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A Note on the Bending of a Cracked Strip(*)

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Abstract. The objective of this note is to describe a technique for calculating the stress intensity factors in a strip under bending by treating the strip as a two-dimensional continuum rather than a simple beam in evaluating the crack surface tractions used for the solution of the perturbation problem.

1. INTRODUCTION

A long strip or beam containing a through crack perpendicular to the sides has been one of the most widely studied problems in linear fracture mechanics (see, for example, references [1-9]). The importance of the problem lies in the fact that its geometry approximates a very common structural component and a standard test specimen. In the case of three or four point bend tests the specimen has an edge crack. Generally, in the existing solutions involving bending it is assumed that the uncracked beam is under a linear stress distribution, thus ignoring the perturbations caused by the supports and the loading fixtures.

The main objective of this note is in a "long" beam to examine the deviation from linearity in the stress field for a given state of

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external loads and to study the effect of this deviation on the stress intensity factors for internal and edge cracks.

2. STRESSES IN UNCRACKED BEAM

Consider the infinite strip or the beam show in Figure 1. Let the external forces be

$$-\sigma^{1}_{yy}(x,0) = p_1(x) = p_1(-x) \quad , \quad \sigma^{1}_{xy}(x,0) = 0 \quad , \tag{1}$$

$$-\sigma^{1}_{yy}(x,h) = p_2(x) = p_2(-x) \quad , \quad \sigma^{1}_{xy}(x,h) = 0 \quad . \tag{2}$$

where h is the height of the beam p_1 and p_2 are known functions satisfying

$$\int_0^{\infty} p_1(x)dx = \int_0^{\infty} p_2(x)dx \quad . \tag{3}$$

Ignoring the crack and using the standard Fourier transform technique (see, for example, [10]), after some routine manipulations the stress component of primary interest may be expressed as

$$\sigma^{1}_{xx}(x,y) = \frac{2}{\pi}\int_0^{\infty} S_x(\alpha,y)\cos\alpha x \, d\alpha \quad , \tag{4}$$

where

$$S_x(\alpha,y) = \frac{e^{-\alpha y}}{D(\alpha)} \{(1-2\alpha^2h^2-2\alpha h-e^{2\alpha h})S_1(\alpha)$$

$$+ [(1+3\alpha h)e^{\alpha h} - (1+\alpha h)e^{-\alpha h}]S_2(\alpha)$$

$$+ \alpha y(-1+2\alpha h+e^{2\alpha h})S_1(\alpha)$$

$$- \alpha y[(1+2\alpha h)e^{\alpha h} - e^{-\alpha h}]S_2(\alpha)\}$$

$$+ \frac{e^{\alpha y}}{D(\alpha)} \{(1+2\alpha h-2\alpha^2 h^2 - e^{-2\alpha h})S_1(\alpha)$$

$$+ [(-1+\alpha h)e^{\alpha h} + (1-3\alpha h)e^{-\alpha h}]S_2(\alpha)$$

$$+ \alpha y(1+2\alpha h-e^{-2\alpha h})S_1(\alpha)$$

$$+ \alpha y[-e^{\alpha h} + (1-2\alpha h)e^{-\alpha h}]S_2(\alpha)\} \quad , \tag{5}$$

$$D(\alpha) = e^{2\alpha h} + e^{-2\alpha h} - 4\alpha^2 h^2 - 2 \quad , \tag{6}$$

$$S_1(\alpha) = \int_0^\infty p_1(x)\cos\alpha x \, dx \quad , \tag{7}$$

$$S_2(\alpha) = \int_0^\infty p_2(x)\mathrm{cox}\alpha x \, dx \quad . \tag{8}$$

3. THE CRACK PROBLEM

To obtain the solution of the crack problem shown in Figure 1 under the external loads given by (1) and (2) one has to superimpose on the solution obtained in the previous section for the uncracked strip a disturbed stress state found from a cracked strip in which the following self-equilibrating crack surface tractions are the only external loads:

$$\sigma_{xx}(0,y) = -\sigma^i_{xx}(0,y) \quad , \quad \sigma_{xy}(0,y) = 0 \quad , \quad a<y<b \quad , \tag{9}$$

where σ^1_{xx} is given by (4). The formulation of this problem too is relatively straightforward. Following, for example, the technique described in [6] or [7], the problem may be reduced to the following integral equation:

$$\int_a^b \left[\frac{1}{t-y} + k(y,t)\right]f(t)dt = -\pi \frac{1+\kappa}{4\mu} \sigma^1_{xx}(0,y) \quad , \quad a<y<b \quad , \tag{10}$$

subject to

$$\int_a^b f(t)dt = 0 \tag{11}$$

where the input function σ^1_{xx} is obtained from (4), μ is the shear modulus, $\kappa=3-4\nu$ for plane strain (e.g., cylindrical bending of a plate with a long surface crack), $\kappa=(3-\nu)/(1+\nu)$ for plane stress (e.g., a beam), ν is the Poisson's ratio, crack extends along the y axis from a to b, the unknown function f is defined by

$$f(y) = \frac{\partial}{\partial y} u(0,y) \quad , \tag{12}$$

u is the x-component of the displacement vector, and the kernel $k(y,t)$ is given by

$$k(y,t) = \int_0^\infty [(e^{\alpha h}-e^{-\alpha h})^2-4\alpha^2h^2]^{-1} \{[-1-2\alpha h+e^{2\alpha h}$$

$$-2\alpha t(e^{2\alpha h}-1)][\tfrac{3}{2}+\alpha(h-y)+\tfrac{1}{2}e^{-2\alpha(h-y)}]e^{-\alpha(t+y)}$$

$$+ [2\alpha h-1+e^{-2\alpha h}-4th\alpha^2][-\tfrac{3}{2}+\alpha(h-y)-\tfrac{1}{2}e^{2\alpha(h-y)}]e^{-\alpha(t-y)}$$

$$+ [1+2\alpha h-e^{-2\alpha h}+2\alpha(h-t)(e^{2\alpha h}-1)][\tfrac{3}{2}+\alpha y$$

$$+ \frac{1}{2} e^{-2\alpha y}]e^{-\alpha(2h-t-y)} + [2\alpha h-1+e^{-2\alpha h}$$

$$- 4(h-t)\alpha^2 h][\frac{3}{2} - \alpha y + \frac{1}{2} e^{2\alpha y}]e^{-\alpha(y-t)} \} d\alpha \ . \qquad (13)$$

For an internal crack $0<a<b<h$, the kernel $k(y,t)$ is bounded in the closed domain $a\leq(y,t)\leq b$, and the integral equation may be solved quite simply by using the technique described in [11]. The stress intensity factors are then defined and calculated from

$$k(a) = \lim_{y\to a} \sqrt{2(a-y)}\ \sigma_{xx}(y,0) = \lim_{y\to a} \sqrt{2(y-a)}\ \frac{4\mu}{1+\kappa} f(y) \ , \qquad (14)$$

$$k(b) = \lim_{y\to b} \sqrt{2(y-b)}\ \sigma_{xx}(y,0) = -\lim_{y\to b} \sqrt{2(b-y)}\ \frac{4\mu}{1+\kappa} f(y) \ . \qquad (15)$$

However, in the case of an edge crack $0=a<b<h$, the kernel $k(x,t)$ is no longer bounded in the closed domain $0\leq(y,t)\leq b$. The singular part of k which becomes unbounded as y and t approach the end point $a=0$ may be separated by considering the asymptotic behavior of the integrand in (13). Expressing

$$k(y,t) = k_s(y,t) + k_f(y,t) \qquad (16)$$

where k_f is bounded in $a\leq(y,t)\leq b$, the singular part k_s is found to be

$$k_s(y,t) = \int_0^\infty (2-\alpha y-3\alpha t+2yt\alpha^2)e^{-\alpha(t+y)}d\alpha$$

$$= -\frac{1}{t+y} + \frac{6y}{(t+y)^2} - \frac{4y^2}{(t+y)^3} \ . \qquad (17)$$

Noting that the integral equation (10) is still valid for $a=0$, after separating k_s it may be solved by following the procedure described in [6].

4. AN EXAMPLE AND SOME RESULTS

As an example consider the beam supported at two points $x=\mp d$ on the side $y=0$ and subjected to uniform pressure on $y=h$, $-c<x<c$ (see Figure 1). In reality all loading problems involving solid grips and fixtures are contact problems. For example, the supports at $x=\mp d$, $y=0$ are generally rollers often with different elastic properties. In the three or four point bending problems the same is true for the loading fixtures on $y=h$. In this case the contact areas and the distribution of contact pressures as well as the crack surface displacement are unknown and the problem is a coupled contact-crack problem which is somewhat more difficult to formulate. However, if the contact areas are small and are sufficiently far from the crack region, one may replace the unknown contact stresses with statically equivalent known tractions gaining considerable simplicity in the formulation of the problem without sacrificing too much in the accuracy of the results. Thus, in the present example it will be assumed that the reactions at the supports $y=0$, $x=\mp d$ are concentrated forces of magnitude P (per unit thickness). In the three point bending problem the pressure under the loading pin will be assumed to be constant with $c=0.05\,h$. The external loads and their Fourier transforms may then be expressed as follows (see equations 1 to 4):

$$p_1(x) = P\delta(x-d) \quad , \tag{18}$$

$$p_2(x) = \begin{cases} P/c \quad , & 0 \le x < c \quad , \\ 0 \quad , & c < x < \infty \quad , \end{cases} \tag{19}$$

$$S_1(\alpha) = P \cos(d\alpha) \quad , \tag{20}$$

$$S_2(\alpha) = \frac{P}{c} \frac{\sin(c\alpha)}{\alpha} \quad . \tag{21}$$

In evaluating the input function σ^1_{xx} from (4)-(8) one runs into convergence difficulty for values of y around 0 and h. To avoid this difficulty the damping of the integrand as $\alpha \to \infty$ is increased by separating the asymptotic value of $S_x(\alpha,y)$ and integrating it in closed form. Equation (4) may then be expressed as

$$\sigma^1_{xx}(x,y) = \frac{2}{\pi} \int_0^\infty [S_x(\alpha,y) - S_\infty(\alpha,y)] \cos\alpha x \, d\alpha + s_\infty(x,y) \quad , \tag{22}$$

$$S_\infty(\alpha,y) = (\alpha y - 1) e^{-\alpha y} S_1(\alpha) - (\alpha y + 1 - \alpha h) e^{-\alpha(h-y)} S_2(\alpha) \quad , \tag{23}$$

$$s_\infty(x,y) = \frac{2}{\pi} \int_0^\infty S_\infty(\alpha,y) \cos\alpha x \, d\alpha \quad , \tag{24}$$

$$s_\infty(0,y) = P \left[- \frac{y}{y^2+d^2} + \frac{y(y^2-d^2)}{(y^2+d^2)^2} + \frac{h-y}{c^2+(h-y)^2} - \frac{1}{c} \tan^{-1} \left(\frac{c}{h-y}\right) \right] \quad . \tag{25}$$

The stress distribution $\sigma^1_{xx}(0,y)$ obtained from the elasticity theory as described by equations (20) to (25) and that obtained from the simple beam theory for the loading condition shown in Figure 1 is given in Table 1. The normalization stress σ_0 used in this and in the subsequent tables is the surface stress given by the beam theory which may be expressed as

$$\sigma_0 = \frac{6P}{h^2} (d - \frac{c}{2}) \quad . \qquad (26)$$

The table shows that in the elasticity results there is some deviation from that of linear beam theory. The indication is that the sign as well as the magnitude of the deviations depend on details of the loading condition. However, at least for the particular loading considered here, the relative magnitude of the deviation does not appear to be very significant (approximately 6 percent at y=0).

Table 1. The stress profile $\sigma^1_{xx}(0,y)/\sigma_0$ for the uncracked beam calculated from (22-25) for various dimensions c and d. The normalization factor: $\sigma_0 = 6P(d-c/2)h^2$.

y/h	c/h = 1 d/h = 1	c/h = 0.05 d/h = 2	c/h = 0.05 d/h = 4	Beam Theory (h/2 - y)
0	1.0589	0.9435	0.9399	1
0.1	0.7949	0.7276	0.7384	0.8
0.2	0.5595	0.5363	0.5491	0.6
0.3	0.3486	0.3615	0.3680	0.4
0.4	0.1566	0.1971	0.1921	0.2
0.5	-0.0233	0.0386	-0.0192	0
0.6	-0.1993	-0.1179	-0.1528	-0.2
0.7	-0.3801	-0.2767	-0.3259	-0.4
0.8	-0.5750	-0.4452	-0.5039	-0.6
0.9	-0.7938	-0.6723	-0.7109	-0.8
0.95	-.09153	-0.9991	-0.9204	-0.9

Table 2 gives the stress intensity factors for the internal crack shown in Figure 1. In this example the crack tip a was fixed at a=0.1h and b was varied until the stress intensity factor at b, k(b) became (approximately) zero or negative. Note that for a very small crack (i.e., a=0.1 h, b=0.1001 h) $k(a) \cong k(b) \cong \sigma^1_{xx}(0,0.1h)\sqrt{\ell}$ which is the expected result for an infinite plane with line crack of length 2ℓ=b-a

Table 2. Stress intensity factors in a beam with an internal crack under the loading condition shown in Figure 1. $\sigma_o = 6P(d-c/2)/h^2$, $\ell = (b-a)/2$, and a=0.1h=constant.

$\frac{b}{h}$	c/h=1, d/h=1		c/h=0.05, d/h=2		c/h=0.05, d/h=4		Beam Theory [7] $\sigma^1_{xx}=\sigma_o(-y+h/2)$	
	$\frac{k(a)}{\sigma_o\sqrt{\ell}}$	$\frac{k(b)}{\sigma_o\sqrt{\ell}}$	$\frac{k(a)}{\sigma_o\sqrt{\ell}}$	$\frac{k(b)}{\sigma_o\sqrt{\ell}}$	$\frac{k(a)}{\sigma_o\sqrt{\ell}}$	$\frac{k(b)}{\sigma_o\sqrt{\ell}}$	$\frac{k(a)}{\sigma_o\sqrt{\ell}}$	$\frac{k(b)}{\sigma_o\sqrt{\ell}}$
0.1001	0.7949	0.7948	0.7275	0.7274	0.7384	0.7383		
0.2	0.7684	0.6425	0.7102	0.6070	0.7228	0.6204		
0.3	0.7627	0.5091	0.7134	0.5014	0.7271	0.5122		
0.4	0.7520	0.3759	0.7131	0.3926	0.7265	0.3966		
0.5	0.7278	0.2386	0.7009	0.2765	0.7123	0.2706	0.7728	0.2829
0.6	0.6873	0.0953	0.6737	0.1531	0.6814	0.1354		
0.7	0.6314	-0.0577	0.6317	0.0217	0.6342	-0.0090	0.6800	-0.0419

pressurized by σ_{xx}^1 (see the rows y/h=0.1 in Table 1 and b/h=0.1001 in Table 2). For fixed a/h the stress intensity ratio $k(a)/\sigma_0\sqrt{\ell}$ remains approximately constant, meaning that for increasing crack length, k(a) itself increases as $\sqrt{\ell}$. Also note that k(b) is reduced rapidly as the crack tip b moves towards and into the compression region. For the beam theory extensive results are given in [7]. Table 2 shows the results from [7] which coincides with the parameters a/h and b/h in the table.

In the important case of the edge crack (i.e., a=0, b<h in Figure 1) the results for c=0.05h and d=4h are shown in Table 3. The stress intensity factor obtained from the beam theory is reproduced from [7]. For a very short crack one would expect to recover the result of the semi-infinite plane with an edge crack of length b pressurized by $\sigma_{xx}^1(0,0)$ given in Table 1. Indeed for b/h=0.001 it is seen that

$$\frac{k(b)}{\sigma_{xx}^1(0,0)\sqrt{\ell}} = \frac{1.0537}{0.9399} = 1.1211$$

which is the half plane result.(*)

The results shown in Tables 2 and 3 indicate that, depending on the loading condition, the elasticity solution may be different than

(*) For higher values of b/h the convergence of the numerical analysis giving the stress intensity factor is somewhat slow. The results shown in the table are obtained by letting $k(b)=A/n^2+B/n+C$ and extrapolating the results to $n=\infty$, where n is the number of collocation points in the solution of the related integral equation.

Table 3. Stress intensity factor in a beam with an edge crack under the loading condition shown in Figure 1 ($c=0.05h$, $a=0$, $b<h$, $d=4h$), $\sigma_o=6P(d-c/2)/h^2$.

b/h	$k(b)/\sigma_o\sqrt{b}$	
	Elasticity	Beam Th.
0.001	1.0537	1.12
0.1	0.9731	1.05
0.2	0.9750	1.06
0.3	1.0375	1.12
0.4	1.1661	1.26
0.5	1.3907	1.50
0.6	1.7839	
0.7	2.5127	

the beam solution. As to how significant this difference is depends on the degree of accuracy required in the particular application as well as the details of the loading condition. It should only be pointed out that the procedure outlined in this paper is very straightforward and would give the results for any given state of loading to any desired degree of accuracy.

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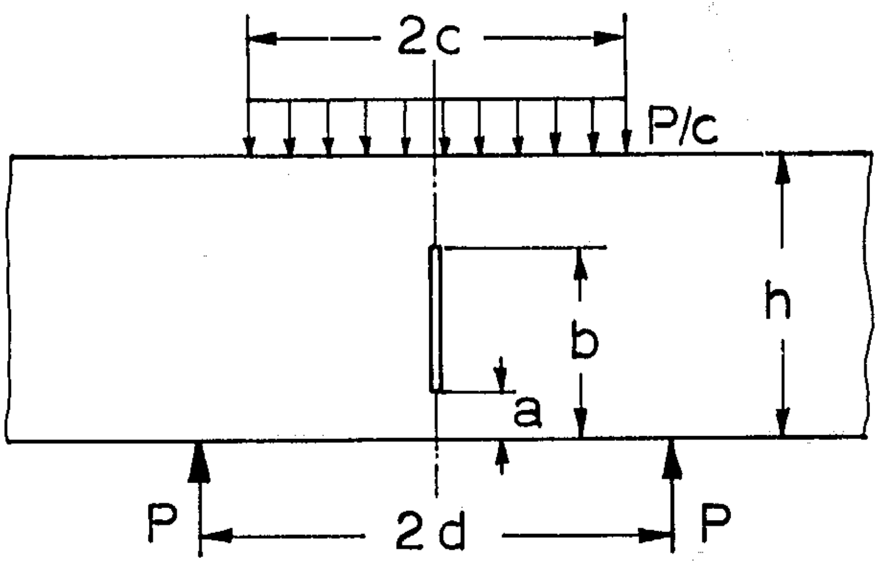

Figure 1. Geometry for a cracked strip under bending.