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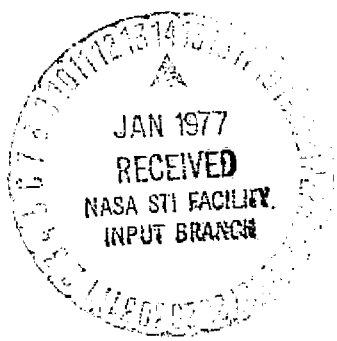
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-786

Transformation From Proper Time on Earth to Coordinate Time in Solar System Barycentric Space-Time Frame of Reference



JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

December 1, 1976

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Transformation From Proper Time on Earth to Coordinate Time in Solar System Barycentric Space-Time Frame of Reference

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Preface

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TABLES

Abstract. An expression is derived for the time transformation $t - \tau$, where t is coordinate time in the solar system barycentric space-time frame of reference and τ is proper time obtained from a fixed atomic clock on earth. This transformation is suitable for use in the computation of high-precision earth-based range and doppler observables of a spacecraft or celestial body located anywhere in the solar system; it can also be used in obtaining computed values of Very Long Baseline Interferometry data types. The formulation for computing range and doppler observables, which is an explicit function of the transformation $t - \tau$, is described briefly.

1. Introduction

This paper derives an expression for the time transformation $t - \tau$, where t is coordinate time in the solar system barycentric space-time frame of reference and τ is proper time obtained from a fixed atomic clock on the surface of the earth.[1] The expression is obtained using general relativity; however, to the accuracy of the retained terms, it is consistent with all viable relativistic theories of gravitation. The expression for $t - \tau$ is suitable for use in the computation of high-precision earth-based range and doppler observables of a spacecraft or celestial body located anywhere in the solar system. It can also be used in obtaining computed values of Very Long Baseline Interferometry data types. The expression for $t - \tau$ can be used in orbit determination programs in which the motion of bodies and light is represented in the solar system barycentric space-time frame of reference with coordinate time t as an independent variable. The errors in computed range and doppler observables due to neglected terms in the expression for $t - \tau$ will not exceed approximately 0.62 m and 2.4×10^{-6} m/s, respectively, per astronomical unit of range to the spacecraft. These figures apply specifically for two-way tracking data (transmitted and received at the same tracking station on earth).

An expression for $t' - \tau$, where t' is coordinate time in the heliocentric space-time frame of reference, was previously obtained by Moyer (1971). However, a term affecting three-way tracking data (transmitted at one station on earth and received at a second station) was inadvertently omitted. This previous expression for $t' - \tau$ was obtained by a straightforward integration of the differential equation for $d\tau/dt'$. Thomas (1975) has shown that the use of integration by parts and a first-order expansion of the gravitational potential simplifies the derivation and provides a clearer understanding of the physical origins of the various terms. The present derivation for $t - \tau$, where t is coordinate time in the solar system barycentric space-time frame of reference, uses the method of Thomas (1975) and produces an expression that includes all of the terms previously obtained by Moyer (1971), with minor

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[1] Coordinate time t is proportional to proper time that would be read by an atomic clock at infinite distance from the solar system and at rest relative to the solar system barycenter.

changes in the coefficients, and the above-mentioned three-way term. The coefficients obtained in this paper are slightly more accurate than those previously obtained. The differences are due to different truncation errors for the two methods. In addition, six new periodic terms are obtained. Two of these are due to long-period variations in the gravitational potential at the earth due to Jupiter and Saturn. The remaining four terms are due to the offset of the solar system barycenter from the center of the sun. Without these four terms, the new expression for $t - \tau$ applies (with slightly reduced accuracy) for coordinate time t' in the heliocentric space-time frame of reference.

The differential equation for $d\tau/dt$ is developed in Sections 2 and 3. It is integrated to give an intermediate expression for $t - \tau$ in Sections 4 and 5. The periodic terms of this intermediate expression, which are integrals or dot products of position and velocity vectors, are converted to sums of sinusoidal functions in Section 6. These terms are collected in Section 7 to give the final expression for $t - \tau$. Auxiliary equations for computing the arguments of the periodic terms of $t - \tau$ are also given. The final expression for $t - \tau$ and the auxiliary equations give $t - \tau$ as a function of time and the earth-fixed coordinates of the atomic clock. Section 8 gives estimates for errors in computed range and doppler observables due to terms neglected in the final expression for $t - \tau$. The effects of the retained terms of $t - \tau$ on these observables are also given.

Section 9 gives an alternate expression for $t - \tau$ which is a function of the position and velocity vectors of the atomic clock and the major bodies of the solar system. This expression is the intermediate expression for $t - \tau$ given in Section 5 with the three terms which are expressed as integrals replaced by functions of position and velocity vectors. In certain circumstances, it may be desirable to compute $t - \tau$ from position and velocity vectors using the equation given in Section 9 instead of computing it as a function of time using the formulation of Section 7.

The notation used is defined in the text. However, the definitions of the symbols used globally throughout this paper are repeated in Appendix A. Numerical values are given for those parameters which appear in the final expression for $t - \tau$. Appendix B gives equations for computing range and

doppler observables. These equations are explicit functions of the time transformation $t - \tau$.

2. Differential Equation for $d\tau/dt$

The invariant interval ds between two events with differences in their space and time coordinates of dx^1, dx^2, dx^3, and dx^4 is given by

$$ds^2 = g_{ij} dx^i dx^j \tag{1}$$

where the repeated indices are summed over the integers 1 through 4. The matrix of coefficients g_{ij} is the metric tensor, obtained by solving Einstein's field equations. A solution for the case of n slowly moving bodies in the weak field approximation is the n-body metric tensor of Eddington and Clark (1938).[2] This solution can be applied to the solar system. For this application, the coordinates x^1, x^2, and x^3 of this solution are nonrotating rectangular components x, y, and z, respectively, of position relative to the solar system barycenter; the coordinate $x^4 = ct$, where c is the speed of light and t is coordinate time. In order to distinguish it from coordinate times of other solutions of the field equations, it will be referred to as coordinate time in the solar system barycentric space-time frame of reference. An interval of proper time $d\tau$ recorded on an atomic clock is related to the interval ds along its world line by

$$d\tau = \frac{ds}{c} \tag{2}$$

Hence, (1) and (2) relate an observed interval of proper time to the changes in the space and time coordinates of the atomic clock.

For the purpose of obtaining an expression for $t - \tau$, where τ is proper time recorded on a fixed atomic clock on earth, and t is coordinate time in the solar system barycentric frame of reference, the n-body metric tensor

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[2] This is the solution previously obtained by de Sitter (1916) except for a correction to one of his terms. The n-body metric tensor corresponds to the n-body Lagrangian given by Eq. 3.3.37 of Infeld and Plebański (1960). The n-body metric tensor (with reversed sign convention) may also be found in Eq. 39.63 of Misner, Thorne, and Wheeler (1973).

is substituted into (1) and terms are retained to order $(1/c)^0$, giving

$$ds^2 = \left(1 - \frac{2U}{c^2}\right) c^2 dt^2 - (dx^2 + dy^2 + dz^2) \tag{3}$$

where U is the Newtonian gravitational potential at the atomic clock, computed using the positive sign convention (i.e., $U = -\phi$). Let the velocity of the atomic clock relative to the solar system barycenter be denoted by $\dot{s}$, which is defined by

$$\dot{s}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \tag{4}$$

Substituting (4) and (2) into (3) gives

$$\frac{d\tau}{dt} = \left[1 - \frac{2U}{c^2} - \left(\frac{\dot{s}}{c}\right)^2 \right]^{1/2} \tag{5}$$

Expanding and retaining terms to order $1/c^2$ gives

$$\frac{d\tau}{dt} = 1 - \frac{U}{c^2} - \frac{1}{2}\left(\frac{\dot{s}}{c}\right)^2 \tag{6}$$

The neglected terms of $t - \tau$ due to the neglected $1/c^4$ terms of (6)[3] have a maximum magnitude of about 10^{-12} s. They affect computed range and doppler observables by a maximum of about 10^{-6} m and 10^{-10} m/s, respectively. Equation (3) is the "Newtonian" approximation to the n-body metric (see paragraph 39.7 of Misner, Thorne, and Wheeler, 1973).[4] All viable relativistic theories of gravitation have the same metric to this level of

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[3] Due to terms neglected in (3) and in expanding (5).

[4] Equation (3) implies the spacelike sections of simultaneity t = constant to be flat Euclidean spaces to first order: the coordinates x, y, and z in them can properly be thought of as Euclidean coordinates, and all of the results of classical Euclidean geometry may be used, e.g., triangle theorems and vector addition laws for slowly moving objects.

approximation. Hence, the expression for $t - \tau$ obtained from (6) is consistent with all viable relativistic theories of gravitation.

Equation (6) is the basic differential equation relating observed atomic time τ to coordinate time t in the solar system barycentric frame of reference. The rate $d\tau/dt$ of an atomic clock relative to uniform coordinate time is a function of the Newtonian gravitational potential U at the clock and the solar system barycentric velocity $\dot{s}$ of the clock.

Equation (6) will be used to obtain an expression for coordinate time t minus proper time obtained from a fixed atomic clock on earth. In the expression to be obtained, the proper time will be specifically International Atomic Time TAI disseminated by the Bureau International de l'Heure (BIH). The TAI second is the SI second (International System of Units), which is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom (Mechtly, 1969). This is the observationally determined average length of the second of coordinate time t.[5] Referring to Equation (6), periodic variations of U and $\dot{s}^2$ from their average values produce periodic terms in $t - \tau$. However, the average values of U and $\dot{s}^2$ for a fixed atomic clock on earth are positive, and the average length of an interval of proper time $d\tau$ is less than the corresponding interval of coordinate time dt. Thus, the average length of the second of proper time τ is greater than the length of the second of coordinate time t. Proper time τ in (6) does not correspond to TAI because the average length of the TAI second is equal to the length of the second of coordinate time t. Let TAI be denoted by τ^*, which differs from τ in the length of the atomic second. A modified form of (6) will be obtained which applies for atomic time τ^*. The average length of the τ^* second must equal the length of the second of coordinate time t. Hence, the

[5] Interpolation of the lunar ephemeris with an observed longitude of the moon gives an "observed" value of the independent variable, coordinate time t. The average number of cycles obtained from a cesium atomic clock per second of coordinate time t was obtained by counting cycles of a cesium atomic clock between two observations of the moon and dividing the observed number of cycles by the difference of the two "observed" values of coordinate time t.

differential equation relating τ^* and t must satisfy the condition that the average value of $d\tau^*$ equals dt.

In (6), $d\tau$ is obtained as

$$d\tau = \frac{dN}{n} \tag{7}$$

where dN is an observed number of cycles obtained from an atomic clock and n (in units of cycles per second) is a conversion factor from cycles to seconds of atomic time. The value of n corresponding to (6) is that value which results in the length of the second of proper time equalling the length of the second of coordinate time when the atomic clock is an infinite distance from the solar system and fixed relative to the solar system barycenter. Equation (6) can be rewritten as

$$\frac{d\tau}{dt} = 1 - \frac{\overline{U}}{c^2} - \frac{1}{2}\frac{\overline{\dot{s}^2}}{c^2} - \frac{U-\overline{U}}{c^2} - \frac{1}{2}\,\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} \tag{8}$$

where $\overline{U}$ is the time average value of U at the atomic clock[6] and $\overline{\dot{s}^2}$ is the time average value of $\dot{s}^2$ for the clock. Ignoring $1/c^4$ terms, this may be written as

$$\frac{d\tau}{dt\left(1 - \frac{\overline{U}}{c^2} - \frac{1}{2}\frac{\overline{\dot{s}^2}}{c^2}\right)} = 1 - \frac{U - \overline{U}}{c^2} - \frac{1}{2}\,\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} \tag{9}$$

Substitute (7) into (9), and let atomic time τ^* be obtained as

$$d\tau^* = \frac{dN}{n^*} \tag{10}$$

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[6] The time average value of U is

$$\overline{U} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} U\,dt$$

where the conversion factor n* is given by

$$n^* = n\left(1 - \frac{\overline{U}}{c^2} - \frac{1}{2}\frac{\overline{\dot{s}^2}}{c^2}\right) \tag{11}$$

Then (9) is given by

$$\frac{d\tau^*}{dt} = 1 - \frac{U-\overline{U}}{c^2} - \frac{1}{2}\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} \tag{12}$$

The average value of $d\tau^*$ equals dt and hence atomic time τ^* is the mathematical representation of International Atomic Time TAI.

Periodic variations of U and $\dot{s}^2$ from their average values result in periodic variations in atomic time τ^* relative to coordinate time t in the solar system barycentric frame of reference. Some of the periodic terms of $t - \tau^*$ for atomic clocks at various locations on earth are not in phase. Thus, one is tempted to conclude that it is not possible to synchronize a world-wide network of atomic clocks which read atomic time τ^*. However, Section 3 shows that such a system of atomic clocks can be synchronized with light signals.

In the conversion factor n* given by Equation (11), the average value $\overline{U}$ varies with the location of the atomic clock on earth. However, the only term of $\overline{U}$ which varies significantly is the gravitational potential U(E) due to the earth. Similarly, the average value $\overline{\dot{s}^2}$ varies with the location of the atomic clock on earth only because of the variation in the square of the geocentric velocity v due to the earth's rotation ($v = u\omega$, where u is the distance of the atomic clock from the earth's spin axis and ω is the inertial rotation rate of the earth). The only significant variable part of n* is thus $-n\left[U(E) + (1/2)v^2\right]/c^2$. The quantity $U(E) + (1/2)v^2$, the sum of the earth's gravitational and centrifugal potentials, is constant on the geoid (mean sea level). The conversion factor n* thus varies with altitude above the geoid; it increases at the approximate rate of 1.00×10^{-3} cycles per second per kilometer. If the defined length of the TAI second (9 192 631 770 cycles from a cesium atomic clock) is taken to apply at mean sea level, the conversion factor n* (cycles from a cesium atomic clock per second of TAI)

is given approximately by

$$n^* = 9\ 192\ 631\ 770 + 0.001\ 00\ h \tag{13}$$

where h is altitude above the geoid in kilometers.

TAI is obtained in practice as a weighted average of times obtained from atomic clocks located at various altitudes. All of the clocks have the same conversion factor from cycles to seconds of atomic time. Prior to computing the weighted average, time obtained from each clock is corrected in value and rate (Bureau International de l'Heure, 1975). The rate corrections remove the variations in the clock rates due to the differing altitudes above mean sea level and other causes. Hence, in the determination of TAI, the altitude-dependent conversion factor n^* is used implicitly, but not explicitly.

In (12), the average lengths of the TAI and coordinate time seconds are the same and are equal to the "observed" length of the coordinate time second. Hence, by using (12), the estimated length of the second of coordinate time becomes its adopted length. This does not produce any errors in observed minus computed residuals for tracking data obtained from a spacecraft, if the n-body ephemeris (for the planets, sun, and moon) is fit to observations using the expression for $t - \tau^*$ obtained from Equation (12). Also, the observed minus computed residuals for the planets, sun, and moon are not degraded.

3. Synchronization of TAI Atomic Clocks

This section shows that a world-wide network of atomic clocks which use the conversion factor n^* given by (11) and read atomic time τ^* = TAI can be synchronized by light signals. This conclusion is reached by considering the following expression for the interval ds which applies in a local region of the geocentric inertial (i.e., nonrotating) frame of reference:

$$ds^2 = \left[1 - \frac{2U(E)}{c^2}\right] c^2 dt^2 - (dx^2 + dy^2 + dz^2) \tag{14}$$

where U(E) is the gravitational potential due to the earth, t is coordinate time in the geocentric inertial frame of reference, and x, y, and z are rectangular components of position in this frame. Equation (14) is a modified form of (3)

applied to a local region of the geocentric inertial frame of reference. The modification consists of replacing the total gravitational potential by the potential due to the earth only. From the Principle of Equivalence, the accelerating earth produces a gravitational field which cancels (in a small region surrounding the earth) the gravitational field due to the sun, moon, and other bodies which accelerate the earth. Thus, in the local geocentric line element (14), the gravitational potential due to all bodies other than the earth does not appear.

Substituting (2) into (14) and retaining terms to order $1/c^2$ gives

$$\frac{d\tau}{dt} = 1 - \frac{U(E)}{c^2} - \frac{1}{2}\left(\frac{\dot{s}}{c}\right)^2 \tag{15}$$

where $\dot{s}$ is the velocity of the atomic clock relative to the geocentric inertial frame of reference. The sum of the gravitational and centrifugal potentials, $U(E)+(1/2)\dot{s}^2$, is constant on the geoid and thus the rate of proper time τ varies with altitude above the geoid. The interval $d\tau$ is obtained as

$$d\tau = \frac{dN}{m} \tag{16}$$

The value of the conversion factor m (cycles per second of atomic time) which corresponds to Equation (15) is that value which results in the length of the second of proper time τ equalling the length of the second of coordinate time t for a fixed atomic clock at infinite distance from the earth. If the length of the atomic second is changed so that

$$d\tau^* = \frac{dN}{m^*} \tag{17}$$

where

$$m^* = m\left(1 - \frac{U(E)}{c^2} - \frac{1}{2}\frac{\dot{s}^2}{c^2}\right) \tag{18}$$

equation (15) reduces to

$$\frac{d\tau^*}{dt} = 1 \tag{19}$$

Thus, using the altitude-dependent conversion factor m* results in atomic time τ^*, which runs at the rate of coordinate time t in the geocentric inertial frame of reference. The constant m in (18) can be selected so that m*=n* given by (11) at mean sea level. However, the altitude-dependent terms in (18) and (11) are proportional to m and n, respectively, which differ by terms of order $1/c^2$. Thus the factor 1.00×10^{-3} in (13) will agree with the corresponding factor obtained from (18) to about 7 digits. The maximum fractional difference between m* and n* is about 10^{-21}, which is negligible in relation to the current stabilities of atomic clocks. Thus, to sufficient accuracy, atomic time τ^* in (19) is the same time scale as τ^* in (12).

From (19), atomic clocks at varying altitudes which read τ^* have the same length for the second of atomic time (in seconds of coordinate time t). These clocks could be synchronized with a master clock by using light signals transmitted by way of an equatorial synchronous satellite. The Newtonian light time (calculated in the geocentric inertial frame of reference) from the master station to any other station is constant. The relativistic contribution to the light time is due to the mass of the earth only; it is less than 0.2 ns and constant. Hence, intervals of reception at the various stations will be identical to intervals of transmission at the master station. Since the various clocks have the same length for the atomic second, the transmission intervals can be synchronized with the seconds pulses of the master clock, and the reception intervals will be synchronized with (but out of phase with) the seconds pulses of the clock at the receiving station. In practice, only one signal is required to synchronize each clock with the master clock. Upon reception of the signal, the clock at the receiving station is set equal to the known transmission time plus the light time calculated in the geocentric inertial frame of reference. The signal can be transmitted by way of any spacecraft in the earth's vicinity or by lunar bounce. For other methods of synchronizing atomic clocks on earth, see Thomas (1975).

4. Integration of Differential Equation

Equation (12) relates an infinitesimal interval of atomic time τ^* obtained from a fixed atomic clock on earth to the corresponding interval of coordinate time t in the solar system barycentric frame of reference, the Newtonian gravitational potential U at the clock, its time average value $\overline{U}$, the square of the solar system barycentric velocity of the clock $\dot{s}^2$, and its time average value $\overline{\dot{s}^2}$.

This section integrates Equation (12) to obtain an intermediate expression for $t - \tau^*$. The terms of this expression, which are integrals or dot products of position and velocity vectors, are converted to sums of sinusoidal functions in Section 6.

This section gives the magnitudes of several discarded terms of $t - \tau$ and their maximum effects on computed range and doppler observables. The effects that are range-dependent were evaluated at a range of 50 astronomical units. The most significant of these errors are included in the error summary in Section 8.

In the derivation of the expression for $t - \tau^*$, the effects of solid earth tides, polar motion, and nutation are ignored. The maximum effects of these phenomena on two-way or three-way range and doppler observables, due to changes in $t - \tau^*$, are given in Table I:

TABLE I

Effects on range and doppler observables due to changes in $t - \tau^*$

Phenomenon	Range, m	Doppler, m/s
Solid earth tides	10^{-3}	10^{-7}
Polar motion	10^{-2}	10^{-7}
Nutation	10^{-1}	10^{-6}

In the evaluation of the terms of $t - \tau^*$ in Section 6, the effects of precession are accounted for. In the final expression for $t - \tau^*$, the coefficients of some of the terms are explicit functions of the earth-fixed coordinates of the atomic clock. Continental drift will result in a slow drift in the values of these coordinates.

In (12), $U - \overline{U}$ excludes the constant potential due to the earth and can be expressed as

$$U - \overline{U} = (U_A - U_E)_p + (U_E)_p \tag{20}$$

where U_A and U_E are the gravitational potentials due to all bodies except the earth, evaluated at the location of the atomic clock and at the center of the earth, respectively. The subscript p indicates that only the periodic terms of the quantity are to be retained.

In the following, $\underline{r}_i^j$, $\dot{\underline{r}}_i^j$, and $\ddot{\underline{r}}_i^j$ will refer to the position, velocity, and acceleration vectors, respectively, of point i relative to point j. The quantity $\dot{s}_i^j$ is the velocity of point i relative to point j. The components of $\underline{r}_i^j$ are space coordinates in the solar system barycentric space-time frame of reference. The dots denote differentiation with respect to coordinate time t in this frame of reference. In (12), $\dot{s}^2$ can be expressed as

$$\dot{s}^2 = \left(\dot{\underline{r}}_A^E + \dot{\underline{r}}_E^C\right) \cdot \left(\dot{\underline{r}}_A^E + \dot{\underline{r}}_E^C\right) \tag{21}$$

where A, E, and C refer to the locations of the atomic clock, the center of the earth, and the solar system barycenter, respectively. Then

$$\dot{s}^2 - \overline{\dot{s}^2} = \left[\left(\dot{s}_A^E\right)^2 + 2\dot{\underline{r}}_E^C \cdot \dot{\underline{r}}_A^E + \left(\dot{s}_E^C\right)^2 \right]_p \tag{22}$$

Substituting (20) and (22) into (12), denoting τ^* by τ, and integrating from an initial epoch (subscript o) to the current time t gives

$$t - \tau = (t - \tau)_o + \frac{1}{2c^2} \int_{t_o}^{t} \left[2U_E + \left(\dot{s}_A^E\right)^2 + \left(\dot{s}_E^C\right)^2 \right]_p dt \tag{23}$$

$$+ \frac{1}{c^2} \int_{t_o}^{t} \left(U_A - U_E + \dot{\underline{r}}_E^C \cdot \dot{\underline{r}}_A^E\right)_p dt$$

In the first integral of (23), $\dot{s}_A^E$ is nearly constant and periodic variations in the square of this quantity can be ignored. In the second integral, the potential difference $U_A - U_E$ may be expanded to first order:

$$\frac{1}{c^2}\int_{t_o}^{t}\left(U_A - U_E\right)_p \, dt = \frac{1}{c^2}\int_{t_o}^{t}\left[\left(\nabla U\right)_E \cdot \underline{r}_A^E\right]_p \, dt \qquad (24)$$

where the gradient $(\nabla U)_E$ is evaluated at the center of the earth. The neglected higher-order terms in this expansion produce terms of $t-\tau$ with magnitudes of 10^{-11}s or less. They affect computed range and doppler observables by less than 10^{-3}m and 10^{-8}m/s, respectively. In the second integral of (23), the dot product of the two velocity vectors is integrated by parts:

$$\frac{1}{c^2}\int_{t_o}^{t}\left(\dot{\underline{r}}_E^C \cdot \dot{\underline{r}}_A^E\right)_p \, dt = \frac{1}{c^2}\left(\dot{\underline{r}}_E^C \cdot \underline{r}_A^E\right)\Big|_{t_o}^{t} - \frac{1}{c^2}\int_{t_o}^{t}\left(\underline{r}_A^E \cdot \ddot{\underline{r}}_E^C\right)_p \, dt \qquad (25)$$

Since terms of order $1/c^4$ are ignored, the acceleration of the earth can be approximated by its Newtonian value:

$$\ddot{\underline{r}}_E^C = \left(\nabla U\right)_E \qquad (26)$$

and the second term of (25) cancels (24). This cancellation of the potential variation $U_A - U_E$ by the earth's acceleration is in accordance with the Principle of Equivalence. According to that principle, in a small region surrounding the freely falling earth, the sum of the gravitational potential due to the bodies accelerating the earth and the inertial potential due to the accelerated earth is constant to a high degree of accuracy. Substituting (24) through (26) into (23) and discarding the term $\left(\dot{s}_A^E\right)^2$ gives

$$t - \tau = (t - \tau)_o$$

$$+ \frac{1}{c^2}\int_{t_o}^{t} \left[U_E + \frac{1}{2}\left(\dot{s}_E^C\right)^2 \right]_P dt \qquad (27)$$

$$+ \frac{1}{c^2}\left(\dot{\underline{r}}_E^C \cdot \underline{r}_A^E \right)\Bigg|_{t_o}^{t}$$

The derivation of (27), starting from (23), is due to Thomas (1975). His Equation (13) corresponds to (27) above.

In the integral in (27), the gravitational potential at the earth due to the moon can be ignored. The neglected periodic terms of $t - \tau$ are smaller than 3 ns and affect computed range and doppler observables by less than 0.1 m and 0.2×10^{-6} m/s, respectively. The potential U_E in (27) can be expressed as

$$U_E = \left(U_E - U_B \right) + U_B \qquad (28)$$

where U_E and U_B are gravitational potentials at the earth and earth-moon barycenter B, respectively, excluding the terms due to the earth and moon. The potential difference $U_E - U_B$ may be expanded to first order:

$$U_E - U_B = \left(\nabla U\right)_B \cdot \underline{r}_E^B \qquad (29)$$

where the gradient is evaluated at the earth-moon barycenter. The neglected terms in this expansion produce terms in $t - \tau$ which are smaller than 10^{-12} s and affect computed range and doppler observables by less than 10^{-4} m and 10^{-9} m/s, respectively. In (27), the square of the solar system barycentric velocity of the earth can be expressed as

$$\left(\dot{s}_E^C\right)^2 = \left(\dot{\underline{r}}_E^B + \dot{\underline{r}}_B^C\right) \cdot \left(\dot{\underline{r}}_E^B + \dot{\underline{r}}_B^C\right)$$

$$= \left(\dot{s}_E^B\right)^2 + 2\dot{\underline{r}}_B^C \cdot \dot{\underline{r}}_E^B + \left(\dot{s}_B^C\right)^2 \tag{30}$$

The periodic terms of $\left(\dot{s}_E^B\right)^2$ can be neglected. They produce terms of $t - \tau$ smaller than 10^{-10}s which affect computed range and doppler observables by less than 10^{-3}m and 10^{-8}m/s, respectively. The dot product in (30) can be expressed as

$$\dot{\underline{r}}_B^C \cdot \dot{\underline{r}}_E^B = \frac{d}{dt}\left(\dot{\underline{r}}_B^C \cdot \underline{r}_E^B\right) - \underline{r}_E^B \cdot \ddot{\underline{r}}_B^C \tag{31}$$

Ignoring relativistic terms and assuming the earth and moon are located at their barycenter gives

$$\ddot{\underline{r}}_B^C = (\nabla U)_B \tag{32}$$

Substituting (28) - (32) except the term $\left(\dot{s}_E^B\right)^2$ of (30) into (27) gives

$$t - \tau = (t - \tau)_o$$

$$+\frac{1}{c^2}\int_{t_o}^{t} \left[U_B + \frac{1}{2}\left(\dot{s}_B^C\right)^2\right]_p dt \tag{33}$$

$$+ \frac{1}{c^2}\left(\dot{\underline{r}}_B^C \cdot \underline{r}_E^B\right)\Bigg|_{t_o}^{t} + \frac{1}{c^2}\left(\dot{\underline{r}}_E^C \cdot \underline{r}_A^E\right)\Bigg|_{t_o}^{t}$$

where U_B excludes the potential due to the earth and moon. The cancellation of the terms of $t-\tau$ due to (29) and the second term of (31) is in accordance with the Principle of Equivalence. This cancellation is associated with the earth-moon barycentric frame of reference and is analogous to the previously described cancellation in the geocentric frame of reference.

In (33), the second term of the integral can be expressed as

$$\frac{1}{c^2}\int_{t_o}^{t}\frac{1}{2}\left[\left(\dot{s}_B^C\right)^2\right]_p dt = \frac{1}{c^2}\int_{t_o}^{t}\left[\frac{1}{2}\left(\dot{s}_B^S\right)^2 + \dot{\underline{r}}_S^C \cdot \dot{\underline{r}}_B^S + \frac{1}{2}\left(\dot{s}_S^C\right)^2\right]_p dt \tag{34}$$

where S refers to the center of the sun. The last term of the integral is negligible. The second term is integrated by parts:

$$\frac{1}{c^2}\int_{t_o}^{t}\left(\dot{\underline{r}}_S^C \cdot \dot{\underline{r}}_B^S\right)_p dt = \frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_B^S\right)\Big|_{t_o}^{t} - \frac{1}{c^2}\int_{t_o}^{t}\left(\underline{r}_B^S \cdot \ddot{\underline{r}}_S^C\right)_p dt \tag{35}$$

Since terms of order $1/c^4$ are ignored, the integral in (35) can be evaluated using Newtonian accelerations. Also, the earth and moon can be considered to be located at the earth-moon barycenter. The result is

$$-\frac{1}{c^2}\int_{t_o}^{t}\left(\underline{r}_B^S \cdot \ddot{\underline{r}}_S^C\right)_p dt = \sum_i \frac{\mu_i}{c^2}\int_{t_o}^{t}\left[\frac{1}{r_i^3}\left(\underline{r}_S^B \cdot \underline{r}_i^S\right)\right]_p dt \tag{36}$$

where the summation over i includes eight planets and the earth - moon barycenter. The quantity $\underline{r}_i^S$ is the heliocentric position vector of planet i, r_i is its magnitude, and μ_i is the gravitational constant of the planet (see Appendix A). When i refers to the earth - moon barycenter, μ_i is the sum of the gravitational constants of the earth and moon. Assuming that the planets move in the ecliptic plane, (36) is given by

$$-\frac{1}{c^2}\int_{t_o}^{t}\left(\underline{r}_B^S \cdot \ddot{\underline{r}}_S^C\right)_p dt = \sum_i \frac{\mu_i}{c^2}\int_{t_o}^{t}\left[\frac{r\cos(\ell-\ell_i)}{r_i^2}\right]_p dt \qquad (37)$$

where r is the magnitude of $\underline{r}_S^B$, ℓ is the true longitude of the sun measured at the earth - moon barycenter, and ℓ_i is the heliocentric true longitude of planet i. Both of these angles are referred to the mean equinox and ecliptic of date. When i refers to the earth - moon barycenter, $r_i = r$ and $\ell - \ell_i = \pm 180°$.

The first term of the integral in (33) can be written as

$$\frac{1}{c^2}\int_{t_o}^{t}(U_B)_p dt = \frac{1}{c^2}\int_{t_o}^{t}\left[U_B(S)\right]_p dt + \sum_i \frac{1}{c^2}\int_{t_o}^{t}\left[U_B(i)\right]_p dt \qquad (38)$$

where $U_B(S)$ and $U_B(i)$ are the contributions to the gravitational potential at the earth - moon barycenter due to the sun S and a planet i, respectively. The summation over i includes all planets except the earth and excludes the moon. For an outer planet i,

$$\frac{1}{c^2}\int_{t_o}^{t} \left[U_B(i)\right]_p dt = \frac{\mu_i}{c^2} \int_{t_o}^{t} \left\{\frac{1}{r_i}\left[1-\left(\frac{r}{r_i}\right)\cos(\ell-\ell_i)\right.\right.$$

$$-\frac{3}{8}\left(\frac{r}{r_i}\right)^3 \cos(\ell-\ell_i) + \frac{3}{4}\left(\frac{r}{r_i}\right)^2 \cos 2(\ell-\ell_i)$$

$$\left.\left.+\frac{1}{4}\left(\frac{r}{r_i}\right)^2 - \frac{5}{8}\left(\frac{r}{r_i}\right)^3 \cos 3(\ell-\ell_i)\right]\right\}_p dt \qquad (39)$$

where all terms to order $(r/r_i)^3$ have been retained, and the planet i is assumed to move in the ecliptic plane. When i refers to an inner planet, r_i and r are interchanged in (39).

For an outer planet, the second term of (39) cancels the term of (37) for that planet. The first term of (39) is retained for Jupiter and Saturn only. The remaining terms of (39) are ignored for all outer planets. For an inner planet, there is no cancellation between (39), after interchanging r and r_i, and the term of (37) for that planet. All terms of (39) are ignored for Mercury and for Venus. The terms of (37) for Mercury, Venus, and the earth - moon barycenter are also ignored. Substituting the surviving terms of (34), (35), (38) and (39) into (33) and replacing solar system barycentric velocity vectors with heliocentric velocity vectors plus $\dot{\underline{r}}_S^C$ gives

$$t - \tau = (t-\tau)_o$$

$$+\frac{1}{c^2}\int_{t_o}^{t}\left[U_B(S) + \frac{1}{2}\left(\dot{s}_B^S\right)^2\right]_p dt + \frac{1}{c^2}\left(\dot{\underline{r}}_B^S \cdot \underline{r}_E^B\right)\Big|_{t_o}^{t}$$

$$+\frac{1}{c^2}\left(\dot{\underline{r}}_E^S \cdot \underline{r}_A^E\right)\Big|_{t_o}^{t} + \frac{1}{c^2}\int_{t_o}^{t}\left(\frac{\mu_J}{r_J}\right)_p dt$$

$$+\frac{1}{c^2}\int_{t_o}^{t}\left(\frac{\mu_{SA}}{r_{SA}}\right)_p dt + \frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_B^S\right)\Big|_{t_o}^{t} + \frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_A^E\right)\Big|_{t_o}^{t} \qquad (40)$$

where μ_J and μ_{SA} are gravitational constants for Jupiter and Saturn, respectively, and r_J and r_{SA} are the heliocentric distances of these bodies. The term

$$\frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_E^B\right)\Big|_{t_o}^{t}$$

is negligible and has been omitted from (40). The neglected terms in Equations (37) and (39) result in a maximum error in $t - \tau$ of about 1.3 μs; the corresponding errors in computed range and doppler observables can be up to 1.1 m and 0.3×10^{-6} m/s, respectively.

For an outer planet, the first term of (39) is the gravitational potential at the sun due to the planet (divided by c^2). The remaining terms are thus the potential at the earth - moon barycenter $U_B(i)$ minus the potential at the sun $U_S(i)$. The second term of (39) is the dot product of the gradient of the potential due to the planet (evaluated at the center of the sun) and the heliocentric position vector of the earth - moon barycenter. The cancellation of this term with the term of (37) due to the same planet is in accordance with the Principle of Equivalence. In a small region surrounding the sun, the variation in the gravitational potential due to a planet is cancelled to high accuracy by the inertial potential arising from the sun's acceleration due to that planet. However, the cancellation of the potential difference $U_B(i) - U_S(i)$ due to an outer planet i is only approximate as evidenced by the higher-order terms in (39). For an inner planet, the cancellation of this potential difference is not even approximate; the term of (37) for an inner planet does not cancel any term of (39) when r and r_i are interchanged in (39).

5. Initial Values

The purpose of this section is to determine the value of $(t - \tau)_o$ in Equation (40). The subscript o indicates that this quantity is to be evaluated at the initial epoch t_o. The value of $(t - \tau)_o$ is station-dependent; that is, it depends upon the location of the atomic clock on earth. However, it will be seen that $(t - \tau)_o$ minus the initial values of all of the periodic terms of (40) has the same value for all stations; it is denoted by ΔT_A.

The conclusion that the constant ΔT_A does not vary with the location of the atomic clock on earth follows from a consideration of the Lorentz transformation. One of the four equations of this transformation is:

$$t = \frac{1}{\sqrt{1 - \frac{\dot{s}^2}{c^2}}}\left(t' + \frac{\dot{s}}{c^2}x'\right) \qquad (41)$$

In this application, t and t' refer to coordinate time in the solar system barycentric and geocentric inertial frames of reference, respectively. The coordinate x' is the component of the geocentric position vector of the atomic clock along the instantaneous direction of the earth's velocity vector relative to the solar system barycenter. The magnitude of this vector is $\dot{s}$.

Section 3 showed that atomic clocks on earth which read International Atomic Time τ^* (denoted as τ starting in Section 4) can be synchronized. It was also shown that this time scale is identical with coordinate time in the geocentric inertial frame of reference. Hence, for two such

synchronized clocks, the interval $\Delta t'$ between corresponding ticks is zero, and (41) gives the interval of coordinate time t between the two "synchronized" ticks:

$$\Delta t = \frac{1}{\sqrt{1 - \frac{\dot{s}^2}{c^2}}} \left(\frac{\dot{s} \Delta x'}{c^2} \right) \approx \frac{\dot{s} \Delta x'}{c^2} \tag{42}$$

where $\Delta x'$ is the difference of the x' coordinates of the two atomic clocks. Thus, fixed observers in the solar system barycentric frame of reference do not agree that the two atomic clocks on earth are synchronized. The difference in the definitions of simultaneity in these two frames of reference is due to their relative velocity $\dot{s}$.

The transformation $t - \tau$ given by (40) must be consistent with (42). This allows the relation between values of ΔT_A for different stations to be determined. Let $(t - \tau)_2$ and $(t - \tau)_1$ refer to values of $t - \tau$ for two different stations on earth at a given epoch τ. Since the atomic clocks at these stations (which read τ) are synchronized, the difference $(t - \tau)_2 - (t - \tau)_1$ is the interval of coordinate time t in the solar system barycentric frame of reference between corresponding ticks of the two atomic clocks on earth. This interval must be equal to Δt given by (42). The sum T of terms four and eight of (40) excluding the values of the functions at the initial epoch t_o is

$$T = \frac{1}{c^2} \left(\dot{\underline{r}}_E^C \cdot \underline{r}_A^E \right) \tag{43}$$

The value of T for station 2 minus the value for station 1 is equal to the approximate form of (42). In order for $(t - \tau)_2 - (t - \tau)_1$ to be equal to this same quantity, the term $(t - \tau)_o$ of (40) minus the initial values of terms four and eight must be the same constant for all stations. The initial values of the remaining terms of (40) are not station-dependent; they can be subtracted from this constant to give a second constant which is the same for all stations. This second constant is ΔT_A, previously defined

as $(t - \tau)_o$ minus the values of all periodic terms of (40) at the initial epoch t_o. Using this notation in (40) and expressing term four as the sum of two terms gives

$$t - \tau = \Delta T_A$$

$$+\frac{1}{c^2}\int \left[U_B(S) + \frac{1}{2}\left(\dot{s}_B^S\right)^2\right]_p dt + \frac{1}{c^2}\left(\dot{\underline{r}}_B^S \cdot \underline{r}_E^B\right)$$

$$+\frac{1}{c^2}\left(\dot{\underline{r}}_E^B \cdot \underline{r}_A^E\right) + \frac{1}{c^2}\left(\dot{\underline{r}}_B^S \cdot \underline{r}_A^E\right) + \frac{1}{c^2}\int \left(\frac{\mu_J}{r_J}\right)_p dt$$

$$+\frac{1}{c^2}\int \left(\frac{\mu_{SA}}{r_{SA}}\right)_p dt + \frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_B^S\right) + \frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_A^E\right) \qquad (44)$$

where the constants of integration for the three indefinite integrals are zero.

All terms of (44) except the first term are periodic. Terms four, five, and nine represent periodic variations in the rate of a specific clock on earth relative to uniform coordinate time t in the solar system barycentric frame of reference. The remaining terms represent periodic variations in the rates of all clocks on earth. At a given epoch, the difference in the sums of terms four, five, and nine for two different stations on earth represents the difference in the definitions of simultaneity for observers on earth and observers fixed in the solar system barycentric frame of reference.

The expressions for $t - \tau$ given in this report apply for ideal International Atomic Time τ, which in general differs from actual International Atomic Time $\tau(A)$. If an expression is desired for $t - \tau(A)$, terms representing $\tau - \tau(A)$ must be added to the expressions for $t - \tau$ given in this report.

It was announced at the 16th General Assembly of the International Astronomical Union (held in Grenoble in August 1976) that the second of TAI

is shorter than the SI second at mean sea level by $1.0 \pm 0.2 \times 10^{-12}$s. A resolution was adopted[7] to increase the length of the TAI second by 1.0×10^{-12}s at January 1, 1977, 0^h TAI. Using $\tau = \tau(A)$ at this epoch, the approximate expression for $\tau - \tau(A)$ which applies from January 1, 1972, 0^h TAI (when the TAI time scale was officially adopted) to January 1, 1977, 0^h TAI is:

$$\tau - \tau(A) = -1 \times 10^{-12} \, t$$

where t (a negative number) is seconds past January 1, 1977, 0^h TAI. The a priori estimate for $\tau - \tau(A)$ which applies after this epoch is zero.

The International Astronomical Union has adopted[7] an atomic time scale which uses the SI second at mean sea level and has the value TAI + 0.0003725 days exactly (32.184 s exactly) at January 1, 1977, 0^h TAI. It has further stated that there be only periodic variations between this time scale and coordinate time in the solar-system barycentric space-time frame of reference. Hence, for $\tau = \tau(A)$ at January 1, 1977 0^h TAI, the International Astronomical Union has adopted 32.184 s (exactly) for the value of the constant ΔT_A.

The value of ΔT_A does not affect observed minus computed residuals for tracking data obtained from a spacecraft if the n-body ephemeris used was fit to observations using the same value for ΔT_A. The value of ΔT_A is an adopted constant used in the expression for $t - \tau$; there is no error in ΔT_A which contributes to errors in computed range and doppler observables.

[7] See Transactions of the International Astronomical Union, Vol. XVI B (1977).

6. Evaluation of Periodic Terms

This section converts the periodic terms in Equation (44) for $t - \tau$ to sums of sinusoidal functions. The various assumptions made and the terms which are discarded are stated but are not justified in this section. However, the resulting errors in computed range and doppler observables are given in Section 8 along with the effects of the retained terms of $t - \tau$. The final expression for $t - \tau$ and the equations for evaluating the arguments of its periodic terms are given in Section 7.

Several of the periodic terms in (44) contain dot products of position and velocity vectors. All of the velocity vectors are inertial velocity vectors: the inertial velocity vector $\dot{\underline{r}}_i^j$ gives the velocity vector of point i relative to a nonrotating coordinate system centered at point j. In evaluating the dot-product terms, the components of each of the two vectors are referred to a nonrotating coordinate system which is aligned with the instantaneous position of a rotating coordinate system. It should be noted that the velocity vectors do not give velocity components relative to the rotating coordinate systems. The two vectors required for the evaluation of a particular term are referred to either the mean earth equator and equinox of date or the mean equinox and ecliptic of date. By using these coordinate systems, the final expression for $t - \tau$ accounts for the precession of the earth's equator.

6.1 ANNUAL TERM

The second term of Equation (44) is

$$(t - \tau)_2 = \frac{1}{c^2} \int \left[U_B(S) + \frac{1}{2} \left(\dot{s}_B^S \right)^2 \right]_p dt \qquad (45)$$

where the constant of integration is zero. This term is evaluated assuming that the heliocentric orbit of the earth-moon barycenter is an ellipse with semi-major axis a, eccentricity e, radial coordinate r, and eccentric anomaly E. The gravitational potential at the earth-moon barycenter due to the sun is

$$U_B(S) = \frac{\mu_S}{r} \qquad (46)$$

where μ_S is the gravitational constant of the sun. The square of the heliocentric velocity of the earth-moon barycenter is

$$\left(\dot{s}_B^S\right)^2 = (\mu_S + \mu_E + \mu_M)\left(\frac{2}{\bar{r}} - \frac{1}{\bar{a}}\right) \approx \mu_S\left(\frac{2}{\bar{r}} - \frac{1}{a}\right) \tag{47}$$

where μ_E and μ_M are the gravitational constants of the earth and moon, respectively. The inverse of r is given by

$$\frac{1}{r} = \frac{1}{a} + \frac{e}{r}\cos E \tag{48}$$

Substituting (46), the approximate form of (47), and (48) into the integrand of (45) and retaining only the periodic terms as specified by the subscript p gives

$$\left[U_B(S) + \frac{1}{2}\left(\dot{s}_B^S\right)^2\right]_p = \frac{2\mu_S e}{r}\cos E \tag{49}$$

The derivative of E with respect to t is

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu_S + \mu_E + \mu_M}{a}} \approx \frac{1}{r}\sqrt{\frac{\mu_S}{a}} \tag{50}$$

Substituting (49) into (45), multiplying the integrand by d E/dt and dividing it by the approximate form of (50) gives

$$(t - \tau)_2 = \frac{2}{c^2}\sqrt{\mu_S a}\; e \sin E \tag{51}$$

Substituting numerical values from Appendix A gives, in units of seconds,

$$(t - \tau)_2 = 1.658 \times 10^{-3} \sin E \tag{52}$$

The coefficient is given to four digits because e is constant to about that many digits from 1950 to 2000. The error due to this approximation is included in the error summary in Section 8. The approximation of the factor $\mu_S+\mu_E+\mu_M$ by μ_S in (47) and (50) is valid since these two quantities agree to six digits.

The term (51) was obtained by Moyer (1965). It was also obtained using an alternate derivation by J. D. Anderson (unpublished) in 1964. The derivation of (51) does not involve an expansion in powers of e. Clemence and Szebehely (1967) used such an expansion and obtained a series of terms, the first of which is (51) with E replaced by the mean anomaly M. Their complete series does convert to the form (51). The annual variation in the rate of atomic clocks on earth corresponding to the term (51) of $t - \tau$ was obtained by Aoki (1964).

6.2 MONTHLY TERM

The third term of Equation (44) is

$$(t - \tau)_3 = \frac{1}{c^2} \left(\dot{\underline{r}}_B^S \cdot \underline{r}_E^B \right) \tag{53}$$

which can be written as

$$(t - \tau)_3 = \frac{1}{c^2(1+\mu)} \left(\dot{\underline{r}}_S^B \cdot \underline{r}_M^E \right) \tag{54}$$

where $\underline{r}_M^E$ is the geocentric position vector of the moon and

$$\mu = \frac{\mu_E}{\mu_M} \tag{55}$$

Both vectors in (54) are evaluated assuming circular orbits, and the 5° inclination of the lunar orbit to the ecliptic is ignored. With rectangular components referred to the mean equinox and ecliptic of date, these vectors are given by

$$\dot{\underline{r}}_S^B = \begin{bmatrix} -\sin L \\ \cos L \\ 0 \end{bmatrix} \dot{s}_c \tag{56}$$

and

$$\underline{r}_M^E = \begin{bmatrix} \cos \mathbb{C} \\ \sin \mathbb{C} \\ 0 \end{bmatrix} a_M \tag{57}$$

where

$\dot{s}_c$ = circular orbit velocity of earth-moon barycenter relative to the sun (see Appendix A)

L = earth-moon-barycentric mean longitude of the sun, referred to the mean equinox and ecliptic of date

a_M = semi-major axis of geocentric orbit of the moon

$\mathbb{C}$ = geocentric mean longitude of the moon, referred to the mean equinox and ecliptic of date

Substituting (56) and (57) into (54) gives

$$(t - \tau)_3 = \frac{\dot{s}_c \, a_M}{c^2 (1+\mu)} \sin D \tag{58}$$

where D, the mean elongation of the moon from the sun, is given by

$$D = \mathbb{C} - L \tag{59}$$

Substituting numerical values from Appendix A gives

$$(t - \tau)_3 = 1.548 \times 10^{-6} \sin D \tag{60}$$

The error summary in Section 8 gives the errors in computed range and doppler observables due to truncating the coefficients of the periodic terms of $t - \tau$.

6.3 DAILY TERM DUE TO MONTHLY MOTION OF EARTH

The fourth term of Equation (44) is

$$(t - \tau)_4 = \frac{1}{c^2} \left(\dot{\underline{r}}^B_E \cdot \underline{r}^E_A \right) \tag{61}$$

which can be expressed as

$$(t - \tau)_4 = - \frac{1}{c^2 (1+\mu)} \left(\dot{\underline{r}}^E_M \cdot \underline{r}^E_A \right) \tag{62}$$

The orbit of the moon is assumed to be circular and in the ecliptic plane. The vectors in (62), with rectangular components referred to the mean equinox and ecliptic of date, are given by

$$\dot{\underline{r}}^E_M = \begin{bmatrix} -\sin \ell \\ \cos \ell \\ 0 \end{bmatrix} \dot{s}_M \tag{63}$$

$$\underline{r}^E_A = \begin{bmatrix} u \cos (\theta_M + \lambda) \\ u \sin (\theta_M + \lambda) \cos \epsilon + v \sin \epsilon \\ -u \sin (\theta_M + \lambda) \sin \epsilon + v \cos \epsilon \end{bmatrix} \tag{64}$$

where

$\dot{s}_M$ = circular orbit velocity of the moon relative to earth (see Appendix A)

ϵ = mean obliquity of the ecliptic

= inclination of the ecliptic plane to mean earth equator of date

θ_M = mean sidereal time = Greenwich hour angle of mean equinox of date

and u, v, and λ are earth-fixed coordinates of the atomic clock. The coordinate u is the distance from the earth's spin axis in kilometers, v is the distance north of the earth's equatorial plane in kilometers, and λ is the east longitude. Substituting (63) and (64) into (62) and using trigonometric identities gives

$$(t - \tau)_4 = -\frac{\dot{s}_M}{c^2(1+\mu)} \left\{ \frac{u}{2} \left[(1+\cos\epsilon) \sin(\theta_M+\lambda-\mathbb{C}) - (1-\cos\epsilon) \sin(\theta_M+\lambda+\mathbb{C}) \right] + v \sin\epsilon \cos\mathbb{C} \right\} \tag{65}$$

The second and third terms of this expression are neglected.

From pages 73-74 of the Explanatory Supplement to the Ephemeris (1961), the relation between mean sidereal time θ_M and universal time UT is

$$\theta_M = UT + R_U - 12^h \tag{66}$$

where UT refers specifically to universal time UT1, hours past midnight, and R_U is the right ascension, measured from the mean equinox of date, of a fictitious point on the equator. The adopted expression for R_U, with units changed from hours to degrees, is

$$R_U = 279°41'27\overset{''}{.}54 + 129\ 602\ 768\overset{''}{.}13\ T_U + 1\overset{''}{.}393\ 5\ T_U^2 \tag{67}$$

where T_U is the number of Julian centuries of 36 525 days of UT1 elapsed since January 0, 1900, 12^h UT1. From page 98 of the above reference, the mean longitude L of the sun, referred to the mean equinox and ecliptic of date, is given by

$$L = 279°41'48''.04 + 129\,602\,768''.13\,T + 1''.089\,T^2 \tag{68}$$

where T is the number of Julian centuries of 36 525 days of ephemeris time (coordinate time t) elapsed since January 0, 1900, 12^h ephemeris time. The constant term of R_U is 20".5 smaller than the corresponding term of L because R_U is corrected for stellar aberration. The difference between R_U and L varies from about 21" at 1950 to about 23" at 2000. Because of this small and nearly constant difference, R_U in (66) is approximated by L which gives, in radians,

$$\theta_M = UT1 + L - \pi \tag{69}$$

If UT1 is given in seconds past January 1, 1950, 0^h UT1, the angle UT1 in radians is computed from

$$UT1\ (\text{radians}) = 2\pi \left\{ \left[\frac{UT1\ (\text{seconds})}{86400} \right]_{\text{decimal part}} \right\} \tag{70}$$

The calendar date for the reference epoch for UT1 (seconds) in (70) is arbitrary, but the time of day must be 0^h UT1. The effects of the approximation in (69) are included in the error summary given in Section 8.

Substituting (69) into the first term of (65) and using (59) gives

$$(t - \tau)_4 = \frac{\dot{s}_M(1+\cos\epsilon)}{2c^2(1+\mu)}\ u \sin(UT1+\lambda-D) \tag{71}$$

Substituting numerical values from Appendix A gives

$$(t - \tau)_4 = 1.33 \times 10^{-13}\ u \sin(UT1+\lambda-D) \tag{72}$$

6.4 DAILY TERMS AND SMALL ANNUAL TERM DUE TO ANNUAL MOTION OF EARTH

The fifth term of Equation (44) is

$$(t - \tau)_5 = \frac{1}{c^2}\left(\dot{\underline{r}}^S_B \cdot \underline{r}^E_A\right) = -\frac{1}{c^2}\left(\dot{\underline{r}}^B_S \cdot \underline{r}^E_A\right) \tag{73}$$

The vectors in (73), with rectangular components referred to the mean earth equator and equinox of date, are given by

$$\dot{\underline{r}}^B_S = \begin{bmatrix} -\sin(\ell - \gamma) \\ \cos(\ell - \gamma)\cos\epsilon \\ \cos(\ell - \gamma)\sin\epsilon \end{bmatrix} \dot{s}_S \tag{74}$$

$$\underline{r}^E_A = \begin{bmatrix} u\cos(\theta_M + \lambda) \\ u\sin(\theta_M + \lambda) \\ v \end{bmatrix} \tag{75}$$

where

$\dot{s}_S$ = velocity of earth-moon barycenter relative to sun

ℓ = earth-moon-barycentric true longitude of the sun, referred to the mean equinox and ecliptic of date

γ = elevation angle of heliocentric velocity vector of earth-moon barycenter, measured from the transverse direction (normal to radius)

From Broucke (1974),

$$\dot{s}_S = \dot{s}_c\left(1 - \frac{1}{4}e^2 + e\cos M + \frac{3}{4}e^2\cos 2M\right) \tag{76}$$

where all terms to order e^2 have been retained, and M is the mean anomaly of the heliocentric orbit of the earth-moon barycenter. The angle $\ell - \gamma$ can be expressed as

$$\ell - \Upsilon = L + (f-M) - \gamma \tag{77}$$

where f is the true anomaly of the heliocentric orbit of the earth-moon barycenter. From Herrick (1971),

$$\Upsilon = \tan^{-1}\left(\frac{e \sin E}{\sqrt{1-e^2}}\right) \approx e \sin E \tag{78}$$

where the approximate form is correct to order e^2. From Smart (1960),

$$\sin E = \sin M + \frac{1}{2} e \sin 2M + \ldots \tag{79}$$

and

$$f - M = 2e \sin M + \frac{5}{4} e^2 \sin 2M + \ldots \tag{80}$$

Substituting (78) - (80) into (77) gives

$$\ell - \Upsilon = L + e \sin M + \frac{3}{4} e^2 \sin 2M \tag{81}$$

which includes all terms to order e^2. Substituting (74) and (75) into (73), using trigonometric identities, and then substituting (69), (81), and (76) gives

$$\begin{aligned}(t - \tau)_5 = {} & \frac{\dot{s}_c (1+\cos\epsilon) u}{2c^2} \left(1 - \frac{1}{4} e^2 + e \cos M + \frac{3}{4} e^2 \cos 2M\right) \\ & \times \sin\left(UT1 + \lambda - e \sin M - \frac{3}{4} e^2 \sin 2M\right) \\ & - \frac{\dot{s}_c (1-\cos\epsilon) u}{2c^2} (1 + e \cos M) \sin (UT1 + \lambda + 2L + e \sin M) \\ & - \frac{\dot{s}_c (\sin\epsilon) v}{c^2} \cos L \end{aligned} \tag{82}$$

where all terms to order e^2, e^1, and e^0 have been retained in terms one, two, and three, respectively. Expanding and retaining terms to the same order of e gives

$$(t - \tau)_5 = \frac{\dot{s}_c(1+\cos\epsilon)u}{2c^2}\left[(1-\tfrac{1}{2}e^2)\sin(UT1+\lambda)\right.$$

$$+ e\sin(UT1+\lambda-M) - \tfrac{1}{8}e^2\sin(UT1+\lambda+2M)$$

$$\left.+ \tfrac{9}{8}e^2\sin(UT1+\lambda-2M)\right]$$

$$- \frac{\dot{s}_c(1-\cos\epsilon)u}{2c^2}\left[\sin(UT1+\lambda+2L)\right.$$

$$\left.+ e\sin(UT1+\lambda+2L+M)\right]$$

$$- \frac{\dot{s}_c(\sin\epsilon)v}{c^2}\cos L \qquad (83)$$

Substituting numerical values from Appendix A gives

$$(t - \tau)_5 = 3.17679 \times 10^{-10}\, u\sin(UT1+\lambda)$$

$$+ 5.312 \times 10^{-12}\, u\sin(UT1+\lambda-M)$$

$$- 1.1 \times 10^{-14}\, u\sin(UT1+\lambda+2M)$$

$$+ 1.00 \times 10^{-13}\, u\sin(UT1+\lambda-2M)$$

$$- 1.3677 \times 10^{-11}\, u\sin(UT1+\lambda+2L)$$

$$- 2.29 \times 10^{-13}\, u\sin(UT1+\lambda+2L+M)$$

$$- 1.3184 \times 10^{-10}\, v\cos L \qquad (84)$$

The term of (83) and (84) with argument (UT1+λ) was first obtained by Anderson (1968); the expression for the coefficient of his term differs slightly from that given in (83). The term with argument UT1 + λ + 2M is not included in the final expression for t-τ; the effects of deleting this term are included in the error summary in Section 8.

6.5 LONG-PERIOD TERMS DUE TO JUPITER AND SATURN

The sixth term of Equation (44) is

$$(t - \tau)_6 = \frac{1}{c^2} \int \left(\frac{\mu_J}{r_J}\right)_p dt \tag{85}$$

where μ_J is the gravitational constant of Jupiter and r_J is the heliocentric radial coordinate of Jupiter. The constant of integration is zero. The seventh term of (44) is (85), with each subscript J replaced by the subscript SA, which refers to Saturn.

The inverse of r_J is given by (48), with a subscript J added to each variable. Substituting $1/r_J$ into (85) and discarding the constant term gives

$$(t - \tau)_6 = \frac{1}{c^2} \int \frac{\mu_J e_J}{r_J} \cos E_J \, dt \tag{86}$$

Multiplying the integrand by dE_J/dt and dividing it by $\dot{E}_J$ given by

$$\dot{E}_J = \frac{1}{r_J} \sqrt{\frac{\mu_S + \mu_J}{a_J}} \tag{87}$$

gives

$$(t - \tau)_6 = \frac{\mu_J e_J}{c^2 \dot{s}_J} \sin E_J \tag{88}$$

where $\dot{s}_J$ is the circular orbit velocity of Jupiter relative to the sun given by

$$\dot{s}_J = \sqrt{\frac{\mu_S + \mu_J}{a_J}} \tag{89}$$

Substituting numerical values from Appendix A into (89) and (88) gives

$$(t - \tau)_6 = 5.21 \times 10^{-6} \sin E_J \tag{90}$$

The corresponding term for Saturn is

$$(t - \tau)_7 = 2.45 \times 10^{-6} \sin E_{SA} \tag{91}$$

6.6 TERMS WITH PERIODS EQUAL TO SYNODIC PERIODS FOR JUPITER AND SATURN

The eighth term of Equation (44) is

$$(t - \tau)_8 = \frac{1}{c^2} \left(\dot{\underline{r}}_S^C \cdot \underline{r}_B^S \right) = \frac{1}{c^2} \left(\dot{\underline{r}}_C^S \cdot \underline{r}_S^B \right) \tag{92}$$

Terms will be obtained for Jupiter and Saturn due to their contribution to $\dot{\underline{r}}_C^S$. The contribution to $\dot{\underline{r}}_C^S$ due to planet i with gravitational constant μ_i is

$$\dot{\underline{r}}_C^S (i) = \frac{\mu_i}{\mu_S^*} \dot{\underline{r}}_i^S \tag{93}$$

where μ_S^* is the gravitational constant of the sun augmented by the gravitational constants of the planets and moon. It is assumed that the heliocentric orbits of the earth-moon barycenter and planet i are circular and that the latter lies in the ecliptic plane. With these assumptions, the vectors $\dot{\underline{r}}_i^S$ and $\underline{r}_S^B$, with rectangular components referred to the mean equinox and ecliptic of date, are

$$\dot{\underline{r}}_i^S = \begin{bmatrix} -\sin L_i \\ \cos L_i \\ 0 \end{bmatrix} \dot{s}_i \tag{94}$$

$$\underline{r}_S^B = \begin{bmatrix} \cos L \\ \sin L \\ 0 \end{bmatrix} a \tag{95}$$

where $\dot{s}_i$ is the circular orbit velocity of planet i relative to the sun (computed from Equation 89 with subscript J replaced by i), and L_i is the heliocentric mean longitude of planet i, referred to the mean equinox and ecliptic of date. Substituting (93) to (95) into (92) gives the contribution to $(t - \tau)_8$ due to planet i:

$$(t - \tau)_{8i} = \frac{\mu_i \dot{s}_i a}{c^2 \mu_S^*} \sin (L - L_i) \tag{96}$$

Substituting numerical values from Appendix A for i = Jupiter gives

$$(t - \tau)_{8J} = 20.73 \times 10^{-6} \sin (L - L_J) \tag{97}$$

For i = Saturn,

$$(t - \tau)_{8SA} = 4.58 \times 10^{-6} \sin (L - L_{SA}) \tag{98}$$

6.7 DAILY TERMS DUE TO MOTION OF SUN RELATIVE TO SOLAR SYSTEM BARYCENTER

The ninth term of Equation (44) is

$$(t - \tau)_9 = \frac{1}{c^2} \left(\dot{\underline{r}}_S^C \cdot \underline{r}_A^E \right) = -\frac{1}{c^2} \left(\dot{\underline{r}}_C^S \cdot \underline{r}_A^E \right) \tag{99}$$

Terms will be obtained for Jupiter and Saturn due to their contribution to $\dot{\underline{r}}_C^S$. The contribution to $(t - \tau)_9$ due to a planet i is obtained by substituting (93), (94), and (64) into (99) and using trigonometric identities:

$$(t-\tau)_{9i} = -\frac{\mu_i \dot{s}_i}{c^2 \mu_S^*} \left\{ \frac{u}{2} \left[(1+\cos\epsilon) \sin(\theta_M + \lambda - L_i) - (1-\cos\epsilon) \sin(\theta_M + \lambda + L_i) \right] + v \sin\epsilon \cos L_i \right\} \quad (100)$$

The second and third terms of this expression are neglected. Substituting (69) into the first term of (100) gives

$$(t-\tau)_{9i} \doteq \frac{\mu_i \dot{s}_i (1+\cos\epsilon)}{2c^2 \mu_S^*} u \sin(UT1 + \lambda + L - L_i) \quad (101)$$

Substituting numerical values from Appendix A for i = Jupiter gives

$$(t-\tau)_{9J} = 1.33 \times 10^{-13} u \sin(UT1 + \lambda + L - L_J) \quad (102)$$

For i = Saturn,

$$(t-\tau)_{9SA} = 2.9 \times 10^{-14} u \sin(UT1 + \lambda + L - L_{SA}) \quad (103)$$

7. Final Expression and Arguments

The periodic terms of $t - \tau$, with analytical expressions for the coefficients, are given by Equations (51), (58), (71), (83), (88) (which applies for Jupiter) and the corresponding term for Saturn, and (96) and (101), each evaluated for Jupiter and for Saturn. Substituting these terms, except the term of (83) with argument UT1 + λ + 2M, into (44) gives

$$
\begin{aligned}
t - \tau = \Delta T_A &+ \frac{2}{c^2}\sqrt{\mu_S a}\, e \sin E + \frac{\dot{s}_c\, a_M}{c^2(1+\mu)} \sin D \\
&+ \frac{\dot{s}_c\,(1+\cos\epsilon)u}{2c^2}\left[\left(1 - \frac{1}{2}e^2\right)\sin(UT1+\lambda)\right. \\
&\left. + e\sin(UT1+\lambda - M) + \frac{9}{8}e^2 \sin(UT1+\lambda - 2M)\right] \\
&- \frac{\dot{s}_c\,(1-\cos\epsilon)u}{2c^2}\left[\sin(UT1+\lambda+2L)\right. \\
&\left. + e\sin(UT1+\lambda+2L+M)\right] \\
&+ \frac{\dot{s}_M\,(1+\cos\epsilon)}{2c^2(1+\mu)}\, u \sin(UT1+\lambda - D) - \frac{\dot{s}_c\,(\sin\epsilon)v}{c^2}\cos L \qquad (104) \\
&+ \frac{\mu_J\, e_J}{c^2\,\dot{s}_J}\sin E_J + \frac{\mu_{SA}\, e_{SA}}{c^2\,\dot{s}_{SA}}\sin E_{SA} \\
&+ \frac{\mu_J\,\dot{s}_J\, a}{c^2\,\mu_S^*}\sin(L-L_J) + \frac{\mu_{SA}\,\dot{s}_{SA}\, a}{c^2\,\mu_S^*}\sin(L-L_{SA}) \\
&+ \frac{\mu_J\,\dot{s}_J\,(1+\cos\epsilon)}{2c^2\,\mu_S^*}\, u \sin(UT1+\lambda + L-L_J) \\
&+ \frac{\mu_{SA}\,\dot{s}_{SA}\,(1+\cos\epsilon)}{2c^2\,\mu_S^*}\, u \sin(UT1+\lambda + L-L_{SA})
\end{aligned}
$$

Substituting numerical values for the coefficients of the periodic terms, obtained from Equations (52), (60), (72), (84), (90), (91), (97), (98), (102), and (103), gives

$$
\begin{aligned}
t - \tau = {} & \Delta T_A + 1.658 \times 10^{-3} \sin E + 1.548 \times 10^{-6} \sin D \\
& + 3.17679 \times 10^{-10}\, u \sin(UT1+\lambda) + 5.312 \times 10^{-12}\, u \sin(UT1+\lambda-M) \\
& + 1.00 \times 10^{-13}\, u \sin(UT1+\lambda-2M) - 1.3677 \times 10^{-11}\, u \sin(UT1+\lambda+2L) \\
& - 2.29 \times 10^{-13}\, u \sin(UT1+\lambda+2L+M) \\
& + 1.33 \times 10^{-13}\, u \sin(UT1+\lambda-D) - 1.3184 \times 10^{-10} v \cos L \qquad (105) \\
& + 5.21 \times 10^{-6} \sin E_J + 2.45 \times 10^{-6} \sin E_{SA} \\
& + 20.73 \times 10^{-6} \sin(L-L_J) + 4.58 \times 10^{-6} \sin(L-L_{SA}) \\
& + 1.33 \times 10^{-13}\, u \sin(UT1+\lambda+L-L_J) \\
& + 2.9 \times 10^{-14}\, u \sin(UT1+\lambda+L-L_{SA})
\end{aligned}
$$

where the coordinates u and v of the atomic clock which reads International Atomic Time τ are in kilometers and $t - \tau$ is in seconds. The coefficients of the daily terms of (105) are proportional to u. For a clock at the equator where u = 6378 km, the magnitude of the coefficients of the daily terms varies from a maximum of 2.026×10^{-6} s to a minimum of 1.85×10^{-10} s. The annual term with argument L has a coefficient proportional to v; its maximum value is 0.838×10^{-6} s.

The last four terms of (105) are due to the offset of the solar system barycenter from the sun. If these terms are deleted in (105), coordinate time t in the solar system barycentric frame of reference is replaced in this equation by coordinate time t′ in the heliocentric frame of reference, and the accuracy of the equation is reduced slightly. The previously derived expression for $t' - \tau$ is Equation (65) of Moyer (1971). In this previous expression, the long-period potential terms due to Jupiter and Saturn (the terms of 105 with arguments E_J and E_{SA}) were omitted along with the term with argument L, which has a significant effect on computed three-way range and doppler observables. The previous expression for $t' - \tau$ consisted of the first nine terms of (105) with minor differences in the coefficients of the periodic terms.

From Smart (1960), the relation between the eccentric and mean anomalies is:

$$E = M + e \sin M + \frac{1}{2} e^2 \sin 2M + \ldots \tag{106}$$

The three eccentric anomalies in (105) can be computed to sufficient accuracy from the following approximations to (106):

$$E = M + e \sin M \tag{107}$$

$$E_J = M_J \tag{108}$$

$$E_{SA} = M_{SA} \tag{109}$$

The errors due to the approximations in (107) - (109) are included in the error summary in Section 8. Expressions are given below for the arguments L, M, D, $(L-L_J)$, $(L-L_{SA})$, M_J, and M_{SA}, which are functions of coordinate time t. The angle UT1 in radians is computed from universal time UT1 in seconds using Equation (70).

The Explanatory Supplement to the Ephemeris (1961) gives polynomials for the mean orbital elements L, M, and D (pp. 98 and 107). The argument for these expressions (quadratics or cubics) is Julian centuries of 36 525 days of ephemeris time ET (coordinate time t) from January 0, 1900, 12^h ET. Linear expressions have been obtained which are tangent to the polynomials at January 1, 1975, 6^h ET (0.75 Julian centuries past the above epoch). In units of radians, the linear expressions are

$$L = 4.888\,339 + 1.991\,063\,83 \times 10^{-7}\, t \tag{110}$$

$$M = 6.248\,291 + 1.990\,968\,71 \times 10^{-7}\, t \tag{111}$$

$$D = 2.518\,411 + 2.462\,600\,818 \times 10^{-6}\, t \tag{112}$$

where t is seconds of ephemeris time past January 1, 1950, 0^h ET.

Seidelmann et al (1974) give quadratic expressions for the heliocentric mean longitudes of the earth, Jupiter, and Saturn (L_E, L_J, and L_{SA} respectively) and for the longitudes of perihelion for Jupiter and Saturn

($\tilde{\omega}_J$ and $\tilde{\omega}_{SA}$, respectively), referred to the mean equinox and ecliptic of 1950.0. Linear expressions were obtained which are tangent to the quadratic expressions for these mean orbital elements at January 1, 1975, 6^h ET. The argument was changed from tropical centuries of 36 524.219 88 ephemeris days past 1950.0 to ephemeris seconds past January 1, 1950, 0^h ET. The expressions for the arguments $L-L_J$, $L-L_{SA}$, M_J, and M_{SA} were obtained from the following combinations of the linear expressions for L_E, L_J, L_{SA}, $\tilde{\omega}_J$, and $\tilde{\omega}_{SA}$:

$$L - L_J = L_E - L_J \pm \pi \tag{113}$$

$$L - L_{SA} = L_E - L_{SA} \pm \pi \tag{114}$$

$$M_J = L_J - \tilde{\omega}_J \tag{115}$$

$$M_{SA} = L_{SA} - \tilde{\omega}_{SA} \tag{116}$$

The resulting expressions, in radians, are

$$L - L_J = 5.652\,593 + 1.823\,136\,37 \times 10^{-7}\, t \tag{117}$$

$$L - L_{SA} = 2.125\,474 + 1.923\,399\,23 \times 10^{-7}\, t \tag{118}$$

$$M_J = 5.286\,877 + 1.678\,506\,3 \times 10^{-8}\, t \tag{119}$$

$$M_{SA} = 1.165\,341 + 0.675\,855\,8 \times 10^{-8}\, t \tag{120}$$

The mean longitudes on the left-hand side of (113) and (114) are referred to the mean equinox and ecliptic of date. The mean longitudes on the right-hand side of these equations are those of Seidelmann et al which are referred to the mean equinox and ecliptic of 1950.0. The equinox difference does not affect the computed longitude differences $L-L_J$ and $L-L_{SA}$. The difference in the reference planes does produce errors in Equations (117) and (118); however, because of the small inclinations of the orbit planes of the earth, Jupiter, and Saturn to the ecliptic of 1950.0, the errors are negligible.

The errors in computed range and doppler observables due to the errors in the linear approximations (110) - (112) and (117) - (120) are completely negligible for the time period 1950 - 2000.

8. Effect of Retained and Neglected Terms of $t - \tau$

The primary purpose of this section is to give estimates for errors in computed range and doppler observables due to errors in Equation (105), the final expression for $t - \tau$. The secondary purpose is to give the effect of the retained terms of $t - \tau$ on these observables.

Periodic terms of $t - \tau$ have a direct and an indirect effect on computed range observables. These two effects can be understood by considering the general procedure for computing a range observable. The light time solution produces solar system barycentric position vectors of the transmitting station on earth at the transmission time t_1, the spacecraft at the reflection time t_2, and the receiving station on earth at the reception time t_3. The reception time is known in atomic time τ; addition of $t - \tau$ from (105) converts it to coordinate time t (in the solar system barycentric space-time frame of reference). The light time solution yields the epochs t_2 and t_1 in coordinate time. The epoch t_3 in coordinate time is directly affected by the magnitude of the periodic terms of $t - \tau$ evaluated at t_3; the epochs t_2 and t_1 in coordinate time are affected by the same amount to an accuracy of 3 or 4 digits (depending upon the range rate of the spacecraft).

The values of t_3 and t_1 in the UT1 time scale are also obtained. The former is not affected by the terms of $t - \tau$; the latter is affected but not significantly. The indirect effect of a periodic term of $t - \tau$ on a computed range observable is due to the effect of the term on the epochs t_3, t_2, and t_1 and hence on the position vectors computed at these epochs. The direct effect is due to the values of the term at t_3 and t_1 which appear explicitly in the equation for computing range observables (Appendix B, Equation B4). Doppler observables are computed from the difference of two computed range observables divided by the difference in the two reception times. The direct and indirect effects of a term of $t - \tau$ on these two range observables produce a corresponding direct and indirect effect on the doppler observable.

Appendix B gives the formulation for computing two-way or three-way range and doppler observables. An analysis of these equations produced the approximate equations given below for the maximum direct effect of a term of

$t - \tau$ on range and doppler observables. The direct effect of a term of $t - \tau$ depends upon the form of the term. All of the retained terms and almost all of the neglected terms have one of the following forms. The terms can be divided into daily terms (period $P \approx 1$ day) and long-period terms (P ranging from roughly half a month to several years). The daily terms contain the longitude of the atomic clock in the argument; the long-period terms do not. The coefficients of the daily terms are proportional to the spin-axis distance u of the atomic clock. The coefficients of some of the long-period terms are proportional to the distance v of the atomic clock north of the earth's equatorial plane. The only neglected terms of $t - \tau$ which do not fit into these categories are some of the terms due to nutation, polar motion, and solid-earth tides.

Let $\delta\rho$ and $\delta\dot{\rho}$ refer to the maximum possible value of the direct effect of a term of $t - \tau$ on range and doppler observables, expressed as the tracking station to spacecraft range and range-rate, respectively. The effects of a daily term of $t - \tau$ on two-way or three-way data are:

$$\delta\rho = Mc \tag{121}$$

$$\delta\dot{\rho} = M\left(\frac{2\pi}{P}\right)c \tag{122}$$

where M is the magnitude of the coefficient of the term and c is the speed of light. For three-way data, the spacecraft range ρ at which these maximum effects occur depends upon the separation in longitude of the transmitting and receiving stations; the separation can be adjusted so that the maximum occurs at any range. For two-way data, the maximum effects at a given range are given by Equations (121) and (122) multiplied by $\sin\left(\frac{2\pi}{P} \cdot \frac{\rho}{c}\right)$. Hence, the maximum two-way effects increase with range, reaching a peak at a range of 43.2 astronomical units.

The following equations give the effects of a long-period term of $t - \tau$ on two-way data. They also give the effect of a long-period term on three-way data if the coefficient of the term is not proportional to v.

$$\delta\rho = M\left(\frac{2\pi}{P}\right)\rho \tag{123}$$

$$\delta\dot{\rho} = M\left(\frac{2\pi}{P}\right)^2\rho \tag{124}$$

If the coefficient of a long-period term is proportional to v, the effects of the term on three-way data are given by (121) and (122), where M is the maximum possible magnitude of the coefficient of the term. These maximum effects are independent of the range to the spacecraft.

The indirect effects of a term of $t - \tau$ on two-way or three-way range and doppler observables will not exceed the following approximate values:

$$\delta\rho_I = \dot{\rho}M \tag{125}$$

$$\delta\dot{\rho}_I = aM \tag{126}$$

where $\dot{\rho}$ is the tracking station to spacecraft range rate, and a is the inertial acceleration of the spacecraft. The highest spacecraft acceleration likely to be encountered is 25 m/s^2, which occurs near the surface of Jupiter. Higher accelerations occur closer than 3.3 solar radii from the center of the sun, but it is unlikely that a spacecraft would enter this region.

An extensive error analysis has been performed to determine estimates for the maximum possible errors in computed range and doppler observables due to errors in Equation (104) or (105) (which contain analytical and numerical coefficients, respectively) for $t - \tau$. The results of this analysis are presented in Table II. The second column lists the errors committed in the derivation of Equation (105). If an error is associated with a term of (104) or (105), that term is identified in column 1. The magnitudes of the coefficients of the error terms are given in column 3. The maximum values of the direct effects of the error terms on computed range and doppler observables are given in columns 4 and 5, respectively. When one figure is given for the effect of an error, it applies for two-way or three-way data. When two figures are given, the figure in parentheses is the error for three-way data and the other figure applies for two-way data. For daily terms, the maximum

TABLE II

Error summary

Term	Source of error	Magnitude of error term, μs	Direct effect on range, m	Direct effect on range rate, μm/s
Argument E	Assuming e constant from 1950 to 2000	1.0	1.5	0.3
Argument E	Neglected e^2 term in Equation (107)	0.3	0.7	0.3
Argument E	Ignoring periodic variations in heliocentric orbital elements of earth-moon barycenter	12.0	15.0	4.0
Argument D	Ignoring e_M and e	0.2	2.3	10.0
Coefficients with factor $\dot{s}_c (1+\cos \epsilon)$	Ignoring e^3 terms	0.00	0.01	0.8
Coefficients with factor $\dot{s}_c (1-\cos \epsilon)$	Ignoring e^2 terms	0.00	0.02	1.2
All with UT1 + λ in argument	Approximation in Equation (69)	0.00	0.08	5.5

TABLE II

Error summary (contd)

Term	Source of error	Magnitude of error term, μs	Direct effect on range, m	Direct effect on range rate, μm/s
Argument UT1 + λ + 2M	Deleting this term from Equation (84)	0.00	0.02	1.5
All with factor $\dot{s}_{cu}$ or $\dot{s}_{cv}$ in coefficient	Ignoring periodic variations in heliocentric orbital elements of earth-moon barycenter	0.00	0.4 (0.6)	30.0
Argument UT1 + λ - D	Ignoring e_M and second term of Equation (65)	0.00	0.03	1.9
Argument UT1 + λ - D	Ignoring third term in Equation (65)	0.00	0.01 (0.11)	0.02 (0.3)
Argument	Ignoring e	0.02	0.04 (4.2)	0.02 (1.7)
Arguments E_J and E_{SA}	Neglected e terms in Equations (108) and (109)	0.2	0.04	0.00
	Neglected terms in Equations (37) and (39)	1.3	1.1	0.3
	Neglected gravitational potential at earth due to moon in Equation (27)	0.00	0.1	0.2

TABLE II

Error summary (contd)

Term	Source of error	Magnitude of error term, μs	Direct effect on range, m	Direct effect on range rate, μm/s
Arguments $L - L_J$ and $L - L_{SA}$	Ignoring e, e_J and e_{SA}	2.1	2.3	0.5
Argument $L - L_i$	Ignoring terms for Mercury, Venus, Mars, Uranus, Neptune, and Pluto, including effects due to eccentricities	1.3	1.8	0.4
	Omitting the term $\frac{1}{c^2}\left(\dot{\underline{r}}_S^C \cdot \underline{r}_E^B\right)$ in Equation (40)	0.00	0.02	0.05
Argument $UT1 + \lambda + L - L_t$	Ignoring second and third terms of Equation (100) for all planets, the first term for all planets except Jupiter and Saturn, and ignoring eccentricities of planetary orbits for retained and neglected terms	0.00	0.06 (0.2)	3.6

TABLE II

Error summary (contd)

Term	Source of error	Magnitude of error term, μs	Direct effect on range, m	Direct effect on range rate, μm/s
Arguments D, UT1 + λ - D, L - L_J, L - L_{SA}, UT1 + λ + L - L_J, and UT1 + λ + L - L_{SA}	Ignoring inclination of orbits of moon, Jupiter, and Saturn to ecliptic plane	0.02	0.2	1.5
Coefficients with factor 1 + cos ϵ, 1 - cos ϵ or sin ϵ	Assuming ϵ and e are constant from 1950 to 2000	0.00	0.02 (0.06)	1.4
	Ignoring nutation, polar motion, and solid-earth tides	0.00	0.1	3.0
All	Roundoff of coefficients	0.2	0.3	0.7
	Miscellaneous (see text)	4.3	3.8	7.8
	Totals	23.0	30.0 (34.6)	75.0 (77.0)

values of the errors are listed. This is also true for three-way errors due to long-period terms which have coefficients proportional to v. The errors in two-way data due to long-period terms and in three-way data due to long-period terms whose coefficients are not proportional to v were computed for a spacecraft range of 50 astronomical units.

A large number of the errors listed in Table II are due to terms neglected in (104) and (105). Another large group of errors is due to ignoring eccentricities of elliptical orbits or ignoring terms above a certain power of the eccentricity. Some of the largest errors are due to ignoring periodic variations in the heliocentric orbital elements of the earth-moon barycenter. These variations, in the form of longitudinal and radial perturbations, were obtained from Newcomb (1898). The effects of periodic variations in the geocentric orbital elements of the moon and in the heliocentric orbital elements of Jupiter and Saturn have not been analyzed. The miscellaneous error listed at the end of Table II is an allowance for these errors and other minor unanalyzed errors.

Estimates have been obtained for the maximum possible direct errors in computed range and doppler observables as a function of the range to the spacecraft. For two-way data, these upper limits are

$$\delta\rho_2 = (0.62 \text{ m})\,\text{AU} \tag{127}$$

$$\delta\dot{\rho}_2 = (2.4 \times 10^{-6} \text{ m/s})\,\text{AU} \tag{128}$$

where AU is the range to the spacecraft in astronomical units. These figures are based upon the constant derivatives of long-period errors with respect to range and the derivatives of the daily errors with respect to range evaluated at a range of zero. Since daily errors are proportional to the sine of a multiple of the spacecraft range, these upper limits are high at large spacecraft ranges. For three-way data, the upper limits are

$$\delta\rho_3 = (0.59 \text{ m})\,\text{AU} + 5.5 \text{ m} \tag{129}$$

$$\delta\dot{\rho}_3 = (0.49 \times 10^{-6}\ \text{m/s})\ \text{AU} + 53 \times 10^{-6}\ \text{m/s} \qquad (130)$$

From Table II, the estimate of the maximum error in the magnitude of $t - \tau$ is 23μs. Using this figure and a spacecraft range rate of 30 km/s, Equation (125) gives a maximum indirect error in computed range observables of 0.69 m. For a spacecraft inertial acceleration of 25 m/s^2, (126) gives an upper limit to the indirect error in computed doppler observables of 575×10^{-6} m/s. This error is almost eight times as large as the maximum value of the direct error.

To an accuracy of 0.01 μs, the error in the magnitude of $t - \tau$ is due entirely to long-period terms. This error could be represented by a curve which is a function of a small number of parameters. When fitting to tracking data obtained from a spacecraft which is near a planet, the values of these parameters could be estimated, thereby eliminating most of the indirect errors in computed observables due to neglected terms of $t - \tau$.

Table III gives the same information for the retained terms of $t-\tau$ as given in Table II for the neglected terms. It is seen that the maximum value of the direct error in a computed range observable due to neglected terms of $t - \tau$ is about 1% of the maximum value of the direct effect of the retained terms of $t - \tau$. The corresponding figure for a doppler observable is 0.16%. The maximum error in the magnitude of $t - \tau$ is about 1.36% of the maximum value of the sum of the periodic terms of $t - \tau$. Hence, the maximum value of the indirect error in a computed observable is about 1.36% of the maximum value of the indirect effect of the retained terms.

Approximate maximum values for the direct effects of the retained terms of $t - \tau$, as a function of the spacecraft range AU (in astronomical units), are

$$\delta\rho_2 = (74.3\text{m})\ \text{AU} \qquad (131)$$

$$\delta\dot{\rho}_2 = (1.72 \times 10^{-3}\text{m/s})\ \text{AU} \qquad (132)$$

TABLE III

Maximum effect of retained terms

Argument of term	Magnitude of term μs	Direct effect on range, m	Direct effect on range rate, μm/s
E	1658.	2476.	493.
D	1.548	28.6	70.4
UT1 + λ	2.026	607.4	44173.
UT1 + λ - M	0.0339	10.2	739.
UT1 + λ - 2M	0.000 64	0.19	13.9
UT1 + λ + 2L	0.0872	26.2	1902.
UT1 + λ + 2L + M	0.0015	0.44	31.8
UT1 + λ - D	0.000 85	0.26	18.5
L	0.838	1.25 (251.)	0.25 (50.0)
E_J	5.21	0.66	0.01
E_{SA}	2.45	0.13	0.00
$L - L_J$	20.73	28.4	5.17
$L - L_{SA}$	4.58	6.61	1.27
UT1 + λ + L - L_J	0.000 85	0.26	18.5
UT1 + λ + L - L_{SA}	0.000 19	0.06	4.03
Totals	1696.	3187. (3437.)	47 471. (47 521.)

$$\delta \rho_3 = (51\text{m})\ \text{AU} + 896\ \text{m} \tag{133}$$

$$\delta \dot{\rho}_3 = (0.012 \times 10^{-3}\text{m/s})\ \text{AU} + 47.0 \times 10^{-3}\ \text{m/s} \tag{134}$$

For a spacecraft range rate of 30 km/s and inertial acceleration of 25 m/s^2, the maximum values of the indirect effects of the retained terms of $t - \tau$ on computed range and doppler observables are 51 m and 42×10^{-3} m/s, respectively.

9. Alternate Expression for $t - \tau$

This section gives an alternate equation for $t - \tau$ which is a function of position and velocity vectors of the atomic clock and the major bodies of the solar system. The equation is obtained from Equation (44) by replacing the second, sixth, and seventh terms, which are expressed as integrals, with functions of position and velocity vectors derived in this section.

The second term of Equation (44) is given by Equation (51). For the heliocentric elliptical orbit of the earth-moon barycenter,

$$\dot{r}r = \dot{\underline{r}}^S_B \cdot \underline{r}^S_B = \sqrt{(\mu_S+\mu_E+\mu_M)\, a}\; e \sin E$$

$$\approx \sqrt{\mu_S a}\; e \sin E \qquad (135)$$

Substituting the approximate form of (135) into (51) gives the desired form for the second term of Equation (44):

$$(t - \tau)_2 = \frac{2}{c^2}\left(\dot{\underline{r}}^S_B \cdot \underline{r}^S_B\right) \qquad (136)$$

The sixth term of Equation (44) is given by Equation (88). The equation analogous to (135) which applies for the heliocentric orbit of Jupiter is:

$$\dot{\underline{r}}^S_J \cdot \underline{r}^S_J = \sqrt{(\mu_S+\mu_J)\, a_J}\; e_J \sin E_J \qquad (137)$$

Substituting $e_J \sin E_J$ from (137) into (88) and using (89) gives

$$(t - \tau)_6 = \frac{\mu_J}{c^2(\mu_S+\mu_J)}\left(\dot{\underline{r}}^S_J \cdot \underline{r}^S_J\right) \qquad (138)$$

The corresponding term for Saturn (SA) is the seventh term of Equation (44):

$$(t-\tau)_7 = \frac{\mu_{SA}}{c^2(\mu_S+\mu_{SA})}(\dot{\underline{r}}^S_{SA}\cdot\underline{r}^S_{SA}) \tag{139}$$

Replacing terms two, six, and seven of Equation (44) with Equations (136), (138), and (139), combining the three terms which contain $\underline{r}^E_A$, and reordering the terms gives the alternate expression for $t-\tau$:

$$t-\tau = \Delta T_A + \frac{2}{c^2}(\dot{\underline{r}}^S_B\cdot\underline{r}^S_B) + \frac{1}{c^2}(\dot{\underline{r}}^C_E\cdot\underline{r}^E_A) + \frac{1}{c^2}(\dot{\underline{r}}^S_B\cdot\underline{r}^B_E) + \frac{1}{c^2}(\dot{\underline{r}}^C_S\cdot\underline{r}^S_B) + \frac{\mu_J}{c^2(\mu_S+\mu_J)}(\dot{\underline{r}}^S_J\cdot\underline{r}^S_J) + \frac{\mu_{SA}}{c^2(\mu_S+\mu_{SA})}(\dot{\underline{r}}^S_{SA}\cdot\underline{r}^S_{SA}) \tag{140}$$

If the term $\frac{1}{c^2}(\dot{\underline{r}}^C_S\cdot\underline{r}^B_E)\Big|_{t_o}^{t}$, which was omitted in Equation (40), was reinstated, $\dot{\underline{r}}^S_B$ in the fourth term of (140) would be replaced by $\dot{\underline{r}}^C_B$.

Appendix A: Notation and Numerical Values for Parameters

This appendix gives the definitions for the parameters used globally throughout the text. Numerical values are given for those parameters which appear in the final expression for $t - \tau$.

Subscripts and Superscripts

A = location of atomic clock on earth which reads International Atomic Time τ

E = earth

B = earth-moon barycenter

M = moon

S = sun

C = solar system barycenter

J = Jupiter

SA = Saturn

p This subscript indicates that periodic terms of the quantity are to be retained and constant terms are to be discarded

o This subscript indicates the quantity is evaluated at the initial epoch t_o

A bar over a quantity indicates the time average value of the quantity.

Position, Velocity, and Acceleration

$\underline{r}_i^j$, $\dot{\underline{r}}_i^j$, $\ddot{\underline{r}}_i^j$ = position, velocity, and acceleration vectors of point i relative to point j. The dots denote differentiation with respect to coordinate time t

$\dot{s}_i^j$ = velocity of point i relative to point j

$\dot{s}$ = velocity of fixed atomic clock on earth relative to solar system barycenter

Time

τ = proper time obtained from an atomic clock on earth

τ^* = International Atomic Time (TAI) obtained from an atomic clock on earth. Starting in Section 4, τ^* is denoted by τ

t = coordinate time in solar system barycentric space-time frame of reference

n = conversion factor from cycles obtained from an atomic clock to seconds of atomic time τ

n^* = conversion factor from cycles obtained from a cesium atomic clock to seconds of International Atomic Time

ΔT_A = constant term in expression for $t - \tau^*$

UT1 = observed universal time, corrected for polar motion. Equation (70) converts UT1 from seconds to radians.

Physical Constants

The physical constants were obtained from Standish et al (1976) or were computed from quantities obtained from this reference.

c = speed of light = 299 792.458 km/s

A_E = the number of kilometers per astronomical unit (AU)

= 149 597 871.410 56 km/AU

μ_i = gravitational constant of body i, km^3/s^2

= Gm_i, where G is the universal constant of gravitation and m_i is the mass of body i.

The gravitational constant of the sun is computed from A_E using Equation (104) of Moyer (1971):

$$\mu_S = \frac{k^2 A_E^3}{(86400)^2} = 1.32712442 \times 10^{11} \text{ km}^3/\text{s}^2$$

where

k = the Gaussian gravitational constant

= 0.017 202 098 95 $AU^{3/2}$/day (exactly)

Using μ_S and mass ratios obtained from Standish et al (1976) gives

μ_E = 398 600.5 km^3/s^2

μ_M = 4 902.79 km^3/s^2

μ_J = $1.267\ 120 \times 10^8$ km^3/s^2

μ_{SA} = $3.793\ 410 \times 10^7$ km^3/s^2

μ = μ_E/μ_M = 81.300 7

μ_S^* = gravitational constant of the sun augmented by the gravitational constants of the planets and the moon

= $1.328\ 906 \times 10^{11}$ km^3/s^2

Gravitational Potential

U = Newtonian gravitational potential, using the positive sign convention (i.e., $U = -\phi$)

U_i = U at location i

$U(j)$ = U due to body j

$U_i(j)$ = U at location i due to body j

$(\nabla U)_i$ = gradient of U at location i

Station Location

The earth-fixed coordinates u, v, and λ of the atomic clock which reads International Atomic Time are:

u = distance from earth's spin axis, km

v = distance north of earth's equatorial plane, km

λ = east longitude

θ_M = mean sidereal time = Greenwich hour angle of mean equinox of date

Heliocentric Orbit of Earth-Moon Barycenter

Numerical values were obtained from page 98 of the Explanatory Supplement to the Ephemeris (1961).

a = semi-major axis = 1.000 000 23 AU

= 149 597 906 km (obtained using A_E given above)

e = eccentricity = 0.0167 2. From 1950 to 2000, the last digit changes from 3 to 1.

r = radial coordinate

$\dot{s}_S$ = velocity

$\dot{s}_c$ = circular orbit velocity

$$= \sqrt{\frac{\mu_S+\mu_E+\mu_M}{a}} = 29.784734 \text{ km/s}$$

M = mean anomaly

E = eccentric anomaly

f = true anomaly

L, ℓ = mean and true longitudes, respectively, of the sun, measured at the earth-moon barycenter. These angles are referred to the mean equinox and ecliptic of date.

γ = elevation angle of velocity vector, measured from the transverse direction (normal to radius)

ϵ = mean obliquity of the ecliptic

= inclination of ecliptic plane to mean earth equator of date

$\cos \epsilon$ = 0.917 46 (from 1950 to 2000, the last digit changes from 4 to 8)

$\sin \epsilon$ = 0.397 83 (from 1950 to 2000, the last two digits change from 88 to 78)

Heliocentric Orbit of Planet i

a_i = semi-major axis. From Seidelmann et al (1974),

$$a_J = 5.202\ 833\ 481 \text{ AU} = 7.783\ 328 \times 10^8 \text{ km}$$

$$a_{SA} = 9.538\ 762\ 055 \text{ AU} = 1.426\ 978 \times 10^9 \text{ km}$$

e_i = eccentricity. From Seidelmann et al (1974), evaluated at 1975.0,

$$e_J = 0.048\ 284$$

$$e_{SA} = 0.056\ 038$$

These figures are constant to three significant digits from 1950 to 2000.

r_i = radial coordinate

$\dot{s}_i$ = circular orbit velocity

$$= \sqrt{\frac{\mu_S + \mu_i}{a_i}}$$

$$\dot{s}_J = 13.064\ 13 \text{ km/s}$$

$$\dot{s}_{SA} = 9.645\ 16 \text{ km/s}$$

M_i = mean anomaly

E_i = eccentric anomaly

L_i, ℓ_i = heliocentric mean and true longitudes, respectively, of planet i, referred to the mean equinox and ecliptic of date.

Geocentric Orbit of Moon

a_M = semi-major axis = 384 399.1 km

This value is obtained from the above values of μ_E and μ_M and the observed mean motion of the moon using Equation (106) of Moyer (1971), which is a modified version of Kepler's third law.

e_M = eccentricity = 0.054 9 (used in error analysis only)

$\dot{s}_M$ = circular orbit velocity

$$= \sqrt{\frac{\mu_E + \mu_M}{a_M}} = 1.024\ 548 \text{ km/s}$$

$\mathbb{L}$ = geocentric mean longitude of the moon, referred to the mean equinox and ecliptic of date

D = $\mathbb{C}$ - L = mean elongation of the moon from the sun

Miscellaneous

$(t - \tau)_i$ = term i of Equation (44)

Appendix B: Computation of Range and Doppler Observables

This appendix gives a brief description of simplified versions of the formulas used to obtain the computed values of range and doppler observables obtained by the Deep Space Network of the Jet Propulsion Laboratory. The purpose is to show how the time transformation $t - \tau$ is used in the computation of these observables. The simplifications made to the formulation do not change the effects of $t - \tau$. For further details of the formulation, see Moyer (1971).

The computation of two-way or three-way range observables is described first. A signal is transmitted from a tracking station on earth at time t_1, received and retransmitted at the spacecraft at time t_2, and received at the same tracking station on earth (two-way data) or at a different station (three-way data) at time t_3. The definition of the range observable R is

$$R = t_3\,(TAI) - t_1\,(TAI) \tag{B1}$$

where t_3 (TAI) and t_1 (TAI) are the reception and transmission times, respectively, in International Atomic Time TAI. The "time tag" associated with each range observable is the known reception time t_3 (TAI).

The first step in computing a range observable is to obtain the light time solution. The epoch t_3 (TAI) is converted to coordinate time in the solar system barycentric space-time frame of reference (denoted here as ephemeris time ET) using

$$t_3(ET) = t_3\,(TAI) + (ET-TAI)_{t_3} \tag{B2}$$

where $(ET-TAI)_{t_3}$ is $(t-\tau)$ given by Equation (105), evaluated at t_3. The light time solution gives the epochs t_2 (ET) and t_1 (ET) and the position vectors $\underline{r}_3^C\,(t_3)$, $\underline{r}_2^C\,(t_2)$, and $\underline{r}_1^C\,(t_1)$; these are the solar system barycentric position vectors of the receiving station at t_3, the spacecraft at t_2, and the transmitting station at t_1, respectively. The epochs t_3 and t_1 are also obtained in the UT1 time scale; they are used along with the ET values of these epochs to compute the geocentric position vectors of the receiving and transmitting

stations at t_3 and t_1, respectively. These vectors are used in forming the above solar system barycentric vectors. The following vector magnitudes are computed:

$$r_i = \left| \underline{r}_i^C (t_i) \right| \qquad i = 1, 2, \text{ and } 3$$

$$r_{12} = \left| \underline{r}_2^C (t_2) - \underline{r}_1^C (t_1) \right|$$

$$r_{23} = \left| \underline{r}_3^C (t_3) - \underline{r}_2^C (t_2) \right|$$

The epochs t_3, t_2, and t_1 satisfy the following equations:

$$t_3(ET) - t_2(ET) = \frac{r_{23}}{c} + \frac{2\mu_S}{c^3} \ln \left(\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}} \right) \qquad \begin{matrix} 3 \to 2 \\ 2 \to 1 \end{matrix} \tag{B3}$$

The first term on the right-hand side is the Newtonian light time; the second term is an approximate expression for the contribution to the light time from general relativity.

Given the light time solution, the range observable is computed from

$$R = \frac{r_{12}}{c} + \frac{2\mu_S}{c^3} \ln \left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \right) + \frac{r_{23}}{c} + \frac{2\mu_S}{c^3} \ln \left(\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}} \right)$$

$$- (ET\text{-}TAI)_{t_3} + (ET\text{-}TAI)_{t_1} + \Delta \tag{B4}$$

The sum of the first four terms is the round-trip light time in ET (see Equation B3). The next two terms convert this interval to an interval of TAI. The last term includes corrections for the effects of the troposphere and charged particles. If the light time solution has a tolerance δt for the determination of t_2 and t_1, observables computed from (B4) will have a corresponding error of roughly $(\dot{r}_{12} + \dot{r}_{23})\delta t/c$. If range observables were computed from (B1) using t_1(TAI) obtained from the light time solution, the error would be about δt, which is approximately four orders of magnitude larger.

It is seen that the time tranformation $(t - \tau)$ affects range observables directly through terms 5 and 6 of (B4). It also affects the epochs $t_3(ET)$, $t_2(ET)$, and $t_1(ET)$ and hence the quantities r_{12}, r_{23}, r_1, r_2, and r_3 in (B4); this is the indirect effect of $(t - \tau)$.

Doppler observables are derived from a signal being continuously transmitted from the transmitting station on earth to the receiving station via the spacecraft. A particular observable is associated with an interval of reception T_c (in seconds of TAI) at the receiving station. Typical values of T_c are 60 s and 600 s. It can be shown that the value of a doppler observable obtained by the Deep Space Network is equal to

$$F = \frac{f_T}{T_c} (R_e - R_s) \tag{B5}$$

where f_T is the frequency of the transmitted signal, and R_e and R_s are pseudo round-trip range observables, defined by (B1), with reception times equal to the end (e) and start (s), respectively, of the reception interval T_c. Computed values of two-way or three-way doppler observables are obtained from (B5) using pseudo two-way or three-way range observables computed from (B4). The computed doppler observables are in error only because of errors in the computed pseudo range observables.

When (B4) is used to obtain the computed value of a true range observable, the contribution to Δ due to charged particles is positive because the ranging signal travels at the group velocity ($<c$). When pseudo range observables are computed (which are differenced to form doppler observables), the sign of the charged particle correction is negative because the doppler signal travels at the phase velocity ($>c$). These two corrections, which have opposite signs, have the same magnitude.

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