APPLICATIONS OF CATASTROPHE THEORY IN MECHANICS

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INTRODUCTION

Consider a system under the influence of control parameters c. It may happen that for some values of c the system has more than one stable equilibrium state and consequently a continuous change in control may cause a discontinuous change from one equilibrium state to another. This occurs, for example, in the "snap-through" of a compressed beam under transverse loading. This kind of abrupt transition between stable equilibrium states - a branching or bifurcation - has been the subject of much study (ref. 1 to 4) and recently the French topologist René Thom developed a theory which presents seven standard types of discontinuous behavior (ref. 5 to 6), called elementary catastrophes, and proved that any discontinuous behavior in systems controlled by not more than four variables is one of these seven elementary catastrophes. Thom's theorem is remarkable for providing a classification of discontinuous behavior but it is also useful as an aid to visualizing phenomena of this sort. The proof of the theorem is difficult but its results are easy to understand and to use in problems involving bifurcation.

Applications of Thom's theory to problems in mechanics are just beginning to appear. The first problem solved appears to have been an example by Zeeman (ref. 7) and his co-workers. This example has recently been generalized by Woodcock and Poston so that it can describe higher order catastrophes.

The most extensive studies come from the group of researchers that work with J. M. T. Thompson of University College, London. Thompson and Hunt (ref. 8) correlate their own theories of elastic stability for discrete systems with the work of Thom and suggest possible fields in which the theory will give significant insights. Troger (ref. 9) suggests the nature of such insights in his study of von Mises truss and a shallow arch from the point of view of catastrophe theory, and Fowler (ref. 10) in his paper on the Riemann-Hugoniot shock does the same.

Chillingworth and Guckenheimer apply the theory to continuous systems. Chillingworth (ref. 11) uses a generalization of Morse's Lemma to Hilbert spaces to reduce the study of the buckling of a beam to a problem in finite dimensions; Guckenheimer (ref. 12) discusses catastrophes and Hamiltonian systems.

The papers by Schulman (ref. 13) on phase transitions, Kozak and Benham (ref. 14) on denaturation, and Mehra and Blum (ref. 15) on the ignition of paper provide examples in the realm of thermodynamics. Detailed bibliographies of catastrophe theory and its applications to problems in other areas can be found in reference 16 .

STRUCTURAL STABILITY OF POTENTIAL FUNCTIONS

In this paper we will describe a method, using Thom's classification of catastrophes, for the analysis of stability of systems whose static behavior is derived from a potential function. Examination of the stability of singular points of potential functions will serve to illustrate the nature of the elementary catastrophes, which can also arise in non-conservative dynamical systems as well as in the static case of potential theory.

The first step in examining the stability of systems admitting discontinuous transitions is to clarify the notion of stable state. Early work of Poincaré (ref. 17), and Pontryagin and Andronov (ref. 18) developed the notion of structural stability which expresses two key ideas. First, equilibrium states of a system are characterized by their topological type; it is the general shape of a state which is important and not numerical values which it might take on. In the case of potential functions the topological type is given by the number of singular points. Second, discontinuous behavior of a system occurs for those (critical) values of control parameters at which the equilibrium state changes its topological type. Let $C(=R^p)$ be the space of control variables c, and $X(=R)$ the state space. The potential function is a smooth map, $V(x,c)$, $V:X \times R^p \to R$. A point x_0 is a singular point of V if $D_xV(x_0,c) = 0$. The collection of control points and their associated singular (state) points form a manifold, called the catastrophe manifold,

$$M = \{(x,c) \in X \times R^p \mid D_xV(x,c) = 0\} \tag{1}$$

The dimension of M is p. Figure 1 illustrates M for a quartic potential. For a fixed value of c, there is a fixed potential function $V_c(x)$ with a fixed number of singular points. As this number changes with c it stratifies (or subdivides) the control space into open and dense regions in which this number is constant, separated by boundaries across which it changes. Such a change will occur whenever the manifold M has a tangent parallel to X, i.e. when $D_x^2V(x) = 0$. A singular point x_0 is said to be structurally stable when $D_x^2V(x_0) \neq 0$. The set of points which are not structurally stable appears as a fold F in the manifold M.

$$F = \{(x,c) \in X \times R^p \mid D_x^2V(x,c) = 0\} \tag{2}$$

These are points at which the map projecting M onto C is singular. The set of critical control variables at which the number of singular points changes (or equivalently which have structurally unstable singular points) is called the bifurcation set B. This set is given by eliminating x from (1) and (2):

$$B = M \cap F$$

In Figure 1, B appears as the cusp in the c-plane.

In the neighborhood of a structurally stable point $x_0 (D_x^2V(x_0) \neq 0)$ the potential is quadratic, that is there is a curvilinear coordinate system $\overline{x}$ in which $V(x) - V(x_0) = \overline{x}^2$. To investigate the behavior of the potential in a neighborhood of a structurally unstable point Thom developed the notion of a universal unfolding of a singularity. Consider a perturbation of the potential $V \rightarrow V + \delta V$ where δV and all of its derivatives are small. Two possibilities arise - either the perturbation gives rise to an infinite number of different topological types of the potential or only a finite number. In the latter case the variation of V can be parameterized by a finite number of variables which can be identified with the control variables, as

$$\delta V = c_1 h_1(x) + c_2 h_2(x) + \dots \quad c_p h_p(x) \tag{3}$$

This variation is universal in the sense that any variation of V depending on p-parameters can be obtained by a transformation of (3). For example suppose we begin with a cubic potential $V(x) = x^3$, so that 0 is a structurally unstable point. If this potential is perturbed by $\delta V = ax$ the topological character of $V + \delta V$ is described by the value of the parameter a as follows: for $a \geq 0$, V has one root, an inflection point, and for $a < 0$, V has 3 roots, thus one maximum and one minimum, c.f. Figure 2. The importance of this result of Thom's work is that for all potentials with the same singularity type, perturbations need depend on only one parameter, and their behavior is of the fold type illustrated in the following examples. The number of parameters involved in the variation of V is called the codimension of the singularity. All singularities of codimension ≤4 have been analyzed by Thom. There are four potentials depending on one state variable and these have the following form:

$$x^{\ell} + a_{\ell-2}x^{\ell-2} + a_{\ell-1}x^{\ell-1} + \dots \quad a_1 x$$

We now summarize these results by stating a version of Thom's Theorem that we will use in the examples of the next section. This version is given by Chow, Hale and Mallet-Paret in reference 4 .

Thom's Transversality Theorem and Catastrophes

Let $V(x,c) : X \times R^p \rightarrow R$ and $f(x,c) \equiv dV/dx$, so that the singular points of V are given by $f(x,c) = 0$. If $x = 0$ is a singular point of V, f can be expanded in the form

$$f(x,0) = Ax^k + 0\left(|x|^{k+1}\right)$$

where $A \neq 0$ and k gives the order of the singular point. Expand the derivatives of f with respect to the parameters:

$$\frac{\partial f}{\partial c_i}(x,0) = \sum_{j=0}^{k-2} A_{ij}\frac{x^j}{j!} + 0|x|^{k-1} \qquad i = 1,2,\dots p$$

Then when $p \leq k-1$ and

$$\text{rank}\ (A_{ij}) = k-1$$

there exists a smooth transformation of coordinates

$$\bar{\lambda}_i = h_i(c_1 \ldots c_p) \qquad i = 1, \ldots p$$

$$\bar{x} = h_0(x, c_1 \ldots c_p)$$

such that

$$\bar{f}(\bar{x}, \bar{\lambda}) = \bar{x}^k + \bar{\lambda}_1 + \bar{\lambda}_2 \bar{x} + \ldots \quad \bar{\lambda}_{k-1} \bar{x}^{k-2}$$

APPLICATION OF CATASTROPHE THEORY TO DISCRETE SYSTEMS WITH ONE STATE VARIABLE

In this section we will concentrate on the simple case of potentials depending upon one state variable and two control parameters; problems of more generality are approached in a similar manner. The physical problems we have studied are traditional in elastic stability: an imperfection-senstive strut and a truss that can experience snap-through. These two problems contain many of the features of more general problems, and the results obtained can be displayed clearly in a graphical form. Similar problems have been treated by Koiter, Thompson and Hunt, Sewell and Ziegler.

Application 1: A Strut With Imperfection Sensitivity

Consider the rigid hinged bar of length 1 that is held in a vertical position by a linear spring, with spring constant k, that is loaded by a vertical force P with an eccentricity $e = \mu \ell$ (see fig. 3). The spring is attached to the strut at a distance h from the base and is supported on its other end so that the spring remains horizontal. The coordinate θ, which is measured between a vertical line and the axis of the bar, specifies the state of the system. The dimensionless parameters $\lambda = \frac{P\ell}{kh^2}$ and μ are the controls.

The force function f is the gradient of the internal and external potentials:

$$f = f(\theta; \lambda, \mu) = \frac{kh^2}{2} \left[\sin\theta\cos\theta - \lambda(\sin\theta + \mu\cos\theta)\right]$$

We begin by finding the surface $f = 0$, which is the catastrophe manifold, and the points of structural instability $f_{,\theta} = 0$. Upon solving these two equations in three unknowns we find

$$\lambda_c = (1 + \mu^{2/3})^{-3/2}$$

$$\theta_c = \tan^{-1}(\mu^{1/3})$$

where the subscript c denotes the critical condition of structural instability.

Next we prepare to use Thom's Theorem. We expand f about the critical value of the state variable and note the leading term. Here we see that in the case where $\mu \neq 0$, f (expanded as required) is of the order two in the variable $x = \theta-\theta_c$. If $\mu = 0$, f expanded is of order three.

Let us first consider the case where f is of order two. The index k equals 2 and n, the number of control parameters, is also two. Therefore the inequality in the theorem is satisfied. We note also that f evaluated at the critical point vanishes, a further preliminary of the theorem. In order to determine the nature of the catastrophe manifold along this portion of the bifurcation set, we must find the rank of the matrix A which is defined in the theorem. Let f* be f expanded about the critical point in terms of x. Now

$$f^*_{;\lambda}(x;\lambda_c,\mu) = \tfrac{1}{2}kh^2[(\sin\theta_c+\mu\cos\theta_c)+(\cos\theta_c-\mu\sin\theta_c)x + \ldots]$$

$$f^*_{;\mu}(x;\lambda_c,\mu) = \tfrac{1}{2}kh^2[\ \lambda_c\cos\theta_c+\lambda\ \sin\theta_c x + \ldots]$$

and therefore

$$A = \begin{pmatrix} \frac{kh^2}{2}\ [\sin\theta_c+\mu\cos\theta_c] \\ \\ -\frac{kh^2}{2}\ \lambda_c\cos\theta_c \end{pmatrix}$$

The rank of A is one; the conditions of the theorem are satisfied. The singularities are locally equivalent to a fold at points along the bifurcation set away from ($\theta_c=0$; $\lambda_c=1$, $\mu=0$). This behavior is identical to that of the cubic potential discussed earlier.

If we consider this latter case of $\theta_c=0$, we find that the function f is locally equivalent to some form of a cusp, the case where k=3 in Thom's Theorem. In order to identify the normal and splitting factors for the manifold (see fig. 1 for the meaning of these terms), and to display the canonical form of the polynomial, we expand f about the point ($\theta=0$; $\lambda=1$, $\mu=0$). We need only retain terms to the third order since the manifold is a cusp in this neighborhood.

$$f = \frac{kh^2}{4}[-\theta^3 + 2(1-\lambda)\theta + 2\lambda\mu]$$

If we place this expansion in the canonical form

$$\frac{f}{-kh^2/4} = \theta^3 + \bar{\lambda}_2\theta + \bar{\lambda}_1$$

we find that $\bar{\lambda}_1 = -2\lambda\mu$ is the normal factor and $\bar{\lambda}_2 = -2(1-\lambda)$ is the splitting factor. The force function for this example is of the same differential type as a cusp but the negative multiplier causes the loci of maxima and minima for the related potential function to be interchanged. This type of force function is the <u>dual</u> cusp and the behavior of the system on the catastrophe manifold is altogether different from that on the manifold of a regular cusp (fig. 1).

The bifurcation set in the control plane is described by $27\overline{\lambda}_1^2 = 4\overline{\lambda}_2^3$. This relation is an imperfection-sensitivity curve and has the familiar two-thirds power form. The equilibrium surface and the bifurcation set are shown in Figure 1. Notice the effect of the imperfection. It lowers the value of the load at which instability occurs. The area of the catastrophe manifold where $\overline{\lambda}_1 > 0$ is composed entirely of unstable points; it is not accessible to the system. The bifurcation set and a visualization of the equilibrium surface can also be presented as in Figure 4. This presentation is possible because the equilibrium surface is a ruled surface: for each value of the state variable vector, the equilibrium equation is an affine equation in the control parameters. The bifurcation set is the envelope of the projection of these lines onto the control space. The three-dimensionality of these figures can be enhanced by a stereographic technique that is described in Woodcock and Poston (ref. 19).

Application 2: An Essential Modification of the Strut With Imperfection Sensitivity

We will now modify the structure in Figure 3 so that the spring is attached to a fixed point at a distance h from the level of the pivot and is fastened to the rigid bar with a sleeve that allows the spring to remain horizontal. The catastrophe manifold near the structurally unstable point ($\theta=0$; $\lambda=1$, $\mu=0$) has the form

$$\frac{f}{\frac{3}{2}kh^2} = \theta^3 + \frac{2}{3}(1-\lambda)\theta - \frac{2}{3}\lambda\mu$$

In this case the catastrophe manifold is locally equivalent to a cusp with normal factor $\overline{\lambda}_1 = -\frac{2}{3}\lambda\mu$ and splitting factor $\overline{\lambda}_2 = \frac{2}{3}(1-\lambda)$. Note the difference in behavior between trajectories along this cusp and those along the dual cusp.

Application 3: A Symmetric Truss With Moveable Supports

In this example we consider a modification of the well-studied symmetric structure that exhibits snap-buckling (fig. 5). The structure consists of two linear-spring elements of unstretched length ℓ_0 and spring constant k that have a horizontal projection of 2x and that are subjected to a downward load P. The location of the tip of the truss with respect to a horizontal line through its end points is denoted by y. We will analyze the behavior of this structure in much the same manner as we did in example 1.

The force function f is

$$f = f(z;a,b) = z(1 - 1/(z^2+a^2)^{1/2}) + b$$

where $b = P/k\ell_0$
$a = x/\ell_0$
and $z = y/\ell_0$

The solution for the structurally unstable points of the mapping yields the critical set of points whose projection on the control space is the bifurcation set. An investigation of the behavior of the system on the bifurcation set away from the special point (z=0; a=1, b=0) shows that the singularities are folds locally. A similar investigation in the neighborhood of the special point indicates the expected cusp there.

In order to determine precisely the normal and splitting factors in the neighborhood of the cusp, we expand the force function about the tip of the cusp retaining only terms as high as cubic. We find that the force function can be rewritten in the canonical form

$$2f = \overline{u}^3 + \overline{\lambda}_1 + \overline{\lambda}_2\overline{u}$$

if

$$\overline{u} = \frac{z}{a}$$

$$\overline{\lambda}_1 = 2b$$

$$\overline{\lambda}_2 = 2(a-1)$$

Therefore near the cusp tip ($\overline{\lambda}_1=0$, $\overline{\lambda}_2=0$) $\overline{\lambda}_1 = 2b$ is the normal factor and $\overline{\lambda}_2 = 2(a-1)$ is the splitting factor.

In this example the portion of the equilibrium surface behind the cusp is accessible to the system. Deformations of the system can occur that will take the state variable from values on the top of the cusp surface to values on the bottom without the occurrence of a jump.

CONCLUSION

It is clear from these examples that catastrophe theory and the methods of adjacent equilibrium and energy (given dynamical significance by their embedding in the theory of Lyapunov) lead to similar results and require many of the same calculations. Qualitative features of the singular behavior of systems, including a unique visualization of discontinuous processes, can be gained quickly from the representation of the catastrophe manifold. Catastrophe theory provides an exhaustive classification of structural instabilities in systems with as many as four control variables and clarifies the nature of the controls. A consistent set of controls must satisfy the rank condition of the transversality theorem. This requirement pinpoints controls that are redundant and suggests the need for additional ones; for example, it would have forced the introduction of the imperfection parameter in application 1 had it been omitted. There still remains a good deal of work to be done before a unified theory of bifurcation is developed and Thom's theory provides a useful set of ideas in this direction.

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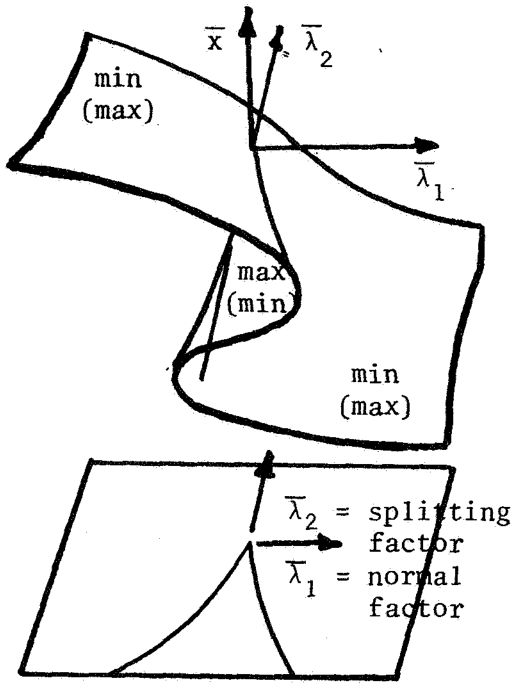

Figure 1.- Quartic potential: cusp (dual cusp).

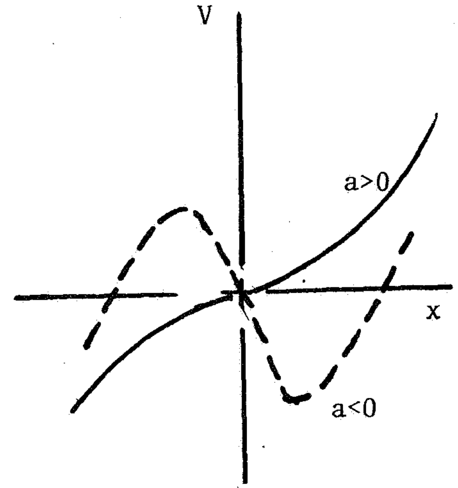

Figure 2.- Cubic potential.

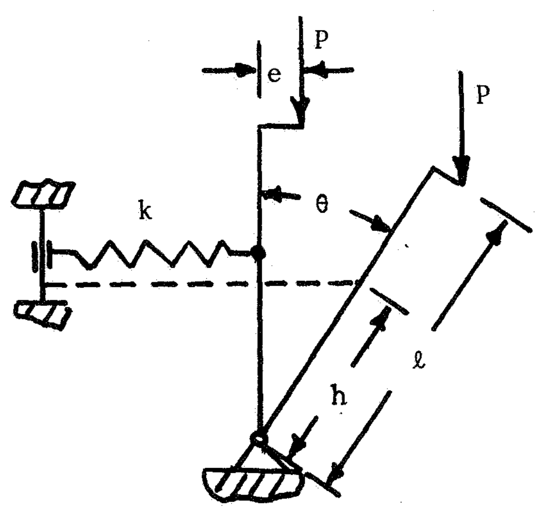

Figure 3.- Imperfection-sensitive bar.

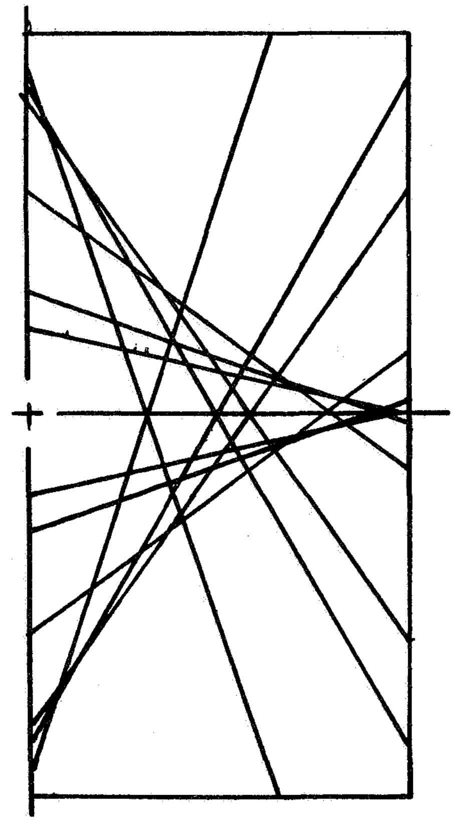

Figure 4.- Ruled surface projections (cusp).

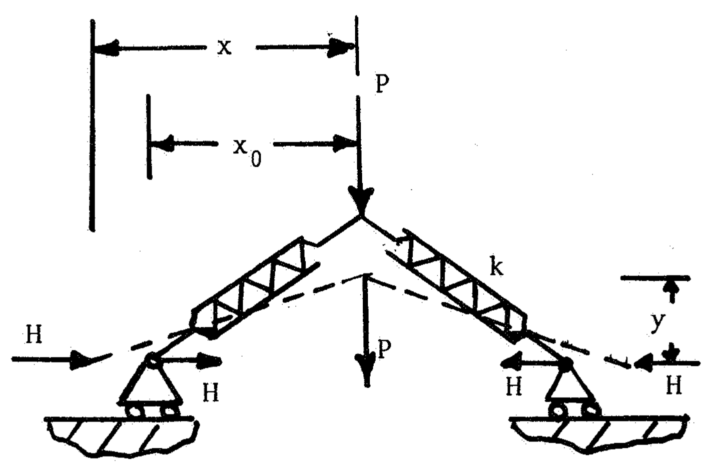

Figure 5.- Snap-through structure (symmetric).