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ICASE REPORT

THE SLOW RECIRCULATING FLOW NEAR THE REAR STAGNATION POINT OF A WAKE

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ABSTRACT

A model is suggested in which some of the important features of the circulating flow inside the two-dimensional near wake are derived by assuming a slow viscous flow. The theory considers the flow away from the body base. It is found that there is a region of constant speed merging, as we go downstream, into a region of stagnation-apex flow. The velocity returning from the rear stagnation point along the center streamline is shown to be a slowly varying function of the "wedge-angle" of the wake and to be roughly one half the velocity at the edge of the shear layers driving the wake-cavity flow. These results seem to be in agreement with experimental data.

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1. Introduction

A complete description of the structure of the two dimensional near wake behind a body in motion depends on knowledge of the whole flow field. In particular one has to know, or derive, the way the boundary layer separates at the base and becomes a shear layer. This shear layer is driven by the inviscid free stream and in turn drives the recirculating flow confined inside the near-wake cavity.

Experiments at low to moderate (Re<200) Reynolds numbers [1] indicate that the recirculating flow field in the cavity is composed of three parts: The flow very near the base, the central region, and the flow field near the rear stagnation point where the cavity closes on itself and where the flow, driven by the shear layers, turns around towards the base. The data show that in the central region the velocity on the stagnation streamline (connecting the rear stagnation point to the center of the base) is constant, independent of the Reynolds number and is about 1/5 the free stream value.

Numerical studies for supersonic wake and Re≃500 [2] indicate that in the near wake the viscous and pressure terms dominate while a short distance downstream of the rear stagnation point the two merged shear layers have boundary-layer-like structure.

The above observations suggest that one may be able to model the flow in the central and rear stagnation regions of the recirculating flow field by considering a slow viscous fluid bounded by converging streamlines which "represent" the bounding shear layers. Away from the apex, i.e. the rear stagnation point, the velocities along the outer and center streamlines are to be constant. Nearer the apex the velocities will depend linearly on the distance from the stagnation point and very close to it the variation will be parabolic - as suggested by Heimenz (See for example Batchlor [3]).

In section 2 the Stokes' equation will be solved for the central region, and it will be found that the returning velocity along the center stream line cannot be assigned arbitrarily but that it is dictated by the constraint that on the center line the pressure gradient normal to it must vanish.

In section 3 the linear stagnation region is considered. It is found that in it the pressure gradient must be radial. Therefore, the solution in the constant velocity region found in section 2 and the solution for this stagnation region can be matched without changing the wedge angle.

In section 4 we examine the final parabolic approach to stagnation. It is found that, as in the central (constant velocity) region, there is a constraint on the returning velocity along the centerline. This condition, however matches the one found in section 2 only for limiting small apex angles. Thus in general this region will produce a discontinuity in the slopes of the streamlines. However, because our model problem possesses no natural reference length, this final stagnation region can be made as small as desired and will not affect the solution elsewhere. An interesting result in this immediate neighborhood of the stagnation apex is that the boundary streamlines cannot form an apex angle of $2\theta_0 = \pi$ radians because the returning velocity along the centerline ($\theta = 0$) is proportional to $\sec\theta_0$ ---i.e. it diverges as θ_0 approaches $\pi/2$.

2. The constant (radial) velocity region

In this region the geometry is as follows:

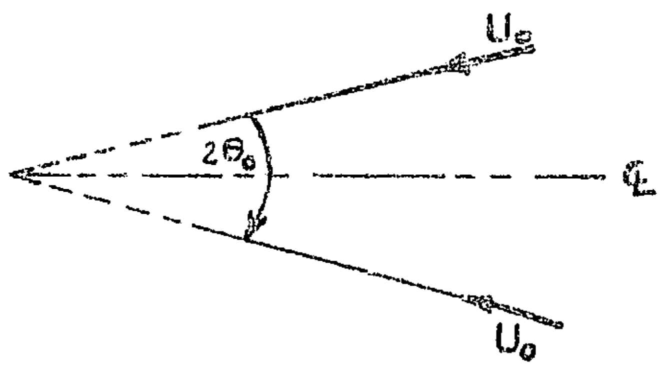

FIGURE 1

The boundary conditions are $u_r = -U_0$ on the streamlines $\theta = 0$, $\theta = 2\theta_0$ and $u_\theta = 0$ on the same streamlines. Since we assumed the inertia forces to be nugatory the equation to be solved is Stokes', which is bi-harmonic for the streamfunction,

$$\nabla^4\psi = 0. \tag{2.1}$$

With the above boundary conditions, however, we shall get a sink-type flow. We must stipulate that along the center line the fluid is moving away from the apex (which is not in our region of consideration now). Therefore we consider the following problem:

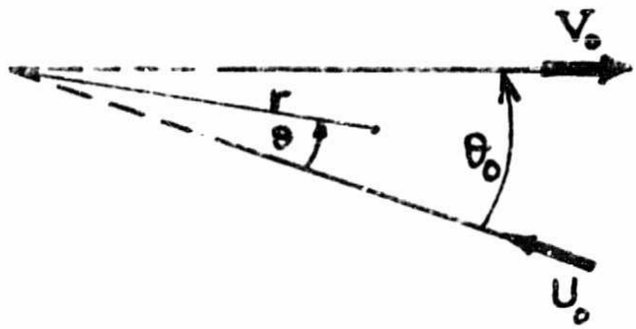

FIGURE 2

It is clear that in polar coordinates the solution to (2.1) with its attendant boundary conditions must be of the form

$$\psi = rf(\theta)$$

from which

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = f'(\theta),$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -f(\theta),$$

and equation (2.1) for the stream function becomes

$$r^3\nabla^4\psi = f^{(iv)} + 2f'' + f = 0. \tag{2.2}$$

The boundary conditions for the half-wedge (see figure 2) are

$$\begin{aligned} f(0) &= 0 \qquad & f(\theta_o) &= 0 \\ f'(0) &= -U_o \qquad & f'(\theta_o) &= V_o \end{aligned} \tag{2.3}$$

The solution of (2.2) subject to (2.3) is

$$f(\theta) = A_o(\sin\theta - \theta\cos\theta) + B_o\theta\sin\theta - U_o\theta\cos\theta \tag{2.4}$$

where

$$A_o = \frac{U_o}{\theta_o^2 - \sin^2\theta_o}\left\{\left(\frac{V_o}{U_o}\right)\theta_o\sin\theta_o - \theta_o^2\right\} \tag{2.5}$$

and

$$B_o = \frac{U_o}{\theta_o^2 - \sin^2\theta_o}\left\{\left(\frac{V_o}{U_o}\right)(\theta_o\cos\theta_o - \sin\theta_o) + \theta_o - \sin\theta_o\cos\theta_o\right\} \tag{2.6}$$

If we wish the center line $\theta = \theta_o$ to be a true line of symmetry for the complete wedge it is not sufficient to have $u_\theta(\theta_o) = -f(\theta_o) = 0$. We must also require that the pressure gradient normal to that streamline vanish; i.e. $(\partial p/\partial\theta)_{\theta=\theta_o} = 0$. Now

$$\frac{\partial p}{\partial\theta} = \frac{\mu}{r}(f''+f) \tag{2.7}$$

where μ is the coefficient of viscosity. The symmetry requirement leads therefore to

$$(A_o+U_o)\sin\theta_o + B_o\cos\theta_o = 0 \tag{2.8}$$

Since A_o and B_c depend on θ_o, U_o and V_o/U_o we find that for a given driving velocity U_o and a given semi-apex angle θ_o equation (2.8) yields a value for V_o/U_o. The variation of $\beta_o \equiv V_o/U_o$ with θ_o is shown in figure 3. Note that for $0\leq\theta_o\leq\pi/2$, β_o varies monotonically between 0.5 and $2/\pi \cong .637$. Thus for most values of θ_o that one expects to encounter, in experiments for example, the value of $\beta_o \equiv V_o/U_o$ will be confined to fairly narrow bounds. This seems to be in agreement with experiments [1] where at different (low) Reynolds number the same constant value of u_r/U_∞ was measured along the stagnation streamline. The measured value of $|u_r/U_\infty| \cong 0.2$ would give our results the right order of magnitude if U_o, the velocity characteristic of the "bottom" edge of the driving shear-layer, is taken to be about 1/3 the local free stream value.

3. The (linear) stagnation region

We assume that from a certain distance, r_o, the velocity along the bounding streamlines begins to decelerate linearly in r. This is not an unreasonable assumption and is in line with Heimenz' results [3] for stagnation point flow. Thus we set in this region $(r \leq r_o)$

$$\psi = r^2 f(\theta).$$

Consequently

$$u_r = rf'(\theta),$$

$$u_\theta = -2rf(\theta),$$

and the bi-harmonic equation reduces to

$$r^2\nabla^4\psi = f^{(iv)} + 4f'' = 0 \tag{3.1}$$

The boundary conditions are:

$$\begin{aligned} f(0) &= 0 \qquad & f(\theta_o) &= 0 \\ f'(0) &= -U_1 \qquad & f'(\theta_o) &= V_1 \ . \end{aligned} \tag{3.2}$$

Since r has not been scaled yet we shall say that this stagnation region is defined by $r \leq r_o = 1$. Then in order for the velocity along the bounding streamline to be continuous in the two regions $r \geq 1$ and $r \leq 1$ we must set $U_1 = U_o$ and $V_1 = V_o$.

The solution to the system (3.1), (3.2) is

$$f = A_1(\sin 2\theta - 2\theta) + B_1(\cos 2\theta - 1) - U_o\theta \tag{3.3}$$

where

$$A_1 = \frac{U_o}{4} \cdot \frac{(1+\beta_1)\tan\theta_o - 2\theta_o}{\theta_o - \tan\theta_o} \tag{3.4}$$

$$B_1 = \frac{U_o}{4} \cdot \frac{(\sin\theta_o\cos\theta_o)(1+\beta_1+2\theta_o\tan\theta_o) - (1+\beta_1)\theta_o}{(\theta_o - \tan\theta_o)(\sin\theta_o\cos\theta_o)} \tag{3.5}$$

$$\beta_1 = \frac{V_1}{U_1} = \frac{V_o}{U_o} \tag{3.6}$$

It is easily established that for the present case $\partial p/\partial\theta = 0$ everywhere and hence there is no constraint on the value of β_1. It follows that it was legitimate to require continuity of the radial velocity on the bounding streamlines at $r = 1$; i.e., to set, as we did, $\beta_1 = \beta_o$.

It is interesting to note that the choice of $\psi = r^2 f(\theta)$ is the only one leading to $\partial p/\partial\theta = 0$ everywhere. This is seen by examining the expression for the pressure gradient:

$$\frac{1}{\mu r}\frac{\partial p}{\partial \theta} = \nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \tag{3.7}$$

If u_θ and u_r are calculated from $\psi = r^m f(\theta)$ we find

$$\frac{1}{\mu r}\frac{\partial p}{\partial \theta} = r^{m-3}(2-m)(f''+m^2 f) \tag{3.8}$$

It is seen from (3.8) that $\partial p/\partial\theta$ vanishes everywhere for $m = 2$. This is fortunate since otherwise we would have $\beta_1 \neq \beta_0$ and would not have been able to match the velocities on the center streamline of the regions at $r = r_0 = 1$, or at any other station. We are now is a position to combine the results of this section with the previous section to specify a flow-field of a slow viscous recirculating fluid in a wedge-like region (with the possible exception of the very near neighborhood of the apex):

$$\psi = \begin{cases} r[A_0(\sin\theta-\theta\cos\theta) + B_0\theta\sin\theta - U_0\theta\cos\theta] & r\geq 1,\ 0\leq\theta\leq\theta_0 \\ r^2[A_1(\sin 2\theta-2\theta) + B_1(\cos 2\theta-1) - U_0\theta] & 0\leq r\leq 1,\ 0\leq\theta\leq\theta_0 \end{cases} \tag{3.9}$$

with A_0, B_0, A_1 and B_1 given by (2.5), (2.6), (3.4) and (3.5) respectively. There remains the question of the continuity of ψ and its derivatives, at $r = 1$, at all θ and not just along the bounding streamlines, $\theta = 0$ and $\theta = \theta_0$, where the boundary conditions were chosen to assure us of continuity. Clearly there is no analytic matching uniformly valid in θ. Numerical calculations of f and f' as functions of θ for various θ_o are shown in figures 4 and 5. It is seen that the matching is quite good.

4. Final approach to the stagnation apex

For $0 \leq r \leq r_2 << 1$ we might have a parabolic appraoch to stagnancy, i.e.,

$$\psi = r^3 f(\theta) \tag{4.1}$$

This small r_2 is likely to be of the order of the diffusion length proper to this problem, $L_d \simeq o(\mu/\rho U_o)$, where we assumed tacitly that $L_d << 1$ and hence

$$1 >> R_e = \frac{U_o}{\nu} >> \frac{U_o L_d}{\nu} \quad .$$

From (4.1) we have

$$u_r = r^2 f'(\theta)$$

$$u_\theta = -3r^2 f(\theta)$$

and the bi-harmonic equation for this case becomes

$$r^3 \nabla^4 \psi = f^{(iv)} + 10f'' + 9f = 0 \tag{4.2}$$

with boundary conditions

$$\begin{aligned} f(0) &= 0 & f(\theta_o) &= 0 \\ f'(0) &= -U_2 & f'(\theta_o) &= V_2 \end{aligned} \tag{4.3}$$

where continuity of velocity along the bounding streamline $(\theta=0)$ demands that $U_1 = r_2 U_2$ or $U_2 = U_o/r_2$. Solving (4.2) and (4.3) for $f(\theta)$ we find

$$f = A_2(\sin\theta - \tfrac{1}{3}\sin 3\theta) + B_2(\cos\theta - \cos 3\theta) - \frac{U_2}{3}\sin 3\theta \tag{4.4}$$

where

$$A_2 = \frac{3}{4}U_2 \frac{\frac{2}{3}\sin^2\theta_o+\beta_2\cos\theta_o-1}{\sin^2\theta_o} \qquad (4.5)$$

$$B_2 = \frac{U_2}{4} \cdot \frac{2\cos\theta_o-\beta_2}{\sin\theta_o} \qquad (4.6)$$

$$\beta_2 = \frac{V_2}{U_2} \qquad (4.7)$$

From (3.3) it follows that the symmetry requirement ($\partial p/\partial\theta = 0$ at $\theta = \theta_o$) takes the form

$$f''(\theta_o) + 9f(\theta_o) = 0 \qquad (4.8)$$

Using (4.4), (4.5) and (4.6) we find that β_2 is constrained to take the value

$$\beta_2 = \frac{1}{2\cos\theta_o} \qquad (4.9)$$

Evidently $\beta_2(\theta_o) \neq \beta_o(\theta_o)$; in fact for the range of interest, $0\leq\theta_o\leq{}^{\pi}/_2$, $\beta_2(\theta_o) \geq \beta_o(\theta_o)$ with equality taking place only at $\theta_o = 0$. Thus in the present model there is no continuous matching possible between the linear and parabolic stagnation regions. Either, if we match the velocities, there is a discontinuity in the slopes of the bounding and other streamlines, or -- if we leave the wedge shape undisturbed -- there is a discontinuity in the velocity field. This was to be expected in a relatively crude model as the one employed here. However, the extent of the parabolic region is very small and its effect on the rest of the flow field should be small. The main point is that the two major regions -- that of constant velocity and the linear stagnation one -- do match well to yield a flow field with properties characteristic of available low Reynolds numbers data.

It is interesting to note that the expression for $\beta_2(\theta_0)$ effectively excludes the possibility of a blunted wedge shaped cavity with closure by streamlines perpendicular to the center line, i.e., $\theta_0 = \pi/2$. In fact, one would guess that at the apex the closure is cusped.

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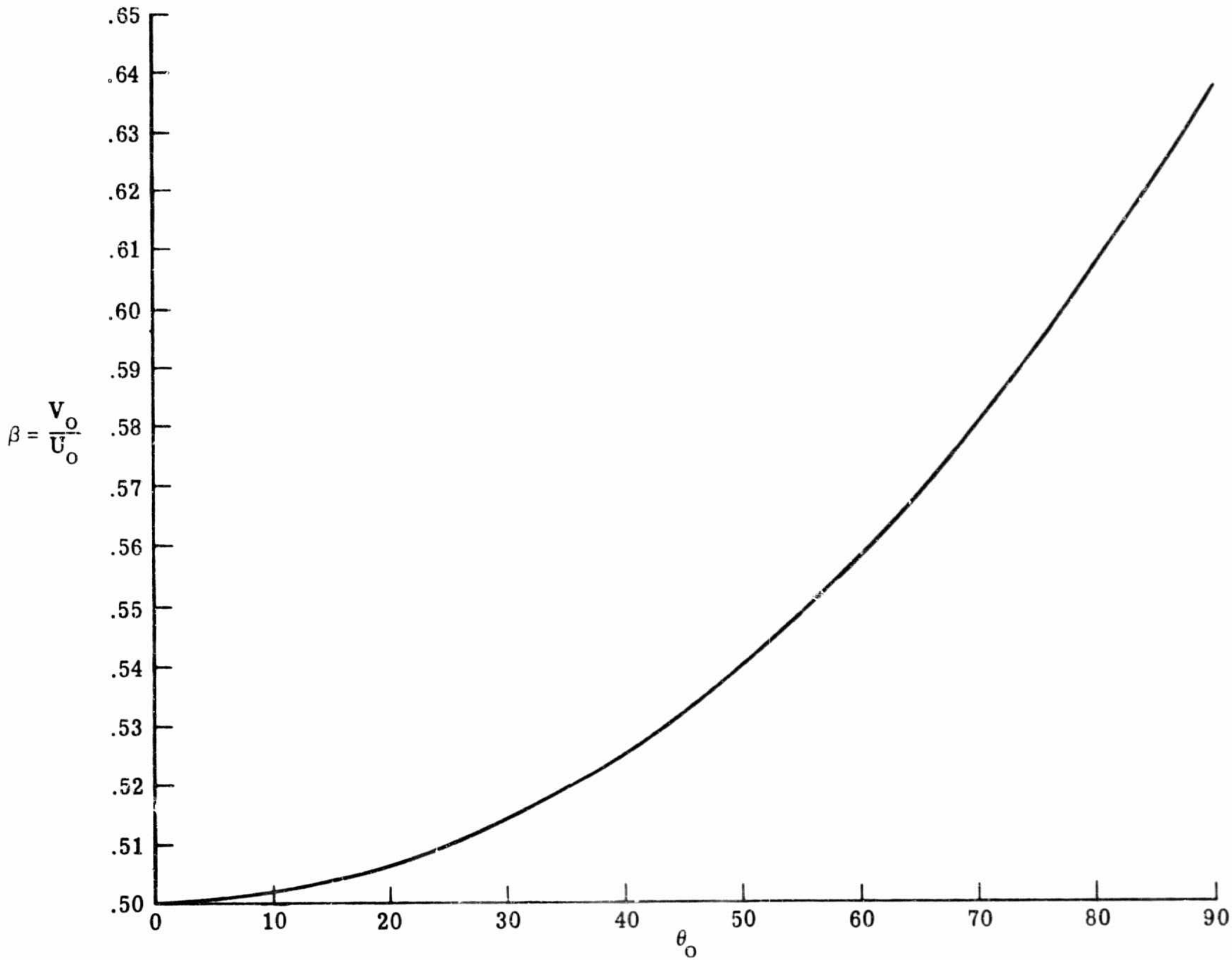

Figure 3.- Symmetry constraint of $\beta = \frac{V_o}{U_o}$ versus θ_o .

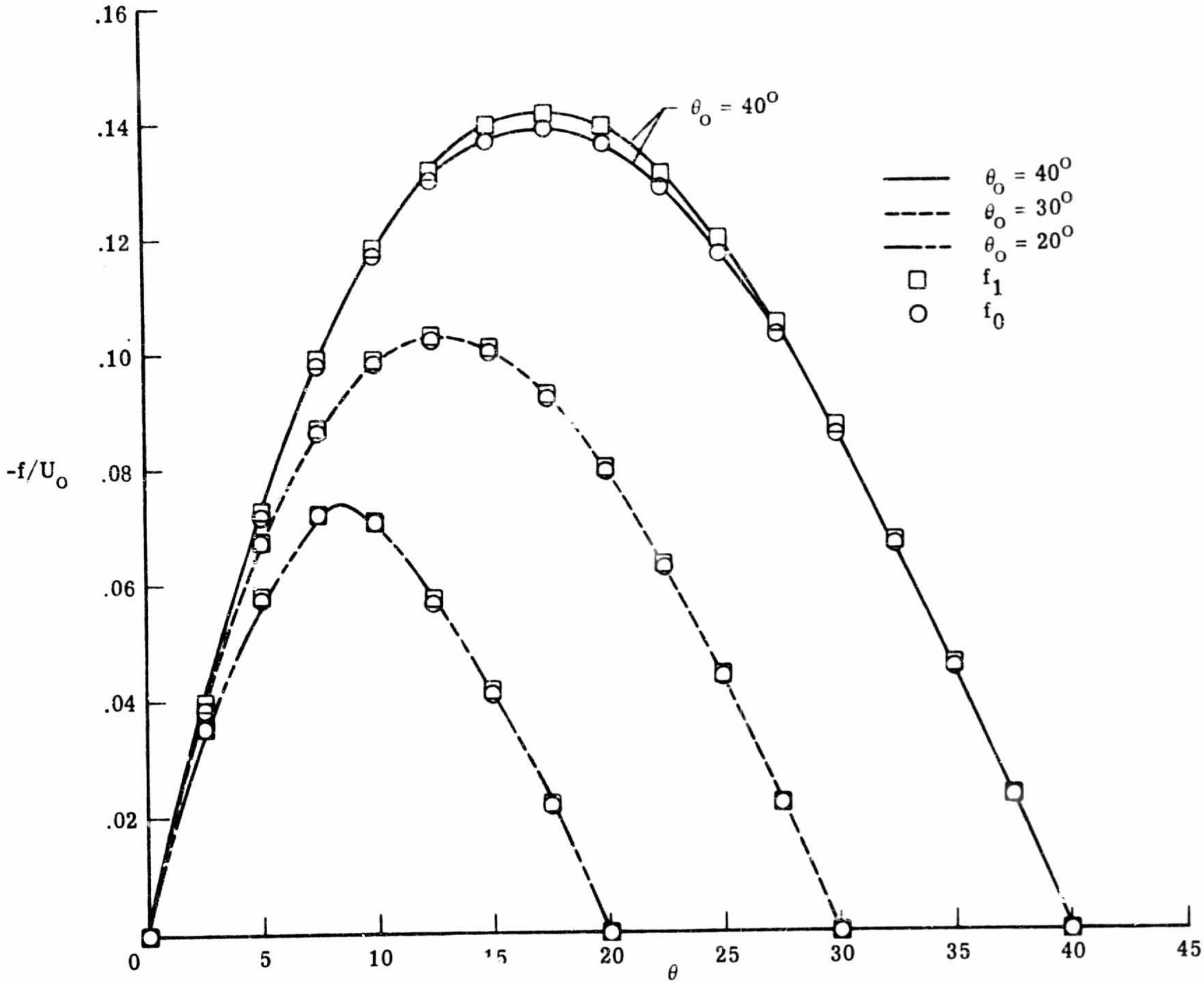

Figure 4.- Matching of stream function distribution $\frac{f}{U_o}$ versus θ.

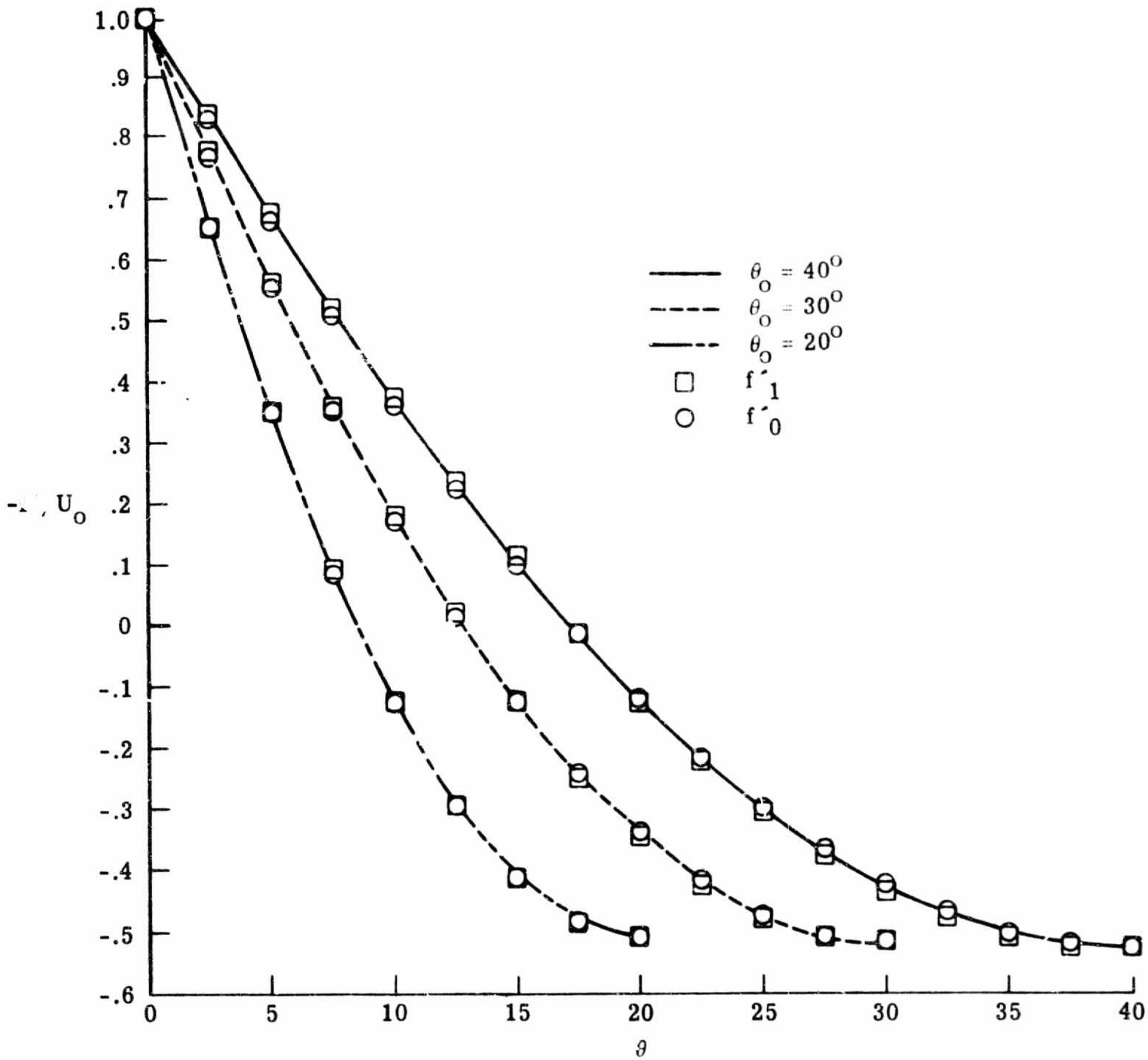

Figure 5.- Matching of the radial velocity distribution $\frac{f'}{U_o}$ versus θ .

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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