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by

C. S. Gardner

RRL Publication No. 481

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ABSTRACT

A theoretical model for the range error covariance is derived by assuming that the residual refraction errors are due entirely to errors in the meteorological data which are used to calculate the atmospheric correction. The properties of the covariance function are illustrated by evaluating the theoretical model for the special case of a dense network of weather stations uniformly distributed within a circle.

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I. INTRODUCTION

When it is necessary to measure the relative positions of numerous widely spaced points on the earth's surface, it can be economical to place a laser ranging system on board a satellite and measure the range to cube corners on the earth. This may be the case for the large arrays which would be used to monitor crustal motion. These arrays could consist of up to 400 survey points uniformly spaced within a square measuring 500 km on a side. To minimize cost and maintenance only a few of the survey points would be instrumented to provide the meteorological data required by the range correction formulas. Consequently, for a satellite based ranging system the range error caused by atmospheric refraction can be considerably different from the error for a ground based system.

The basic problem in satellite based ranging is to measure the relative vertical and horizontal positions of the cube corners in the array. In the case of two retroreflectors, this is accomplished by making "m" range measurements from the spacecraft to one cube corner and "n" range measurements to the other cube corner, using these ranges to recover the vertical and horizontal positions. The overall measurement precision is a function of both instrumentation errors and channel effects. The expected precision can be estimated if the statistical properties of the system accuracy are known [1]. In particular the effects of errors in the atmospheric correction can be determined if the covariance matrix is known for the residual refraction errors associated with the m + n range measurements to the two retroreflectors.

The covariance can be calculated directly by ray tracing. However, this is a cumbersome process requiring radiosonde data from multiple locations throughout the array of survey points. The covariance matrix will have to be calculated for each different satellite pass and array configuration and will be valid only for the region from which the radiosonde data were collected.

As an alternative, a theoretical model for the range error covariance can be derived if we make some simplifying assumptions about the residual refraction errors. This approach is discussed in the following sections. The range error is related to errors in the meteorological data which are used to calculate the atmospheric correction. A theoretical model for the covariance function is derived and its properties examined.

II. RESIDUAL REFRACTION ERRORS

The range error due to atmospheric refraction is defined as the difference between the optical path length R_o and the straight-line path length R_s. The error can be written in the form

$$\Delta R = R_o - R_s = S + G$$

$$G = \sum_{k=1}^{\infty} G_K \quad . \tag{1}$$

S is the most significant term and corresponds to the error introduced by a spherically symmetric atmosphere. It is on the order of 13 meters at 10° elevation. The G_k terms are the errors introduced by horizontal refractivity gradients. The effects of these terms were investigated by ray tracing through three-dimensional refractivity profiles [2, 3]. The profiles were constructed from radiosonde data gathered in Project Haven Hop I [4]. The results indicate that only the linear gradients contribute significantly to the range error at elevation angles above 10°. At 10° elevation linear gradients contribute up to 3 or 4 centimeters to the range error while quadratic and higher order variations contribute less than 2 to 3 millimeters [2]. The contribution due to quadratic and higher order variations decreases to less than a millimeter above 20° elevation. Therefore, the range error can be accurately expressed using only the first two terms in Equation (1).

$$\Delta R \simeq S + G_1 \tag{2}$$

Analytic expressions for the range error can be derived by evaluating the integral of the group refractivity and its horizontal gradient along the optical path. In general, the actual refractivity and gradient profiles will not be known. However, accurate estimates of the range error can be calculated by using suitable theoretical models.

Gardner and Rowlett [2] used Marini and Murray's [5, 6] approach to obtain an estimate of S in terms of the surface pressure, temperature and relative humidity at the laser site.

$$SC = f(\lambda) \frac{A}{\sin E + \cfrac{B/A}{\sin E + \cfrac{C/B}{\sin E + 0.1}}} \tag{3}$$

$$A = \frac{1}{F(\theta,H)} \left[0.002357\, P_s + 0.000141\, e_s\right] + 1.0842 \times 10^{-8}\, P_s T_s K_s - 9.4682 \times 10^{-8} \frac{P_s^2}{T_s}$$

$$B = 1.0842 \times 10^{-8}\, P_s T_s K_s + 4.7343 \times 10^{-8} \frac{P_s^2}{T_s} \frac{2}{3 - 1/K_s}$$

$$C = 1.4961 \times 10^{-13}\, P_s T_s^2 \frac{K_s^2}{2 - K_s}$$

$$K_s = 1.163 + 0.00968 \cos 2\theta - 0.00104\, T_s + 0.00001435\, P_s$$

$$f(\lambda) = 0.965 + 0.0164/\lambda^2 + 0.000228/\lambda^4$$

θ = colatitude of laser site

H = altitude of laser site (km)

P_s = surface pressure at laser site (mb)

T_s = surface temperature at laser site (°K)

e_s = surface partial pressure of water vapor at laser site (mb)

λ = wavelength of laser radiation (μm)

The accuracy of this spherical correction formula was investigated by comparing it with ray trace corrections. The formula is an unbiased estimator of S. This is illustrated by the ray tracing data in Table I which was taken from Gardner and Rowlett's report [2]. RT_1 is the range correction obtained by ray tracing through a spherically symmetric refractivity profile. The profiles were generated from the Haven Hop I radiosonde data. If the radiosonde profiles and the ray tracing procedure accurately describe the atmospheric refraction, then RT_1 will be equal to S.

TABLE I

COMPARISON OF THE SPHERICAL CORRECTION FORMULA AND THE SPHERICALLY SYMMETRIC RAY TRACE

	$SC - RT_1$ (31 Data Sets)	
Elevation Angle	Mean (cm)	Standard Deviation (cm)
10°	-0.03	0.46
20°	0.06	0.25
40°	0.00	0.14
80°	0.01	0.09

RT_1 = spherically symmetric ray trace correction

SC = spherical correction formula

The standard deviation of the difference between SC and RT_1 arises from two factors: errors in the formula for SC and errors in the measured values of surface pressure, temperature and relative humidity which are used to calculate SC. The dominant effect is pressure (see Section III). A 1 mb pressure error introduces approximately 14 mm error in SC at 10°

elevation. The surface pressure used to make the comparison in Table I was calculated by fitting a regression polynomial to the pressure measurements obtained by eight weather stations surrounding the laser site. The pressure error is a function of the order of the regression polynomial, the location of the weather stations and the measurement error. For the data in Table I the effective pressure error is estimated at 0.6 mb which gives a value of 8.4 mm for the error in SC at 10° elevation. The estimated error is larger than the standard deviation of RT_1 - SC listed in Table I. Either the estimated pressure error is too large, or there is some correlation between the pressure error effects in RT_1 and SC. We believe the latter explanation is more likely since the surface pressure measured by the radiosonde balloons is also used to generate the refractivity profile. Consequently, it seems reasonable to assume that the formula errors in SC are negligible compared to the measurement errors in the meteorological data used to evaluate SC.

Gardner and Hendrickson [3] obtained an estimate of G_1 in terms of the horizontal pressure and temperature gradients at the laser site

$$GC_1 = \frac{C}{\sin E \tan E}\, \underline{n} \cdot \nabla(P_s T_s K_s) + \frac{D(1 + \frac{1}{2}\cos^4 E)}{\sin^3 E \tan E}\, \underline{n} \cdot \nabla\left(\frac{P_s\,{}_s^{2} K_s^{2}}{2 - K_s}\right) \quad (4)$$

where

$C = 5.915 \times 10^{-2}\ f(\lambda)$

$D = -6.362 \times 10^{-7}\ f(\lambda)$

$\underline{n} = \sin\alpha\ \underline{x} + \cos\alpha\ \underline{y}$

α = satellite azimuth angle

$\underline{x}$ and $\underline{y}$ are the east and north unit vectors.

The accuracy of the gradient correction formula was also investigated by comparing it with ray trace corrections. The results indicate that GC_1 is nearly an unbiased estimator of G_1. This is illustrated in Figures 1 and 2 where the means of the corrected and uncorrected ray trace data are plotted versus azimuth. RT_3 is the range correction calculated by ray tracing through three-dimensional refractivity profiles (see reference [2]). RT_3 contains the effects of horizontal gradients and depends on both azimuth and elevation angles. RT_1 is the range correction obtained by ray tracing through a spherically symmetric profile. The gradient effects are isolated by calculating the term $RT_3 - RT_1$ which will be equal to G_1. Any effects not compensated by GC_1 are given by the error term $RT_3 - RT_1 - GC_1$.

The standard deviation of $RT_3 - RT_1 - GC_1$ is plotted in Figures 3 and 4. It arises from errors in the formula for GC_1 and errors in the meteorological data which are used to calculate GC_1. The magnitude of the latter effect can be estimated from an analysis of the standard error of the regression coefficients which were used to obtain a least squares fit of the surface data. Gardner and Hendrickson's [3] results indicate that a 1 to 1.3° K temperature error could account for almost the entire residual error in $RT_3 - RT_1 - GC_1$.

In summary, the ray trace comparisons indicate that the spherical correction formula SC and the gradient correction formula GC_1 are nearly unbiased estimators of the range error due to atmospheric refraction. The uncorrected residual error appears to be caused almost entirely by errors in the surface meteorological data which are used to calculate SC and GC_1. In the following section this assumption is used to derive a theoretical model for the range residual covariance function.

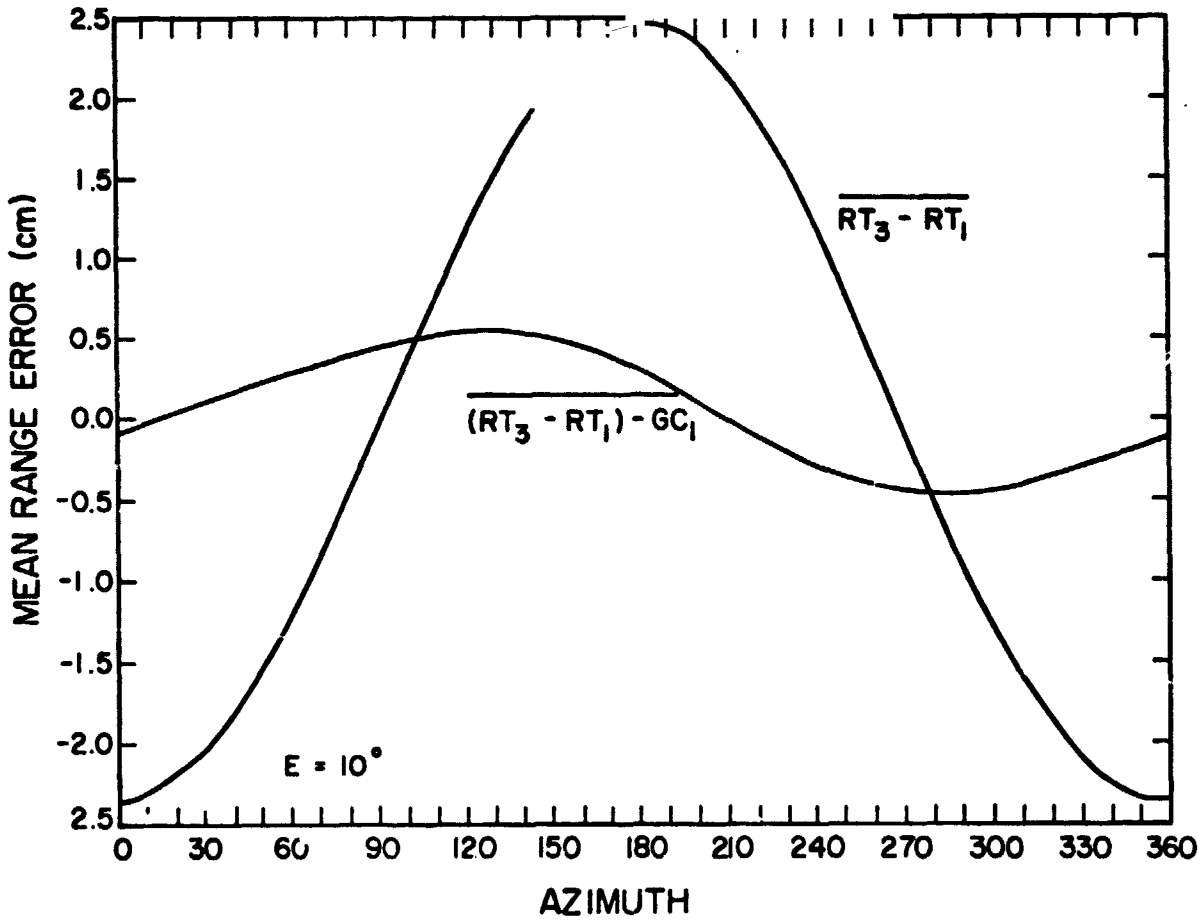

Figure 1. Mean of the uncorrected (RT_3 - RT_1) and corrected (RT_3 - RT_1 - GC_1) gradient error versus azimuth. The elevation angle is 10°. The data are from Gardner and Hendrickson's report, page 26 [3].

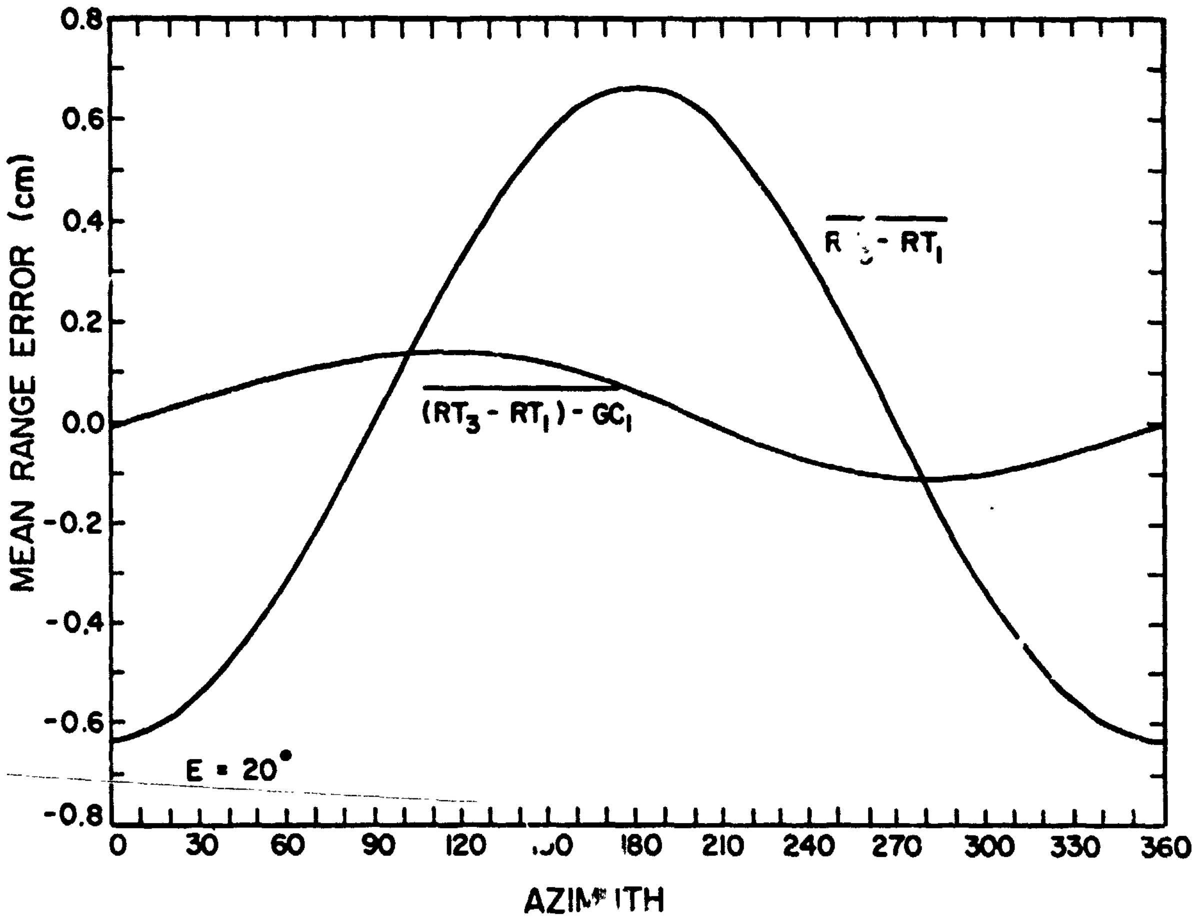

Figure 2. Mean of the uncorrected $(RT_3 - RT_1)$ and uncorrected $(RT_3 - RT_1 - GC_1)$ gradient error versus azimuth. The elevation angle is 20°. The data are from Gardner and Hendrickso 's report, page 28 [3].

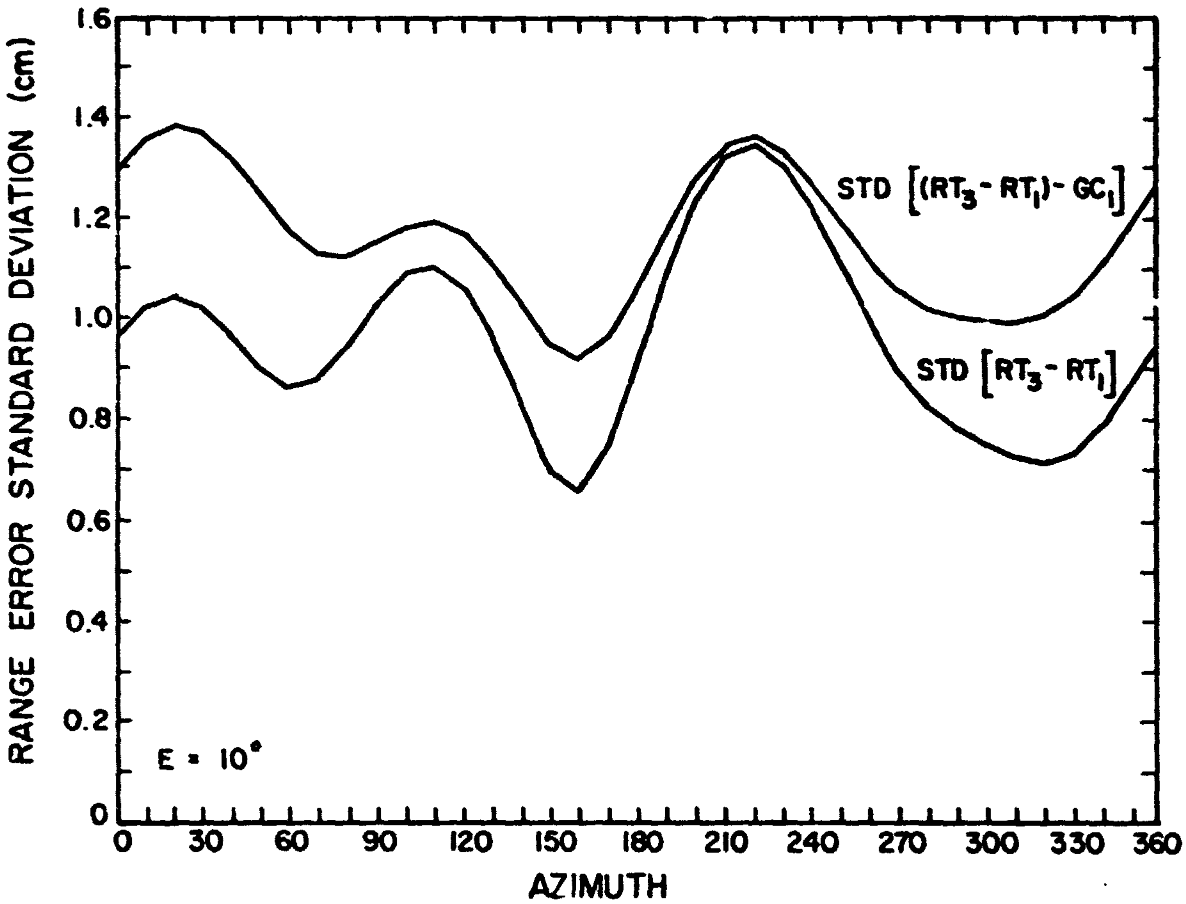

Figure 3. Standard deviation of the uncorrected (RT_3 - RT_1) and corrected (RT_3 - RT_1 - GC_1) gradient error versus azimuth. The elevation angle is 10°. The data are from Gardner and Hendrickson's report, page 27 [3].

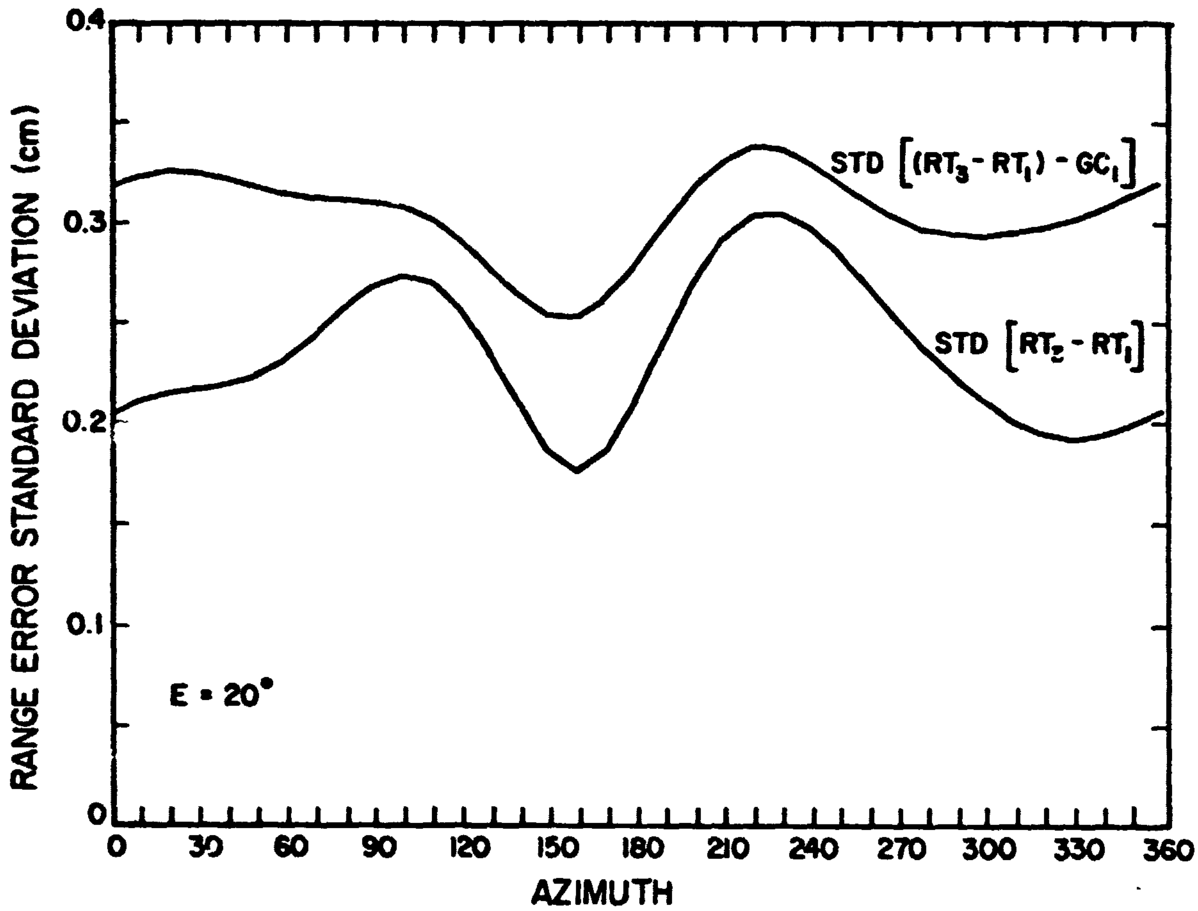

Figure 4. Standard deviation of the uncorrected (RT_3 - RT_1) and corrected (RT_3 - RT_1 - GC_1) gradient error versus azimuth. The elevation angle is 20°. The data are from Gardner and Hendrickson's report, page 29 [3].

III. RANGE ERROR COVARIANCE FUNCTION

The problem is to calculate the covariance between the refraction errors inherent in the range measurements to two different points on the earth's surface. If both the spherical and gradient correction terms are applied to the ranging data, the residual refraction error will be given by

$$\Delta R = R_o - R_s - (SC + GC_1) \simeq \Delta SC + \Delta GC_1 \tag{5}$$

where ΔSC and ΔGC_1 are the errors in the correction formulas. We will assume that ΔSC and ΔGC_1 are due entirely to errors in the measured values of surface pressure, temperature and relative humidity. ΔSC is estimated by taking derivatives of SC with respect to the meteorological parameters

$$\Delta SC = \frac{\partial SC}{\partial P}\,\Delta P + \frac{\partial SC}{\partial T}\,\Delta T + \frac{\partial SC}{\partial Rh}\,\Delta Rh$$

$$\frac{\partial SC}{\partial P} \simeq \frac{f(\lambda)}{F(\theta,H)}\,\frac{2.357 \times 10^{-3}}{\sin E}$$

$$\frac{\partial SC}{\partial T} \simeq -\,\frac{f(\lambda)1.084 \times 10^{-8}\ P_s K_s}{\sin^3 E} \tag{6}$$

$$\frac{\partial SC}{\partial Rh} \simeq \frac{f(\lambda)}{F(\theta,H)}\,\frac{8.615 \times 10^{6}}{\sin E}\,\exp\left[\frac{17.27(T_s - 273.15)}{237.15 + (T_s - 273.15)}\right]$$

When ΔP is measured in mb, ΔT in °K and ΔRh in percent, ΔSC is given in meters. The derivatives are plotted versus elevation angle in Figure 5. Since the measurement errors are statistically independent, the total

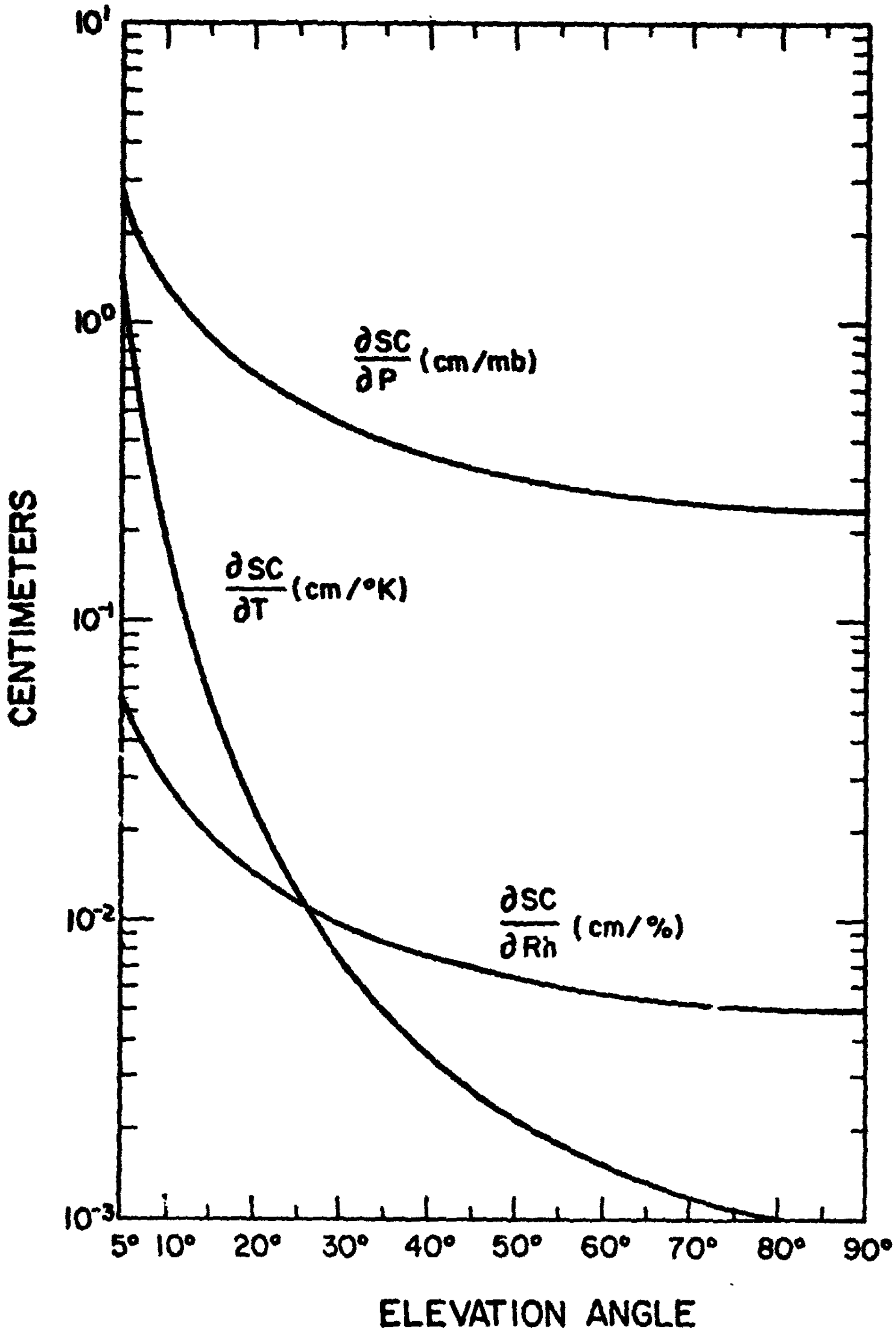

Figure 5. Variation of the spherical correction formula (Equation (13)) with respect to pressure, temperature and relative humidity.

error in SC is given by

$$\sigma_{SC} = \left[\left(\frac{\partial SC}{\partial P} \sigma_P \right)^2 + \left(\frac{\partial SC}{\partial T} \sigma_T \right)^2 + \left(\frac{\partial SC}{\partial Rh} \sigma_{Rh} \right)^2 \right]^{1/2} \quad (7)$$

where

σ_{SC} = STD(ΔSC)

σ_P = STD(ΔP)

σ_T = STD(ΔT)

σ_{Rh} = STD(ΔRh).

Typical measurement errors are 0.5 to 1.0 mb for pressure, 0.7 to 1.5° K for temperature and 5 to 10 percent for relative humidity. Consequently, above 20° elevation pressure errors are the dominant factor in ΔSC

$$\Delta SC \approx \frac{\partial SC}{\partial P} \Delta P \quad . \quad (8)$$

To minimize cost and maintenance only a few of the survey points on the earth's surface would be instrumented to provide the meteorological measurements required by the range correction formulas. Pressure, temperature and relative humidity at the survey points would be interpolated from measurements obtained at existing weather stations. In this case the error in the range correction formula would be related to the interpolation technique and the locations of the surface weather stations. To illustrate, we will calculate the expected error when a least squares regression polynomial is used to interpolate the surface data. This approach was used by Gardner and Hendrickson [3] to investigate the accuracy of the gradient correction formula.

Although the weather stations in a realistic network will be located at different altitudes, we will consider the simpler case where the stations are all located at the same altitude. The meteorological parameters are individually expanded in a two-dimensional m^{th} order regression polynomial of the form

$$F = \sum_{k=1}^{m} \beta_k X_k \qquad (9)$$

where the X_k variables are the horizontal coordinate polynomials and the β_k factors are the regression coefficients. Since the weather station network and survey points extend over a region several hundred km in length, the X_k variables should be expressed in geocentric coordinates. For linear regression we have

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \theta \\ x_3 &= \phi \sin \theta \ . \end{aligned} \qquad (10)$$

ϕ is the longitude and θ is the colatitude. θ is proportional to horizontal displacement in the north-south direction and $\phi \sin \theta$ is proportional to horizontal displacement in the east-west direction.

The regression coefficients are calculated by measuring F at $n \geq m$ different weather stations. Let F_i be the i^{th} measurement which is made at the coordinates $(X_{i1}, X_{i2}, \ldots, X_{im})$. Define $\underline{X}$ as the $n \times m$ matrix containing x_{ij} in its i^{th} row j^{th} column.

$$\underline{X} = \begin{bmatrix} X_{11} \ldots X_{1m} \\ \vdots \\ X_{n1} \ldots X_{nm} \end{bmatrix} \qquad (11)$$

Also define the column vectors

$$\begin{aligned} \underline{F} &= [F_1 F_2 \ldots F_n]^T \\ \underline{\beta} &= [\beta_1 \beta_2 \ldots \beta_m]^T \ . \end{aligned} \qquad (12)$$

The weighted least squares estimate for $\underline{\beta}$ is [7]

$$\hat{\underline{\beta}} = (\underline{X}^T \, \underline{\Sigma}_F^{-1} \, \underline{X})^{-1} \, \underline{X}^T \, \underline{\Sigma}_F^{-1} \, \underline{F} \tag{13}$$

$\underline{\Sigma}_F$ is the diagonal covariance matrix of the measured values of F. When the weather stations are all located at the same altitude, the measurement error is probably the same at each station so that $\underline{\Sigma}_F$ is just the identity matrix multiplied times the standard deviation of F. The m × m variance-covariance matrix for the regression coefficients is given by

$$\underline{C}_F = \begin{bmatrix} C_{11} \cdots\cdots C_{1m} \\ \vdots \qquad \vdots \\ C_{m1} \cdots\cdots C_{mm} \end{bmatrix} = (\underline{X}^T \, \underline{\Sigma}_F^{-1} \, \underline{X})^{-1} \tag{14}$$

$$C_{kl} = \text{cov}\ (\beta_k, \beta_l)$$

$$C_{kk} = \text{var}\ (\beta_k) \quad .$$

If we let $\underline{Z} = (Z_1 \;\; Z_2 \; \ldots Z_m)^T$ denote the position coordinate for a survey point, then the estimated value of F at the survey point is

$$\hat{F} = \hat{\underline{\beta}}^T \underline{Z} \quad . \tag{15}$$

The standard error in $\hat{F}$ is

$$\text{STD}(\Delta\hat{F}) = \left[\sum_{k=1}^{m} \sum_{\ell=1}^{m} C_{k\ell} Z_k Z_\ell \right]^{1/2} = (\underline{Z}^T \underline{C}_F \underline{Z})^{1/2} \ . \tag{16}$$

The covariance between the errors in the estimate of F at two different survey points is

$$\text{cov}(\Delta\hat{F}_1, \; \Delta\hat{F}_2) = \underline{Z}_1^T \; \underline{C}_F \underline{Z}_2 = \underline{Z}_2^T \; \underline{C}_F \underline{Z}_1 \ . \tag{17}$$

The standard error in SC is calculated using Equations (8) and (17)

$$\sigma_{SC} = \left[\left(\frac{\partial SC}{\partial P}\right)^2 \underline{Z}^T \underline{C}_p \underline{Z}\right]^{1/2} \tag{18}$$

where $\underline{C}_p$ is the pressure measurement variance - covariance matrix.

GC_1 is evaluated by calculating the horizontal gradients of the parameters $P_sT_sK_s$ and $\frac{P_sT_s^2K_s^2}{2 - K_s}$. This can be accomplished by expanding the parameters in a regression polynomial and then taking the derivative of the expansion

$$\underline{n} \cdot \nabla\hat{F} = \underline{n} \cdot \nabla(\underline{\hat{\beta}}^T\underline{Z}) = \underline{\hat{\beta}}^T\underline{n} \cdot \nabla\underline{Z}. \tag{19}$$

The errors in $\underline{n} \cdot \nabla\hat{F}$ can be related to the errors in the measured values of F

$$\Delta(\underline{n} \cdot \nabla\hat{F}) = \underline{n} \cdot \nabla(\Delta\hat{F})$$

$$\mathrm{STD}\left[\Delta(\underline{n} \cdot \nabla\hat{F})\right] = \left[(\underline{n} \cdot \nabla\underline{Z})^T\underline{C}_F(\underline{n} \cdot \nabla\underline{Z})\right]^{1/2} \tag{20}$$

The covariance between the gradient errors at two different survey points is

$$\mathrm{cov}\left[\Delta(\underline{n}_1 \cdot \nabla_1\hat{F}_1),\ \Delta(\underline{n}_2 \cdot \nabla_2\hat{F}_2)\right] = (\underline{n}_1 \cdot \nabla_1)(\underline{n}_2 \cdot \nabla_2)\mathrm{cov}(\Delta\hat{F}_1,\ \Delta\hat{F}_2)$$

$$= (\underline{n}_2 \cdot \nabla_2\underline{Z}_2)^T\underline{C}_F(\underline{n}_1 \cdot \nabla_1\underline{Z}_1). \tag{21}$$

The unit vectors, gradients and position coordinates are evaluated at survey points 1 and 2 as indicated by the subscripts.

It is not difficult to show that the magnitude of the error in the second term comprising GC_1 (see Equation (4)) is less than 20 percent of the error in the first term at 10° elevation. The second term error decreases to less than 0.5 percent of the first term error at zenith.

If we neglect this small contribution from the second term comprising GC_1, the residual refraction error can be approximated as

$$\Delta R \approx \frac{\partial SC}{\partial P} \Delta P + \frac{C}{\sin E \tan E} \Delta \left[\underline{n} \cdot \nabla (P_s T_s K_s) \right] \quad . \tag{22}$$

The covariance between the refraction errors to two different survey points on the earth's surface is given by

$$\begin{aligned} \mathrm{cov}(\Delta R_1, \Delta R_2) &= \frac{\partial SC}{\partial P_1} \frac{\partial SC}{\partial P_2} \mathrm{cov}(\Delta P_1, \Delta P_2) \\ &+ \frac{\partial SC}{\partial P_1} \frac{C}{\sin E_2 \tan E_2} \mathrm{cov}\left\{ \Delta P_1, \Delta[\underline{n}_2 \cdot \nabla_2 (P_s T_s K_s)_2] \right\} \\ &+ \frac{\partial SC}{\partial P_2} \frac{C}{\sin E_1 \tan E_1} \mathrm{cov}\left\{ \Delta P_2, \Delta[\underline{n}_1 \cdot \nabla_1 (P_s T_s K_s)_1] \right\} \\ &+ \frac{C^2}{\sin E_1 \tan E_1 \sin E_2 \tan E_2} \mathrm{cov}\left\{ \Delta[\underline{n}_1 \cdot \nabla_1 (P_s T_s K_s)_1], \Delta[\underline{n}_2 \cdot \nabla_2 (P_s T_s K_s)_2] \right\} \end{aligned} \tag{23}$$

The pressure covariance can be calculated from Equation (17) and the gradient covariance can be calculated from Equation (21). The covariance between the pressure and gradient errors is calculated by first rewriting the covariance function

$$\mathrm{cov}\left\{ \Delta P_1, \Delta[\underline{n}_2 \cdot \nabla_2 (P_s T_s K_s)_2] \right\} = (\underline{n}_2 \cdot \nabla_2) \mathrm{cov}[\Delta P_1, \Delta (P_s T_s K_s)_2] . \tag{24}$$

$\Delta(P_s T_s K_s)_2$ is a function of both pressure and temperature errors

$$\Delta(P_s T_s K_s)_2 = \left(T_s K_s + P_s T_s \frac{\partial K_s}{\partial P_2} \right)_2 \Delta P_2 + \left(P_s K_s + P_s T_s \frac{\partial K_s}{\partial T_2} \right)_2 \Delta T_2 \ . \tag{25}$$

The variation of K_s with respect to pressure is small and can be neglected. Since ΔP_1 and ΔT_2 are uncorrelated, the covariance is given by

$$\mathrm{cov}\left\{\Delta P_1,\ \Delta[\underline{n}_2 \cdot \nabla_2(P_sT_sK_s)_2]\right\} = (\underline{n}_2 \cdot \nabla_2)\mathrm{cov}[\Delta P_1,\ (T_sK_s)_2\Delta P_2]$$

$$= \underline{Z}_1^T\underline{C}_p(\underline{n}_2 \cdot \nabla_2\underline{Z}_2)(T_sK_s)_2 \quad . \tag{26}$$

Substituting Equations (17), (21), and (26) into (23) gives

$$\mathrm{cov}(\Delta R_1,\ \Delta R_2) = \frac{\partial SC}{\partial P_1}\frac{\partial SC}{\partial P_2}\,\underline{Z}_1^T\underline{C}_p\underline{Z}_2$$

$$+ \frac{\partial SC}{\partial P_1}\frac{C(T_sK_s)_2}{\sin E_2 \tan E_2}\,\underline{Z}_1^T\underline{C}_p(\underline{n}_2 \cdot \nabla_2\underline{Z}_2)$$

$$+ \frac{\partial SC}{\partial P_2}\frac{C(T_sK_s)_1}{\sin E_1 \tan E_1}\,\underline{Z}_2^T\underline{C}_p(\underline{n}_1 \cdot \nabla_1\underline{Z}_1)$$

$$+ \frac{C^2}{\sin E_1 \tan E_1 \sin E_2 \tan E_2}(\underline{n}_1 \cdot \nabla_1\underline{Z}_1)^T\,\underline{C}_{PTK}(\underline{n}_2 \cdot \nabla_2\underline{Z}_2) \tag{27}$$

where

$$\underline{C}_p = (\underline{X}^T\,\underline{\Sigma}_p^{-1}\,\underline{X})^{-1}$$

$$C_{PTK} = (\underline{X}^T\,\underline{\Sigma}_{PTK}^{-1}\,\underline{X})^{-1} \ .$$

All the parameters for the covariance function are completely specified in the Appendix. To illustrate the basic properties of the covariance function, Equation (27) is evaluated in the following section for the special case of a dense weather station network surrounding the survey points.

IV. PROPERTIES OF THE RANGE ERROR COVARIANCE FUNCTION

The basic properties of the range residual covariance function can be illustrated by considering the special case of a dense network of weather stations uniformly distributed within a circle of radius R (see Figure 6). We will assume that the measurement precision is the same for all the weather stations. In this case the variance - covariance matrix for the regression coefficients is given by

$$\underline{C}_F = (\underline{X}^T \underline{\Sigma}_F^{-1} \underline{X})^{-1} = (\underline{X}^T \underline{X})^{-1} \sigma_F^2 \quad . \tag{28}$$

For simplicity the regression coordinates will be expressed in polar coordinates (rather than geocentric coordinates) with the origin located at the center of the weather station network (Figure 6).

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \rho \cos \alpha \\ x_3 &= \rho \sin \alpha \\ x_4 &= \rho^2 \cos \alpha \sin \alpha \\ x_5 &= \rho^2 \cos^2 \alpha \\ x_6 &= \rho^2 \sin^2 \alpha \end{aligned} \tag{29}$$

ρ is the radial distance from the center of the circle and α is the azimuth which is measured clockwise from the north-south line.

The element in the i^{th} row j^{th} column of the $\underline{X}^T\underline{X}$ matrix is

$$\sum_{k=1}^{n} X_{ki} X_{kj} \tag{30}$$

where x_{ki} is the i^{th} regression coordinate evaluated at the k^{th} weather station. The summation in (30) can be approximated as an integral if the number of weather stations is large and if the stations are uniformly

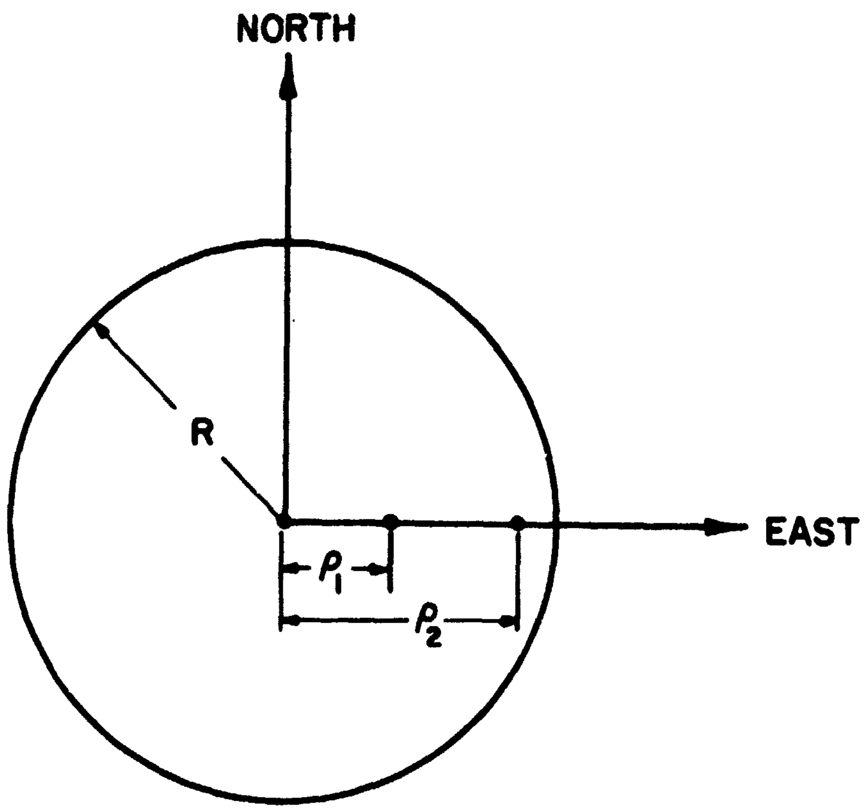

Figure 6. Geometry of the weather station network and survey points.

distributed throughout the circle. Under these conditions, the $\underline{X}^T\underline{X}$ and $\underline{C}_F$ matrices for a quadratic regression fit become

$$\underline{X}^T X \approx n \begin{bmatrix} 1 & 0 & 0 & 0 & R^2/4 & R^2/4 \\ 0 & R^2/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & R^2/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & R^4/24 & 0 & 0 \\ R^2/4 & 0 & 0 & 0 & R^4/8 & R^4/24 \\ R^2/4 & 0 & 0 & 0 & R^4/24 & R^4/8 \end{bmatrix} \tag{31}$$

$$\underline{C}_F \approx \frac{\sigma_F^2}{n} \begin{bmatrix} 4 & 0 & 0 & 0 & -6/R^2 & -6/R^2 \\ 0 & 4/R^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4/R^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24/R^4 & 0 & 0 \\ -6/R^2 & 0 & 0 & 0 & 18/R^4 & 6/R^4 \\ -6/R^2 & 0 & 0 & 0 & 6/R^4 & 18/R^4 \end{bmatrix} \tag{32}$$

The range residual covariance function can now be calculated by substituting the covariance matrix given by (32) into Equation (27).

From Equation (27) we see that the variance of the range error ΔR is a function of the pressure error variance, the $P_sT_sK_s$ gradient error variance and covariance between the pressure and gradient errors. The relative magnitudes of these three factors will depend on the location of the survey points and the number and locations of the weather stations used to obtain the regression fit. The standard deviations of these three factors were calculated for the dense weather station network illustrated in Figure 6. The results are plotted in Figures 7 and 8, versus ρ the distance of the survey point from the center of the circle. A linear regression fit was used to calculate the curves plotted in Figure 7 and a

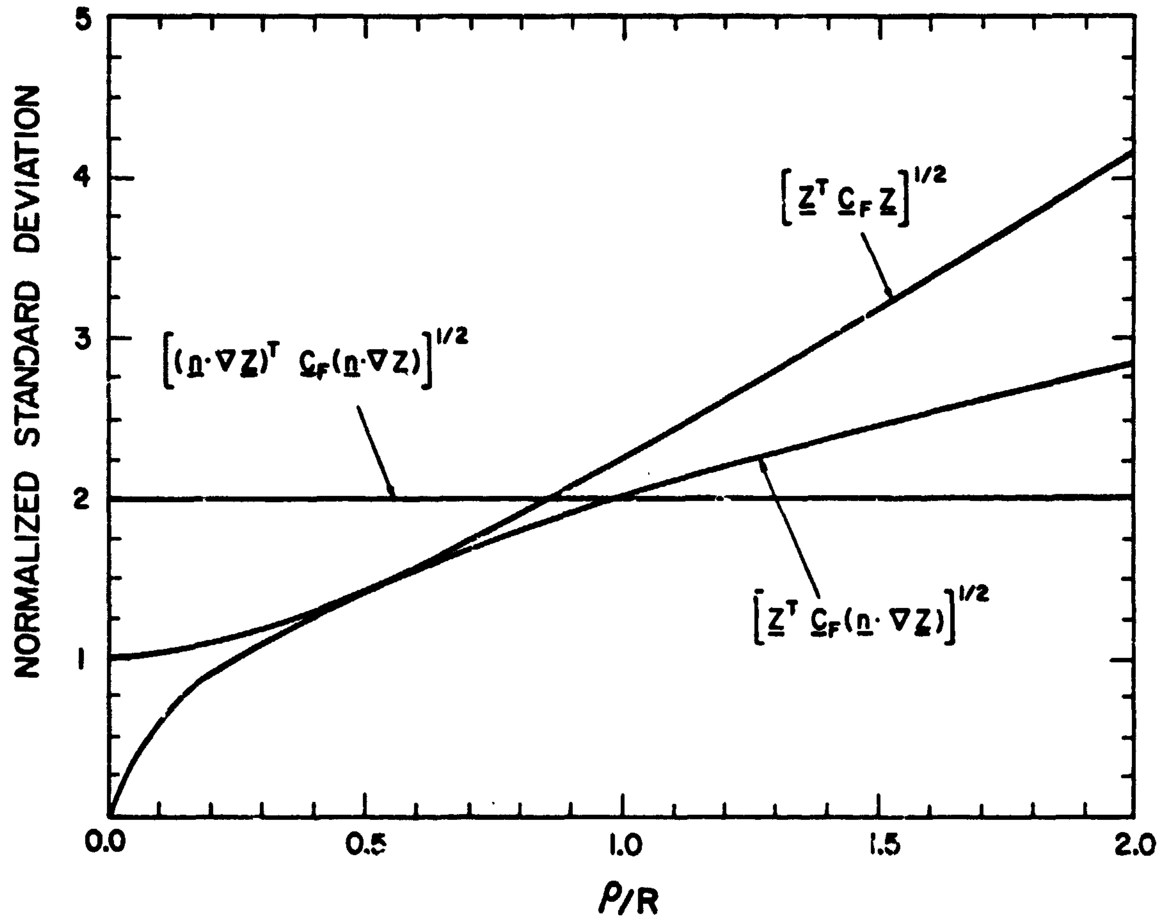

Figure 7. Normalized standard de ˈ˺tion of the measurement error $[\underline{Z}^T\underline{C}_F\underline{Z}]^{1/2}$, gradient error $[(\underline{n} \cdot \nabla Z)^T\underline{C}_F(\underline{n} \cdot \nabla\underline{Z})]^{1/2}$ and co-variance between the measurement and gradient errors $[\underline{Z}^T\underline{C}_F(\underline{n} \cdot \nabla\underline{Z})]^{1/2}$. The normalization factors are $\sigma_F/\sqrt{n}$ $\sigma_F/\sqrt{nR^2}$ and $\sigma_F/\sqrt{nR}$ respectively. The data were calculated for the case where a linear regression polynomial was used to interpolate the measurements.

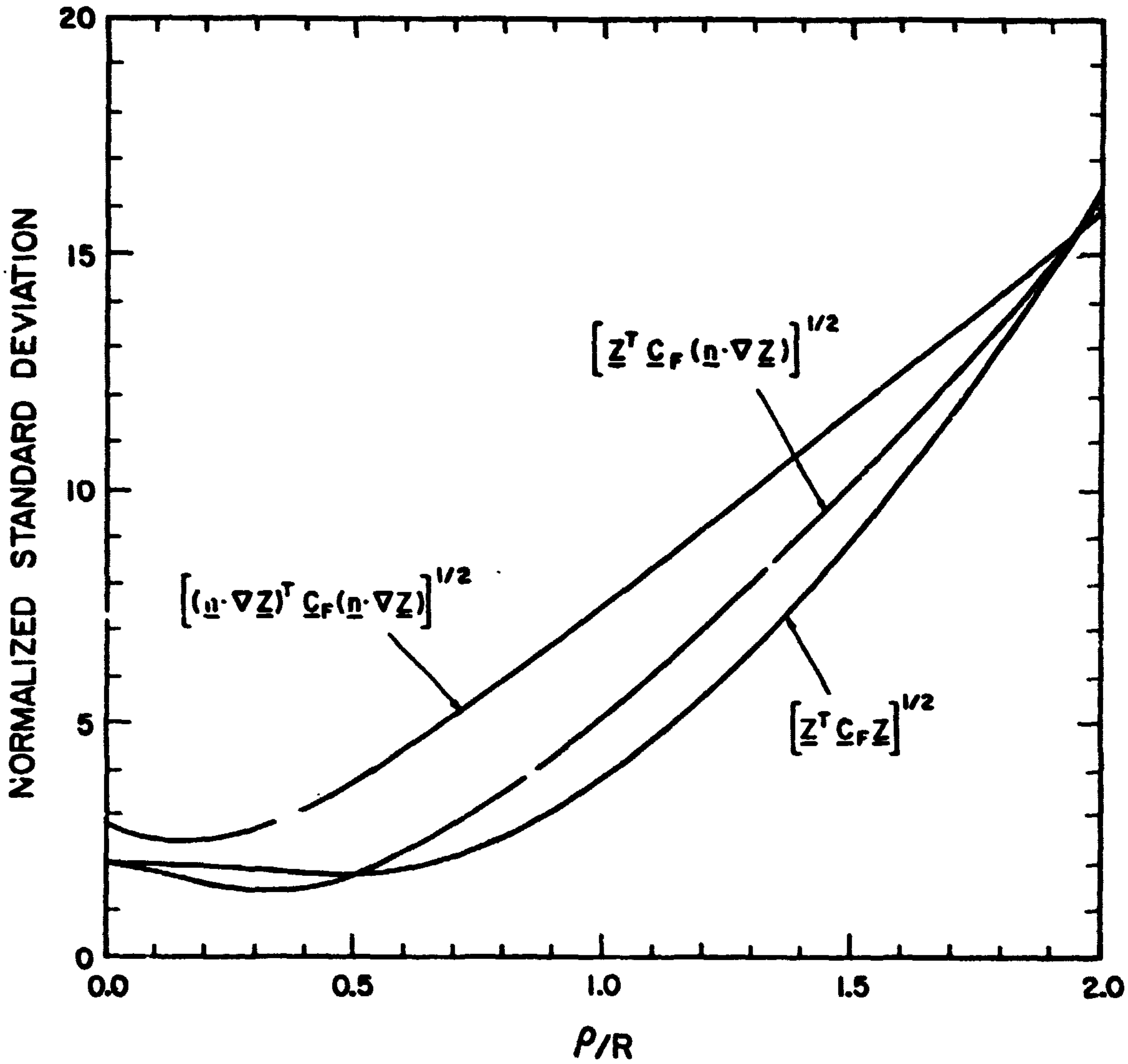

Figure 8. Normalized standard deviation of the measurement error $[\underline{Z}^T \underline{C}_F \underline{Z}]^{1/2}$, gradient error $[(\underline{n} \cdot \nabla\underline{Z})^T \underline{C}_F (\underline{n} \cdot \nabla\underline{Z})]^{1/2}$ and covariance between the measurement and gradient errors $[\underline{Z}^T \underline{C}_F (\underline{n} \cdot \nabla\underline{Z})]^{1/2}$. The normalization factors are $\sigma_F \sqrt{n}$, $\sigma_F / \sqrt{nR^2}$ and $\sigma_F / \sqrt{n}R$ respectively. The data were calculated for the case where a quadratic regression polynomial was used to interpolate the measurements.

quadratic regression fit was used to calculate the curves plotted in Figure 8. The gradient error is constant for the linear fit because the gradient is independent of position. In all other cases the error increases, particularly for ρ larger than R. This behavior is typical of regression polynomials. The weather stations are all located within the circle ($\rho < R$) and in this region the error is relatively small. Outside the circle where there are no weather stations, the error increases substantially. In general, the higher the order of the regression polynomial, the more rapidly the error increases outside the region covered by the weather station network.

For a quadratic regression fit the variance of ΔR is calculated by substituting (32) into (27) and letting $z_1 = z_2$

$$\begin{aligned} \mathrm{var}(\Delta R) \approx\ & \frac{0.222}{\sin^2 E} \frac{\sigma_p^2}{n} \left(1 - 2\hat{\rho} + \frac{9}{2}\hat{\rho}^4\right) \\ & + \frac{3.51 \times 10^3}{\sin^2 E \tan E} \frac{\sigma_p^2}{nR} \left(1 - 2\hat{\rho} - \frac{3}{2}\hat{\rho}^2 + 9\hat{\rho}^3\right) \\ & + \frac{1.32 \times 10^8}{\sin^2 E \tan^2 E} \frac{\sigma_T^2 + 0.21\sigma_p^2}{nR^2} \left(1 - 3\hat{\rho} + 9\hat{\rho}^2\right) \end{aligned} \tag{33}$$

where $\hat{\rho} = \rho/R$. The variance is given in units of cm^2 when R is in meters, σ_T is in °K and σ_p is in mb. The standard deviation of ΔR is plotted versus ρ in Figure 9 for the case where $\sigma_p = 1.0$ mb and $\sigma_T = 1.5°$ K. These are probably worst case estimates for the measurement errors. As expected, the residual refraction error is small whenever $\rho < R$ and increases significantly whenever the observation point moves outside the region covered by the weather station network.

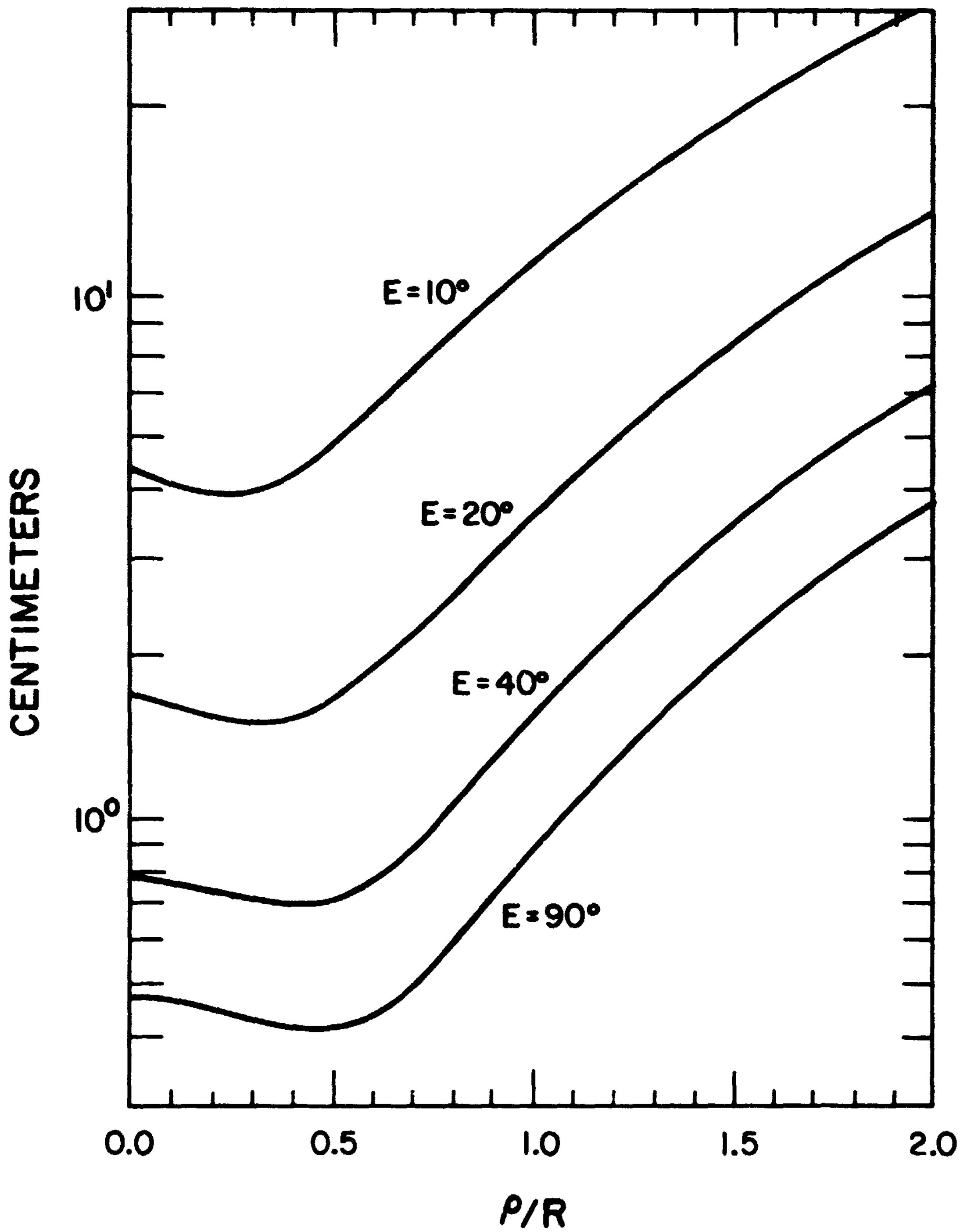

Figure 9. Standa.d deviation of the residual refraction error. The curves are normalized with respect to $1/\sqrt{n}$ where n is the number of weather stations. The results were calculated for σ_P = 1.0 mb and σ_T = 1.5° K and correspond to the case where the surface meteorological is interpolated using a quadratic regression polynomial.

There are situations where ΔR is determined primarily by either the pressure error or the temperature error. For example, at the high elevation angles and when R is large, only the first term in Equation (33) is significant so that ΔR is primarily a function of the pressure error. The conditions under which either pressure or temperature errors are dominant are illustrated in Figure 10. The curves define the points where the pressure and temperature errors contribute equally to the ΔR variance. In the region above the curves, the pressure errors dominate and below the curves the temperature errors dominate. The data plotted in Figure 10 were calculated for the case where the survey point is located at the center of the weather station network and a quadratic regression polynomial is used to interpolate the surface data. The corresponding curves for a linear regression polynomial are similar. R will probably be on the order of a few hundred kilometers for a practical weather station network. In this case, the results in Figure 10 indicate that pressure errors will be the most important factor determining the magnitude of the residual refraction errors.

The range residual correlation function is plotted in Figures 11 and 12 for the case where a linear regression polynomial is used to interpolate the surface data. The corresponding curves for a quadratic regression polynomial are plotted in Figures 13 and 14. In all cases, the azimuth and elevation angles of the laser beam trajectory are assumed to be the same at both survey points (ρ_1 and ρ_2). In Figures 11 and 13 one survey point is located at the center of the weather station network and the other survey point is located at a distance ρ from the center. In Figures 12 and 14 the survey points are symmetrically located on opposite sides of the line intersecting the center of the network. Whenever the

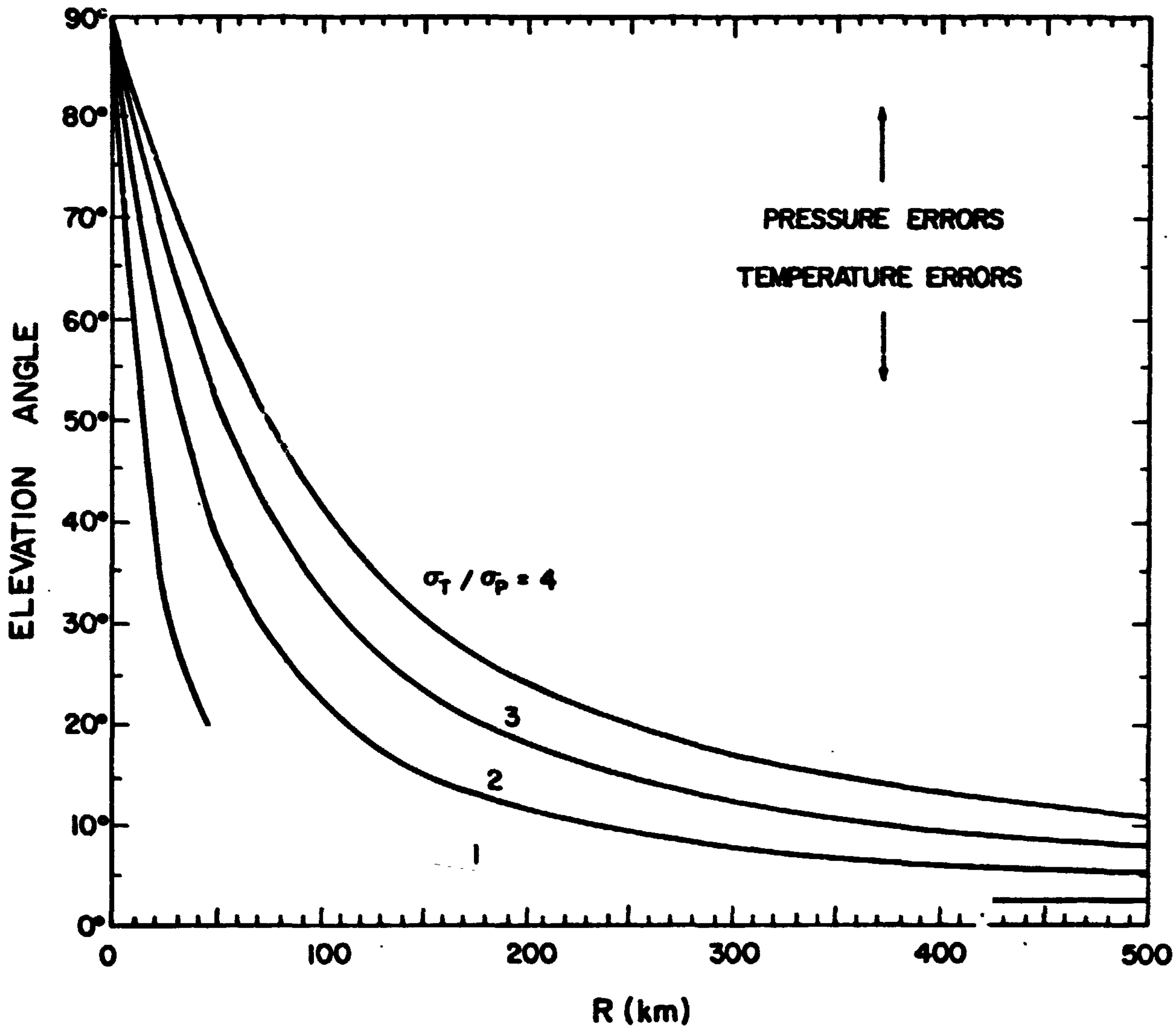

Figure 10. Refraction error regions. The curves define the points where the pressure and temperature errors contribute equally to the residual refraction errors. In the region above the curves the pressure errors dominate and below the curves the temperature errors dominate.

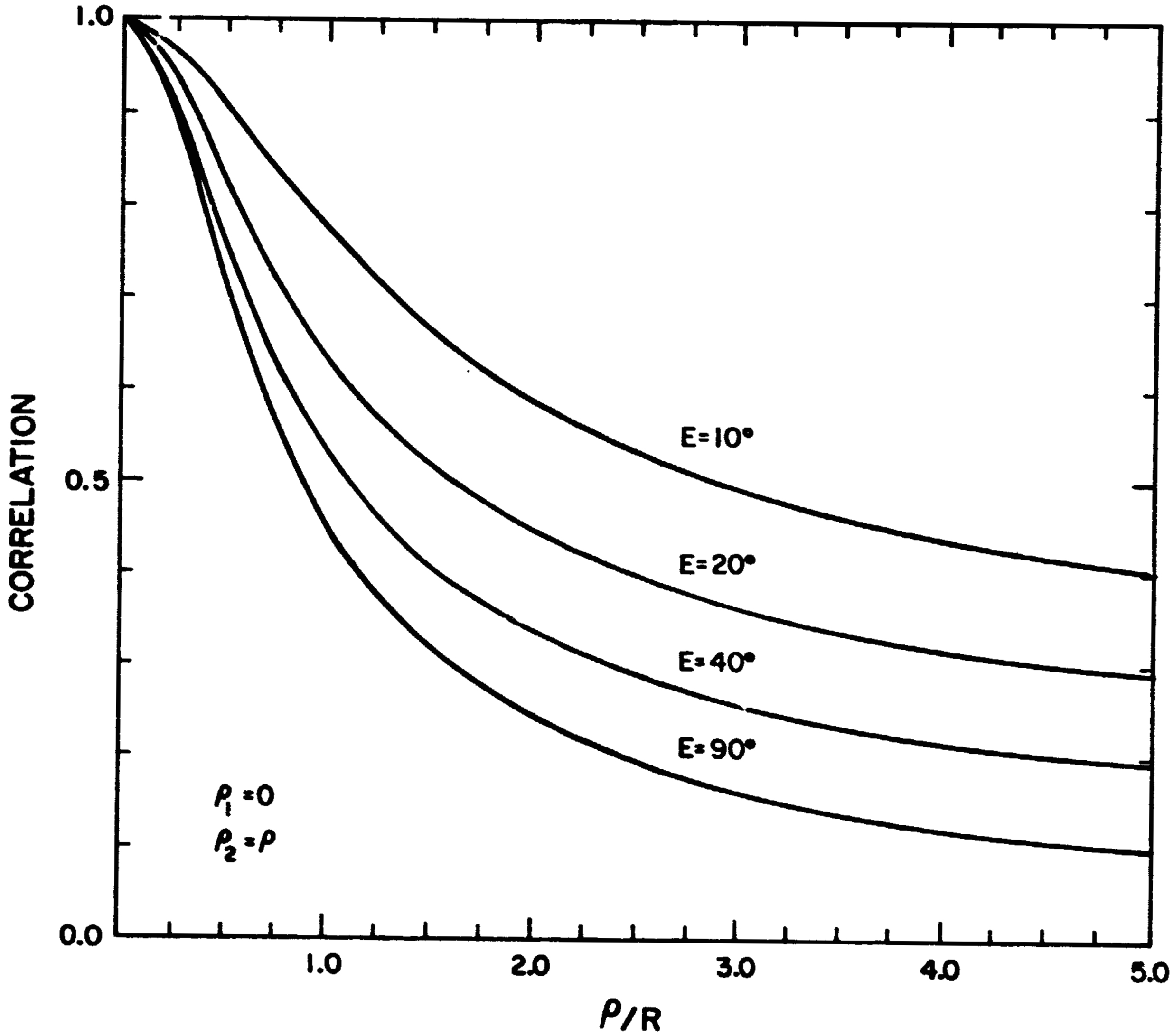

Figure 11. Range error correlation function for the case where a linear regression polynomial is used to interpolate the surface data.

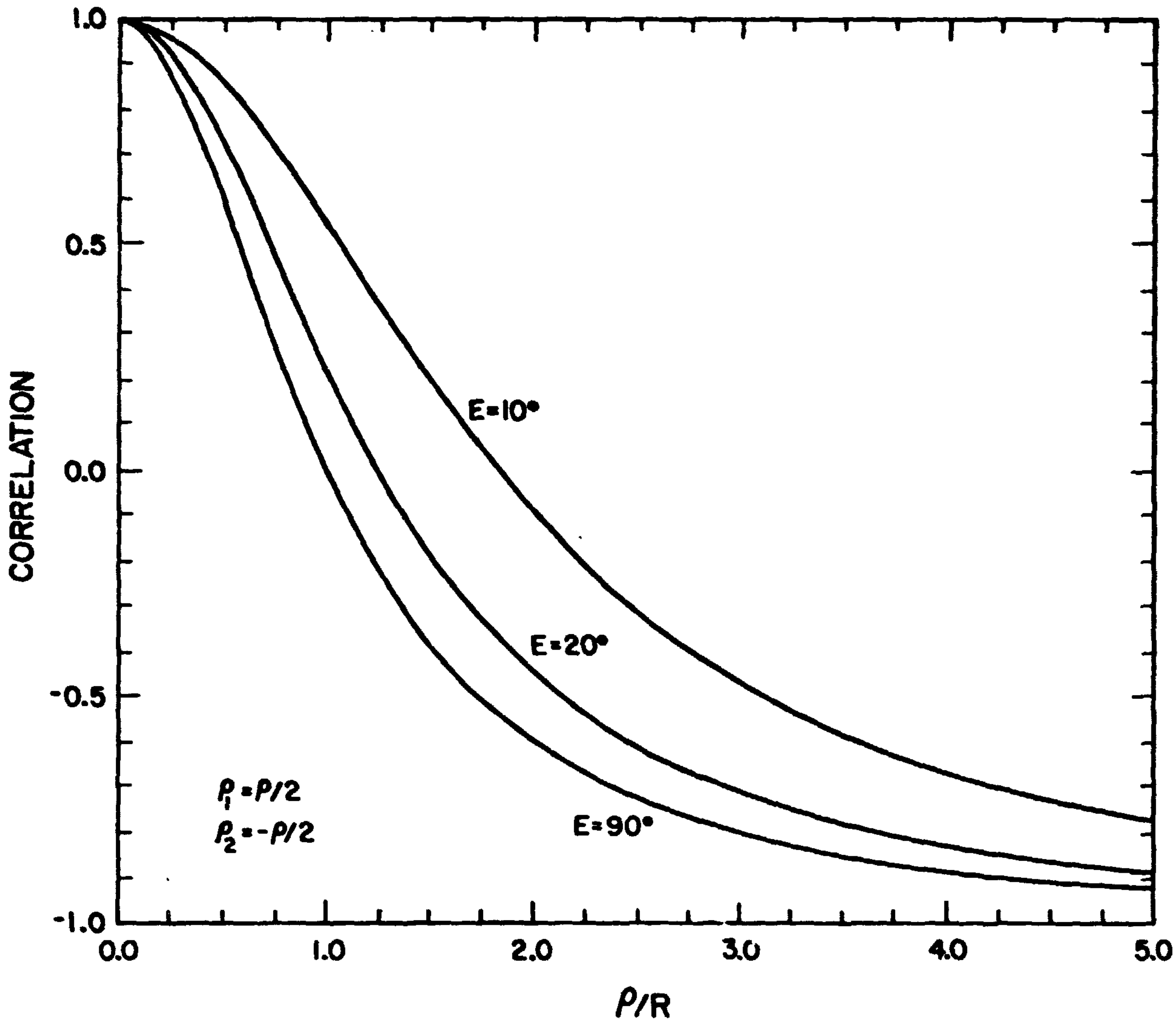

Figure 12. Range error correlation function for the case where a linear regression polynomial is used to interpolate the surface data.

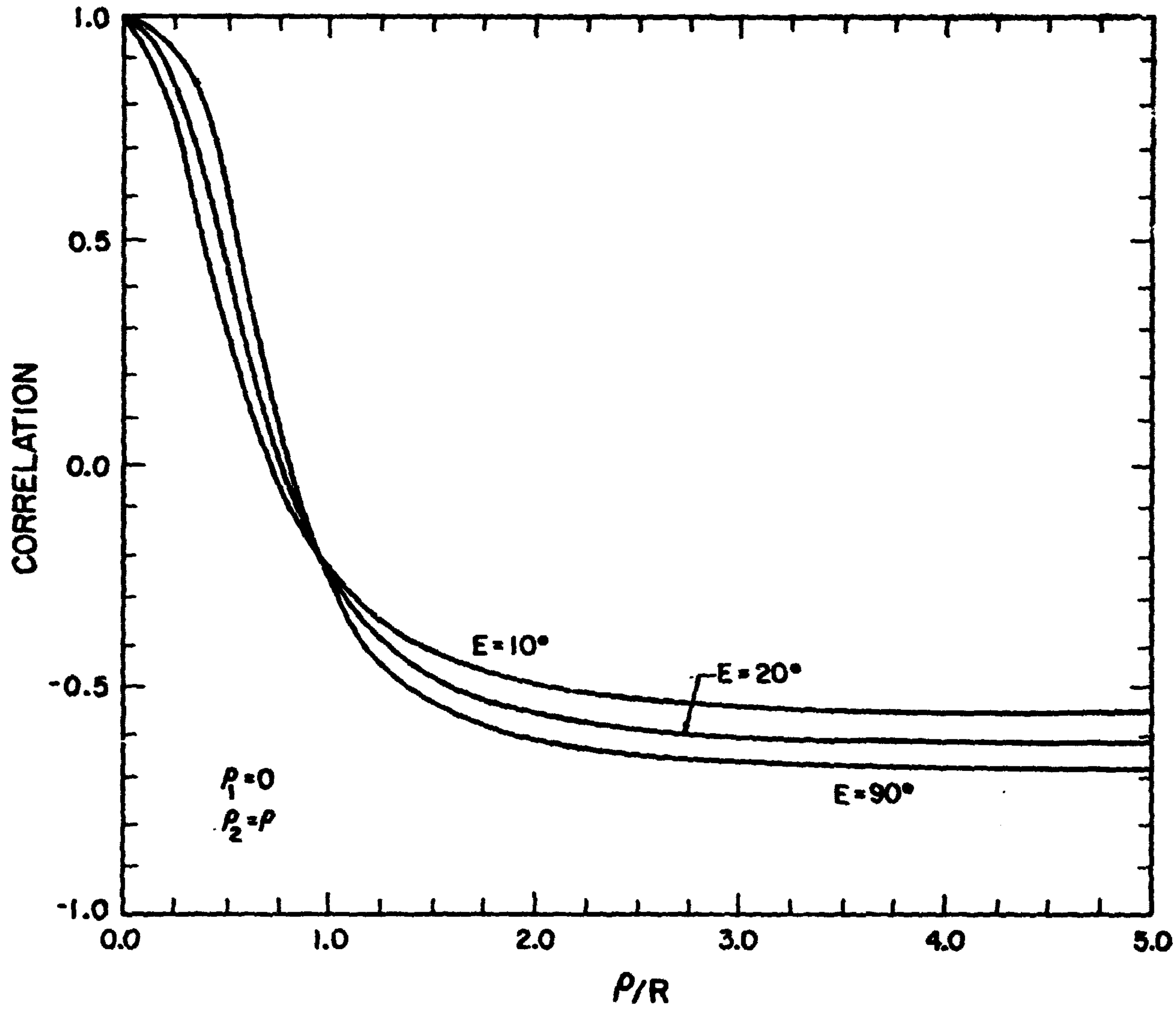

Figure 13. Range error correlation function for the case where a quadratic regression polynomial is used to interpolate the surface data.

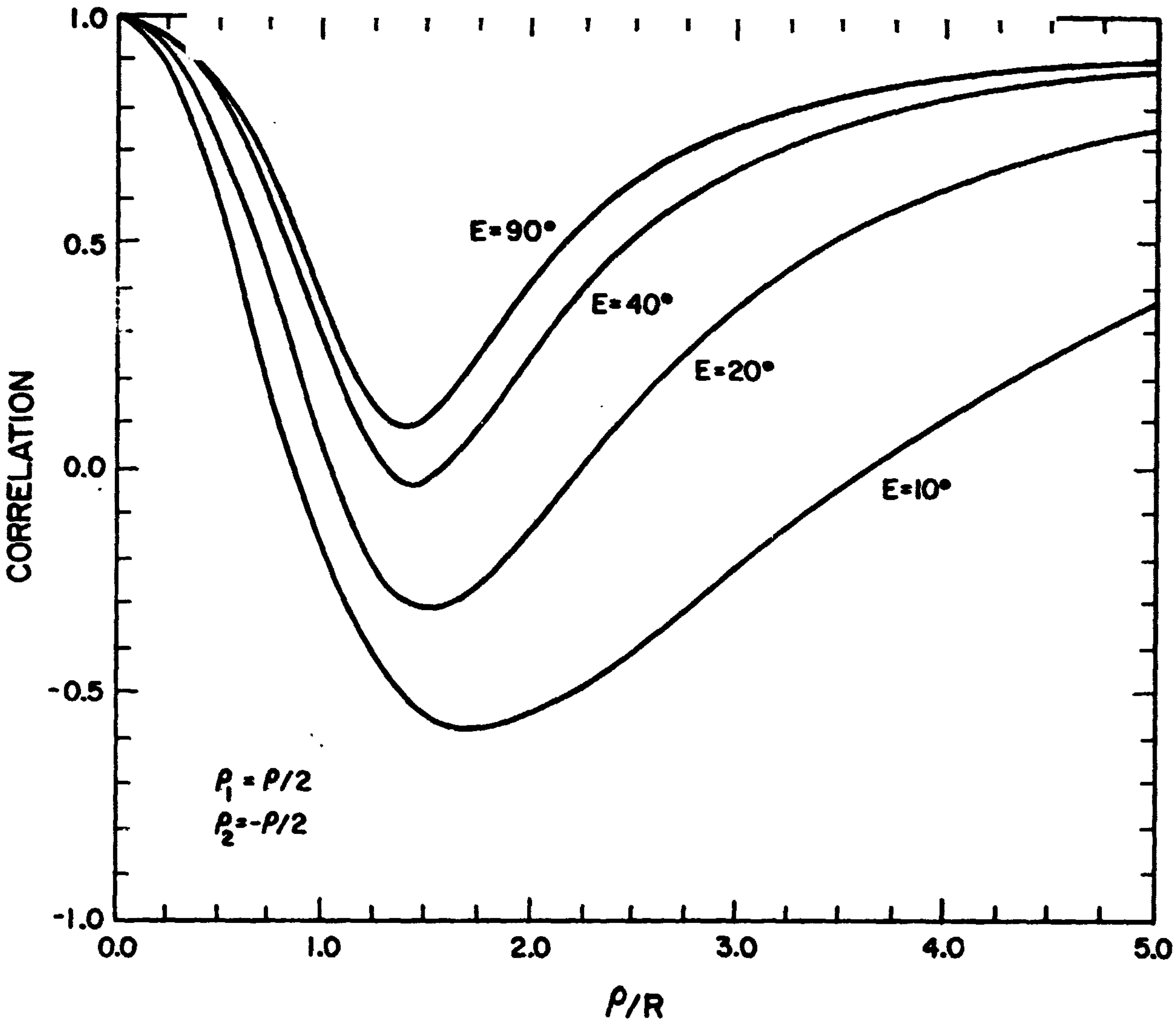

Figure 14. Range error correlation function for the case where a quadratic regression polynomial is used to interpolate the surface data.

distance separating the two survey points is small compared to R, the correlation is close to one. The residual refraction errors become uncorrelated when the separation distance is on the order of R.

V. CONCLUSIONS

Ray trace comparisons using the Haven Hop radiosonde data indicate that the spherical correction (SC) and gradient correction (GC_1) formulas are nearly unbiased estimators of the range error due to atmospheric refraction. The uncorrected residual error is approximately 5 mm at 20° elevation and decreases to approximately 1 mm at zenith. At this level the residual refraction error appears to be caused almost entirely by errors in the meteorological data which are used to calculate SC and GC_1. Other error sources apparently contribute much less than 5 mm at 20° elevation and much less than 1 mm at zenith. A theoretical model for the range residual covariance function was derived under this assumption. Since the surface meteorological data are interpolated from weather station measurements, the residual error and error covariance are a function of the number and location of the stations.

To illustrate its properties, the covariance function was evaluated for the special case of a dense network of weather stations uniformly distributed within a circle of radius R. The standard deviation of the residual refraction error is inversely proportional to the square root of the number of weather stations. It is relatively small provided the survey point is located within the circle covered by the network. Outside the network the standard deviation increases substantially particularly when high order regression polynomials are used to interpolate the weather data. At low elevation angles (< 20°) and when the network extends over a small region ($R < 100$ km), the range error is primarily a function of the temperature error. At the higher elevation angles (> 20°) and when the weather station network extends over a large region

(R > 100 km), range error is primarily a function of the pressure error. The refraction errors inherent in the range measurements to two different survey points will be highly correlated if the distance between the points is small compared to R. The errors are uncorrelated when the separation distance is on the order of R.

In general, the residual refraction error decreases as the size of the network and number of weather stations increase. If the weather station network is large enough, the effects of pressure and temperature errors may be reduced to the point where other error sources dominate. The covariance model derived in this report includes only the effects of pressure and temperature errors. These other error sources arise from the departure of the atmosphere from hydrostatic equilibrium, from the neglect of quadratic and higher order terms in the horizontal variation of refractivity, and from inadequate regression models for surface pressure and temperature. Based upon our analysis of the Haven Hop ray tracing data, we believe the effects contribute less than a few millimeters error above 20° elevation. If the surface temperature and pressure do not vary significantly among the survey points, the atmospheric refractivity will be fairly homogeneous. Under these conditions we expect the errors caused by these other sources to be highly correlated between each pair of survey points. The validity of the theoretical model is now being investigated by ray tracing using the Haven Hop radiosonde data.

The analysis in this report did not consider the problem of extrapolating weather measurements to different altitudes. This could potentially be a major error source, particularly in mountainous regions such as California where weather stations may be located at widely different altitudes. This problem is also being investigated.

APPENDIX

RANGE RESIDUAL COVARIANCE FUNCTION

ΔR_1 = range error at cube corner 1

ΔR_2 = range error at cube corner 2

$$\overline{\Delta R_1 \Delta R_2} = \frac{A^2 \sigma_p^2}{\sin E_1 \sin E_2} \underline{Z}_1^T (\underline{X}^T \underline{X})^{-1} \underline{Z}_2$$

$$+ \frac{AB\ T_S K_s\ \sigma_p^2}{\sin E_1 \sin E_2 \tan E_2} \underline{Z}_1^T (\underline{X}^T \underline{X})^{-1}\ \underline{n}_2 \cdot \nabla_2 \underline{Z}_2$$

$$+ \frac{AB\ T_S K_s\ \sigma_p^2}{\sin E_2 \sin E_1 \tan E_1} \underline{Z}_2^T (\underline{X}^T X)^{-1}\ \underline{n}_1 \cdot \nabla_1 \underline{Z}_1$$

$$+ \frac{B^2 \left[T_S^2 K_s^2 \sigma_p^2 + (K_s - \zeta T_S)^2 P_S^2 \sigma_T^2 \right]}{\sin E_1 \tan E_1 \sin E_2 \tan E_2} (\underline{n}_1 \cdot \nabla_1 \underline{Z}_1)^T (\underline{X}^T X)^{-1} \underline{n}_2 \cdot \nabla_2 \underline{Z}_2$$

$A = \dfrac{f(\lambda)}{F(\theta,H)}\ 0.002357$

$B = f(\lambda)\ 6.915 \times 10^{-2}$

$\zeta = 1.04 \times 10^{-3}$

$f(\lambda) = 0.9650 + \dfrac{0.0164}{\lambda^2} + \dfrac{0.000228}{\lambda^4}$

λ = laser wavelength (μm)

$F(\theta,H) = 1 + 0.0026 \cos(2\theta) - 0.00031\ H$

θ = colatitude of cube corner = 90° − latitude

H = altitude of cube corner (km)

$K_s = 1.163 + 0.00968 \cos(2\theta) - 0.00104\ T_S + 0.00001435\ P_s$

T_S = surface temperature (°K)

P_S = surface pressures (mb)

σ_P = rms pressure error

σ_T = rms temperature error

$$\underline{Z}_1 = \begin{bmatrix} 1 \\ \theta_1 \\ \phi_1 \sin \theta_1 \\ \theta_1\phi_1 \sin \theta_1 \\ \theta_1^2 \\ \phi_1^2 \sin^2 \theta_1 \end{bmatrix} \qquad \underline{Z}_2 = \begin{bmatrix} 1 \\ \theta_2 \\ \phi_2 \sin \theta_2 \\ \theta_2\phi_2 \sin \theta_2 \\ \theta_2^2 \\ \phi_2^2 \sin^2 \theta_2 \end{bmatrix}$$

$$\underline{n}_1 \cdot \nabla_1 \underline{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \theta_1 \\ 0 \\ 2\phi_1 \sin \theta_1 \end{bmatrix} \frac{\sin \alpha_1}{r_e} - \begin{bmatrix} 0 \\ 1 \\ \phi_1 \cos \theta_1 \\ \phi_1 \sin \theta_1 + \theta_1\phi_1 \cos \theta_1 \\ 2\theta_1 \\ \phi_1^2 \sin 2\theta_1 \end{bmatrix} \frac{\cos \alpha_1}{r_e}$$

$$\underline{n}_2 \cdot \nabla_2 \underline{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \theta_2 \\ 0 \\ 2\phi_2 \sin \theta_2 \end{bmatrix} \frac{\sin \alpha_2}{r_e} - \begin{bmatrix} 0 \\ 1 \\ \phi_2 \cos \theta_2 \\ \phi_2 \sin \theta_2 + \theta_2\phi_2 \cos \theta_2 \\ 2\theta_2 \\ \phi_2^2 \sin 2\theta_2 \end{bmatrix} \frac{\cos \alpha_2}{r_e}$$

θ_1 = colatitude of cube corner 1 = 90° - latitude

θ_2 = colatitude of cube corner 2 = 90° - latitude

ϕ_1 = longitude of cube corner 1

ϕ_2 = longitude of cube corner 2

E_1 = satellite elevation angle as seen from cube corner 1

E_2 = satellite elevation angle as seen from cube corner 2

α_1 = satellite azimuth angle as seen from cube corner 1

α_2 = satellite azimuth angle as seen from cube corner 2

r_e = earth radius (m)

$$\underline{X} = \begin{bmatrix} 1 & \tilde{\theta}_1 & \tilde{\phi}_1 \sin \tilde{\theta}_1 & \tilde{\theta}_1 \tilde{\phi}_1 \sin \tilde{\theta}_1 & \tilde{\theta}_1^2 & \tilde{\phi}_1^2 \sin^2 \tilde{\theta}_1 \\ 1 & \tilde{\theta}_2 & \tilde{\phi}_2 \sin \tilde{\theta}_2 & \tilde{\theta}_2 \tilde{\phi}_2 \sin \tilde{\theta}_2 & \tilde{\theta}_2^2 & \tilde{\phi}_2^2 \sin^2 \tilde{\theta}_2 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 1 & \tilde{\theta}_n & \dot{\phi}_n \sin \tilde{\theta}_n & \tilde{\theta}_n \tilde{\phi}_n \sin \tilde{\phi}_n & \theta_n^2 & \tilde{\phi}_n^2 \sin^2 \tilde{\theta}_n \end{bmatrix}$$

$\tilde{\theta}_i$ = colatitude of the i^{th} weather station

$\tilde{\phi}_i$ = longitude of the i^{th} weather station

Note: This derivation assumes that all weather stations and both cube corners are located at the same altitude. The measurement errors at the weather stations are assumed to be statistically independent and identically distributed random variables. Although the $\underline{X}$ and $\underline{Z}$ matrices are given for the case where the surface pressure and temperature are expanded in quadratic regression polynomials, the results can easily be extended for higher or lower expansions.

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