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APPLIED GROUP THEORY

APPLICATIONS IN THE ENGINEERING (PHYSICAL, CHEMICAL, AND MEDICAL), BIOLOGICAL, SOCIAL, AND BEHAVIORAL SCIENCES AND IN THE FINE ARTS

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SUMMARY

A generalized "applied group theory" is developed and it is shown that phenomena from a number of diverse disciplines may be included under the umbrella of a single theoretical formulation based upon the concept of a "group" consistent with the usual definition of this term.

INTRODUCTION

The essence of the "group" concept as used herein is contained in the three terms, element, transformation and invariance, and it may be shown that they are included in the various analyses discussed in this paper. More formally, the mathematical definition of a group generally includes the "inverse" operation (however defined) and also an "identity" operation (also variously though consistently defined). These may be brought into the discussion of this report without difficulty, as will be shown, although the main emphasis will be placed upon the element, transformation and invariance properties of the groups being considered.

It must be noted at the outset that the various terms, quantities and operations will have different forms for the different disciplines considered. In some cases they take a mathematical form; in others they appear as curves, or as sounds or as visual entities. However, despite these differences, it will be shown that the requirements of the "group" representation will be satisfied in each case and in this sense all of the disciplines discussed fall within the overall province of the group concept.

The following manner of presentation will be utilized. In the next section a Table will be presented in which the entire theory will be summarized. All of the group requirements will be listed for the different disciplines considered in this paper. Others may be included, without difficulty, if desired.

After this an example from each discipline will be discussed in greater detail.

THE GROUPS

A concise, detailed general classification scheme for the underlying theory is contained in Table 1.

Note especially how all of the formal requirements of group representation are satisfied - although these vary from group to group.

In particular two distinct typical types of group elements are shown: (1) tensor or (2) events. There appears to be a connection between these seemingly separate group types in that many "event" groups may, in fact, be "tensors". A discussion of this point in connection with "Behavior" is presented later in the paper and current continuing analyses indicate that this duality may be a general property of many event phenomena.

The transformation corresponding to the tensors is a rotation of axes. The transformation corresponding to an event is an alteration or change of the phenomenon caused by a change in the particular activity variable involved in the phenomenon. The invariants (which imply conservation of the structure of the element during the transformation) are the tensor invariants for tensor and the single equation or curve or other phenomenon representing events. All of the above will be explained in greater detail in the next section.

DISCUSSION OF TYPICAL GROUPS IN THE VARIOUS DISCIPLINES

In this section, one typical group from each of the disciplines will be described in more detail than given in Table 1. The references cover the subject in even greater detail.

Engineering-Physics Groups

A typical group element of many engineering-physical problems is the tensor - zero, first and second order (ref. 1). Zero order tensors are scalars, first order tensors are vectors and second order tensors are usually called "tensors".

Some typical familiar examples of tensors are the stress tensor and the inertia tensor. In three dimensional x-y-z space these may be shown in a 3x3 matric form, with each term of the matrix representing either a stress component or a moment of inertia with respect to x-y-z axes.

These tensors may be <u>transformed</u> by rotating the x-y-z axes arbitrarily about the origin of the axial system. If this is done, then it can be shown there are <u>three invariants</u>, that is, quantities whose values are not changed by this rotation. In addition, the tensor itself is an invariant since it can be expressed without regard to axial orientation. Furthermore, the inverse operation and the identity (unit) tensor may be defined and we have, therefore, all second order tensors as elements of the group.

In an analogous manner, we may discuss a particular physical event (ref. 2) - an infinite straight-sided wedge impacting with constant velocity on an infinite ocean, with time $t=0$ the instant the point of the wedge touches the surface. At any time $t>0$, the wedge and water surface will be at particular locations, and each of these will be different for different times. The representation of the wedge and ocean, at any time $t=t$, corresponds to the element of the group. If, into this phenomenon, we introduce a change of coordinates, $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$, then the entire phenomenon, for all $t>0$, may be shown on a single map (the invariant) in the ξ,η plane. The time, t, is the transformation coordinate, since for each different value of t the event transforms to new wedge and water positions.

The fundamental behavior in the above group is the collapsing of multi-curve data (the elements) by a suitable change in coordinates to a single curve (the invariant) valid for all the separate elements for all values of the transformation coordinate, t. This concept is the basis for many of the group representations considered in the present paper.

We may define as a group, a set of objects, quantities, happenings or other items which, by means of a mathematical relation is transformed into a single event, this being the invariant representation of the separate items or phenomena. The separate items are called the elements of the group. The variable which transforms or alters the event is called the activity variable and the single equation or visual representation of the event is called the invariant of the group.

The identity relation for these phenomena is either

1) unity, a multiplier of the mathematical equation,

or

2) a transparent sheet placed over the curve such that the curve shows through unchanged.

The inverse relation for these phenomena is either

1) the negative equation, which when added to the original equation gives zero,

or

2) an obliterating cover sheet which annihilates the given curve, resulting in a blank sheet.

Chemical Groups

In reference 3 an experimental study is reported of the sensitivity of the DNA-RNA hybrid obtained from the CSCI density gradient to ribonuclease A and to fraction A (the transformation variables). Six different curves (elements) were drawn corresponding to six different sets of transformation variables.

As shown in reference 4 all six curves can be collapsed to a single curve (and mathematical equation), the invariant, in terms of a suitable change of variables. The details are presented in the reference.

The unity and inverse statements are as in the engineering-physical groups, case b.

Biological-Medical Groups

Orentreich and Selmanowitz, (ref. 5) discuss results of experiments dealing with healing of wounds in dogs and men. Their report shows curves of healing of originally 40 sq cm wounds on men of 20, 30 and 40 years indicating wound healing in relation to age (the activity variable).

In reference 3 it is shown that all three curves (the elements) can be collapsed into a single equation or curve (the invariant) by means of a suitable change of coordinates. The details are given in reference 3. The inverse and identity statements are equivalent to those of case b, engineering-physical groups.

Social Groups

A social application occurs in connection with a study reported by Sherman (ref. 6), dealing with total food intake of children from birth to age 13-15. His results are presented in a chart showing the food allowances (in calories) for children of about average weight for their age. The data is given separately for girls and for boys (the activity variables), these being the group elements. By means of a suitable change of variables (as shown in ref. 3) it is possible to collapse both sets of data to a single mathematical equation and curve - the invariant. The identity and inverse statements are again as in case b, engineering-physical groups.

Behavior Groups

Just as in the case of engineering-physical applications, in the area of behavior there appear to be two different types of group representation - the "tensoral" and the "event" forms.

As an example of the "tensoral" behavior group, the author (in an as yet unpublished report) developed a theory in which it was hypothesized that certain variables related to behavior may be interpreted as tensors, satisfying the same transformation and other relations that engineering-physical tensors satisfy.

As a check against the hypothesis experimental data presented in a report (ref. 7) was used, dealing with a number of subjects who imagined happy, sad and angry situations. Different patterns of facial muscle activity were produced (the elements) and these were measured by electromyography. The facial expressions were recorded for depressed and for non-depressed subjects and, suitably calibrated, were presented in bar graph form. The subjects were tested on the Zung Self Rating Depression Scale and scored accordingly. These scores corresponded to the transformation variable. Complete details are given in the unpublished report.

It was shown that quantities satisfying the tensor transformations could be established for this one test, at least. A fair check on the hypothesis was obtained and, subject to further verification, it seems possible that many of the phenomena in the field of behavior may be treated utilizing tensor theory. If this is in fact true, it will permit one to predict by extrapolation various new relations in behavior theory which themselves may be capable of experimental verification. Also by modelling suitable mathematical tensoral equations one may be able to correlate measured behavior quantities with fundamental measurable central nervous system responses.

A typical "event" type of behavior group occurred (ref. 8) in an experimental study of the swimming ability of new-born rats treated with hormones, the activity variable. Three different groups of rats were studied and three separate curves were obtained. As shown, (ref. 3) it is possible, by means of a suitable change of coordinates, to collapse all three curves, the elements, to a single curve, the invariant. The identity and inverse statements are similar to the ones shown for case b, engineering-physical groups.

Music Groups

Several different types of music groups may occur. A particular arrangement of notes (as for example Ravel's "Bolero" or the Schönberg "twelve tone music") is a typical element. A discussion of the entire range of music composition, as it relates to "groups" including such factors as pitch, repetition, sequential treatment, counterpoint, loudness, etc., is clearly beyond the limits of this paper. One may, however, consider Ravel's Bolero as an example. In this composition we have the repetition of a single theme (the element), representable by means of a musical equation (the notes), being transformed while being performed by means of a continuing gradual crescendo into a composition (the invariant) all shown as a symbolic mathematical equation. It is also possible to represent musical forms in matrix equations. The identity and inverse statements may be taken as shown in Table 1. Reference 9 lists a number of additional studies in this area.

Art-Architectural Groups

In the art-architectural field one may think of piastre band treatments as being typical of group phenomena. In these cases one may have a series of "figures" (gargoyles or Saints or Kings or windows for example) in a "band" going along one side of the building, or completely around the building. These may be identical (as in the case of windows and possibly the human or other figures) or they may vary from one to the other as in the case of human and gargoyle figures.

It is possible to reduce these band figures, the elements, to a single mathematical equation or to collapse the different figures to a single visual quantity, as follows:

For, say, the identical windows, we have

(window)x(function of spacing) = n (identical windows)

For, say, the figures, we have (with a suitable definition of the summation process)

$$\Sigma(\text{figure})_n \text{x} \ \frac{(\text{face})n}{(\text{face})_o} \cdots \text{x} \ \frac{(\text{clothes})n}{(\text{clothes})_o} \cdots = n(\text{identical figures})_o$$

in which all figures are transformed to an identical figure by means of the alterations noted. From these equations the invariant - identical window or identical figure - may be determined.

A somewhat different approach is presented in reference 9.

Poetry Groups

In the case of poetry (reference 10 for example) one deals with terms such as metre, rhyme, image, texture, triolet, stanza, etc. It is possible to indicate rhyming schemes by means of letters. As a typical example, the rondeau which may consist of ten lines has a rhyming scheme as follows:

abbaabRabbaR

In this, R, the refrain, is frequently simply a tail and may be the first word of the opening stanza.

The above scheme may be put in a rather more symmetrical matrix form (symmetry is desirable in some theories of composition),

$$\begin{pmatrix}(aba) & (ab)\end{pmatrix}\begin{pmatrix}\begin{pmatrix}b\\a\\bR\end{pmatrix}\\ \\ \begin{pmatrix}b\\aR\end{pmatrix}\end{pmatrix} = \text{rondeau}$$

in which the usual rules of matrix multiplication are used and "rondeau" is the invariant. One may, conceivably, invent new poetic forms by performing various matrix operations - an "inverse rondeau", for example. A much more elaborate treatment of this topic is presented in reference 9 with particular emphasis on its application to Russian literature.

CONCLUDING REMARKS

It was shown that the general mathematical definition of "group" may be applied to phenomena occurring in many different disciplines. The basic terms of the theory - element, invariant, identity, transformation and inverse - all have counterparts in the different fields considered, subject to suitable alterations as required, for example, with visual or tonal or other characteristic phenomena.

In some of the disciplines discussed, by using the group concept and developing the group invariant, new relations are obtained which permit one to predict new engineering, biological, etc. phenomena that are capable of experimental verification.

Finally, it is conceivable that some of the general theorems and properties of "mathematical group theory" may - by suitable modification - be applicable to the different disciplines considered, thereby permitting one to obtain new fundamental insights and knowledge in these fields.

REFERENCES

1. Borg, S.F.: Matrix - Tensor Methods in Continuum Mechanics, D. Van Nostrand Co., Inc. Princeton, N.J. 1963.

2. Borg, S.F.: Some Contributions to the Wedge-Water Entry Problem. Journal of the Engineering Mechanics Division, Proc. ASCE, April 1957, 1214-1 through 1214-28.

3. Stein, H. and P. Hansen, 1969 Science 166; 393-395

4. Borg, S.F.: Similarity Solutions in the Engineering, Physical-Chemical, Biological-Medical and Social Sciences, Proceedings of the Symposium on Symmetry, Similarity and Group Theoretic Methods in Mechanics, Eds. P.G. Glockner and M.C. Singh, The University of Calgary, Aug. 19-21, 1974, pp. 263-282.

5. Orentreich, N. and V.J. Selmanowitz, Levels of Biological Functions with Aging, Trans. N.Y. Acad. Sci. 31, 1966, p. 992.

6. Sherman, H.C. Chemistry of Food and Nutrition, 8th Ed. The Macmillan Co., New York, N.Y. 1952, p. 563.

7. Schwartz, Gary E. et al., 30 April 1976, Science 192, p. 489.

8. Schapiro, S., Salas, M., Vucovich, K., 1970, Science 168, pp. 147-151.

9. Shubnikov, A.V. and Koptsik, V.A., Symmetry in Science and Art, translated from the Russian Edition (Moscow 1972) by G.D. Archard. David Harker, Ed., Plenum Press, New York, London, 1974.

10. Deutch, B.: Poetry Handbook, 2nd. Ed. Funk and Wagnalls Co., New York, 1957, p. 136.

TABLE I - THE GROUPS

	THE DISCIPLINE	TYPICAL ELEMENTS	TYPICAL INVARIANT	TYPICAL TRANSFORMATION	TYPICAL IDENTITY STATEMENT	TYPICAL INVERSE	REFERENCE NUMBER
1	ENGINEERING-PHYSICAL	a) tensor components	a) tensor invariants. The tensor	a) a rotation of axes	a) the unit tensor	a) the inverse tensor	a) 1
		b) events	b) a single mathematical equation or collapse of multi-curve data to a single curve	b) alteration caused by a particular "activity variable" for the given phenomenon	b) unity or a transparent sheet	b) the negative equation or the obliterating sheet	b) 2
2	CHEMICAL			The same as 1b			3,4
3	BIOLOGICAL-MEDICAL			The same as 1b			5,3
4	SOCIAL			The same as 1b			6,3
5	BEHAVIOR	a) tensor components	a) tensor invariants The tensor	a) a rotation of axes	a) the unit tensor	a) the inverse tensor	7
		b)		The same as 1b			8,3
6	MUSIC	tones, melodies, time, etc. the events	a musical composition reduced to a single mathematical equation	tonal, melodic, loudness, etc. changes	unity	the negative equation or silence	9
7	ART-ARCHITECTURE	visual figures, parts of a strip or band, the events	a visual sequence reduced to a single mathematical equation representing the entire band	spatial or visual-spatial rearrangement of the elements	the transparent sheet	the obliterating sheet-blank band	9

TABLE I - THE GROUPS (CONCLUDED)

	THE DISCIPLINE	TYPICAL ELEMENTS	TYPICAL INVARIANT	TYPICAL TRANSFORMATION	TYPICAL IDENTITY STATEMENT	TYPICAL INVERSE	REFERENCE NUMBER
8	POETRY	syllables, lines, words stanzas, rhymes, etc. The events	a matrix equation representing the poetic scheme	rearrangement of the elements new words, etc.	unit matrix	the negative matrix-blank page	9,10