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SOUND PROPAGATION IN CHOKED DUCTS

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16. Abstract

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The results of a numerical study, for the case of plane waves propagating in a one-dimensional converging-diverging duct, show that the sound field is adequately described by the linearized equations only when the axial mean flow Mach number at the duct throat $M_{th}<0.6$. For $M_{th}>0.6$, the numerical results showed that acoustic energy flux was not conserved. An attempt was made to extend the study to include the nonlinear behavior of the sound field. Meaningful results were not obtained due, primarily, to numerical difficulties.

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DEFINITION OF SYMBOLS

Symbol	Definition
$A(x)$	duct cross-sectional area at location x
$c(x)$	speed of sound at location x
f,g	non-dimensionalized acoustic density and velocity functions respectively defined by Eqn. (40)
G	
i	imaginary number $\sqrt{-1}$
K_s	sound source wave number $(=\omega_s/c_o)$
$2L$	duct converging-diverging length
M_x	duct mean Mach number $(=V_x/c)$
m'	acoustic Mach number $(=u_x'/c)$
$\bar{p},p'$	mean and acoustic pressure respectively
$\underline{Q}$	duct vector velocity $(=\underline{V}+\underline{u}')$
R	sound energy reflection coefficient defined by Eq. (69)
r	cylindrical radial coordinate
T	sound energy transmission coefficient defined by Eq. (69)
t	time
u'	acoustic velocity
V	duct mean velocity
x	duct axial coordinate
α	see Eq. (24); also Eq. (62)
β	see Eq. (25), also (62)
γ,δ	defined by Eq. (62)
θ	a cylindrical azimuthal coordinate
ζ	$\frac{1}{G(x)}\,{dM}/{dx}$
λ	$G(x=0)/G(x)$

Symbol	Definition
ν	A_{th}/A_o
$\bar{\rho}, \rho'$	mean and acoustic density
ω	sound radian frequency
$\mathcal{L}$	defined by Eq. (62)
ψ	phase angle
ϕ	column matrix defined by Eqn. (21)

Subscripts

r, θ, x	denotes radial, azimuthal, or axial cylindrical coordinates
o,in	denotes duct section at x = 0
$_2$	denotes nondimensionalized duct section at x = 2
r,t	denotes sound reflection and transmission respectively
p,u	denotes acoustic pressure and velocity respectively
s	denotes sound source
th	denotes duct throat
T	denotes total or stagnation values
$\underline{(\ )}$	denotes vector quantity
R,I	denotes real and imaginary parts respectively

Superscripts

$\bar{\bar{(\ )}}$	denotes matrix quantity
$\bar{(\ )}$	denotes mean value

1.0 INTRODUCTION

Jet engine inlet whine is a particularly troublesome and persistent source of community noise pollution. A promising new technique to reduce or eliminate this noise is the use of variable area inlets to choke the flow. The idea here is that the choked flow acts as an effective barrier to sound transmission through the inlet. Recent experimental sonic inlet noise suppression studies[1,2] demonstrate the feasibility of this concept. The success of these studies have prompted a number of theoretical studies dealing with sound propagation through variable area ducts containing flow[3-5]. All of these studies, however, share a common deficiency - they assume that the usual linearized convected wave equations adequately describe the bahavior of sound waves in an accelerating flow.

The principal objective of this study is to show that the linearized sound propagation equations are not valid for sonic or near sonic mean flows. The linearized sound propagation equations are singular when the mean flow is sonic. The singularity is removed only when nonlinear terms previously ignored are retained. A second objective is to understand the details of how the sonic inlet reduces the transmission of sound through the inlet. As we understand it, there are two effects that act to reduce the sound transmission. One is related to the fluid mean velocity at the throat (called convection) and the other to flow inhomogeneities (i.e., mean velocity and density gradients).

The reduction of inlet sound transmission by convection is explained as follows. The rate at which sound energy propagates through a duct containing inhomogeneous media is called the group velocity V_g. For simplicity consider a highly idealized sonic inlet consisting of a constant area duct containing a uniform flow. For this case, the group velocity simplifies to

$$V_g = c-V$$

where c is the local speed of sound and V is the duct mean flow. It is clear that the *rate* at which acoustic energy is transmitted out of the inlet *decreases* with *increasing* speed. When $V=c$, there will be no transmission of sound out the inlet; all the internally generated sound will be reflected back into the sound generating interior region of the duct. A physical explanation of the breakdown of the linearized sound propagation equation when the duct mean flow is sonic can be made in terms of group velocity.
The reduction of the rate at which the sound energy is transmitted out the duct inlet can be thought of as a piling-up of the sound at the throat. When the duct flow is choked, all of the sound piles-up; thus the sound pressure and density increase at the throat violating the original linearized assumptions of small disturbances.

It follows that under these circumstances nonlinear terms, previously ignored, must be retained.

When the flow contains inhomogeneities in the form of mean velocity and density gradients, the situation becomes very complicated. It is no longer easy to describe analytically the group velocity. The distinction is made here between gradients that cause refraction and gradients that cause reflection (or scattering). Gradients perpendicular to the wave vector refract or bend the wave and gradients that are in the same direction as the wave vector reflect or scatter the wave. In this study, only scattering by mean axial gradients will be considered.

The report is organized as follows. The behavior of sound waves propagating in a converging-diverging choked duct are described in Section 2. The Mach number range within which the linearized sound propagation equations are valid is explored numerically in Section 3. The report closes with a summary of the results of the study and describes aspects of the behavior of sound fields in accelerating duct flows requiring further study.

2.0 GOVERNING SOUND PROPAGATION EQUATIONS

The equations describing the propagation of sound in a variable area duct containing a mean flow will be derived in its most general form. The objective here is to show that the linearized sound propagation equations are singular when the duct mean flow is sonic. The (variable area) duct mean flow $\underline{V}$ is assumed to contain cylindrical components (V_r, V_θ, V_x) where the subscripts (r,θ,x) represent the radial, azimuthal, and axial directions respectively as defined in Figure 1. In general, $\underline{V}$ is a function of all three components. The flow is assumed to be inviscid and nonheat-conducting so that the fluid pressure and density are adiabatically related. Further, the mean flow is assumed to be independent of the sound field. In the following derivation the quantities (p,ρ,c) represent mean fluid pressure, density and sound speed and $(p',\rho',\underline{u}')$ represent acoustic pressure, density, and velocity respectively. Vector quantities will be denoted by a single horizontal bar located below the symbol $(\underline{\ })$ and matrix quantities by a double horizontal bar located above the symbol $(\bar{\bar{\ }})$.

The equations describing the conservation of mass and momentum are respectively

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{Q}) = 0 \tag{1}$$

and, in component form,

$$\rho\left(\frac{\partial Q_r}{\partial t} + Q_r \frac{\partial Q_r}{\partial r} + \frac{Q_\theta}{r} \frac{\partial Q_r}{\partial \theta} - \frac{Q_\theta^2}{r} + Q_x \frac{\partial Q_r}{\partial x}\right) + c^2 \frac{\partial \rho}{\partial r} = 0 \tag{2}$$

$$\rho\left(\frac{\partial Q_\theta}{\partial t} + Q_r \frac{\partial}{\partial r} Q_\theta + \frac{Q_\theta}{r}\frac{\partial Q_\theta}{\partial r} + \frac{Q_r Q_\theta}{r} + Q_x \frac{\partial}{\partial x} Q_\theta\right) + \frac{c^2}{r}\frac{\partial \rho}{\partial \theta} = 0 \tag{3}$$

$$\rho\left(\frac{\partial Q_x}{\partial t} + Q_r \frac{\partial}{\partial r} Q_x + \frac{Q_\theta}{r}\frac{\partial}{\partial \theta} Q_x + Q_x \frac{\partial}{\partial x} Q_x\right) + c^2 \frac{\partial \rho}{\partial x} = 0 \tag{4}$$

where

$$\underline{Q} = \left(Q_r, Q_\theta, Q_x\right) = \underline{V} + \underline{u}' \tag{5}$$

is the total flow velocity. In Eqns. (3)-(5), the variations of pressure was replaced by the adiabatic condition that

$$\partial p/\partial \rho = c^2 \tag{6}$$

where c is the fluid local sound speed. Now assume that the flow field can be written as the sum of the mean and fluctuating quantities so that

$$p(r,\theta,x,t) = \overline{p(r,\theta,x)} + p'(r,\theta,x,t) \tag{7}$$

$$\rho(r,\theta,x,t) = \overline{\rho(r,\theta,x)} + \rho'(r,\theta,x,t) \tag{8}$$

$$\underline{Q}(r,\theta,x,t) = \overline{\underline{V}(r,\theta,x)} + \underline{u}'(r,\theta,x,t) \tag{9}$$

It is further assumed that the time-averages

$$\overline{p'} = \overline{\rho'} = \overline{u'} = 0 \tag{10}$$

where $(\bar{\ }) = \frac{1}{T}\int_0^T (\ )dt$ and T is the sound wave period. Substituting Eqns. (7-9) and retaining all terms yields

$$\frac{\partial \rho'}{\partial t} + (\nabla\cdot\underline{V})\rho' + (\bar{\rho}+\rho')(\nabla\cdot\underline{u}') + (\underline{u}'\cdot\nabla)(\bar{\rho}+\rho') + (\underline{V}\cdot\nabla)\rho' = 0 \tag{11}$$

and

$$\frac{\partial u_r'}{\partial t} + (\underline{u}'\cdot\nabla)(V_r+u_r') + (\underline{V}\cdot\nabla)u_r' - \frac{1}{r}(2V_\theta + u_\theta')u_\theta'$$
$$+\frac{\bar{c}^2}{\bar{\rho}}(1-\rho'/\bar{\rho})\frac{\partial \rho'}{\partial r} - \frac{\bar{c}^2}{\bar{\rho}^2}\rho'\frac{\partial \bar{\rho}}{\partial r} = 0 \tag{12}$$

$$\frac{\partial u_\theta'}{\partial t} + (\underline{u}'\cdot\nabla)(V_\theta+u_\theta') + (\underline{V}\cdot\nabla)u_\theta' + (V_r+u_r')\frac{1}{r}u_\theta' + \frac{V_\theta u_r'}{r}$$
$$+\frac{\bar{c}^2}{r}(1-\rho'/\bar{\rho})\frac{\partial \rho'}{\partial \theta} - \frac{1}{r}\frac{\bar{c}^2}{\bar{\rho}^2}\rho'\frac{\partial \bar{\rho}}{\partial \theta} = 0 \tag{13}$$

$$\frac{\partial u_x'}{\partial t} + (\underline{u}'\cdot\nabla)(V_x+u_x') + (\underline{V}\cdot\nabla)u_x' + \bar{c}^2(1-\rho'/\bar{\rho})\frac{\partial \rho'}{\partial x}$$
$$-\frac{\bar{c}^2}{\bar{\rho}^2}\rho'\frac{\partial \bar{\rho}}{\partial x} = 0 \tag{14}$$

where the operator ∇ equals

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial x}\right) \tag{15}$$

The notation used to express Eqns. (11-14) is considerably simplified using matrix notation. Letting

$$
\bar{\bar{M}}_1 = \begin{bmatrix}
\dfrac{\partial V_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial V_r}{\partial \theta} - 2V_\theta - u'_\theta\right) & \dfrac{\partial V_r}{\partial x} & -\left(\dfrac{\bar{c}}{\bar{\rho}}\right)^2 \dfrac{\partial \bar{\rho}}{\partial r} \\[2ex]
\dfrac{1}{r}\dfrac{\partial}{\partial r}(rV_\theta) & \dfrac{1}{r}\left(\dfrac{\partial V_\theta}{\partial \theta} + V_r + u'_r\right) & \dfrac{\partial V_\theta}{\partial x} & -\dfrac{1}{r}\left(\dfrac{\bar{c}}{\bar{\rho}}\right)^2 \dfrac{\partial \bar{\rho}}{\partial \theta} \\[2ex]
\dfrac{\partial V_x}{\partial r} & \dfrac{1}{r}\dfrac{\partial V_x}{\partial \theta} & \dfrac{\partial V_x}{\partial x} & -\left(\dfrac{\bar{c}}{\bar{\rho}}\right)^2 \dfrac{\partial \bar{\rho}}{\partial x} \\[2ex]
\dfrac{\partial \bar{\rho}}{\partial r} & \dfrac{1}{r}\dfrac{\partial \bar{\rho}}{\partial \theta} & \dfrac{\partial \bar{\rho}}{\partial x} & \nabla\cdot\left(\underline{V} + \underline{u}'\right)
\end{bmatrix} \tag{16}
$$

$$
\bar{\bar{M}}_2 = \begin{bmatrix}
(V_r + u'_r) & 0 & 0 & \dfrac{\bar{c}^2}{\bar{\rho}}\left(1 - \rho'/\bar{\rho}\right) \\[2ex]
0 & (V_r + u'_r) & 0 & 0 \\[2ex]
0 & 0 & (V_r + u'_r) & 0 \\[2ex]
\bar{\rho} & 0 & 0 & (V_r + u'_r)
\end{bmatrix} \tag{17}
$$

$$
\bar{\bar{M}}_3 = \begin{bmatrix}
(V_\theta + u'_\theta) & 0 & 0 & 0 \\[2ex]
0 & (V_\theta + u'_\theta) & 0 & \dfrac{\bar{c}^2}{\bar{\rho}}\left(1 - \rho'/\bar{\rho}\right) \\[2ex]
0 & 0 & (V_\theta + u'_\theta) & 0 \\[2ex]
0 & \bar{\rho} & 0 & (V_\theta + u'_\theta)
\end{bmatrix} \tag{18}
$$

and

$$\bar{\bar{M}}_4 = \begin{bmatrix} (V_x + u_x') & 0 & 0 & 0 \\ 0 & (V_x + u_x') & 0 & 0 \\ 0 & 0 & (V_x + u_x') & \frac{\bar{c}^2}{\bar{\rho}}(1 - \rho'/\bar{\rho}) \\ 0 & 0 & \bar{\rho} & (V_x + u_x') \end{bmatrix} \tag{19}$$

Substituting the matrices defined by Eqns. (16-19) into Eqns. (11-14), the governing equations become

$$\left[\bar{\bar{I}} \frac{\partial}{\partial t} + \bar{\bar{M}}_1 + \bar{\bar{M}}_2 \frac{\partial}{\partial r} + \bar{\bar{M}}_3 \frac{1}{r} \frac{\partial}{\partial \theta} + \bar{\bar{M}}_4 \frac{\partial}{\partial x} \right] \bar{\bar{\phi}} = 0 \tag{20}$$

where $\bar{\bar{I}}$ is the identity matrix and $\bar{\bar{\phi}}$ is the column matrix representing the acoustic field defined below as

$$\bar{\bar{\phi}} = \begin{bmatrix} u_r' \\ u_\theta' \\ u_x' \\ \rho' \end{bmatrix} \tag{21}$$

Equation (20) may be rewritten as

$$\frac{\partial \bar{\bar{\phi}}}{\partial x} = -(\bar{\bar{M}}_4)^{-1} \left[\bar{\bar{I}} \frac{\partial}{\partial t} + \bar{\bar{M}}_1 + \bar{\bar{M}}_2 \frac{\partial}{\partial t} + \frac{\bar{\bar{M}}_3}{r} \frac{\partial}{\partial \theta} \right] \bar{\bar{\phi}} \tag{22}$$

where the inverse matrix $(\bar{\bar{M}}_4)^{-1}$ can be shown to equal

$$\left(\bar{\bar{M}}_4\right)^{-1} = \frac{1}{\alpha^2(\alpha^2-\bar{c}^2\beta)}\begin{bmatrix} \alpha(\alpha^2-\bar{c}^2\beta) & 0 & 0 & 0 \\ 0 & \alpha(\alpha^2-\bar{c}^2\beta) & 0 & 0 \\ 0 & 0 & \alpha^3 & -\frac{\bar{c}^2}{\bar{\rho}}\alpha^2\beta \\ 0 & 0 & -\bar{\rho}\alpha^2 & \alpha^3 \end{bmatrix} \tag{23}$$

Here the quantities α and β are defined as

$$\alpha = V_x + u_x' \tag{24}$$

$$\beta = 1 - \rho'/\bar{\rho} \tag{25}$$

We are now in a position to demonstrate a principal conclusion of this report - namely, that the well known linearized wave propagation equations that are often used to model the behavior of sound waves in choked ducts become singular when the duct mean flow is sonic. The matrix $(\bar{\bar{M}}_4)^{-1}$ in Eqn. (24) contains the terms $[\alpha^2(\alpha^2-c^2\beta)]^{-1}$. Substituting from Eqns. (24) & (25) for the quantities α and β yields the term

$$\left\{ \bar{c}^4(M_x+m_x')^2\left[(M_x^2-1)+\left(2M_x m_x' + \rho'/\bar{\rho} + m_x'^2\right)\right]\right\}^{-1}$$

where $M_x = V_x/\bar{c}$ and $m'_x = u_x'/\bar{c}$. Now recall that in the derivation of Eqn. (23), all terms were retained including the nonlinear terms. If the nonlinear terms were ignored, there the above expression simplifies to

$$\left[\bar{c}^4\, M_x^2(M_x^2-1)\right]^{-1}$$

which is clearly singular at $M_x = 1$. It is also clear that the nonlinear terms must be retained at $M_x = 1$ to remove the singularity. This conclusion is quite general because the flow field within the duct is fully three-dimensional and unsteady.

3.0 NUMERICAL EXAMPLE

A numerical study has been undertaken to map out, for a particular duct geometry, the Mach number range for which the *linearized* sound propagation equations are valid. To simplify the analysis, only plane waves propagating in a one-dimensional duct are considered. The duct geometry, shown in Figure 2, consists of two infinitely long ducts of equal area connected by a contraction-expansion (constriction) region. The use of infinitely long ducts avoids the complexities associated with end reflections. Mean flow is introduced into the duct as shown and is accelerated to sonic or near sonic speeds in the throat. Sound propagating in the direction of the mean flow (see Fig. 2a) is called the downstream case and sound propagating against the mean flow (see Fig. 2b) is the upstream case.

The duct mean flow properties are assumed to be steady, one-dimensional and isentropic. They are described by known functions of the duct axial coordinate x, the origin of which is chosen as shown in Fig. 2. The constriction region is specified to be symmetrical about the throat, the area change occurring over the distance 2L. The duct cross-sectional area is described by the expression

$$\frac{A(x)}{A(0)} = \begin{cases} 1\,; \left(\frac{x}{L}\right) \le 0 \\ 1-(1-\nu)\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]; 0 \le \left(\frac{x}{L}\right) \le 1 \\ 5-(1-\nu)\left[2\left(\frac{x}{L}\right)^3 - 9\left(\frac{x}{L}\right)^2 + 12\left(\frac{x}{L}\right)\right] - 4\nu; 1 \le \left(\frac{x}{L}\right) \le 2 \\ 1\,; \left(\frac{x}{L}\right) \ge 2 \end{cases} \tag{26}$$

where $\nu = A(L)/A(0)$ is the ratio of the duct constriction throat area (at $x = L$) to inlet area (at $x = 0$).

3.1 Derivation of Governing Equations

The governing linearized sound propagation equations describing the behavior of the sound field in the duct are derived below starting from the basic mass and momentum conservation equations. Although they could be derived starting from the more general form given by Eqn. (22), the highly simplifying assumptions of a one-dimensional duct and plane sound waves motivates the rederivation of the governing equations.

The derivation starts from the equations describing the conservation of mass and momentum in a one-dimensional variable area duct

$$\frac{\partial \rho}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}(\rho Q A) = 0 \tag{27}$$

and

$$\rho \frac{\partial Q}{\partial t} + \rho Q \frac{\partial Q}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{28}$$

where p, ρ and Q are the one-dimensional counterparts of the three-dimensional flow field described in Section 2. Assuming that the flow field can be described by a steady state and a fluctuating part, the equations describing the conservation of mass and momentum simplify to

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial u'}{\partial x} + \left(\frac{d\bar{\rho}}{dx}\right)u' + V\frac{\partial \rho'}{\partial x} + \left(\frac{dV}{dx}\right)\rho' + \left(\frac{1}{A}\frac{dA}{dx}\right)\left(V\rho' + \bar{\rho}u'\right) = 0 \tag{29}$$

and

$$\bar{\rho}\frac{\partial u'}{\partial t} + \bar{\rho}\frac{dV}{dx}\cdot u' + V\frac{dV}{dx}\rho' + \bar{\rho}V\frac{\partial u'}{\partial x} + \frac{\partial p'}{\partial x} = 0 \tag{30}$$

Implicit in this derivation is that the steady state terms satisfy independently of the acoustic terms the time-average conservation of mass and momentum. Equations (29) and (30) are further simplified by incorporating the following

$$\partial p'/\partial \rho' = c^2(x) \longrightarrow p' = c^2(x)\rho' \tag{31}$$

$$\bar{\rho}(x) = \bar{\rho}_T\left[1 + \left(\frac{\gamma-1}{2}\right)M^2(x)\right]^{-\left(\frac{1}{\gamma-1}\right)} \equiv \rho_T\left[G(x)\right]^{-\left(\frac{1}{\gamma-1}\right)} \tag{32}$$

$$V(x) = \bar{c}_T \, M G^{-\frac{1}{2}} \tag{33}$$

and

$$\frac{1}{A}\,\frac{dA}{dx} = -\,\frac{(1-M^2)}{GM}\,\frac{dM}{dx} \tag{34}$$

where the subscript T denotes fluid stagnation or total values. Equation (31) represents the adiabatic relationship between the sound pressure and density while Equations (32), (33) and (34) follow from one-dimensional steady state, isentropic flow. Substituting Eqns. (31-34) into Eqns. (29) and (30) yields

$$\frac{\partial \rho'}{\partial t} + \left[\frac{\rho_T}{G^{(1/\gamma-1)}}\right]\frac{\partial u'}{\partial x} - \left[\frac{\rho_T}{M}\,\frac{dM}{dx}\,\frac{1}{G^{\gamma/\gamma-1}}\right]u' + \left[\frac{M c_T}{G^{1/2}}\right]\frac{\partial \rho'}{\partial x} + \left[\frac{c_T M^2}{G^{3/2}}\,\frac{dM}{dx}\right]\rho' = 0 \tag{35}$$

and

$$\frac{\partial u'}{\partial t} + \left[\frac{c_T M}{G^{1/2}}\right]\frac{\partial u'}{\partial x} + \left[\frac{c_T}{G^{3/2}}\,\frac{dM}{dx}\right]u' + \left[\frac{c_T^{\,2}}{\rho_T}\,G^{\left(\frac{3-2\gamma}{\gamma-1}\right)}\frac{dM}{dx}\right]\rho' + \left[\frac{G^{\left(\frac{2-\gamma}{\gamma-1}\right)}c_T^{\,2}}{\rho_T}\right]\frac{\partial \rho'}{\partial x} = 0 \tag{36}$$

Equations (35) and (36) are non-dimensionalized by scaling ρ', u', x and t as follows. Let

$$\rho' = \rho_s \rho^*; \quad u' = u_s u^*; \quad x = K_s^{-1} x^*; \quad t = \omega_s^{-1} t^* \tag{37}$$

where ρ_S and u_S represent the magnitude of the *known* acoustic source density and velocity introduced into the duct far upstream of the constriction, K_S is the acoustic wave number ($K_S = \omega_S/\bar{c}_0$) where ω_S is the sound radian frequency and $\bar{c}_0$ is the local speed of sound at $x = 0$. Substituting Eqn. (37) into Eqns. (35) and (36) yields

$$(K_sL)\frac{\partial\rho^*}{\partial t^*}+\left(\frac{G_o}{G}\right)^{1/2}\frac{\partial u^*}{\partial x^*}-\left[\left(\frac{G_o}{G}\right)^{\frac{1}{\gamma-1}}\frac{1}{GM}\frac{dM}{dx^*}\right]u^*$$

$$+M\left(\frac{G_o}{G}\right)^{1/2}\frac{\partial\rho^*}{\partial x^*}+\left[\left(\frac{G_o}{G}\right)^{1/2}\frac{M^2}{G}\frac{dM}{dx^*}\right]\rho^*=0 \tag{38}$$

and

$$(K_sL)\frac{\partial u^*}{\partial t^*}+\left[\left(\frac{G_o}{G}\right)^{1/2}M\right]\frac{\partial u^*}{\partial x^*}+\left[\left(\frac{G_o}{G}\right)^{1/2}\frac{1}{G}\frac{dM}{dx^*}\right]u^*$$

$$+\left[\left(\frac{G_o}{G}\right)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}\frac{1}{G}\frac{dM}{dx^*}\right]\rho^*+\left(\frac{G_o}{G}\right)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}\frac{\partial\rho^*}{\partial x^*}=0 \tag{39}$$

To solve Eqns. (38) and (39), the following form of the solutions are assumed,

$$\rho^*(x^*,t^*)=e^{it^*}g^*(x^*)\;;\;u^*(x^*,t^*)=e^{it^*}f^*(x^*) \tag{40}$$

Substituting Eqn. (40) into Eqns. (38) and (39) yields

$$\left[i(K_sL)+\left(\frac{G_o}{G}\right)^{1/2}\frac{M^2}{G}\frac{dM}{dx^*}\right]g^*+M\left(\frac{G_o}{G}\right)^{1/2}\frac{dg^*}{dx^*}-\left[\left(\frac{G_o}{G}\right)^{\frac{1}{\gamma-1}}\frac{1}{GM}\frac{dM}{dx^*}\right]f^*$$

$$+\left(\frac{G_o}{G}\right)^{\frac{1}{\gamma-1}}\frac{df^*}{dx^*}=0 \tag{41}$$

and

$$\left[i(K_sL)+\left(\frac{G_o}{G}\right)^{1/2}\frac{1}{G}\frac{dM}{dx^*}\right]f^*+M\left(\frac{G_o}{G}\right)^{1/2}\frac{df^*}{dx^*}+\left[\left(\frac{G_o}{G}\right)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}\frac{1}{G}\frac{dM}{dx}\right]g^*$$

$$+\left(\frac{G_o}{G}\right)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}\frac{dg^*}{dx^*}=0 \tag{42}$$

The non-dimensional functions f and g (the ( )* notation is deleted for convenience) are complex and are written

$$f(x) = f_R(x) + i\, f_I(x) \tag{43}$$

and

$$g(x) = g_R(x) + i\, g_I(x) \tag{44}$$

Substituting Eqns. (43) and (44) into Eqns. (41) and (42) yields for the derivatives

$$\begin{bmatrix} \frac{dg_R}{dx} \\ \frac{dg_I}{dx} \\ \frac{df_R}{dx} \\ \frac{df_I}{dx} \end{bmatrix} = \frac{1}{1-M^2} \begin{bmatrix} -\alpha M^2(1-M) & -M\lambda^{-1/2} & -2\alpha & \lambda^{\left(\frac{2-\gamma}{\gamma-1}\right)} \\ M\lambda^{-1/2} & -\alpha M(1-M^2) & -\lambda^{\left(\frac{2-\gamma}{\gamma-1}\right)} & -2\alpha M\lambda^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} \\ 0 & \lambda^{-\left(\frac{1}{\gamma-1}\right)} & \alpha(1+M^2) & -M\lambda^{-1/2} \\ -\lambda^{-\left(\frac{1}{\gamma-1}\right)} & 0 & M\lambda^{-1/2} & \alpha(1+M^2) \end{bmatrix} \begin{bmatrix} g_R \\ g_I \\ f_R \\ f_I \end{bmatrix} \tag{45}$$

where $\lambda = G_o/G$, $G = 1 + \left(\frac{\gamma-1}{2}\right)M^2(x)$, $G_o = 1 + \left(\frac{\gamma-1}{2}\right)M^2(o)$, and $\alpha = \frac{1}{G}\frac{dM}{dx}$.
Here again, Eqn. (45) is singular at M = 1.

3.2 Boundary Conditions

The boundary conditions are particularly simple because the duct cross-section and mean flow are uniform for $|x| \geq L$. The boundary conditions are different for the cases of upstream and downstream sound propagation.

3.2.1 Downstream Propagation

At x = 0 (see Fig. 2a), both incident and reflected waves exist. The incident wave is assumed known and generated far upstream of the constriction. The reflected wave, generated by the mean flow inhomogeneities in the duct constriction is unknown. At x = 2, only transmitted waves propagating to the right (i.e., right running waves) are assumed.

For values of $x \leq 0$, both the duct cross-sectional area and the mean flow velocity are constant. Thus the sound propagation equations reduce to the well-known (dimensional) solutions

$$\rho'(x,t) = \rho_s' \, e^{i\omega\left(t - \frac{x}{c+V}\right)} + \rho_r' \, e^{i\omega\left(t + \frac{x}{c-V}\right) + i\psi_o} \tag{46}$$

and

$$u'(x,t) = u_s' \, e^{i\omega\left(t - \frac{x}{c+V}\right)} + u_r' \, e^{i\omega\left(t + \frac{x}{c-V}\right) + i\psi_o} \tag{47}$$

Here ρ_s' and u_s' are the assumed known source strengths of the incident acoustic density ρ' and velocity u' respectively, ρ_r' and u_r' are the unknown strengths of the reflected sound wave, and ψ_o is the unknown phase of the reflected wave relative to the incident wave.

Equations (46) and (47) may be suitably non-dimensionalized using Eqn. (37) yielding for the (non-dimensional) functions f and g,

$$f(x) = e^{-\frac{ix}{1+M}} + \Gamma_o \, e^{\frac{ix}{1-M} + i\psi_o} \tag{48}$$

and

$$g(x) = e^{-\frac{ix}{1+M}} - \Gamma_o \, e^{\frac{ix}{1-M} + i\psi_o} \tag{49}$$

where

$$\Gamma_o = \frac{u_r'}{u_s'} = -\frac{\rho_r'}{\rho_s'} \tag{50}$$

Equation (50) follows by substituting the solution given by Eqns. (48) and (49) into Eqn. (29) (the linearized continuity Eqn.). Thus the boundary conditions at $x = 0$ are

$$f(0) = 1 + \Gamma_o \, e^{i\psi_o} = \left[1 + \Gamma_o \cos\psi_o\right] + i\left[\Gamma_o \sin\psi_o\right] \tag{51}$$

and

$$g(0) = 1 - \Gamma_0 e^{i\psi_0} = \left[1 - \Gamma_0 \cos\psi_0\right] - i\left[\Gamma_0 \sin\psi_0\right] \tag{52}$$

For values of $x \geq 2L$, (see Fig. 2b), only transmitted (right running waves) are permitted. They are described by the solution

$$\rho'(x,t) = \rho'_t \, e^{i\omega\left(t - \frac{x}{c+V}\right) + i\psi_2} \tag{53}$$

and

$$u'(x,t) = u'_t \, e^{i\omega\left(t - \frac{x}{c+V}\right) + i\psi_2} \tag{54}$$

where ρ'_t, u'_t are constants (it can be shown that $\rho'_t = u'_t$ by substituting Eqn. (53) and (54) into Eqn. (29)) and ψ_2 is a unknown phase shift due to nonhomogeneities generated by the duct constriction. Non-dimensionalizing Eqns. (53) and (54) using Eqn. (37) yields the boundary condition at $x = 2L$ (or $x^* = 2K_sL$)

$$f(2K_sL) = g(2K_sL) = \Gamma_2 e^{i\psi_2^*} = \Gamma_2 \cos\psi_2^* + i\,\Gamma_2 \sin\psi_2^* \tag{55}$$

where

$$\psi_2^* = \psi_2 - \frac{2K_sL}{1 + M(0)} \tag{56}$$

3.2.2 Upstream Propagation

The boundary conditions for upstream sound propagation are very similar to the downstream case. Here, the sound approaches the constriction as shown in Fig. 2b. At $x = 2L$, it is straight-forward to show that the incident and reflected wave may be written

$$f(2K_sL) = \underbrace{e^{i\frac{2K_sL}{1-M_0}}}_{\text{Incident Wave}} + \underbrace{\Gamma_2 e^{i\left(\psi_2 - \frac{2K_sL}{1-M_0}\right)}}_{\text{Reflected Wave}} \tag{57}$$

and

$$g(2K_sL) = \underbrace{-e^{\,i\frac{2K_sL}{1-M_0}}}_{\text{INCIDENT WAVE}} + \underbrace{\Gamma_2\, e^{\,i\left(\psi_2 - \frac{2K_sL}{1-M_0}\right)}}_{\text{REFLECTED WAVE}} \tag{58}$$

at $x = 0$, the transmitted wave is

$$f(0) = - \quad = \left(\Gamma_0 \cos\psi_0\right) + i\left(\Gamma_0 \sin\psi_0\right) \tag{59}$$

Thus there are four equations g_R, g_I, f_R, f_I and four unknowns ($\Gamma_0, \Gamma_2, \psi_0, \psi_2$), the solution of which completely describes the problem.

3.3 Numerical Integration Scheme

The solution to the four simultaneous differential equations described by Eqn. (45) requires that four (4) constants (of integration) be specified. The four unknowns ($\Gamma_0, \Gamma_2, \psi_0, \psi_2$) will be solved for directly by taking full advantage of the linearity of the solution. The solution starts by observing that the four unknown functions (g_R, g_I, f_R, f_I) may be written

$$\begin{bmatrix} g_R \\ g_I \\ f_R \\ f_I \end{bmatrix} = g_R \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + g_I \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + f_R \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + f_I \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{60}$$

Using linearity, the starting values defined by the column matrix at $x = 0$,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{integrates to the value} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \quad \text{at } x = 2K_sL$$

Consider first the downstream case. Combining Eqn. (60) with the unknown initial conditions described by Eqns. (51), (52) and (55), the solution to Eqn. (45) may be written

$$
\begin{aligned}
(1-\alpha)a_{11} & - \beta a_{12} + (1+\alpha)a_{13} + \beta a_{14} = \gamma \\
(1-\alpha)a_{21} & - \beta a_{22} + (1+\alpha)a_{23} + \beta a_{24} = -\delta \\
(1-\alpha)a_{31} & - \beta a_{32} + (1+\alpha)a_{33} + \beta a_{34} = \gamma \\
(1-\alpha)a_{41} & - \beta a_{42} + (1+\alpha)a_{43} + \beta a_{44} = -\delta
\end{aligned}
\tag{61}
$$

where the quantities α, β, γ and δ are defined as

$$
\alpha = \Gamma_o \cos\psi_o \; ; \quad \beta = \Gamma_o \sin\psi_o \; ; \quad \gamma = \Gamma_o \cos\psi_2^* \; ; \quad \delta = \Gamma_2 \sin\psi_2^* \tag{62}
$$

The solution to the four unknowns α, β, γ, δ follows immediately by rewriting Eqn. (61) as

$$
\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}
= -
\begin{bmatrix}
(a_{13}-a_{11}) & (a_{14}-a_{12}) & -1 & 0 \\
(a_{23}-a_{21}) & (a_{24}-a_{22}) & 0 & 1 \\
(a_{33}-a_{31}) & (a_{34}-a_{32}) & -1 & 0 \\
(a_{43}-a_{41}) & (a_{44}-a_{42}) & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
(a_{11}+a_{13}) \\ (a_{21}+a_{23}) \\ (a_{31}+a_{33}) \\ (a_{41}+a_{43})
\end{bmatrix}
\tag{63}
$$

This approach is very efficient. It avoids the shooting approach used by Davis and Johnson[3] and the particularly restrictive approach used by Eisenberg and Kao[4]. It is also more accurate that the characteristic method used by King and Karamcheti[5]. The work of Eisenberg and Kao consists of transforming the coupled acoustic equations governing the propagation of sound in a variable area, one dimensional duct into two separate equations containing variable coefficients. By an appropriate choice of duct shape, the variable coefficients are made constant permitting relatively easy solutions of the separate equations for particular duct shapes. Physical variables (such as pressure and velocity perturbations) are found by tracing back all functional transformations. A serious draw back to their approach is the severe restriction in duct shape. King and Karamcheti also adopted the quasi-one-dimensional assumptions in their solution of the behavior of sound in flow. They used the method of characteristics to solve for the sound field. Because of the variation of mean flow quantities in the duct, the characteristics are curved. An iterative scheme of second order accuracy in mesh spacing is used to account for curvature when determining the intersection of characteristics of different families from adjacent mesh points. This scheme encounters difficulty in maintaining reasonable accuracy.

To verify that the solutions given by Eqn. (63) are correct, the conservation of acoustic energy flux will be *independently* evaluated at $x = 0$ and $2(K_sL)$. Using the form for the energy flux W given by Cantrell and Hart.[6]

$$W(x) = \frac{1}{2}\, A(x)\left[\left(1 + M^2(x)\right)|p'(x)|\,|u'(x)|\cos(\Theta_p - \Theta_u) + M(x)\left(\frac{|p'(x)|^2}{\bar{\rho}(x)\bar{c}(x)} + \bar{\rho}(x)\bar{c}(x)|u'(x)|^2\right)\right] \tag{64}$$

where Θ_p and Θ_u represent the phase of the acoustic pressure and velocity respectively.

Evaluating Eqn. (64) at $x = 0$ and $x = 2(K_sL)$ and denoting by W_i^+, W_r^- and W_t^+ the incident energy flux (at $x = 0$) the reflected energy flux (at $x = 0$) and the transmitted energy flux (at $x = 2(K_sL)$ respectively (for the downstream case), then these quantities may be written as

$$W_i^+ = \frac{1}{2} A_o \bar{C}_o^2 |\rho_s'|\,|u_s'|(1 + M_o)^2 \tag{65}$$

$$W_r^- = \frac{1}{2} A_o \bar{C}_o^2 |\rho_r'|\,|u_r'|(1 - M_o)^2 \tag{66}$$

$$W_t^+ = \frac{1}{2} A_o \bar{C}_o^2 |\rho_t'|\,|u_t'|(1 + M_o)^2 \tag{67}$$

where $M_o = M(x=0) = M(x=2K_sL)$ by symmetry.

Conservation of acoustic energy requires that

$$W_i^+ = W_r^- + W_t^+ \tag{68}$$

Upon substitution of the various quantities into Eqn. (68), the desired form of the solution is

$$\underbrace{\left(\frac{1-M_o}{1+M_o}\right)^2 |\Gamma_o|^2}_{\substack{\text{Reflected Energy} \\ \text{Coefficient (R)}}} + \underbrace{|\Gamma_2|^2}_{\substack{\text{Transmitted Energy} \\ \text{Coefficient (T)}}} = 1 \tag{69}$$

where from Eqn. (62), $|\Gamma_o|^2 = \alpha^2 + \beta^2$ and $|\Gamma_2|^2 = \gamma^2 + \delta^2$

The solution of the upstream case is quite similar to that of the downstream case. By combining Eqn. (60) with the boundary conditions for the upstream case defined by Eqns. (57), (58) and (59), the solution is

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} (a_{13}-a_{11}) & (a_{14}-a_{\ 4}) & -1 & 0 \\ (a_{23}-a_{21}) & (a_{24}-a_{22}) & 0 & -1 \\ (a_{33}-a_{31}) & (a_{34}-a_{32}) & -1 & 0 \\ (a_{43}-a_{41}) & (a_{44}-a_{42}) & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -\cos\left(\frac{2K_sL}{1-M_o}\right) \\ -\sin\left(\frac{2K_sL}{1-M_o}\right) \\ \cos\left(\frac{2K_sL}{1-M_o}\right) \\ \sin\left(\frac{2K_sL}{1-M_o}\right) \end{bmatrix} \tag{70}$$

Again using the definition of Cantrell and Hart, the conservation of acoustic energy flux may be written as

$$\underbrace{|\Gamma_o|^2}_{T} + \underbrace{\left(\frac{1+M_o}{1-M_o}\right)^2 |\Gamma_2|^2}_{R} = 1 \tag{71}$$

3.4 Results

Numerical results for selected duct configurations having throat to inlet area ratios (ν) equal to 0.305, 0.420, 0.558 and 0.746 and for the case of $K_sL = 1$ are summarized in Figures 3-6. The mean flow velocity distribution through the duct is specified by Eqn. (26). Figure 3 summarizes the upstream sound propagation solutions and Fig. 4, the downstream solutions. The solutions are presented in terms of the sound energy reflection coefficient R and the total non-dimensionalized acoustic energy flux R + T. In Figs. 3 and 4, energy is conserved (i.e., T + R = 1) for throat Mach numbers less than about $M_{th} < 0.6$; this is true for all duct area ratios considered. However, for $M_{th} > 0.6$, it is clear that acoustic energy flux T + R is not conserved, the departure of T + R from unity increasing as the throat Mach number approaches unity.

The departure of T + R from unity for $M_{th} > 0.6$ is due to the singular behavior of the linearized sound propagation equations. For values of M_{th} near unity, the linearized sound propagation equations are inadequate to account for the conservation of acoustic mass and momentum.

Figures 3 and 4 also show that the sound energy reflection coefficient (and hence the transmission coefficient) is very sensitive to the duct throat to inlet area ratio ν. For a fixed value of M_{th}, the increase of R (decrease of T) with the decrease of ν is related to sound reflection or scattering generated by the axial velocity gradients of the duct mean flow. This is easily shown by restricting the duct mean Mach number at the throat to values less than 0.6 so that the effects of compressibility can be ignored. It is straight forward to show that the duct mean flow velocity gradient defined below as

$$\frac{V_{th}-V_{IN}}{L} = \frac{V_{th}}{L}\left(1-\nu\right)$$

depends only upon the parameter ν for a fixed duct geometry and throat speed. The large increase in R observed for both the upstream (Fig. 3) and downstream (Fig. 4) cases of sound propagation suggests that the reduction in sound transmission arises primarily because of scattering effects generated by the duct mean flow velocity gradients. This is more clearly shown in Fig. 5 which combines the reflection coefficient R from Figs. 3 and 4. Figure 5 shows that for values of $M_{th} < 0.6$, there are large and significant differences for the various duct throat-to-inlet area ratios shown. The effect, however, on R of upstream or downstream sound propagation is small. These results have important application to our understanding of the manner in which the sonic inlet acts as an effective barrier to internally generated sound. The contribution of the mean flow velocity gradients within the duct may contribute more to reduction of sound transmission than was previously thought.

Figure 6 is a rather dramatic example of the effect of the duct mean flow on the sound field. The distribution along the duct of the gradient of the imaginary part of the density f_I' is plotted for three different throat Mach numbers. The singular nature of the sound field is evident.

4.0 NONLINEAR NUMERICAL EXAMPLE

The numerical study undertaken in Section 3 showed that the solution of the linearized sound propagation equations failed to conserve acoustic energy flux (T + R) for throat Mach numbers $M_{th} > 0.6$. The linearized solution is extended below to include the previously ignored nonlinear terms. The purpose here is to determine numerically, for the test case treated in Section 3, if the accuracy of the solution can be extended up to or near $M_{th} = 1$ by including nonlinear terms.

4.1 Derivation of Governing Equations

The derivation assumes that the mean flow is independent of the sound field so that

$$\frac{d}{dx}\left(\rho VA\right) = 0$$

and

$$\rho V \frac{dV}{dx} + \frac{dp}{dx} = 0$$

With this understanding, the nondimensional sound propagation equations are

$$(K_sL)\frac{\partial \rho}{\partial t} + \left(\frac{G_o}{G}\right)^{1/2}\frac{\partial u}{\partial x} - \left[\left(\frac{G_o}{G}\right)^{\frac{1}{\gamma-1}} \frac{\frac{dM}{dx}}{GM}\right] u + M\left(\frac{G_o}{G}\right)^{1/2}\frac{\partial \rho}{\partial x}$$

$$\left[\left(\frac{G_o}{G}\right)^{1/2}\frac{M^2}{G}\frac{dM}{dx}\right]\rho = -\epsilon\left\{\frac{\partial}{\partial}\left(\rho u\right) + \rho u \frac{\partial}{\partial x}\ln A\right\} \tag{72}$$

and

$$(K_sL)\frac{\partial u}{\partial t} + \left[\left(\frac{G_o}{G}\right)^{1/2} M\right]\frac{\partial u}{\partial x} + \left[\left(\frac{G_o}{G}\right)^{1/2}\frac{1}{G}\frac{dM}{dx}\right] u + \left[\left(\frac{G_o}{G}\right)^{-\left(\frac{3-\gamma}{\gamma-1}\right)}\frac{1}{G}\frac{dM}{dx}\right]\rho$$

$$+\left(\frac{G_o}{G}\right)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}\frac{\partial \rho}{\partial x} = -\epsilon\left\{(K_sL)\rho\frac{\partial u}{\partial t} + \left(\frac{G_o}{G}\right)^{1/2}\rho\frac{\partial u}{\partial x} + \left(\frac{G_o}{G}\right)^{1/2}\frac{1}{G}\frac{dM}{dx}\left(\rho u\right) + u\frac{\partial u}{\partial x}\right\} \tag{73}$$

where $\epsilon = \frac{1}{\gamma}(P_s/P_o)$ represents approximately the ratio of the amplitude of the incident source sound pressure to the mean flow static pressure at x = 0. Equations (72) and (73) have been written with the nonlinear terms on the RHS. To solve these equations, we seek solutions of the form

$$\rho(x,t) = \sum_{n=-N}^{N} g_n(x) e^{int} \tag{74}$$

and

$$u(x,t) = \sum_{n=-N}^{N} f_n(x) e^{int} \tag{75}$$

The idea behind the assumed form of the solutions is that given a sound wave with radian frequency ω, nonlinear effects will generated higher and lower harmonics of frequencies ranging from -Nω, -(N-1)ω,... -2ω, -ω, o, ω, 2ω,... (N-1)ω, Nω for fixed N. Since we are interested only in determining the importance of retaining the nonlinear terms, we select N = 2. Thus we will have to solve five simultaneous differential eqns. with variable coefficients.

To demonstrate the details of the proposed approach, we will consider the first terms on the LHS and RHS of Eqn. (72).

$$(K_s L) \frac{\partial \rho}{\partial t} = -\left(\frac{1}{\gamma} \frac{P_s}{P_o}\right) \frac{\partial}{\partial x}(\rho u)$$

Substituting Eqns. (74) and (75) yields

$$i(K_s L) \sum_{n=-2}^{2} n g_n(x) e^{int} = -\epsilon \frac{\partial}{\partial x} \sum_{K=-2}^{2} \sum_{j=-2}^{2} g_j(x) f_K(x) e^{i(j+K)t}$$

Multiplying both sides by e^{-imt} and integrating over a period T yields

$$i(K_s L) \sum_{n=-2}^{2} n g_n(x) \frac{1}{T} \int_0^T e^{i(n-m)t} dt = -\epsilon \frac{d}{dx} \sum_{K=-2}^{2} \sum_{j=-2}^{2} g_j(x) f_K(x) \frac{1}{T} \int_0^T e^{i(j+K-m)t} dt \tag{76}$$

All terms on the LHS vanish except the term n · m and similarily all combinations of terms on the RHS vanish except those summing to m (i.e., j + k = m). Thus Eqn. (76) may be written

$$i(K_sL)m\,g_m(x) = -\epsilon \frac{d}{dx}\sum_{j=-2}^{2} g_j(x)\, f_{m-j}(x) \tag{77}$$

To compare directly with the linear case, we select m = 1 so that the contribution of the nonlinear term to the fundamental solution is

$$i(K_sL)g_1(x) = -\epsilon \frac{d}{dx}\left[g_2(x)f_{-1}(x) + g_1(x)f_0(x) + g_0(x)f_1(x) + g_{-1}(x)f_2(x)\right] \tag{78}$$

Thus to correct the linear solution for nonlinear effects requires the solution to the functions f_n and g_n for all n up to ± 2.

The functions f_0 and g_0 require special comment. They correspond to time independent or steady state solutions (the so-called acoustical steaming solutions). We believe that these solutions require at least two-dimensionality to exist because the streamlines are closed (i.e., there is no mass addition). Thus they vanish in the present one-dimensional application.

Incorporating the nonlinear terms, the governing differential equation becomes

$$\begin{bmatrix} g'_{r,n} \\ g'_{i,n} \\ f'_{r,n} \\ f'_{i,n} \end{bmatrix} = \frac{\delta(n-1)}{1-M^2}\begin{bmatrix} M_L \end{bmatrix}\begin{bmatrix} g_{r,n} \\ g_{i,n} \\ f_{r,n} \\ f_{i,n} \end{bmatrix} + \frac{\epsilon}{1-M^2}\begin{bmatrix} M\lambda^{-1/2} S_{r,n} - \lambda^{\left(\frac{3-\gamma}{\gamma-1}\right)} T_{r,n} \\ M\lambda^{-1/2} S_{i,n} - \lambda^{\left(\frac{3-\gamma}{\gamma-1}\right)} T_{i,n} \\ M\, T_{r,n} - \lambda^{-\left(\frac{1}{\gamma-1}\right)} S_{r,n} \\ M\, T_{i,n} - \lambda^{-\left(\frac{1}{\gamma-1}\right)} S_{i,n} \end{bmatrix} \tag{79}$$

where the matrix $[M_L]$ is the same as the linear matrix defined by Eqn. (45), $\delta(n-1)$ is the kroncker delta, and the quantities S and T are nonlinear source terms defined as

$$S_n = S_{r,n} + i\, S_{i,n} = \underbrace{\sum_m \sum_k}_{m+k=n} \left\{ f'_m g_k + f_m g'_k - \left(\frac{1-M^2}{RM}\right)\frac{dM}{dx} f_m g_k \right\} \tag{80}$$

$$T_n = T_{r,n} + i\,T_{i,n} = \sum_{m}\sum_{\substack{K \\ m+K=n}} \Bigg\{ i(K_sL)K g_m f_K + \left(\frac{R_o}{R}\right)^{\frac{1}{\gamma-1}} f_m f_K'$$

$$+\, m\left(\frac{R_o}{R}\right)^{1/2} g_m f_K' + \frac{1}{R}\frac{dM}{dX}\left(\frac{R_o}{R}\right)^{1/2} g_m f_K \Bigg\} \qquad (81)$$

Subscripts m and k denote respectively the m-th and k-th mode of the sound wave. The subscript n ranges from -2 to +2. It is important to realize that the solution to Eqn. (79) requires that the functions f_n and g_n be known for all n (in the present application n = +2).

4.2 Boundary Conditions

The boundary conditions specified in Section 3.2 for the linearized sound propagation equations can be extended, in a straight-forward manner, to apply to the nonlinear equation described by Eqn. (79). The boundary conditions are different for the cases of downstream and upstream sound propagation.

4.2.1 Downstream Propagation

At x=0, both incident and reflected waves exist. The incident wave is assumed known, of frequency corresponding to n = 1, and generated far upstream of the constricted. Reflected waves having (complex) frequencies corresponding to n = -2, -1, 2 are generated by the constriction. Also, transmitted waves are generated at x = $2K_sL$ corresponding to values of n = -2, -1, 1, 2. The corresponding functions $f_n(0)$ and $g_n(0)$ are

$$f_n(0) = \underbrace{\delta(n-1)}_{\text{Incident Wave}} + \underbrace{\Gamma_{o,n} e^{i\Psi_{o,n}}}_{\text{Reflected Wave}} \qquad (82)$$

and

$$g_n(0) = \underbrace{\delta(n-1)}_{\text{Incident Wave}} - \underbrace{\Gamma_{o,n} e^{i\Psi_{o,n}}}_{\text{Reflected Wave}} \qquad (83)$$

where $\delta(n-1)$ is used to specify the non-dimensional source sound introduced far upstream of the duct constricted. The corresponding functions $f_n(2K_sL)$ and $g_n(2K_sL)$ are

$$f_n(2K_sL) = g_n(2K_sL) = \Gamma_{2,n}\; e^{i\Psi^{*}_{2,n}} \qquad (84)$$

where

$$\psi^{*}_{2,n} = \psi_{2,n} - \frac{2nK_sL}{1+M_o} \tag{85}$$

The equation describing the conservation of energy flux for the downstream propagation case is,

$$\sum_{n=-N}^{N} \left\{ \underbrace{\left(\frac{1-M_o}{1+M_o}\right)^2 |\Gamma_{o,n}|^2}_{\text{Reflected}} + \underbrace{|\Gamma_{2,n}|^2}_{\text{Transmitted}} \right\} = 1 \tag{86}$$

4.2.2 Upstream Propagation

Analogous to the downstream sound propagation, the upstream boundary conditions are

$$f_n(2K_sL) = \underbrace{\delta(n-1)e^{\frac{2iK_sL}{1-M_o}}}_{\text{Incident Wave}} + \underbrace{\Gamma_{2,n}e^{i\psi^{*}_{2,n}}}_{\text{Reflected Wave}} \tag{87}$$

and

$$g_n(2K_sL) = \underbrace{-\delta(n-1)e^{\frac{2iK_sL}{1-M_o}}}_{\text{Incident Wave}} + \underbrace{\Gamma_{2,n}e^{i\psi^{*}_{2,n}}}_{\text{Reflected Wave}} \tag{88}$$

where

$$\psi^{*}_{2,n} = \psi_{2,n} - \frac{2nK_sL}{1-M_o} \tag{89}$$

The equation describing the conservation of energy flux, for the upstream sound propagation case, is

$$\sum_{n=-N}^{N} \left\{ \underbrace{\left(\frac{1+M_o}{1-M_o}\right)^2 |\Gamma_{2,n}|^2}_{\text{Reflected}} + \underbrace{|\Gamma_{o,n}|^2}_{\text{Transmitted}} \right\} = 1 \tag{90}$$

4.3 Integration Scheme

The value of N is restricted to N = 2. Consider first the downstream case. There are a total of 16 unknowns corresponding to values of N of (-2, -1, 1, 2). From Eqn. (79), there are 16 equations to be solved. To solve these equations, the nonlinear source terms are assumed known. Thus the governing equations are linear and the general solution is a linear combination of the homogeneous and particular solutions for each value of n. The solution proceeds much like that described in Section 3 except that the contribution to the particular solution must be included. For the downstream case, the n^{th} solution may be written

$$\left[\delta(n-1)-\alpha_n\right]\begin{vmatrix} a_{11}^n \\ a_{21}^n \\ a_{31}^n \\ a_{41}^n \end{vmatrix} - \beta_n \begin{vmatrix} a_{12}^n \\ a_{22}^n \\ a_{32}^n \\ a_{42}^n \end{vmatrix} + \left[\delta(n-1)+\alpha_n\right]\begin{vmatrix} a_{13}^n \\ a_{23}^n \\ a_{33}^n \\ a_{43}^n \end{vmatrix} + \beta_n \begin{vmatrix} a_{14}^n \\ a_{24}^n \\ a_{34}^n \\ a_{44}^n \end{vmatrix} + \begin{vmatrix} g_{R,P}^n \\ g_{I,P}^n \\ f_{R,P}^n \\ f_{I,P}^n \end{vmatrix} = \begin{vmatrix} \gamma_n \\ -\delta_n \\ \gamma_n \\ -\delta_n \end{vmatrix} \tag{91}$$

where

$$\alpha_n = \Gamma_{o,n}\cos\Psi_{o,n}\ ,\quad \beta_n = \Gamma_{o,n}\sin\Psi_{o,n},\ \gamma_n = \Gamma_{2,n}\cos\Psi_{2,n}^*,\ \delta_n = -\Gamma_{2,n}\sin\Psi_{2,n}^* \tag{92}$$

and

$g_{R,P}^n$, $g_{I,P}^n$, $f_{R,P}^n$, and $f_{I,P}^n$ represent the particular solutions of g and f. The solution to Eqn. (91) for values of n = -2, -1, 1, 2 yields the values of the 16 unknowns. The solution may be written as

$$\begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \\ \delta_n \end{bmatrix} = - \begin{bmatrix} (a_{13}^n - a_{11}^n) & (a_{14}^n - a_{12}^n) & -1 & 0 \\ (a_{23}^n - a_{21}^n) & (a_{24}^n - a_{22}^n) & 0 & 1 \\ (a_{33}^n - a_{31}^n) & (a_{34}^n - a_{32}^n) & -1 & 0 \\ (a_{43}^n - a_{41}^n) & (a_{44}^n - a_{42}^n) & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \delta(n-1)(a_{11}^n + a_{13}^n) + g_{R,P}^n \\ \delta(n-1)(a_{21}^n + a_{23}^n) + g_{I,P}^n \\ \delta(n-1)(a_{31}^n + a_{33}^n) + f_{R,P}^n \\ \delta(n-1)(a_{41}^n + a_{43}^n) + f_{I,P}^n \end{bmatrix} \tag{93}$$

The solution to the upstream sound propagation case is similar to that derived above and is written below as

$$\begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \\ \delta_n \end{bmatrix} = - \begin{bmatrix} (a_{13}^n - a_{11}^n) & (a_{14}^n - a_{12}^n) & -1 & 0 \\ (a_{23}^n - a_{21}^n) & (a_{24}^n - a_{22}^n) & 0 & 1 \\ (a_{33}^n - a_{31}^n) & (a_{34}^n - a_{32}^n) & -1 & 0 \\ (a_{43}^n - a_{41}^n) & (a_{44}^n - a_{42}^n) & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \delta(n-1)\cos\left(\frac{2nK_sL}{1-M_o}\right) + g_{R,P}^n \\ \delta(n-1)\sin\left(\frac{2nK_sL}{1-M_o}\right) + g_{I,P}^n \\ -\delta(n-1)\cos\left(\frac{2nK_sL}{1-M_o}\right) + f_{R,P}^n \\ -\delta(n-1)\sin\left(\frac{2nK_sL}{1-M_o}\right) + f_{I,P}^n \end{bmatrix} \tag{94}$$

In order for this scheme to be successful, the various source terms for S_n and T_n must be known for each value of n. The following approach is used to solve for S_n and T_n:

(1) Initially, the *homogeneous* solution n = 1 is solved. From Eqns. (80) and (81), the source terms for n = 2 is seen to depend only upon the n = 1 solutions. This follows because f_0 and g_0 are identically zero leaving only the terms m = k = 1 summing to m = k = 2). Thus the n = 1 homogeneous solution is used to estimate the f_2 and g_2 solutions.

(2) With the homogeneous n = 1 and the nonhomogeneous solutions known, the n = -2 and n = -1 solutions remain to be estimated. The nonlinear n = -2 solution is estimated directly from the nonlinear n = 2 solution by assuming that $\mathrm{Re}(f_n) = \mathrm{Re}(f_{-n})$ and $\mathrm{I_m}(f_n) = \mathrm{I_m}(f_{-n})$ with similar expressions for g_n. This assumption follows from the physical constraint that the solutions for f_n and g_n must be real. Thus knowing the solution for n = +2, the solution for n = -2 is also known.

(3) The solution for n = -1 can be estimated from the n = -2 and n = 1 solutions. This follows from Eqn. (79) where now m + k = -1; the only combination permitted is m = -2 and j = 1.

(4) With the solutions for the cases n = -1, 2 and -2 known, the correction to the linear (i.e., homogeneous) n = 1 solution is obtained from Eqn. (79) where here the various combinations summing to m + j = 1 are m = 2, j = -1, and m = -1, j = 2.

(5) Steps (1) through (4) above should be repeated until satisfactory convergence is obtained.

4.4 Results

The results of this approach are summarized in Figures 7 and 8 for the cases $k_sL = 1$ and $\nu = 0.746$. Figure 7 represents the downstream sound propagation solution and Figure 8, the upstream solution.

The results are discouraging in the sense that none of the solutions adequately conserved acoustic energy flux.

Figures 7 and 8 represents the results of only a single integration pass. No iterations were attempted because of contract cost limitations. Thus convergence was not explored, the accuracy of the proposed solution has not been adequately pursued. For this reason, there will be no further interpretation of the nonlinear solution. This does not mean that the approach is invalid -- it only means that more than one iteration is required before satisfactory convergence is achieved.

5.0 CONCLUDING REMARKS

The *linearized* equations describing the propagation of sound in variable area ducts have been shown to be singular when the duct mean axial flow component is sonic. The singularity is removed when previously ignored nonlinear terms are retained. This conclusion is quite general because the flow field within the duct is fully three-dimensional (both mean and fluctuating parts) and unsteady. The only assumptions made are that the fluid is inviscid and non-heat conducting.

A numerical study was conducted to map out, for the case of a plane wave propagating in a one-dimensional converging-diverging duct, the axial mean flow Mach number range at the throat M_{th} for which the *linearized* sound propagation equations are valid. The results of the study showed, in terms of the conservation of acoustic energy flux, that the linearized sound propagation equations are valid only when the mean flow Mach number $M_{th} < 0.6$ at the duct throat. For values of $M_{th} > 0.6$, the acoustic energy flux is not conserved. Since considerable care has been taken to minimize numerical errors by varying integration step size, integration algorithm, etc., it is concluded (at least for this example) that for $M_{th} > 0.6$, the *linearized* sound propagation equations do not adequately model the conservation of acoustic mass and momentum within the duct.

The numerical study also suggested for $M_{th} < 0.6$, that sound transmission within a converging-diverging duct is reduced primarily by scattering due to the duct mean velocity gradients. This does not preclude the possibility that other velocity gradients, say in the radial direction, may also significantly reduce sound transmission. Recall that this study considers only plane waves propagating in a one-dimensional mean flow.

These conclusions have an important bearing on our present understanding of how the sonic inlet acts as an effective noise barrier to internally generated sound. The large observed decreases of the sound transmission in a sonic inlet may arise in large measure because of scattering due to mean flow gradients within the inlet. Previously, it was believed that most of the sound transmission reduction was due to the mean sonic flow at the throat opposing, thereby preventing, the propagation of sound through the throat.

An attempt was made to study the nonlinear behavior of sound waves propagating in variable area ducts containing sonic or near sonic flow. Meaningful results were not obtained. This does not necessarily mean that our approach is invalid, but rather that more effort is required to assess its validity. It would be of immense value to continue the nonlinear solution described herein. This would provide a preliminary understanding of the acoustic behavior of the sonic inlet.

REFERENCES

1. F. Klujber, "Development of Sonic Inlets for Turbofan Engines", Journ. of Aircraft, Vol. 10, no. 10, October 1973, pp. 579-586.

2. D. Chestnut, "Noise Reduction by Means of Inlet-Guide-Vane Choking in an Axial-Flow Compressor", NASA TN D-4682, July, 1968.

3. Davis,S,S,and Johnson,M,L, "Propagation of Plane Waves in a Variable Area Duct Carrying a Compressible Subsonic Flow". presented at the 87th ASA meeting,April,1974, N.Y.,N.Y.

4. Eisenberg, N.A. and Kao, T.W., "Propagation of Sound Through a Variable-Area Duct with a Steady Compressible Flow", Jour. Acous. Soc. Am., Vol. 49, No. 1, 1971, pp. 169-175.

5. King, L.S. and Karamcheti, K., "Propagation of Plane Waves in the Flow Through a Variable Area Duct", AIAA Paper No. 73-1009.

6. Cantrell, R.H. and Hart, R.W., "Interaction Between Sound and Flow in Acoustic Cavities: Mass, Momentum and Energy Considerations, J. Acoust. Soc. Amer. V. 36, No. 4, 697-706, (April 1964).

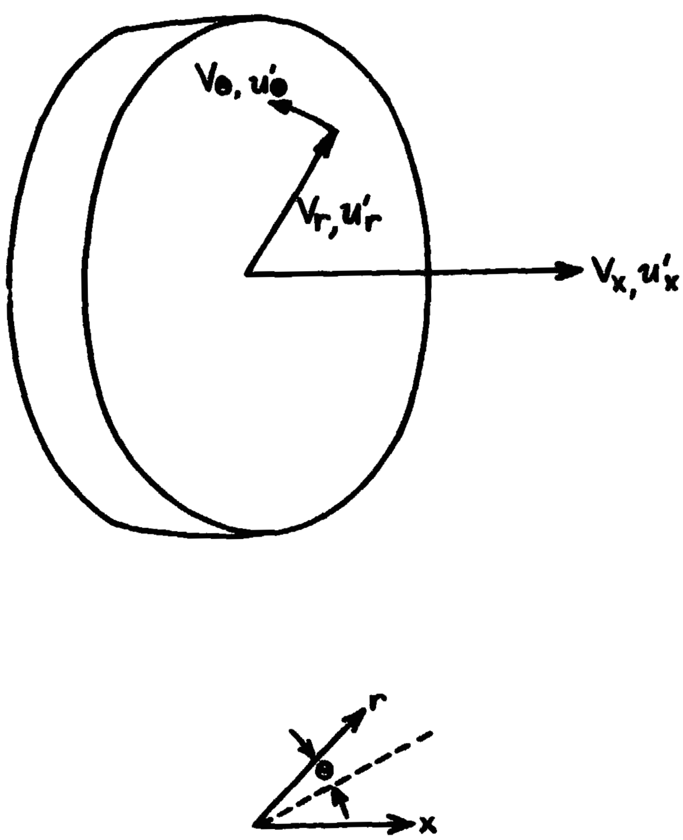

FIGURE 1. CYLINDRICAL COORDINATE SYSTEM DEFINITION

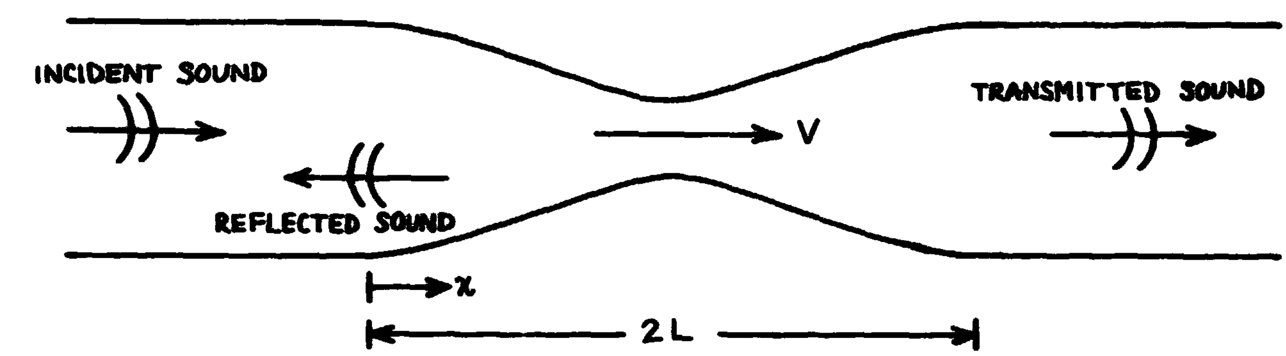

FIGURE 2(a) DOWNSTREAM SOUND PROPAGATION

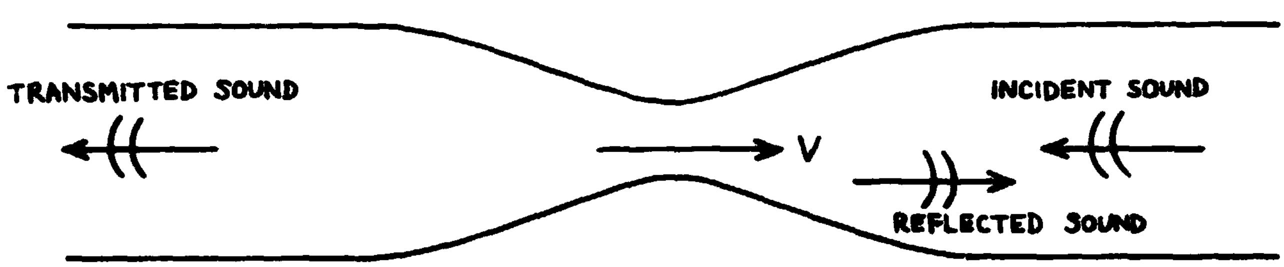

FIGURE 2(b) UPSTREAM SOUND PROPAGATION

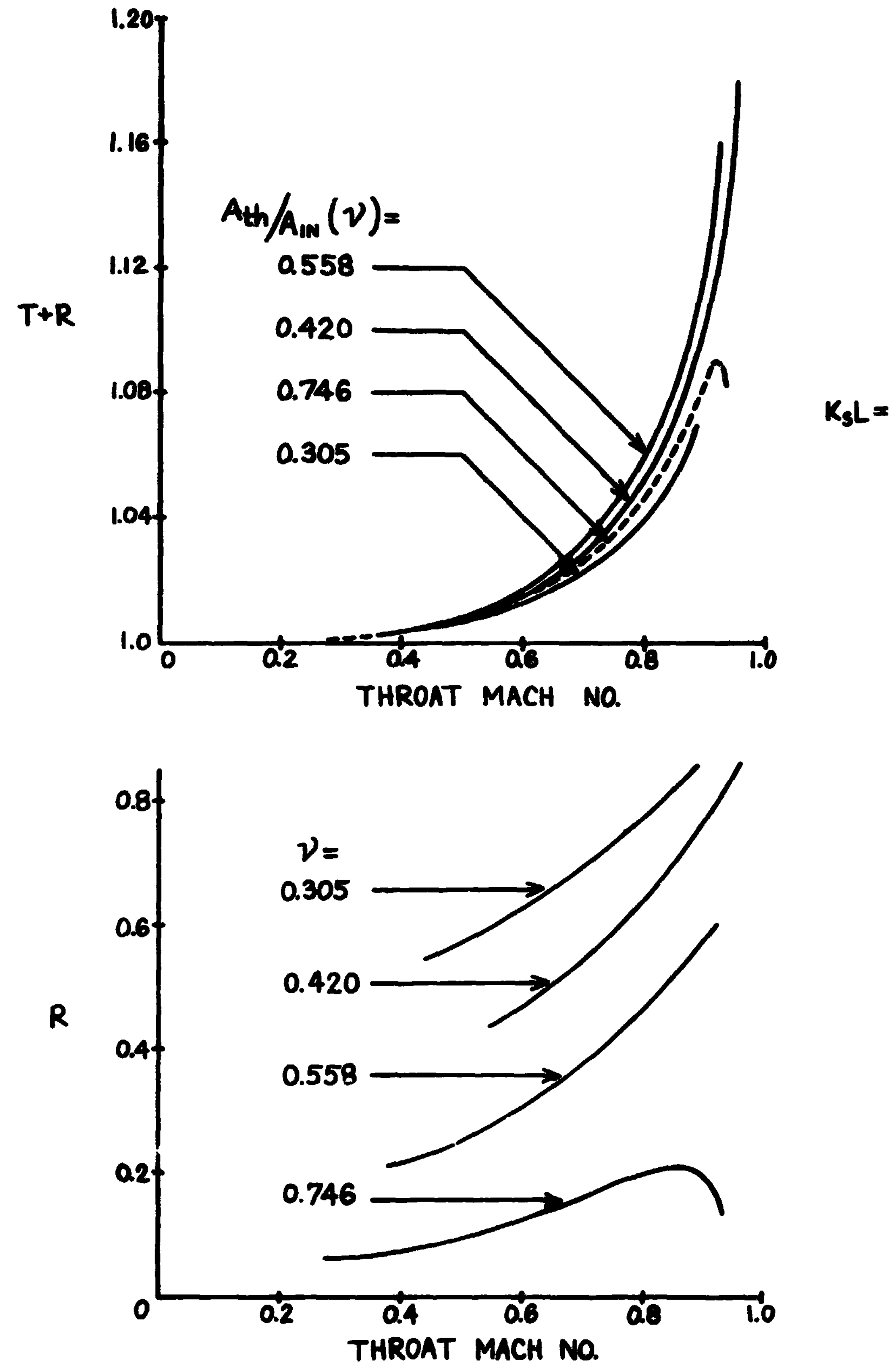

FIGURE 3. EFFECT OF THROAT MACH NO. ON UPSTREAM SOUND TRANSMISSION

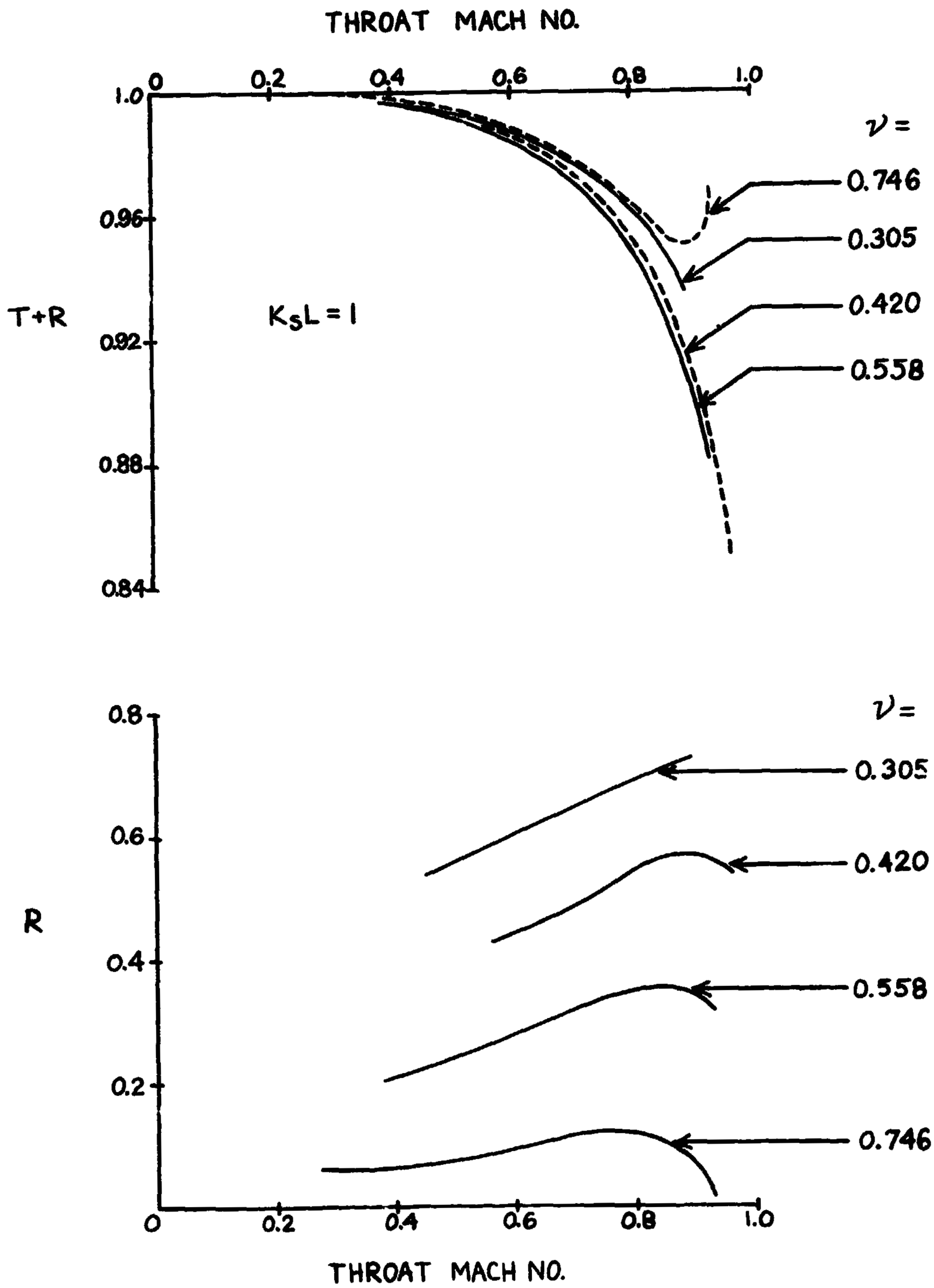

FIGURE 4. EFFECT OF THROAT MACH NO. ON DOWNSTREAM SOUND TRANSMISSION

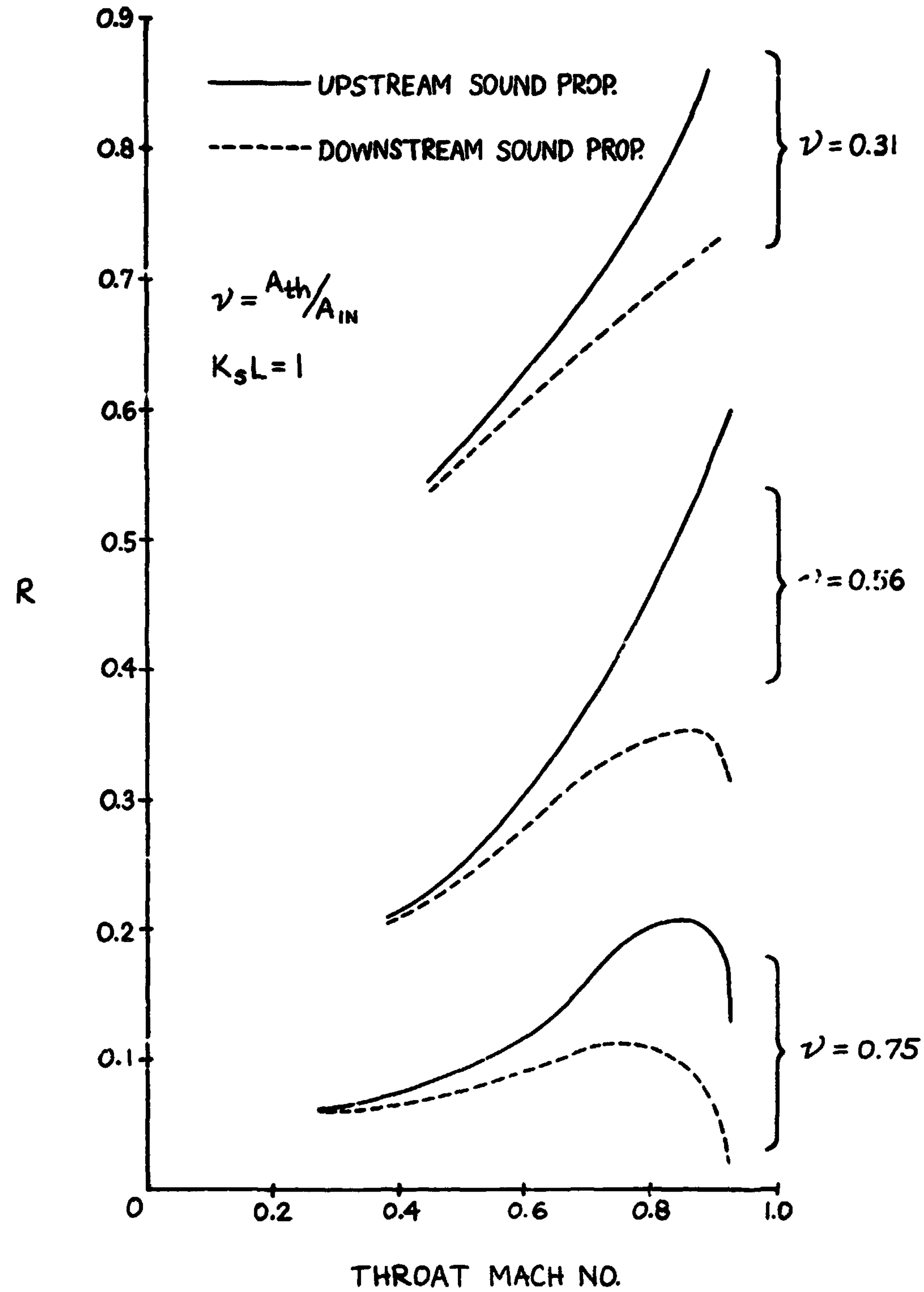

FIGURE 5. EFFECT OF DUCT INLET-TO-THROAT AREA RATIO ON SOUND REFLECTION COEFFICIENT

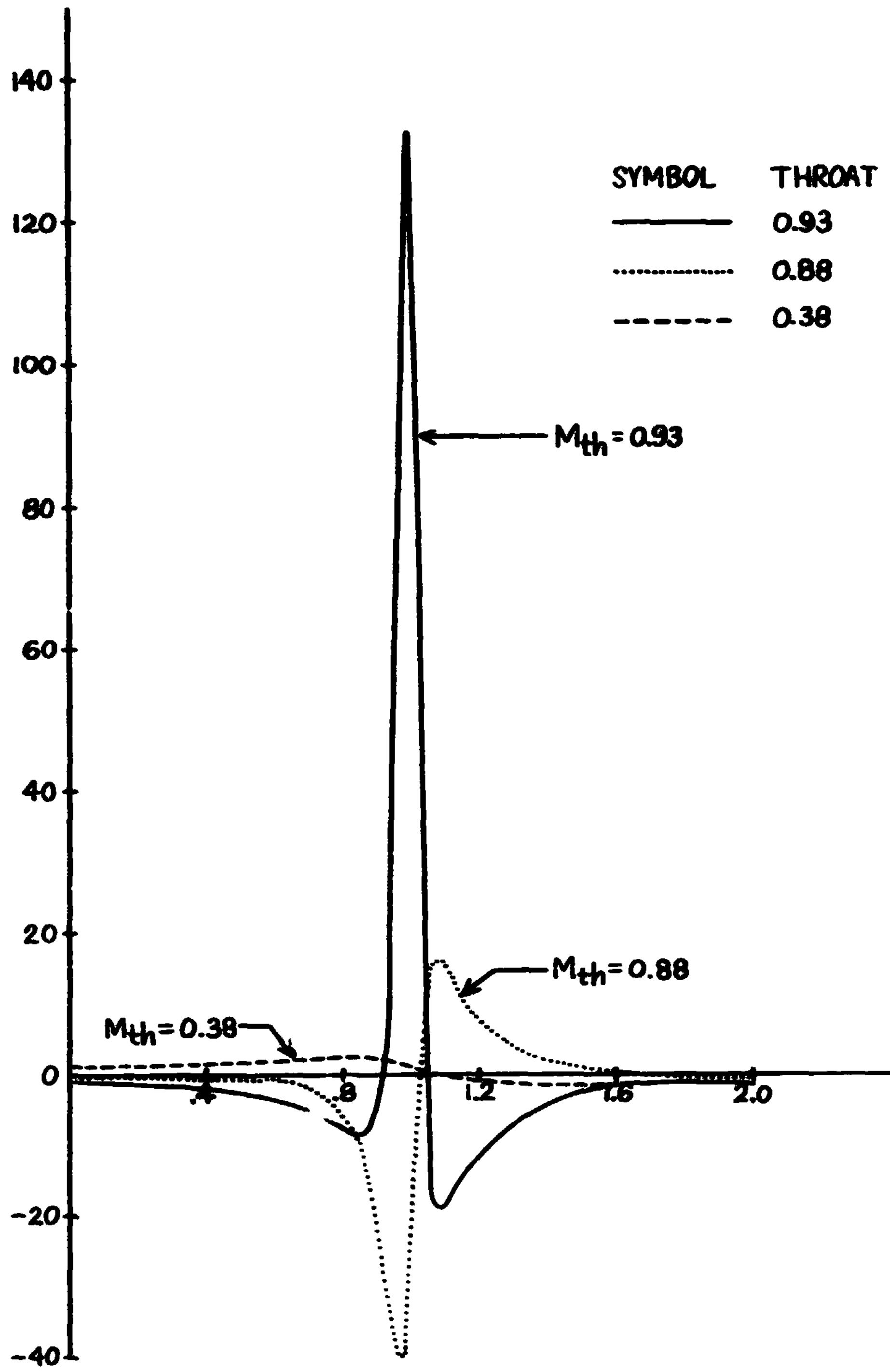

FIGURE 6 EFFECT OF MEAN FLOW ON SOUND BEHAVIOR

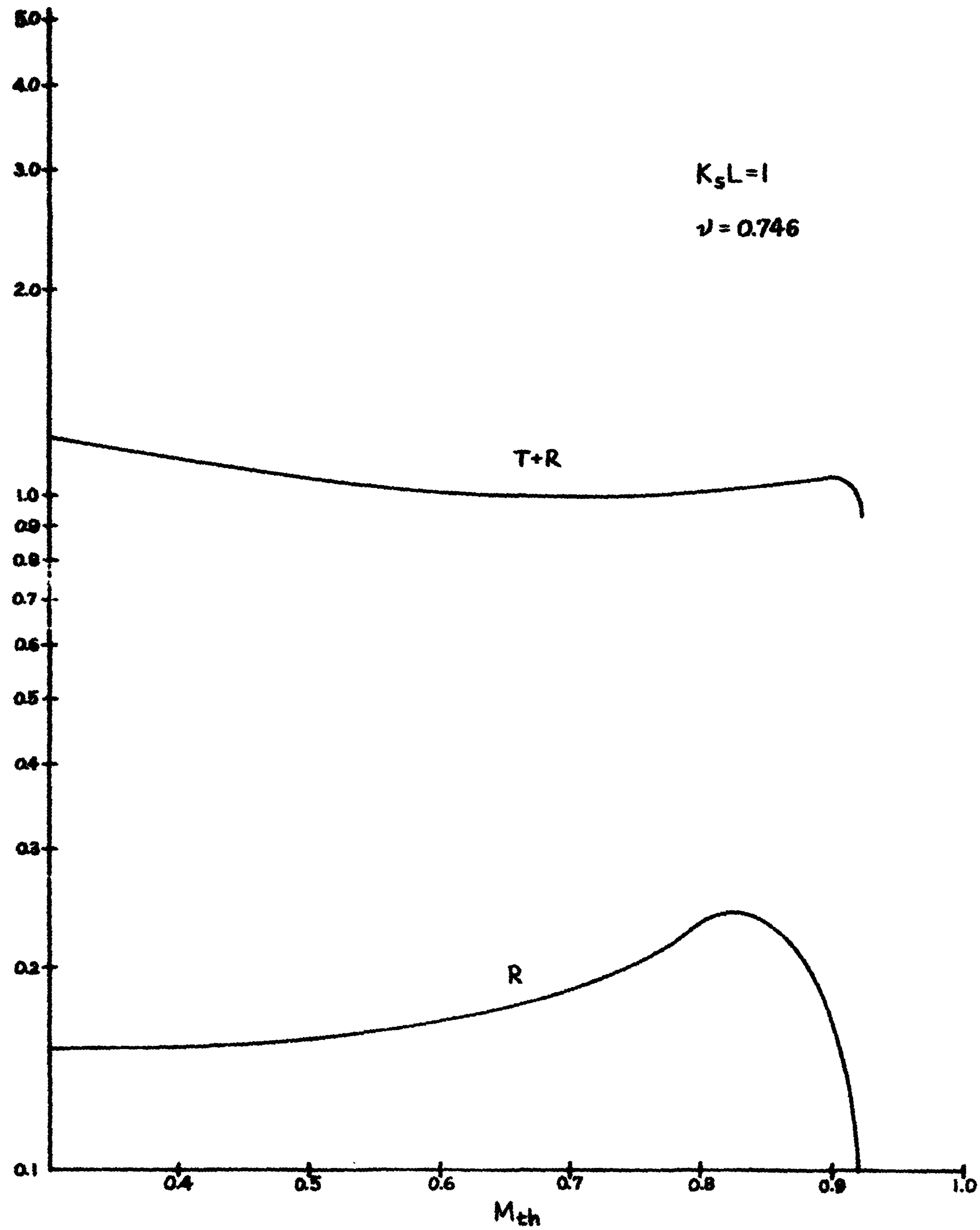

FIGURE 7. NONLINEAR SOLUTION - DOWNSTREAM SOUND PROPAGATION

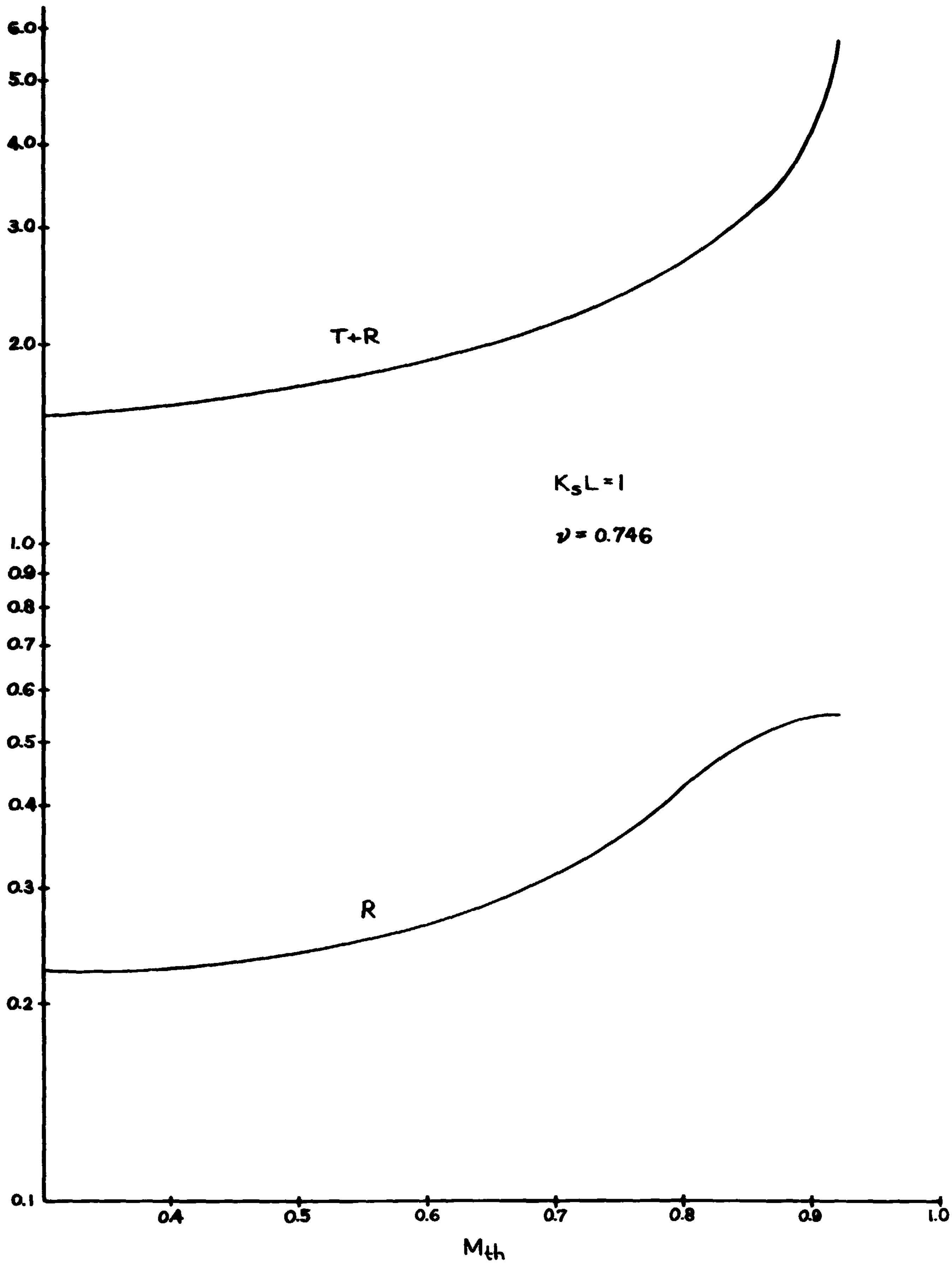

FIGURE 8. NONLINEAR SOLUTION - UPSTREAM SOUND PROPAGATION