) \\ \mu\bar{w}_0(\xi,z) \\ \bar{\sigma}_0(\xi,z) \\ \bar{\tau}_1(\xi,z) \end{Bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{Bmatrix} \mu\bar{u}_1(\xi,0) \\ \mu\bar{w}_0(\xi,0) \\ \bar{\sigma}_0(\xi,0) \\ \bar{\tau}_1(\xi,0) \end{Bmatrix} \tag{11}$$

where the influence functions mapping the initial field quantities into those at an arbitrary depth in the field are given by

$$\left.\begin{aligned}
L_{11} &= L_{44} = \cosh z\xi + [(\lambda+\mu)/(\lambda+2\mu)]z\xi \sinh z\xi \\
L_{12} &= -L_{34} = [\mu \sinh z\xi + (\lambda+\mu)z\xi \cosh z\xi]/(\lambda+2\mu) \\
L_{13} &= -L_{24} = [(\lambda+\mu)/(\lambda+2\mu)]z \sinh z\xi \\
L_{14} &= [1/2(\lambda+2\mu)\xi][(\lambda+3\mu) \sinh z\xi + (\lambda+\mu)z\xi \cosh z\xi] \\
L_{21} &= -L_{43} = [1/(\lambda+2\mu)][\mu \sinh z\xi - (\lambda+\mu)z\xi \cosh z\xi] \\
L_{22} &= L_{33} = \cosh z\xi - [(\lambda+\mu)/(\lambda+2\mu)]z\xi \sinh z\xi \\
L_{23} &= [1/2(\lambda+2\mu)\xi][(\lambda+3\mu) \sinh z\xi - (\lambda+\mu)z\xi \cosh z\xi] \\
L_{31} &= -L_{42} = -2z\xi^2 \sinh z\xi \\
L_{32} &= [2(\lambda+\mu)\xi/(\lambda+2\mu)][\sinh z\xi - z\xi \cosh z\xi] \\
L_{41} &= [2(\lambda+\mu)\xi/(\lambda+2\mu)][\sinh z\xi + z\xi \cosh z\xi]
\end{aligned}\right\} \tag{12}$$

The actual physical quantities are then recovered through the inverse Hankel transform.

APPLICATION TO LAYERED SYSTEM

Consider a layered medium with perfect bonding along all interfaces as shown in figure 2. This implies the continuity of transformed stresses and displacements across each interface. In order to enforce this condition, the first two entries of the state vector which appear in equation (11) are divided by the shear modulus, to produce a new state vector consisting of transformed stresses and displacements. The elements of the new matrix G become $G_{13} = L_{13}/\mu$; $G_{14} = L_{14}/\mu$; $G_{23} = L_{23}/\mu$; $G_{24} = L_{24}/\mu$; $G_{31} = \mu L_{31}$; $G_{32} = \mu L_{32}$; $G_{41} = \mu L_{41}$; and $G_{42} = \mu L_{42}$. The remaining elements of the G matrix are identical to the corresponding elements of the L matrix.

The modified equation (11) can now be written in contracted form as

$$\{\bar{Y}(\xi,z)\} = [G(\lambda,\mu,z,\xi)]\ \{\bar{Y}(\xi,0)\} \tag{13}$$

Applying equation (13) to each interface in turn, in the sequence shown in figure 2, leads to

$$\{\bar{Y}(\xi,h_n)\} = [G(\lambda_n,\mu_n,h_n,\xi)\cdots G(\lambda_1,\mu_1,h_1,\xi)]\ \{\bar{Y}(\xi,0)\} \tag{14}$$

in which the missing initial conditions are determined from boundary conditions. Equation (14) then describes the overall response of the layered system.

Local information consisting of state vectors at interfaces can now be obtained by terminating the matrix multiplication indicated by equation (14) at the appropriate interface. These relations are shown by the block diagrams shown in figures 3 and 4.

The state vector in any arbitrary layer m can now be found by the relation

$$\{\bar{Y}(\xi,z)\} = [G(\lambda_m,\mu_m,z,\xi)\ \prod_{i=1}^{m-1}\ G(\lambda_i,\mu_i,h_i,\xi)]\{\bar{Y}(\xi,0)\} \tag{15}$$

in which the z coordinate is the local depth within the layer m, ranging from zero to h_m. The actual stresses and displacements are given by the inverse Hankel transformation of the state vector.

CONCLUDING REMARKS

In this paper, a transfer matrix method to determine the response of a layered medium subjected to an axially symmetric loading has been presented.

The matrix formulation shows that the need for matching interface conditions explicitly is avoided by imposing the continuity of the state vector across each interface. This is accomplished through the continued multiplication of layer transfer matrices. Therefore, the size of the transfer matrix remains four by four, and is independent of the number of layers contained in the medium. This is the main conceptual as well as computational advantage of the proposed method.

APPENDIX

The transfer matrix is given by the expression $\exp(zA) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$, in which the matrices A, A^2, and A^3 are given by

$$A = \begin{bmatrix} 0 & \xi & 0 & 1 \\ -\lambda\xi/(\lambda+2\mu) & 0 & \mu/(\lambda+2\mu) & 0 \\ 0 & 0 & 0 & -\xi \\ 4(\lambda+\mu)\xi^2/(\lambda+2\mu) & 0 & \lambda\xi/(\lambda+2\mu) & 0 \end{bmatrix}$$

$$(\lambda+2\mu)A^2 = \begin{bmatrix} (3\lambda+4\mu)\xi^2 & 0 & (\lambda+\mu)\xi & 0 \\ 0 & -\lambda\xi^2 & 0 & -(\lambda+\mu)\xi \\ -4(\lambda+\mu)\xi^3 & 0 & -\lambda\xi^2 & 0 \\ 0 & 4(\lambda+\mu)\xi^3 & 0 & (3\lambda+4\mu)\xi^2 \end{bmatrix}$$

$$(\lambda+2\mu)A^3 = \begin{bmatrix} 0 & (3\lambda+4\mu)\xi^3 & 0 & (2\lambda+3\mu)\xi^2 \\ -(3\lambda+2\mu)\xi^3 & 0 & -\lambda\xi^2 & 0 \\ 0 & -4(\lambda+\mu)\xi^4 & 0 & -(3\lambda+4\mu)\xi^3 \\ 8(\lambda+\mu)\xi^4 & 0 & (3\lambda+2\mu)\xi^3 & 0 \end{bmatrix}$$

and the coefficients $a_0, a_1 a_2$, and a_3 are given by the set of relations (10). The elements of the matrix exponential are given explicitly by the expressions (12).

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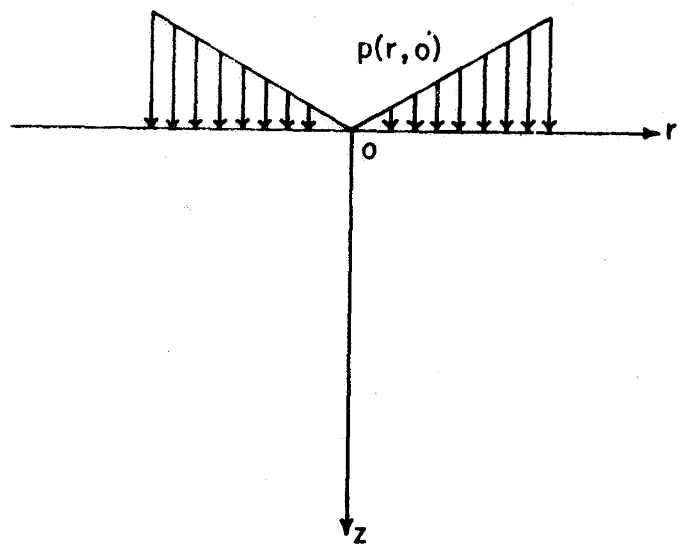

Figure 1.- Semi-infinite elastic medium.

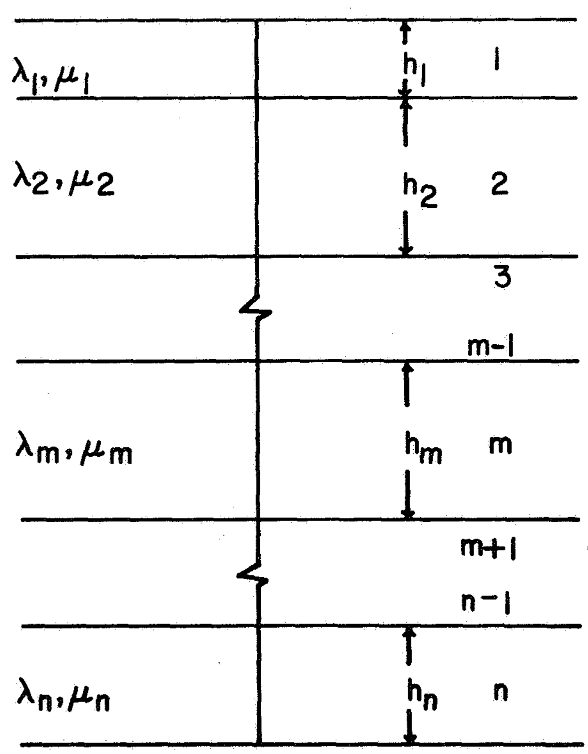

Figure 2.- Axially symmetric layered medium.

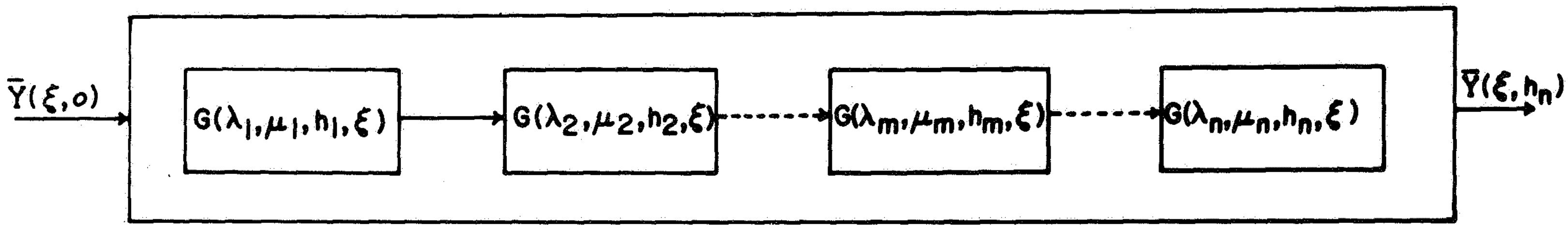

Figure 3.- Overall response of layered medium.

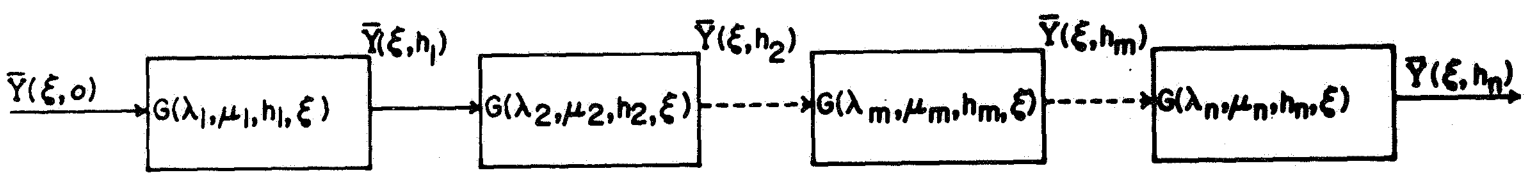

Figure 4.- Local response of layered medium.