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Technical Report 76-T19

ADDITION OF HIGHER ORDER PLATE AND SHELL ELEMENTS INTO NASTRAN COMPUTER PROGRAM

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Final Report

Prepared for the
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Langley Research Center
Hampton, Virginia 23665

Under
Grant NSG 1117
Dr. D. J. Weidman, Technical Monitor
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By

R. Narayanaswami[1]

ABSTRACT

Two higher order plate elements, the linear strain triangular membrane element TRIM6 and the quintic bending element TRPLT1, and a shallow shell element TRSHL, suitable for inclusion into the NASTRAN® (NASA STRuctural ANalysis) program are described in this report. Additions to the NASTRAN Theoretical Manual [NASA SP-221 (03)], the NASTRAN Users' Manual [NASA SP-222 (03)], the NASTRAN Programmers' Manual [NASA SP-223 (03)], and the NASTRAN Demonstration Problem Manual [NASA SP-224 (03)], for inclusion of these elements into the NASTRAN program are presented herein

INTRODUCTION

New higher order plate (nonconforming quintic) and shell elements suitable for inclusion into the NASTRAN (NASA STRuctural ANalysis) program have been developed by Narayanaswami (refs. 1 and 2). The linear strain triangular membrane element developed by Argyris (ref. 3) has proved to be an accurate element for solving for membrane action in plates. Preliminary studies in the NASTRAN environment indicate that these elements are more efficient than the existing plate and shell elements in NASTRAN. These elements have now been added to the level 16.0 version of NASTRAN® and the complete computer listing for the addition has been delivered to the NASTRAN Systems Management Office, NASA Langley Research Center, Hampton, Virginia This report describes the theoretical formulations, information pertaining to their use, programming details and demonstration problems pertaining to the

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three elements, TRIM6, TRPLT1 and TRSHL in update form suitable for incorporation into the respective NASTRAN manuals (refs. 4, 5, 6 and 7).

DETAILS OF ELEMENT FORMULATIONS

The Linear Strain Triangular Membrane Element TRIM6

First proposed by Argyris (ref. 3), this element has six nodes, three at the corners and three at the midpoints of the sides. The element uses a quadratic displacement field. The thickness of the element as well as the temperature distribution within the element are permitted to have bilinear variation; the three constants of the bilinear equation for the same are evaluated by the respective user-specified values at the three corner nodes of the element. The FORTRAN subroutines for stiffness, mass, thermal load vector, and stress data recovery have been coded and tested out in stand-alone computer programs. The updates for incorporating the element into the NASTRAN program have been prepared and checked out in NASTRAN® Level 16.0 versions. The element is currently designed for use with the statics and normal modes rigid formats of NASTRAN.

The Higher Order Triangular Bending Element TRPLT1

This element was developed by Narayanaswami (refs. 1 and 7) as a modification of the high precision triangular plate bending element developed by Cowper et al. (ref. 8). The element has six nodes, three at the corners and three at the midpoints of the sides. A quintic displacement field is chosen for the transverse displacement. Transverse shear flexibility is taken into account in the stiffness formulation. The thickness of the element is permitted to have bilinear variations, the three constants of the bilinear equation for the same are evaluated by the respective user-specified values at the three corner nodes of the element. The FORTRAN subroutines for stiffness, mass, thermal load vector and stress data recovery have been coded and tested out in stand-alone computer programs. The updates for incorporating the element into the NASTRAN program have been prepared and checked out in

NASTRAN® Level 16.0 version. The element is currently designed for use with the statics and normal modes rigid formats of NASTRAN.

The Triangular Shallow Shell Element TRSHL

This element was developed by Narayanaswami (ref. 2). In the element coordinate system, the element has 30 degrees of freedom (d.o f.), viz , the three translations u , v , w in the x , y , z directions and the 2 rotations α and β about the xz and yz planes, at each of the 3 corner nodes and 3 midside nodes of the triangle. The membrane behavior of this element is approximated by the TRIM6 element, the bending behavior is approximated by the TRPLT1 element and the membrane-bending coupling is approximated using shallow shell theory of Novozhilov (ref. 8) The element is currently designed for use with the statics, normal modes and buckling rigid formats of NASTRAN.

ADDITIONS OR MODIFICATIONS TO NASTRAN MANUALS AND SOURCE CODE

The updates to the NASTRAN Manuals for the addition of these elements are given in Appendixes A, B, C and D. The list of subroutines that are being modified or added to the NASTRAN Source Code is given in Appendix E.

CONCLUDING REMARKS

The addition of higher order plate elements, TRIM6 and TRPLT1, and the triangular shallow shell element TRSHL, into the NASTRAN program is completed These elements are added to the Level 16.0 version of NASTRAN®. The demonstration problems indicate the excellent accuracy of these elements for solving plate and shell problems. The availability of these elements in NASTRAN enchances the program's capability in these areas.

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APPENDIX A

Updates to the NASTRAN Theoretical Manual for the addition of TRIM6, TRPLT1, and TRSHL elements

5.8 PLATES

NASTRAN includes two different shapes of plate elements (triangular and quadrilateral) and two different stress systems (membrane and bending) which are, at present, uncoupled. There are in all a total of 13 different forms of plate elements as follows:

1. TRMEM - A triangular element with finite inplane stiffness and zero bending stiffness.

2. TRIM6 - A triangular element with finite inplane stiffness and zero bending stiffness. Uses quadratic polynomial representation for membrane displacements; bilinear variation in terms of the planar coordinates is permitted for the thickness as well as the temperature distribution of the element

3. TRBSC - The basic unit from which the bending properties of the other plate elements are formed. In stand-alone form, it is used mainly as a research tool.

4. TRPLT - A triangular element with zero inplane stiffness and finite bending stiffness. It si composed of three basic bending triangles that are coupled to form a Clough composite triangle; see section 5 8 3.3

5. TRPLT1 - Higher order bending element - a triangular element with zero inplane stiffness and finite bending stiffness. Uses quintic polynomial representation for transverse displacement, bilinear variation in terms of the planar coordinates of the element is permitted for the element thickness

6. TRIA1 - A triangular element with both inplane and bending stiffness It is designed for sandwich plates in which different materials can be referenced for membrane, bending, and transverse shear properties.

7. TRIA2 - A triangular element with both inplane and bending stiffness that assumes a solid homogeneous cross section

8. QDMEM - A quadrilateral membrane element consisting of four overlapping TRMEM elements

9 QDMEM1 - An isoparametric quadrilateral membrane element.

10. QDMEM2 - A quadrilateral membrane element consisting of four nonoverlapping TRMEM elements.

11. QDPLT - A quadrilateral bending element. It is composed of four basic bending triangles.

12. QUAD1 - A quadrilateral element with both inplane and bending stiffness, similar to TRIA1.

13. QUAD2 - A quadrilateral element similar to TRIA2.

Anisotropic material properties may be employed in all plate elements. TRMEM and TRBSC are the basic plate elements from which TRPLT, TRIA1, TRIA2, QDMEM, QDMEM2, QDPLT, QUAD1, and QUAD2 elements are formed. The stiffness matrices of plate elements are formed from the rigorous application of energy theory to a polynomial representation of displacement functions. An important feature in the treatment of bending is that transverse shear flexibility is included.

All of the properties of the elements except those of TRIM6 and TRPLT1 are assumed uniform over their surfaces. For elements TRIM6 and TRPLT1, the thickness can have bilinear variation over their surfaces. In addition, element TRIM6 has bilinear variation over the surface for the temperature distribution.

The detailed discussion of the plate elements is divided into subsections according to the following topics: membrane triangles, TRMEM, QDMEM, QDMEM1, QDMEM2, the basic bending triangle, TRBSC; composite triangles and quadrilaterals, TRPLT1, TRIA1, TRIA2, QDPLT, QUAD1, QUAD2, the treatment of inertia properties; the isoparametric quadrilateral membrane element, QDMEM1, linear strain membrane triangle, TRIM6; higher order bending element, TRPLT1. The accuracy of the bending plate elements in various applications is discussed in section 15.2, the accuracy of the quadrilateral membrane elements is discussed in section 15.3, and the accuracy of the TRIM6 element is discussed in section 15.4.

5.8.6. TRIM6 The Linear Strain Membrane Element

This element was first formulated by J. H. Argyris and is described in references 1 and 2. The present development is based on the derivation in reference 2, and the important characteristics of the element are that:

1. The stresses and strains vary within the element linearly

2. Bilinear variation in the planar coordinates for the thickness of the element is permitted.

3 Bilinear variation in the planar coordinates for the temperature in the element is provided.

4. Differential stiffness and piecewise linear analysis capability are not implemented at present.

The element is compared for accuracy against theoretical results in section 15 4. The calculation of its mass properties is discussed in section 5.8.4

5.8.6.1 Geometry and Displacement Field

The geometry of the element is shown in figure A1. The element has six grid points, three at the vertices and three at the midpoints of the sides.

u and v are components of displacements parallel to the x- and y-axes of the local (element) coordinate system The inplane displacements at the corners of the element are represented by the vector $\{u_e\}$ where

$$\{u_e\}^t = \lfloor u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4\ u_5\ v_5\ u_6\ v_6 \rfloor \tag{1}$$

Let $[K_{ee}]$ be the stiffness matrix referred to the vector $\{u_e\}$, i.e.,

$$[K_{ee}]\{u_e\} = \{f_e\} \tag{2}$$

where the elements of $\{f_e\}$ are the inplane forces at the corners of the element. The stiffness matrix $[K_{ee}]$ is derived by standard finite element procedures.

The u and v displacements are assumed to vary quadratically with position on the surface of the element,

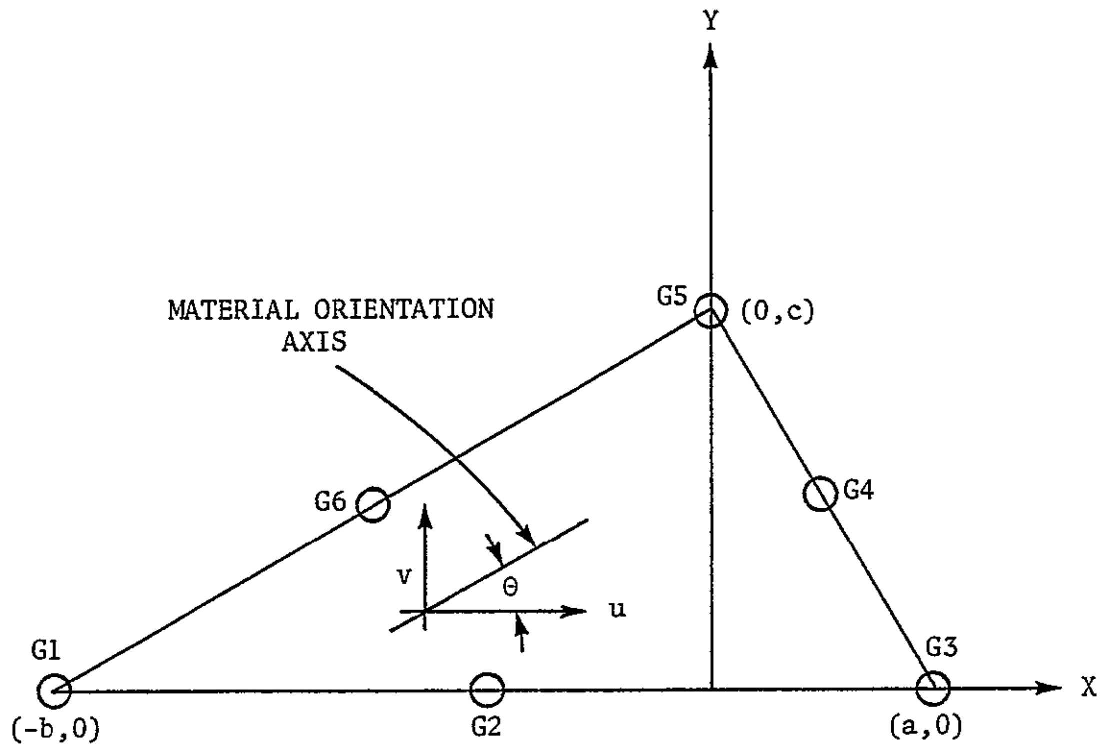

Figure A1. TRIM6 membrane element in element coordinate system.

$$u = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \tag{3}$$

$$v = b_7 + b_8x + b_9y + b_{10}x^2 + b_{11}xy + b_{12}y^2 \tag{4}$$

The quantities a_1 , a_2 , . . . a_6 , b_7 , b_8 , . . . b_{12} may be regarded as generalized coordinates to which the displacements at the corners of the element are uniquely related, i.e , the vector of generalized coordinates is expressed as

$$\{a\}^T = \lfloor a_1 a_2 a_3 a_4 a_5 a_6 b_7 b_8 b_9 b_{10} b_{11} b_{12} \rfloor \tag{5}$$

In concise form equations (3) and (4) are written as

$$u = \sum_{i=1}^{6} a_i x^{m_i} y^{n_i} \tag{6}$$

$$v = \sum_{i=7}^{12} b_i x^{p_i} y^{q_i} \tag{7}$$

For convenience in later calculations, the range of summation is kept as 1 to 12 for expressions for both u and v , i.e.,

$$u = \sum_{i=1}^{12} a_i x^{m_i} y^{n_i} \tag{8}$$

$$v = \sum_{i=1}^{12} b_i x^{p_i} y^{q_i} \tag{9}$$

such that

$$m_1 = 0 \ , \quad m_2 = 1 \ , \quad m_3 = 0 \ , \quad m_4 = 2 \ , \quad m_5 = 1 \ , \quad m_6 = 0$$

$$n_1 = 0 \ , \quad n_2 = 0 \ , \quad n_3 = 1 \ , \quad n_4 = 0 \ , \quad n_5 = 1 \ , \quad n_6 = 2$$

$$a_i = m_i = n_i = 0 \qquad i = 7 \text{ to } 12 \tag{10}$$

$$p_7 = 0 \, , \quad p_8 = 1 \; ; \quad p_9 = 0 \; ; \quad p_{10} = 2 \; ; \quad p_{11} = 1 \; ; \quad p_{12} = 0$$

$$q_7 = 0 \; ; \quad q_8 = 1 \; ; \quad q_9 = 0 \; ; \quad q_{10} = 0 \; ; \quad q_{11} = 1 \; ; \quad q_{12} = 2$$

$$b_i = p_i = q_i = 0 \qquad i = 1 \text{ to } 6 \; . \tag{11}$$

In matrix notation, the vector $\{u_e\}$ is written as

$$\{u_e\} = \left[H\right] \{a\} \tag{12}$$

where the 12 × 12 $\left[H\right]$ matrix can be obtained by substituting the coordinates of the six grid points into equations (8) and (9). Since complete polynomial expressions are chosen for the u and v displacements, the inverse of $\left[H\right]$ matrix exists. Hence $\{a\}$ can be expressed as

$$\{a\} = \left[H\right]^{-1} \{u_e\} \tag{13}$$

Bilinear variation in the x- and y-coordinates is assumed for the thickness t of the element, i.e., the thickness t of the element at any point (x,y) within the element is given by

$$t(x,y) = c_1 + c_2 x + c_3 y \tag{14}$$

In concise form, this is written as

$$t = \sum_{k=1}^{3} c_k x^{r_k} y^{s_k} \tag{15}$$

The thickness of the element at the three vertices is specified as t_1 , t_3 , t_5 . Hence the coefficients c_1 , c_2 , c_3 can be expressed as

$$c_1 = \frac{t_1 a + t_3 b}{(a + b)} \tag{16}$$

$$c_2 = \frac{t_3 - t_1}{a + b} \tag{17}$$

$$c_3 = \frac{1}{c} (t_5 - c_1) \tag{18}$$

where a , b , c are the projected lengths of the triangle on the local x- and y-axes and are obtained from the basic coordinates of the vertices of the triangle as given in section 4.87.21.2 of the Programmers' Manual.

The membrane strains are

$$\varepsilon_x = \frac{\partial u}{\partial x} = \sum_{i=1}^{12} a_i m_i x^{(m_i - 1)} y^{n_i} \tag{19}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \sum_{i=1}^{12} b_i V_i x^{p_i} y^{(q_i - 1)} \tag{20}$$

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum \left(a_i n_i x^{m_i} y^{(n_i - 1)} + b_i p_i x^{(p_i - 1)} y^{q_i} \right) \tag{21}$$

The stress vector $\{\sigma\}$ is related to the strain vector by the two-dimensional elastic modulus matrix, $[G_e]$:

$$\{\sigma\} = [G_e] \{\varepsilon\} \tag{22}$$

The specification of $[G_e]$ for isotropic and anisotropic materials is the same as that given by equations (13), (14), and (15) in section 5.8.4.

The membrane strain energy of the element is

$$E_s = \frac{1}{2} \int \{\sigma\}^T \{\varepsilon\} \, t dx dy \tag{23}$$

By virtue of equation (22) and the symmetry of matrix $[G_e]$,

$$E_s = \frac{1}{2} \int \{\varepsilon\}^T \left[G_e\right] \{\varepsilon\} \, t dxdy \tag{24}$$

Substitution of equation (15) into equation (24) results in

$$E_s = \frac{1}{2} \int \{\varepsilon\}^T \left[G_e\right] \{\varepsilon\} \left(\sum_{k=1}^{3} c_k x^{r_k} y^{s_k} \right) dxdy \tag{25}$$

Expressing the elements of the symmetric portion of the matrix $\left[G_e\right]$ by G_{11} , G_{12} , G_{13} , G_{22} , G_{23} , G_{33} , i.e.,

$$\left[G_e\right] = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ & G_{22} & G_{23} \\ \text{sym} & & G_{33} \end{bmatrix} \tag{26}$$

and performing the matrix multiplication of equation (25), the expression for strain energy becomes

$$E_s = \frac{1}{2} \iint \Big\{ \varepsilon_x{}^2 G_{11} + \varepsilon_y{}^2 G_{22} + \gamma^2 G_{33} + G_{12} \left(\varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_x\right) + G_{13} \left(\varepsilon_x \gamma + \gamma \varepsilon_x\right) + G_{23} \left(\varepsilon_y \gamma + \gamma \varepsilon_y\right) \Big\} \left(\sum_{k=1}^{3} c_k x^{r_k} y^{s_k} \right) dxdy \tag{27}$$

To proceed further it is necessary to have a formula for the integral of the type

$$\iint x^m y^n \, dxdy$$

taken over the area of the element. The value of the integral is given in reference 3 as

$$\iint x^m y^n dxdy = F(m,n)$$

$$= c^{n+1} \left\{ a^{m+1} - (-b)^{m+1} \right\} \frac{m!n!}{(m+n+2)!} \tag{28}$$

Using equations (19), (20), (21), and (28) in (27), the first term of equation (27) becomes

$$\frac{1}{2} \iint \varepsilon_x{}^2 G_{11} \left(\sum_{k=1}^{3} c_k x^{r_k} y^{s_k} \right) dxdy$$

$$= \frac{1}{2} \sum_{i=1}^{12} \sum_{j=1}^{12} \sum_{k=1}^{3} a_i a_j c_k m_i m_j \; F(m_i + m_j + r_k - 2, n_i + n_j + s_k) \tag{29}$$

Similarly the other terms of equation (27) can be expressed in terms of the area integral F . The strain energy, E_s , can also be expressed as

$$E_s = \frac{1}{2} \{a\}^T \left[k_{gen} \right] \{a\} \tag{30}$$

where $\left[k_{gen} \right]$ is the stiffness matrix with respect to generalized coordinates $\{a\}$ Expressing each of the terms of the right-hand side of equation (27) in terms of the area integral F and comparing the same with equation (30), the jth element of the ith row of the generalized stiffness matrix is

$$\begin{aligned} k_{ij} = \sum_{k=1}^{3} c_k \Big[& G_{11} m_i m_j F(m_i + m_j + r_k - 2, n_i + n_j + s_k) \\ & + G_{22} q_i q_j F(p_i + p_j + r_k, q_i + q_j + s_k - 2) \\ & + G_{33} \Big\{ n_i n_j F(m_i + m_j + r_k, n_i + n_j + s_k - 2) \\ & \quad + p_i p_j F(p_i + p_j + r_k - 2, q_i + q_j + s_k) \\ & \quad + n_i p_j F(m_i + p_j + r_k - 1, n_i + q_j + s_k - 1) \\ & \quad + p_i n_j F(m_j + p_i + r_k - 1, n_j + q_i + s_k - 1) \Big\} \end{aligned} \tag{31}$$

(continued)

$$
\begin{aligned}
&+ G_{12}\Big\{ m_j q_i F(m_j + p_i + r_k - 1,\ n_j + q_i + s_k - 1) \\
&\qquad + m_i q_j F(m_i + p_j + r_k - 1,\ n_i + q_j + s_k - 1)\Big\} \\
&+ G_{13}\Big\{ (m_j n_i + m_i n_j) F(m_i + m_j + r_k - 1,\ n_i + n_j + s_k - 1) \\
&\qquad + m_j p_i F(m_j + p_i + r_k - 2,\ n_j + q_i + s_k) \\
&\qquad + m_i p_j F(m_i + p_j + r_k - 2,\ n_i + q_j + s_k)\Big\} \\
&+ G_{23}\Big\{ (p_i q_j + p_j q_i) F(p_i + p_j + r_k - 1,\ q_i + q_j + s_k - 1) \\
&\qquad + n_i q_j F(m_i + p_j + r_k,\ n_i + q_j + s_k - 2) \\
&\qquad + n_j q_i F(m_j + p_i + r_k,\ n_j + q_i + s_k - 2)\Big\}\Big]
\end{aligned}
\tag{31}
$$

(31) (concluded)

Using equation (13), the generalized stiffness matrix $[k_{gen}]$ can be transformed to element stiffness matrix $[k_{ee}]$ as

$$[k_{ee}] = [H^{-1}]^T [k_{gen}][H^{-1}] \tag{32}$$

As a final step, the stiffness matrix is transformed from the local element coordinate system to the basic coordinate system of the grid points and to the global coordinate system. Let the transformation for displacements be

$$\{u_{basic}\} = [E]^T \{u_{element}\} \tag{33}$$

and

$$\{u_{global}\} = [T]\{u_{basic}\} \tag{34}$$

Then,

$$[k_{basic}] = [E][k_{element}][E]^T \tag{35}$$

and

$$[k_{global}] = [T]^T [k_{basic}] [T] \tag{36}$$

Substituting equation (32) in equation (35) and equation (36), the global stiffness matrix becomes

$$[k_{global}] = [T]^T [E] [H^{-1}]^T [k_{gen}] [H^{-1}] [E]^T [T] \tag{37}$$

Equivalent Thermal Load Vector:

Thermal expansion of an element produces equivalent loads at the grid points. Thermal expansion is represented by a vector of thermal strains.

$$\{\varepsilon_t\} = \begin{Bmatrix} \varepsilon_{xt} \\ \varepsilon_{yt} \\ \gamma_t \end{Bmatrix} = \begin{Bmatrix} \alpha_{e1} \\ \alpha_{e2} \\ \alpha_{e3} \end{Bmatrix} (\overline{T} - T_o) = \{\alpha_e\} (\overline{T} - T_o) \tag{38}$$

Where $\{\alpha_e\} = [U]^{-1} \{\alpha_m\}$ is a vector of thermal expansion coefficients, $[U]$ is the strain transformation matrix given in equation (15) in section 5.8.4, and $\{\alpha_m\}$ is the vector of thermal expansion coefficients in the material axis system, T_o is the reference or stress-free temperature of the material, and $\overline{T}$ is the temperature at any point (x,y) in the element and is given by a linear polynomial.

$$\overline{T} = d_1 + d_2 x + d_3 y \tag{39}$$

In concise form, this is written as

$$\overline{T} = \sum_{\ell=1}^{3} d_\ell x^{t_\ell} y^{u_\ell} \tag{40}$$

The temperature T_1 , T_3 , and T_5 at the three vertices of the element will be modified by the reference temperature T_o and used to evaluate the three constants d_1 , d_2 , and d_3 :

$$d_1 = \frac{T_1'a + T_3'b}{(a + b)} \tag{41}$$

$$d_2 = \frac{T_3' - T_1'}{(a + b)} \tag{42}$$

$$d_3 = \frac{1}{c} \left[T_5' - d_1\right] \tag{43}$$

where

$$T_1' = (T_1 - T_o) \ ; \quad T_3' = (T_3 - T_o) \ ; \quad \text{and} \quad T_5' = (T_3 - T_o) \tag{44}$$

An equivalent elastic state of stress that will produce the same thermal strains is

$$\{\sigma_t\} = \left[G_e\right] \{\varepsilon_t\} = \left[G_e\right] \{\alpha_e\} \ (\bar{T} - T_o) \tag{45}$$

An equivalent set of generalized loads $\{P_{gen}\}$ applied to corners of the element is obtained from the relation

$$\begin{aligned} \{a\}^t \ \{P_{gen}\} &= \int_A \ \{\varepsilon\}^t \ \{\sigma_t\} \ tdA \\ &= \iint \ \{\varepsilon\}^t \left[G_e\right] \{\alpha_e\} \left(\sum_{\ell=1}^{3} \ d_\ell x^{t_\ell} y^{u_\ell}\right) \\ &\quad \left(\sum_{k=1}^{3} \ c_k x^{r_k} y^{s_k}\right) \ dxdy \end{aligned} \tag{46}$$

Performing the matrix multiplications in equation (46) and using the following notations, viz

$$G_{11} = G_{11}\alpha_{e1} + G_{12}\alpha_{e2} + G_{13}\alpha_{e3} \tag{47}$$

$$G_{22} = G_{12}\alpha_{e1} + G_{22}\alpha_{e2} + G_{23}\alpha_{e3} \tag{48}$$

$$G_{33} = G_{13}\alpha_{e1} + G_{23}\alpha_{e2} + G_{33}\alpha_{e3} \tag{49}$$

Equation (46) reduces to

$$\{a\}^t \{P_{gen}\} = \iint (\varepsilon_x G'_{11} + \varepsilon_y G'_{22} + \gamma G'_{33}) \left(\sum_{\ell=1}^{3} d_\ell x^{t_\ell} y^{u_\ell}\right) \left(\sum_{k=1}^{3} c_k x^{r_k} y^{s_k}\right) dxdy \tag{50}$$

Performing the integration term by term, the first term in equation (50) becomes

$$\begin{aligned} &\iint \varepsilon_x G'_{11} \left(\sum_{\ell=1}^{3} d_\ell x^{t_\ell} y^{u_\ell}\right) \left(\sum_{k=1}^{3} c_k x^{r_k} y^{s_k}\right) dxdy \\ &= \sum_{i=1}^{12} \sum_{k=1}^{3} \sum_{\ell=1}^{3} G'_{11} a_i m_i c_k d_\ell \iint x^{(m_i + r_k + t_\ell - 1)} \cdot y^{(n_i + s_k + u_\ell)} dxdy \\ &= \sum_{i=1}^{12} \sum_{k=1}^{3} \sum_{\ell=1}^{3} G'_{11} a_i m_i c_k d_\ell F(m_i + r_k + t_\ell - 1, n_i + s_k + u_\ell) \end{aligned} \tag{51}$$

Similarly, the second and third terms of equation (50) reduce to

$$\sum_{i=1}^{12} \sum_{k=1}^{3} \sum_{\ell=1}^{3} G'_{22} b_i q_i c_k d_\ell F(p_i + r_k + t_\ell, q_i + s_k + u_\ell - 1)$$

and

$$\sum_i \sum_k \sum_\ell G'_{33} c_k d_\ell \left\{ a_i n_i F(m_i + r_k + t_\ell, n_i + s_k + u_\ell - 1) \right.$$
$$\left. + b_i p_i F(p_i + r_k + t_\ell - 1, q_i + s_k + u_\ell) \right\} .$$

respectively. From equations (50) and (51), the ith element of the generalized load vector $\{P_{gen}\}$ is

$$\begin{aligned}(P_{gen})_i = \sum_{k=1}^{3} \sum_{\ell=1}^{3} c_k d_\ell \Big[& G'_{11} m_i F(m_i + r_k + t_\ell - 1, n_i + s_k + u_\ell) \\ & + G'_{22} q_i F(p_i + r_k + t_\ell, q_i + s_k + u_\ell - 1) \\ & + G'_{33} \Big\{ n_i F(m_i + r_k + t_\ell, n_i + s_k + u_\ell - 1) \\ & + p_i F(p_i + r_k + t_\ell - 1, q_i + s_k + u_\ell) \Big\} \Big]\end{aligned} \qquad (52)$$

The generalized equivalent load vector $\{P_{gen}\}$ is transformed to load vector $\{P_e\}$ in element coordinate and to $\{P_g\}$, in global coordinates by the following transformations

$$\{P_e\} = \{H^{-1}\}^T \{P_{gen}\} \qquad (53)$$

and

$$\{P_g\} = [T]^T [E] \{P_e\} \qquad (54)$$

After the grid point displacements have been evaluated, stresses in the element are computed by combining the relationships

$$\{u_e\} = [E] [T]^T \{u_g\} \qquad (55)$$

$$\{a\} = [H^{-1}] \{u_e\} \qquad (56)$$

$\{\varepsilon\}$ is evaluated from equations (19), (20), and (21). Stress vector $\{\sigma\}$ is then equal to

$$\{\sigma\} = \left[G_e\right] (\{\varepsilon\} - \{\varepsilon_t\}) \tag{57}$$

The stresses are computed at the three vertices and at the centroid. The principal stresses and the maximum shear force are computed from the elements of $\{\sigma\}$. The direction of the maximum principal stress is referenced to the side joining grid points 1 and 3 of the triangle.

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5.8.7 TRPLT1, higher order bending element

This element was developed by Narayanaswami (refs. 1 and 2) as a modification of the high precision bending element of Cowper, et al. (ref. 3). The element has grid points at the vertices and at the midpoints of the sides of the triangle. At each grid point, there are three degrees of freedom in the element coordinate system, viz, the transverse displacement, w, normal to the X-Y plane, with positive direction outward from the paper, and the rotations of the normal to the plate α and β, with positive directions following from the right-hand rule. The element, thus, has 18 degrees of freedom in the element coordinate system. The transverse displacement, w, at any point within and on the boundaries of the element is assumed to vary as a quintic polynomial. Since the variation of deflection along any edge is a quintic polynomial in the edgewise coordinate, the six coefficients of this polynomial are uniquely determined by deflection and edgewise slope at the three grid points of the edge. Displacements are thus continuous between two elements that have a common edge. The rotation about each edge is constrained to vary cubically; however, since the rotations are defined only at three points along an edge, there is no rotation continuity between two elements that have a common edge. The element thus belongs to the class of nonconforming elements. The requirement that the edge rotation varies cubically along each edge established three constraint equations between the coefficients of the quintic polynomial for w. These equations together with the 18 relations between the grid point degrees of freedom and the polynomial coefficients serve to evaluate uniquely the 21 coefficients a_1 to a_{21} of the quintic polynomial assumed for the transverse displacement.

5.8.7.1 Derivation of element properties

Element geometry: Rectangular Cartesian coordinates are used in the formulation. An arbitrary triangular element is shown in figure A2. X, Y, and Z are the basic coordinates, x, y, and z are the local coordinates. The grid points of the element are numbered in counter-clockwise direction as shown in the figure. The lengths a, b, and c shown in figure A2 can be easily evaluated from the basic coordinate (X_1, Y_1, Z_1), (X_3, Y_3, Z_3), and (X_5, Y_5, Z_5) of the vertices of the triangle.

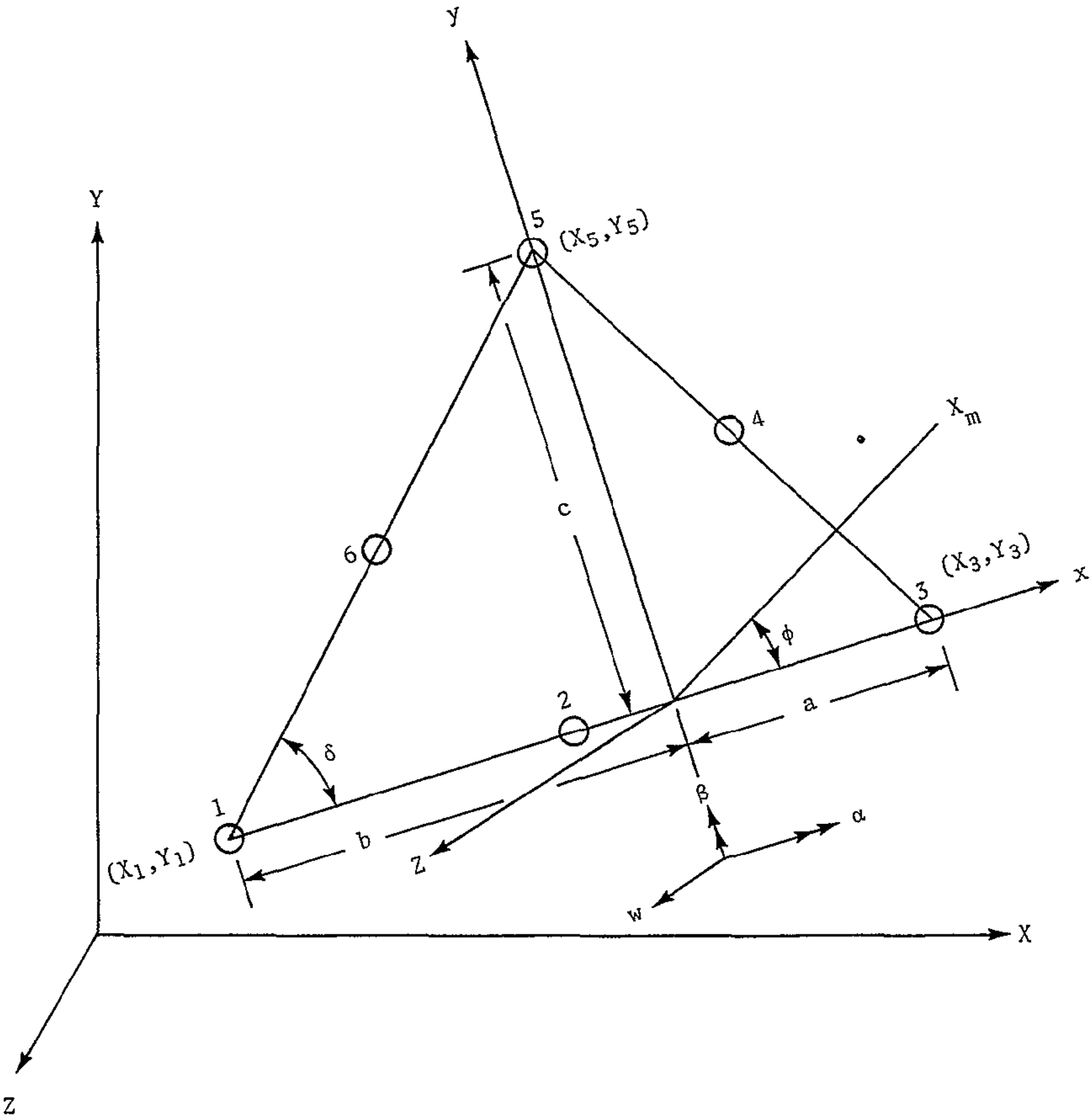


Figure A2. Triangular element TRPLT1 geometry.

<u>Displacement field</u>: The deflection $w(x,y)$ within the triangular element is assumed to vary as a quintic polynomial in the local coordinates, that is,

$$\begin{aligned} w(x,y) = {} & a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 \\ & + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^4 + a_{12}x^3y \\ & + a_{13}x^2y^2 + a_{14}xy^3 + a_{15}y^4 + a_{16}x^5 + a_{17}x^4y \\ & + a_{18}x^3y^2 + a_{19}x^2y^3 + a_{20}xy^4 + a_{21}y^5 \end{aligned} \tag{1}$$

In concise form, this is written as

$$w = \sum_{i=1}^{21} a_i x^{m_i} y^{n_i} \tag{1a}$$

There are 21 independent coefficients, a_1 to a_{21}. These are evaluated by the following procedure.

The element has 18 degrees of freedom; namely, lateral displacement w in the z-direction, rotation α about the x-axis, and rotation β about the y-axis at each of the six grid points. The rotations α and β are obtained from the definitions of transverse shear strains γ_{xz} and γ_{yz}, that is,

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \beta \qquad \gamma_{yz} = \frac{\partial w}{\partial y} - \alpha \tag{2}$$

It is shown later on that γ_{xz} and γ_{yz} and hence α and β at any grid point can be expressed in terms of the coefficients a_1 to a_{21}. Thus, 18 equations relating w, α and β at the grid points to the 21 coefficients are obtained. Three additional relations are required so that the 21 coefficients can be uniquely determined. These relations are obtained by imposing the condition that the edge rotation varies cubically along each edge. It is clear that these three constraint equations involve only the coefficients of the fifth degree terms in equation (1), since the lower degree terms satisfy the condition of cubic edge rotation automatically. Moreover, the condition depends only on the orientation of an edge. Along the edge defined by grid points 1 and 3 (where y = 0), the condition of the cubic edge rotation requires that

$$a_{17} = 0 \tag{3}$$

Along the edge defined by grid points 1 and 5 (inclined at angle δ to the x-axis) the edge rotation r_e is given by

$$\begin{aligned} r_e = \beta \sin\delta + \alpha\cos\delta = &- (5a_{16}x^4 + 4a_{17}x^3y + 3a_{18}x^2y^2 \\ &+ 2a_{19}xy^3 + a_{20}y^4)\sin\delta + (a_{17}x^4 + 2a_{18}x^3y + 3a_{19}x^2y^2 \\ &+ 4a_{20}xy^3 + 5a_{21}y^4)\cos\delta + \ldots \end{aligned} \tag{4}$$

where the dots indicate terms of third or lower degree. Also, along this edge,

$$x = s\cos\delta \qquad y = s\sin\delta \tag{5}$$

where s is the distance along the edge and

$$\cos\delta = b/\sqrt{b^2 + c^2} \qquad \sin\delta = c/\sqrt{b^2 + c^2} \tag{6}$$

By substituting x and y from equation (5) and $\cos\delta$ and $\sin\delta$ from equations (6) into equation (4) and rearranging (so that the leading terms are positive), the condition for cubic variation of rotation about edge 1-5 is

$$\begin{aligned} &5b^4ca_{16} + (4b^3c^2 - b^5)a_{17} + (3b^2c^3 - 2b^4c)a_{18} \\ &\quad + (2bc^4 - 3b^3c^2)a_{19} + (c^5 - 4b^2c^3)a_{20} - 5bc^4a_{21} = 0 \end{aligned} \tag{7}$$

Similarly, the condition for cubic variation of the rotation about the edge defined by grid points 3 and 5 (fig. 1) can be written as

$$5a^4ca_{16} + (-4a^3c^2 + a^5)a_{17} + (3a^2c^3 - 2a^4c)a_{18} + (-2ac^4 + 3a^3c^2)a_{19} + (c^5 - 4a^2c^3)a_{20} + 5ac^4a_{21} = 0 \tag{8}$$

The 18 relations between grid point displacements and the coefficients of the polynomial in equation (6) are written as

$$\{\delta\} = \left[Q\right] \{a\} \tag{9}$$

where $\{\delta\}$ is the vector of grid point displacements, $\left[Q\right]$ is the 18 x 21 matrix involving the coordinates of grid points substituted into the functions w [eq. (1)] and the appropriate expressions of α and β derived in detail later, and $\{a\}$ is the column vector of coefficients a_1 to a_{21}. The $\left[Q\right]$ matrix is now augmented by the three constraint equations (3), (7), and (8) to form a new 21 × 21 matrix $\left[R\right]$ in the following equation·

$$\{\delta_a\} = \left[R\right] \{a\} \tag{10}$$

where

$$\{\delta_a\} = \begin{Bmatrix} \{\delta\} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{10a}$$

For use in the evaluation of the stiffness matrix, $\{a\}$ needs to be expressed in terms of $\{\delta_a\}$; and, hence, it has to be established that the inverse of matrix $\left[R\right]$ exists. The nonsingularity of such a matrix $\left[R\right]$ for the T-15 and T-21 elements of Bell (ref. 4) follows from the completeness of the polynomials for w . For the high precision element, Cowper et al. (ref. 3) give an explicit expression for the determinant of such a matrix and show that the matrix is nonsingular in all practical situations. For this element, a numerical experiment described in reference 1 verifies that R is nonsingular for all practical cases. Hence, equation (10) is inverted to give

$$\{a\} = \left[R\right]^{-1} \{\delta_a\} \tag{11}$$

This equation can also be written as

$$\{a\} = \left[S\right] \{\delta\} \tag{12}$$

where $\left[S\right]$ is a 21 × 18 matrix and consists of the first 18 columns of $\left[R\right]^{-1}$.

From the computational standpoint, it is advantageous to substitute equation (3) into equation (1) and replace coefficients a_{18} to a_{21} by coefficients a_{17} to a_{20} , respectively. The matrix $\left[Q\right]$ then is of size 18 × 20, $\left[R\right]$ becomes 20 × 20 , and $\left[S\right]$ becomes 20 × 18. To add to the clarity of presentation, however, the complete quintic polynomial for w in equation (1) is retained throughout this section, and matrices $\left[Q\right]$, $\left[R\right]$, and $\left[S\right]$ and vector $\{a\}$ will have sizes 18 × 21, 21 × 21, 21 × 18, and 21 × 1, respectively.

<u>Elastic relationships</u> The elastic relationships are obtained from the theory of deformation for plates (ref 5) The curvatures are defined by

$$\begin{Bmatrix} \chi_x \\ \\ \chi_y \\ \\ \chi_{xy} \end{Bmatrix} \qquad \begin{Bmatrix} -\dfrac{\partial\beta}{\partial x} \\ \\ \dfrac{\partial\alpha}{\partial y} \\ \\ \dfrac{\partial\alpha}{\partial x} - \dfrac{\partial\beta}{\partial y} \end{Bmatrix} \tag{13}$$

Bending and twisting moments are related to curvatures by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} x_x \\ x_y \\ x_{xy} \end{Bmatrix} \tag{14}$$

where $[D]$ is, in general, a full symmetric matrix of elastic coefficients. For a solid isotropic plate of uniform thickness t ,

$$[D] = \frac{Et^3}{12(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \tag{15}$$

The thickness t of the element is assumed to vary bilinearly with position over the surface

$$t = c_1 + c_2x + c_3y \tag{16}$$

In concise form, it is written as

$$t = \sum_{k=1}^{3} c_k x^{r_k} y^{s_k} \tag{16a}$$

The thickness of the three vertices of the element t_1 , t_3 , and t_5 will be used to evaluate the constants c_1 , c_2 , and c_3 . It can be shown that

$$c_1 = \frac{t_1a + t_3b}{(a + b)} \tag{17}$$

$$c_2 = \frac{t_3 - t_1}{(a + b)} \tag{18}$$

$$c_3 = \frac{1}{c} [t_5 - c_1] \tag{19}$$

where a , b , c are the length of the element marked in figure B1. For an isotropic plate, $[D]$ becomes

$$[D] = \frac{1}{12}[G_e]\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} c_i c_j c_k x^{r_i+r_j+r_k} y^{s_i+s_j+s_k}\right) \tag{20}$$

where

$$[G_e] = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ \frac{E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \tag{21}$$

For anisotropic materials with the material orientation axis inclined at ϕ to the x-axis, the material elastic modulus matrix $[D_m]$ is transformed to the element elastic modulus matrix by

$$[D] = [U]^T [D_m] [U] \tag{22}$$

where

$$[U] = \begin{bmatrix} \cos^2\phi & \sin^2\phi & \cos\phi\sin\phi \\ \sin^2\phi & \cos^2\phi & -\cos\phi\sin\phi \\ -2\cos\phi\sin\phi & 2\cos\phi\sin\phi & \cos^2\phi - \sin^2\phi \end{bmatrix} \tag{23}$$

The positive sense of bending and twisting moments and transverse shear resultants is shown in figure A3.

The moment equilibrium equations are written as

$$V_x + \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = 0 \tag{24}$$

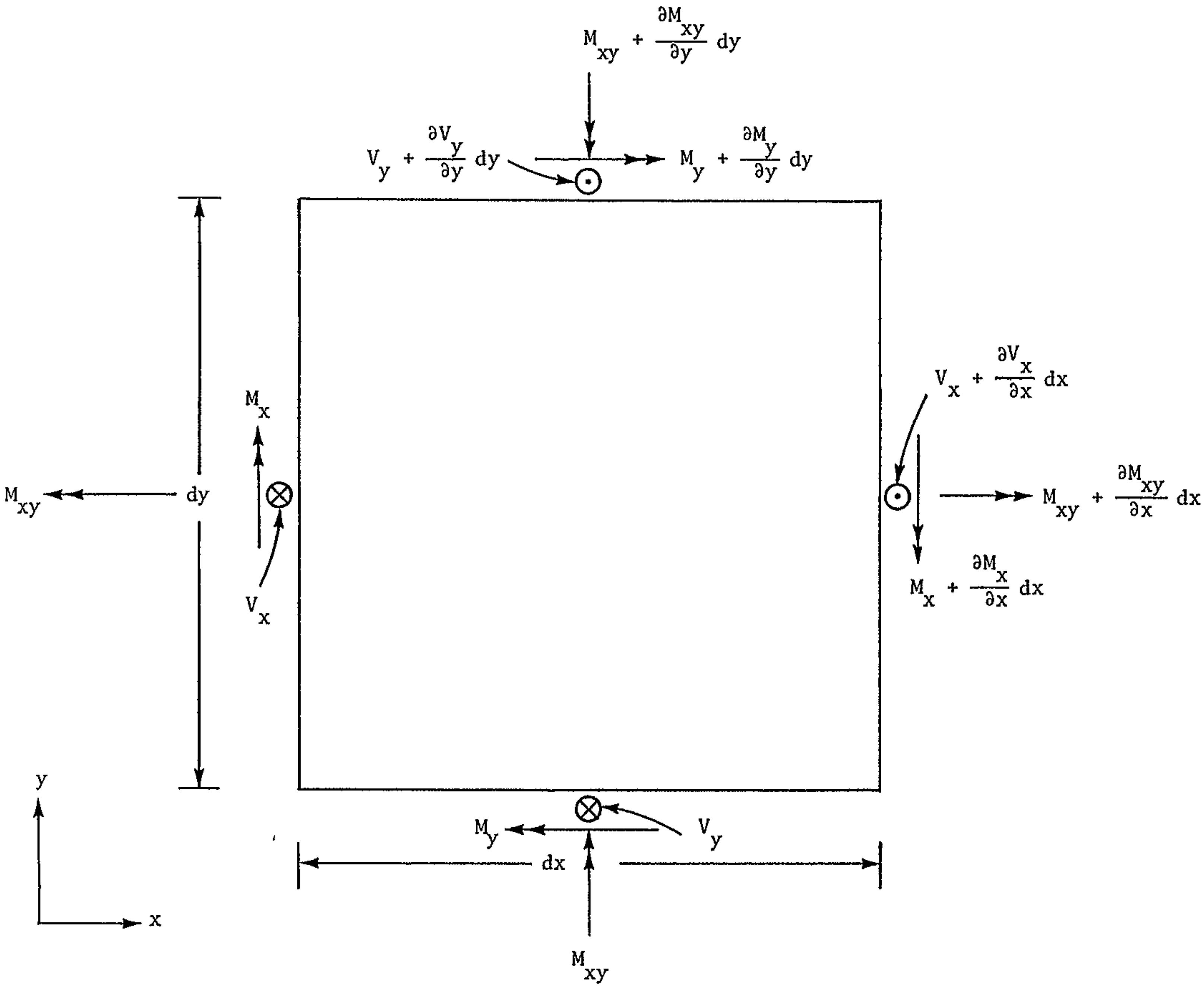


Figure A3. Sign convention for moments and shears.

$$V_y + \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = 0 \qquad (25)$$

Transverse shear strains are related to the shear resultants by

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = [J] \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} \qquad (26)$$

The matrix $[J]$ is, in general, a full symmetric 2 × 2 matrix of elements J_{11}, J_{12} $(J_{21} = J_{12})$ and J_{22}. For a plate with isotropic transverse shear material,

$$[J] = \frac{1}{Gt^*} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (27)$$

where G is the shear modulus and t^* is an "effective" thickness for transverse shear. For a simple case of a plate of uniform thickness t, t^* has the value t.

From equations (24), (25), and (26), it follows that

$$\left.\begin{aligned} \gamma_{xz} &= -J_{11}\left[\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right] - J_{12}\left[\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right] \\ \gamma_{yz} &= -J_{12}\left[\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right] - J_{22}\left[\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right] \end{aligned}\right\} \qquad (28)$$

Performing the partial differentiation with respect to x and y on equation (14), with subscripts on D denoting the elements of $[D]$, results in

$$
\left.
\begin{aligned}
\frac{\partial M_x}{\partial x} &= D_{11}\frac{\partial \chi_x}{\partial x} + D_{12}\frac{\partial \chi_y}{\partial x} + D_{13}\frac{\partial \chi_{xy}}{\partial x} \\
\frac{\partial M_x}{\partial y} &= D_{12}\frac{\partial \chi_x}{\partial y} + D_{22}\frac{\partial \chi_y}{\partial y} + D_{23}\frac{\partial \chi_{xy}}{\partial y} \\
\frac{\partial M_{xy}}{\partial x} &= D_{13}\frac{\partial \chi_x}{\partial x} + D_{23}\frac{\partial \chi_y}{\partial x} + D_{33}\frac{\partial \chi_{xy}}{\partial x} \\
\frac{\partial M_{xy}}{\partial y} &= D_{13}\frac{\partial \chi_x}{\partial y} + D_{23}\frac{\partial \chi_y}{\partial y} + D_{33}\frac{\partial \chi_{xy}}{\partial y}
\end{aligned}
\right\} \tag{29}
$$

where the symmetry of the $[D]$ matrix has been used. By substituting equations (29) into equations (28),

$$
\begin{aligned}
\gamma_{xz} = &- J_{11}\left[D_{11}\frac{\partial \chi_x}{\partial x} + D_{12}\frac{\partial \chi_y}{\partial x} + D_{13}\frac{\partial \chi_{xy}}{\partial x} \right. \\
&\left. + D_{13}\frac{\partial \chi_x}{\partial y} + D_{23}\frac{\partial \chi_y}{\partial y} + D_{33}\frac{\partial \chi_{xy}}{\partial y} \right] \\
&- J_{12}\left[D_{12}\frac{\partial \chi_x}{\partial y} + D_{22}\frac{\partial \chi_y}{\partial y} + D_{23}\frac{\partial \chi_{xy}}{\partial y} \right. \\
&\left. + D_{13}\frac{\partial \chi_x}{\partial x} + D_{23}\frac{\partial \chi_y}{\partial x} + D_{33}\frac{\partial \chi_{xy}}{\partial x} \right]
\end{aligned} \tag{30}
$$

and

$$
\begin{aligned}
\gamma_{yz} = &- J_{12}\left[D_{11}\frac{\partial \chi_x}{\partial x} + D_{12}\frac{\partial \chi_y}{\partial x} + D_{13}\frac{\partial \chi_{xy}}{\partial x} \right. \\
&\left. + D_{13}\frac{\partial \chi_x}{\partial y} + D_{23}\frac{\partial \chi_y}{\partial y} + D_{33}\frac{\partial \chi_{xy}}{\partial y} \right] \\
&- J_{22}\left[D_{12}\frac{\partial \chi_x}{\partial y} + D_{22}\frac{\partial \chi_y}{\partial y} + D_{23}\frac{\partial \chi_{xy}}{\partial y} \right. \\
&\left. + D_{13}\frac{\partial \chi_x}{\partial x} + D_{23}\frac{\partial \chi_y}{\partial x} + D_{33}\frac{\partial \chi_{xy}}{\partial x} \right]
\end{aligned} \tag{31}
$$

Rearranging and writing equations (30) and (31) in matrix notation yields

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} A_{11} A_{12} A_{13} A_{14} A_{15} A_{16} \\ A_{21} A_{22} A_{23} A_{24} A_{25} A_{26} \end{bmatrix} \begin{Bmatrix} \chi_{x,x} \\ \chi_{y,x} \\ \chi_{xy,x} \\ \chi_{x,y} \\ \chi_{y,y} \\ \chi_{xy,y} \end{Bmatrix} \tag{32}$$

where a comma in the subscript denotes partial differentiation and where

$$A_{11} = -(J_{11}D_{11} + J_{12}D_{13}) \tag{33a}$$

$$A_{12} = -(J_{11}D_{12} + J_{12}D_{23}) \tag{33b}$$

$$A_{13} = -(J_{11}D_{13} + J_{12}D_{33}) \tag{33c}$$

$$A_{14} = -(J_{11}D_{13} + J_{12}D_{12}) \tag{33d}$$

$$A_{15} = -(J_{11}D_{23} + J_{12}D_{22}) \tag{33e}$$

$$A_{16} = -(J_{11}D_{33} + J_{12}D_{23}) \tag{33f}$$

$$A_{21} = -(J_{12}D_{11} + J_{22}D_{13}) \tag{33g}$$

$$A_{22} = -(J_{12}D_{13} + J_{22}D_{23}) \tag{33h}$$

$$A_{23} = -(J_{12}D_{13} + J_{22}D_{33}) \tag{33i}$$

$$A_{24} = -(J_{12}D_{13} + J_{22}D_{12}) \quad (33j)$$

$$A_{25} = -(J_{12}D_{23} + J_{22}D_{22}) \quad (33k)$$

$$A_{26} = -(J_{12}D_{33} + J_{22}D_{23}) \quad (33l)$$

From equations (2) and (13), it follows that

$$\left.\begin{aligned} \chi_x &= -\frac{\partial\beta}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial\gamma_{xz}}{\partial x} \\ \chi_y &= \frac{\partial\alpha}{\partial y} = \frac{\partial^2 w}{\partial y^2} - \frac{\partial\gamma_{yz}}{\partial y} \\ \chi_{xy} &= \frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial y} = 2\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial\gamma_{xz}}{\partial y} - \frac{\partial\gamma_{yz}}{\partial x} \end{aligned}\right\} \quad (34)$$

Shear forces (and hence shear strains) are proportional to the third derivatives of the displacements. Since the displacement within the element is assumed to vary as a quintic polynomial, shear strains are expressed by a quadratic polynomial as follows

$$\gamma_{xz} = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 \quad (35)$$

$$\gamma_{yz} = b_7 + b_8x + b_9y + b_{10}x^2 + b_{11}xy + b_{12}y^2 \quad (36)$$

The task now is to express the unknown coefficients b_1 to b_6 and b_7 to b_{12} in terms of the generalized coordinates a_1 to a_{21}. Performing the differentiations on χ_x, χ_y, and χ_{xy} and substituting w, γ_{xz}, and γ_{yz} from equations (1), (35), and (36) into equations (34)

$$\chi_{x,x} = \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 \gamma_{xz}}{\partial x^2} = 6a_7 + 24a_{11}x + 6a_{12}y + 60a_{16}x^2 + 24a_{17}xy + 6a_{18}y^2 - 2b_4 \tag{37}$$

$$\chi_{y,x} = \frac{\partial^3 w}{\partial x \partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial x \partial y} = 2a_9 + 4a_{13}x + 6a_{14}y + 6a_{18}x^2 + 12a_{19}xy + 12a_{20}y^2 - b_{11} \tag{38}$$

$$\chi_{xy,x} = 2\frac{\partial^3 w}{\partial x^2 \partial y} - \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} - \frac{\partial^2 \gamma_{yz}}{\partial x^2} = 4a_8 + 12a_{12}x + 8a_{13}y + 24a_{17}x^2 + 24a_{18}xy + 12a_{19}y^2 - b_5 - 2b_{10} \tag{39}$$

$$\chi_{x,y} = \frac{\partial^3 w}{\partial x^2 \partial y} - \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} = 2a_8 + 6a_{12}x + 4a_{13}y + 12a_{17}x^2 + 12a_{18}xy + 6a_{19}y^2 - b_5 \tag{40}$$

$$\chi_{y,y} = \frac{\partial^3 w}{\partial y^3} - \frac{\partial^2 \gamma_{yz}}{\partial y^2} = 6a_{10} + 6a_{14}x + 24a_{15}y + 6a_{19}x^2 + 24a_{20}xy + 60a_{21}y^2 - 2b_{12} \tag{41}$$

and

$$\chi_{xy,y} = 2\frac{\partial^2 w}{\partial x \partial y^2} - \frac{\partial^2 \gamma_{xz}}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial x \partial y} = 4a_9 + 8a_{13}x + 12a_{14}y + 12a_{18}x^2 + 24a_{19}xy + 24a_{20}y^2 - 2b_6 - b_{11} \tag{42}$$

By substituting equations (35) to (42) into equations (32), the following equations are obtained.

$$
\begin{aligned}
b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 = {} & A_{11}(6a_7 + 24a_{11}x \\
& + 6a_{12}y + 60a_{16}x^2 + 24a_{17}xy + 6a_{18}y^2 - 2b_4) \\
& + A_{12}(2a_9 + 4a_{13}x + 6a_{14}y + 6a_{18}x^2 + 12a_{19}xy + 12a_{20}y^2 - b_{11}) \\
& + A_{13}(4a_8 + 12a_{12}x + 8a_{13}y + 24a_{17}x^2 + 24a_{18}xy + 12a_{19}y^2 \\
& - b_5 - 2b_{10}) + A_{14}(2A_8 + 6a_{12}x + 4a_{13}y + 12a_{17}x^2 + 12a_{18}xy \\
& + 6a_{19}y^2 - b_5) + A_{15}(6a_{10} + 6a_{14}x + 24a_{15}x + 6a_{19}x^2 \\
& + 24a_{20}xy + 60a_{21}y^2 - 2b_{12}) + A_{16}(4a_9 + 8a_{13}x + 12a_{14}y \\
& + 12a_{18}x^2 + 24a_{19}xy + 24a_{20}y^2 - 2b_6 - b_{11})
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
b_7 + b_8x + b_9y + b_{10}x^2 + b_{11}xy + b_{12}x^2 = {} & A_{21}(6a_7 + 24a_{11}x \\
& + 6a_{12}y + 60a_{16}x^2 + 24a_{17}xy + 6a_{18}y^2 - 2b_4) \\
& + A_{22}(2a_9 + 4a_{13}x + 6a_{14}y + 6a_{18}x^2 + 12a_{19}xy + 12a_{20}y^2 \\
& - b_{11}) + A_{23}(4a_8 + 12a_{12}x + 8a_{13}y + 24a_{17}x^2 + 24a_{18}xy \\
& + 12a_{19}y^2 - b_5 - 2b_{10}) + A_{24}(2a_8 + 6a_{12}x + 4a_{13}y + 12a_{17}x^2 \\
& + 12a_{18}xy + 6a_{19}y^2 - b_5) + A_{25}(6a_{10} + 6a_{14}x + 24a_{15}y \\
& + 6a_{19}x^2 + 24a_{20}xy + 60a_{21}y^2 - 2b_{12}) + A_{26}(4a_9 + 8a_{13}x \\
& + 12a_{14}y + 12a_{18}x^2 + 24a_{19}xy + 24a_{20}y^2 - 2b_6 - b_{11})
\end{aligned}
\tag{44}
$$

By comparing coefficients of like powers in x , y , x^2 , xy , and y^2 and constants of equations (43) and (44), the coefficients b_1 to b_6 and b_7 to b_{12} can be expressed in terms of the generalized coordinates a_1 to a_{21} . Thus

$$
\left.
\begin{aligned}
b_2 &= 24A_{11}a_{11} + 6(A_{14} + 2A_{13})a_{12} + 4(A_{12} + 2A_{16})a_{13} + 6A_{15}a_{14} \\
b_3 &= 6A_{11}a_{12} + 4(A_{14} + 2A_{13})a_{13} + 6(A_{12} + 2A_{16})a_{14} + 24A_{15}a_{15} \\
b_4 &= 60A_{11}a_{16} + 12(A_{14} + 2A_{13})a_{17} + 6(A_{12} + 2A_{16})a_{18} + 6A_{15}a_{19} \\
b_5 &= 24A_{11}a_{17} + 12(A_{14} + 2A_{13})a_{18} + 12(A_{12} + 2A_{16})a_{19} + 24A_{15}a_{20} \\
b_6 &= 6A_{11}a_{18} + 6(A_{14} + 2A_{13})a_{19} + 12(A_{12} + 2A_{16})a_{20} + 60A_{15}a_{21} \\
b_1 &= 6A_{11}a_7 + 2(A_{14} + 2A_{13})a_8 + 2(A_{12} + 2A_{16})a_9 + 6A_{15}a_{10} \\
&\quad - 2A_{11}b_4 - (A_{13} + A_{14})b_5 - 2A_{16}b_6 - 2A_{13}b_{10} - (A_{12} + A_{16})b_{11} \\
&\quad - 2A_{15}b_{12}
\end{aligned}
\right\} \quad (45)
$$

$$
\left.
\begin{aligned}
b_8 &= 24A_{21}a_{11} + 6(A_{24} + 2A_{23})a_{12} + 4(A_{22} + 2A_{26})a_{13} + 6A_{25}a_{14} \\
b_9 &= 6A_{21}a_{12} + 4(A_{24} + 2A_{23})a_{13} + 6(A_{22} + 2A_{26})a_{14} + 24A_{25}a_{15} \\
b_{10} &= 60A_{21}a_{16} + 12(A_{24} + 2A_{23})a_{17} + 6(A_{22} + 2A_{26})a_{18} + 6A_{25}a_{19} \\
b_{11} &= 24A_{21}a_{17} + 12(A_{24} + 2A_{23})a_{18} + 12(A_{22} + 2A_{26})a_{19} + 24A_{25}a_{20} \\
b_{12} &= 6A_{21}a_{18} + 6(A_{24} + 2A_{23})a_{19} + 12(A_{22} + 2A_{26})a_{20} + 60A_{25}a_{21} \\
b_7 &= 6A_{21}a_7 + 2(A_{24} + 2A_{23})a_8 + 2(A_{22} + 2A_{26})a_9 + 6A_{25}a_{10} \\
&\quad - 2A_{21}b_4 = (A_{23} + A_{24})b_5 - 2A_{26}b_6 - 2A_{23}b_{10} - (A_{22} + A_{26})b_{11} \\
&\quad - 2A_{25}b_{12}
\end{aligned}
\right\} \quad (46)
$$

If equations (45) and (46) are substituted into equations (35) and (36), the explicit relation between the transverse shear strain and the generalized coordinates (i.e., coefficients of the displacement polynomial) can be obtained in matrix notation as

$$\{\gamma\} = [B_1]\,\{a\} \tag{47}$$

where $[B_1]$ is a 2 × 21 matrix whose nonzero elements are as follows:

$$B_1(1,7) = 6A_{11} \tag{47a}$$

$$B_1(1,8) = 2A_{31} \tag{47b}$$

$$B_1(1,9) = 2A_{32} \tag{47c}$$

$$B_1(1,10) = 6A_{15} \tag{47d}$$

$$B_1(1,11) = 24A_{11}x \tag{47e}$$

$$B_1(1,12) = 6(A_{31}x + A_{11}y) \tag{47f}$$

$$B_1(1,13) = 4(A_{32}x + A_{31}y) \tag{47g}$$

$$B_1(1,14) = 6(A_{15}x + A_{32}y) \tag{47h}$$

$$B_1(1,15) = 24A_{15}y \tag{47i}$$

$$B_1(1,16) = -120(A^2{}_{11} + A_{13}A_{21} - 0.5A_{11}x^2) \tag{47j}$$

$$B_1(1,17) = -24\,[A_{11}(A_{31} + A_{38}) + A_{13}A_{33} + A_{21}A_{39} - 0.5A_{31}x^2 - A_{11}xy] \tag{47k}$$

$$B_1(1,18) = -12(A_{11}A_{32} + A_{13}A_{34} + A_{38}A_{31} + A_{39}A_{33} + A_{11}A_{16} + A_{15}A_{21} - 0.5A_{32}x^2 - A_{31}xy - 0.5A_{11}y^2) \quad (47l)$$

$$B_1(1,19) = -12(A_{11}A_{15} + A_{13}A_{25} + A_{38}A_{32} + A_{39}A_{34} + A_{16}A_{31} + A_{15}A_{33} - 0.5A_{15}x^2 - A_{32}xy - 0.5A_{31}y^2) \quad (47m)$$

$$B_1(1,20) = -24(A_{15}A_{38} + A_{25}A_{39} + A_{16}A_{32} + A_{15}A_{34} - A_{15}xy - 0.5A_{32}y^2) \quad (47n)$$

$$B_1(1,21) = -120(A_{15}A_{16} + A_{15}A_{25} - 0.5A_{15}y^2) \quad (47o)$$

$$B_1(2,7) = 6A_{21} \quad (47p)$$

$$B_1(2,8) = 2A_{33} \quad (47q)$$

$$B_1(2,9) = 2A_{34} \quad (47r)$$

$$B_1(2,10) = 6A_{25} \quad (47s)$$

$$B_1(2,11) = 24A_{21}x \quad (47t)$$

$$B_1(2,12) = 6(A_{33}x + A_{21}y) \quad (47u)$$

$$B_1(2,13) = 4(A_{34}x + A_{33}y) \quad (47v)$$

$$B_1(2,14) = 6(A_{25}x + A_{34}y) \quad (47w)$$

$$B_1(2,15) = 24A_{25}y \quad (47x)$$

$$B_1(2,16) = -120(A_{11}A_{21} + A_{23}A_{21} - 0.5A_{21}x^2) \tag{47y}$$

$$B_1(2,17) = -24(A_{21}A_{31} + A_{11}A_{40} + A_{23}A_{33} + A_{21}A_{34} - 0.5A_{33}x^2 - A_{21}xy) \tag{47z}$$

$$B_1(2,18) = -12(A_{21}A_{32} + A_{23}A_{34} + A_{40}A_{31} + A_{41}A_{33} + A_{26}A_{11} + A_{25}A_{21} - 0.5A_{34}x^2 - A_{33}xy - 0.5A_{21}y^2) \tag{47aa}$$

$$B_1(2,19) = -12(A_{21}A_{15} + A_{23}A_{25} + A_{40}A_{32} + A_{41}A_{34} + A_{26}A_{31} + A_{25}A_{33} - 0.5A_{25}x^2 - A_{34}xy - 0.5A_{33}y^2) \tag{47bb}$$

$$B_1(2,20) = -24(A_{15}A_{40} + A_{25}A_{41} + A_{26}A_{32} + A_{25}A_{34} - A_{25}xy - 0.5A_{34}y^2) \tag{47cc}$$

$$B_1(2,21) = -120(A_{15}A_{26} + A^2_{25} - 0.5A_{25}y^2) \tag{47dd}$$

where A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, and A_{26} are as defined in equations (33) and

$$\left.\begin{aligned} A_{31} &= A_{14} + 2A_{13} \\ A_{32} &= A_{12} + 2A_{16} \\ A_{33} &= A_{24} + 2A_{23} \\ A_{34} &= A_{22} + 2A_{26} \\ A_{35} &= A_{33} + A_{11} \end{aligned}\right\} \tag{48}$$

(continued)

$$\left.\begin{aligned}
A_{36} &= A_{34} + A_{31} \\
A_{37} &= A_{25} + A_{32} \\
A_{38} &= A_{13} + A_{14} \\
A_{39} &= A_{12} + A_{16} \\
A_{40} &= A_{23} + A_{24} \\
A_{41} &= A_{22} + A_{26}
\end{aligned}\right\} \qquad \text{(48) (concluded)}$$

If the plate is assumed to be rigid in transverse shear, the coefficients A_{11} to A_{16} and A_{21} to A_{26} of equations (33) are zero (since $G = \infty$), and, hence, coefficients b_1 to b_6 and b_7 to b_{12} of equations (40) and (41) are zero. Moreover, the transverse shear strains vary linearly with G^{-1} with $\{\gamma\}$ approaching 0 as $G \to \infty$, that is, convergence to the limiting case of zero transverse shear is uniform.

Stiffness matrix. The strain energy for a plate may be written as

$$U = \frac{1}{2} \iint \left(\{M\}^T \{\chi\} + \{V\}^T \{\gamma\} \right) dxdy \tag{49}$$

where $\{M\}$ is the vector of bending and twisting moments per unit length, $\{\chi\}$ is the vector of curvatures, $\{V\}$ is the vector of transverse shear forces per unit length, and $\{\gamma\}$ is the vector of transverse shear strains. Substituting equations (14) and (26) into equation (49), and using the symmetry of $[D]$ and $[J]$ matrices, yields

$$U = \frac{1}{2} \iint \{\chi\}^T [D] \{\chi\} + \{\gamma\}^T [G] \{\gamma\} \, dxdy \tag{50}$$

where

$$[G] = [J]^{-1} \tag{51}$$

With $[K_{gen}]$ denoting the generalized stiffness matrix, that is the stiffness matrix with respect to generalized coordinates (coeffcients of the displacement polynomial) $\{a\}$, the strain energy can also be expressed as

$$U = \frac{1}{2} \{a\}^T [K_{gen}] \{a\} \tag{52}$$

The vector of curvatures $\{\chi\}$ is now rewritten as

$$\{\chi\} = \{\chi_1\} + \{\chi_2\} = \left([B_2] + [B_3]\right) \{a\} \tag{53}$$

where

$$\{\chi_1\} = \begin{Bmatrix} \frac{\partial^2 \omega}{\partial x^2} \\ \frac{\partial^2 \omega}{\partial y^2} \\ 2\frac{\partial^2 \omega}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} \sum a_i m_i (m_i - 1) x^{(m_i-2)} y^{n_i} \\ \sum a_i n_i (n_i - 1) x^{m_i} y^{(n_i-2)} \\ \sum a_i m_i n_i x^{(m_i-1)} y^{(n_i-1)} \end{Bmatrix} \tag{53a}$$

and

$$\{\chi_2\} = \begin{Bmatrix} \frac{-\partial \gamma_{xz}}{\partial x} \\ \frac{-\partial \gamma_{yz}}{\partial y} \\ \frac{-\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} \end{Bmatrix} \tag{53b}$$

It follows that $\{\chi_1\}$ is the vector of curvature in the absence of transverse shear and $\{\chi_2\}$ Is the contribution of transverse shear to the vector of curvatures.

Substituting equation (53) into (50) and comparing the resultant equation with (52), noting that $\{a\}$ is independent of x and y, the generalized stiffness matrix can be obtained as

$$[K_{gen}] = \iint [B_2]^T [D] [B_2] \, dxdy + \iint [B_2]^T [D] [B_3] \, dxdy$$
$$+ \iint [B_3]^T [D] [B_2] \, dxdy + \iint [B_3]^T [D] [B_3] \, dxdy \tag{54}$$
$$+ \iint [B_1]^T [G] [B_1] \, dxdy$$

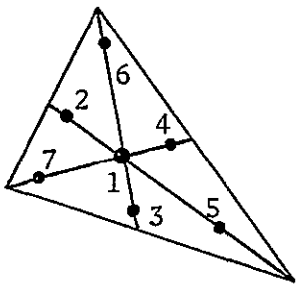
The evaluation of the elements of the generalized stiffness matrix $[k_{gen}]$ in closed form is, though straightforward, very tedious The first term $\iint [B_2]^T [D] [B_2] \, dxdy$ is evaluated in closed form, the other four terms are evaluated by using numerical integration. If the plate is assumed to be rigid in transverse shear, the matrices $[B_1]$ and $[B_3]$ are null, and the last four terms vanish. The numerical integration formulae used are the seven-point integration scheme listed in reference 6 and are given below for easy reference. For a triangle, the integrals of the form

$$I = \int_0^1 \int_0^{1-L_1} f(L_1 L_2 L_3) dL_1 dL_2 \tag{55}$$

can be integrated by using a seven-point numerical integration which can exactly integrate functions up to and including quintic order The value of the integral is given by

$$I = \sum_{k=1}^{7} W_k f_k(L_1, L_2, L_3) \tag{56}$$

where the points and the weighting factors are as follows



Point	Triangular Coordinates L_1, L_2, L_3	Weight, $2W_k$
1	1/3, 1/3, 1/3	0.225
2	α_1 β_1 β_1	
3	β_1 α_1 β_1	0.13239415
4	β_1 β_1 α_1	
5	α_2 β_2 β_2	
6	β_2 α_2 β_2	0.12593918
7	β_2 β_2 α_2	

with

$\alpha_1 = 0.05971588$ $\qquad \beta_1 = 0.47014206$

$\alpha_2 = 0.79742699$ $\qquad \beta_2 = 0.101286505$

Note the error in the value of α_1 as given in reference 6, page 151.

Denoting by G_{11} , G_{12} , G_{13} , G_{22} , G_{23} , and G_{33} the symmetric portion of the G_e matrix of equation (21), it can be shown that the jth element of the ith row of the generalized stiffness matrix $[K_{gen}]$, for the case of a plate infinitely rigid in transverse shear, is given by

$$(K_{ij})_{gen} = \frac{1}{12} \sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} c_{k_1} c_{k_2} c_{k_3}$$

$$\Big[G_{11} m_i m_j (m_i - 1)(m_j - 1) \; F(m_i + m_j + r_{k_1} + r_{k_2}$$

$$+ r_{k_3} - 4, \; n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3})$$

$$+ G_{22} n_i n_j (n_i - 1)(n_j - 1) \; F(m_i + m_j + r_{k_1}$$

$$+ r_{k_2} + r_{k_3}, \; n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3} - 4)$$

$$+ \Big(4G_{33} m_i m_j n_i n_j + G_{12} \{ m_i n_j (m_i - 1)(n_j - 1)$$

$$+ m_j n_i (m_j - 1)(n_i - 1) \} \Big) \; F(m_i + m_j + r_{k_1} \tag{57}$$

$$+ r_{k_2} + r_{k_3} - 2, \; n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3} - 2)$$

$$+ 2G_{13} \{ m_i m_j n_j (m_i - 1) + m_i n_i m_j (m_j - 1) \} \; F(m_i + m_j$$

$$+ r_{k_1} + r_{k_2} + r_{k_3} - 3, \; n_i + n_j + s_{k_1}$$

$$+ s_{k_2} + s_{k_3} - 1) + 2G_{23} \{ m_j n_i n_j (n_i - 1)$$

$$+ m_i n_i n_j (n_j - 1) \} \; F(m_i + m_j + r_{k_1} + r_{k_2}$$

$$+ r_{k_3} - 1, \; n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3} - 3) \Big]$$

All computations involved in evaluating $[K_{gen}]$ for the case of a plate infinitely rigid in transverse shear can be carried out within the computer. For plates with transverse shear flexibility, the contribution of the last four integrals of equation (54) will be evaluated using the numerical integration formula [eq. (56)] and algebraically added on to the closed form expression for $[K_{gen}]$ evaluated by equation (57).

Once the generalized stiffness matrix $[K_{gen}]$ is evaluated, the element stiffness matrix in the local element coordinates $[K_{ee}]$ is obtained as, by virtue of equation (12),

$$[K_{ee}] = [S]^T [K_{gen}] [S] \tag{58}$$

$[K_{ee}]$ can then be transformed to the global coordinate system of the surrounding grid points in the same manner as for all other elements.

Let the transformation for displacements be

$$\{u\}_{basic} = [E]^T \{u\}_{element} \tag{59}$$

and

$$\{u\}_{global} = [T] \{u\}_{basic} \tag{60}$$

Then, the stiffness matrix in global coordinates is

$$[K]_{global} = [T]^T [E] [K_{ee}] [E]^T [T] \tag{61}$$

Equivalent thermal bending loads

The following derivation to obtain the equivalent thermal bending loads is given for the case of different thermal gradients at the three vertices of the element. This capability is not currently operational in NASTRAN. However, the derivation is valid for cases with the same thermal gradients at the vertices, if T_3' and T_5' in equations (83), (74), and (75) are set equal to T_1' .

The stress-free strains developed in a free plate due to a variation of temperature with depth are

$$\{\varepsilon_t\} = \begin{Bmatrix} \varepsilon_x t \\ \varepsilon_y t \\ \varepsilon_t \end{Bmatrix} = \begin{Bmatrix} \alpha_{e1} \\ \alpha_{e2} \\ \alpha_{e3} \end{Bmatrix} (\overline{T} - T_{ref}) = \{\alpha_e\} (\overline{T} - T_{ref}) \tag{62}$$

where $\overline{T}$ is the temperature at any point (x,y,z) of the element, T_{ref} is the reference or stress-free temperature of the material, and $\{\alpha_e\}$ is the vector of thermal expansion coefficients in the element coordinate system

An applied stress vector which would produce the thermal strains is

$$\{\sigma_t\} = [G_e] \{\varepsilon_t\} = [G_e] \{\alpha_e\} (\overline{T} - T_{ref}) \tag{63}$$

where $[G_e]$ is the matrix of elastic coefficients at the point on the cross section

The generalized equivalent thermal load vector $\{p^t_{gen}\}$ is obtained as

$$\{P^t_{gen}\} = \frac{\partial}{\partial\{a\}} \int_V \{\varepsilon\}^T \{\sigma_t\} dv \tag{64}$$

The strains $\{\varepsilon\}$ are related to the curvatures $\{\chi\}$ by

$$\{\varepsilon\} = -z\{\chi\} \tag{65}$$

where z is measured from the neutral surface of the plate Substituting equations (63) and (65) into equation (64),

$$\{P^t_{gen}\} = -\frac{\partial}{\partial\{a\}} \int z\{\chi\}^T [G_e]\{\alpha_e\} (T - T_{ref}) dV \tag{66}$$

The variation over the surface of the element of the mean temperature, T_o, and the thermal gradient at a cross section, T', is assumed to be a bilinear polynomial.

$$T_o = \sum_{i=1}^{3} d_i x^{p_i} y^{q_i} \tag{67}$$

$$T' = \sum_{i=1}^{3} d_i' x^{p_i} y^{q_i} \tag{68}$$

so that the temperature at any point (x,y,z) $\overline{T}$, is

$$\overline{T} = T_o + T'z \tag{69}$$

The constants d_i and d_i' are evaluated from the values at the vertices. Thus,

$$d_1 = \frac{T_{01}'a + T_{03}'b}{(a + b)} \tag{70}$$

$$d_2 = \frac{T_{03}' - T_{01}'}{(a + b)} \tag{71}$$

$$d_3 = \frac{1}{c} \left[T_{05}' - d_1\right] \tag{72}$$

$$d_1' = \frac{T_1'a + T_3'b}{(a + b)} \tag{73}$$

$$d_2' = \frac{T_3' - T_1'}{(a + b)} \tag{74}$$

$$d_3' = \frac{1}{c} \left[T_5' - d_1'\right] \tag{75}$$

where T_{01}' , T_{03}' , and T_{05}' are the difference between the grid point mean temperature T_{01} , T_{03} and T_{05} , at grid points 1, 3, and 5, respectively, and the reference temperature, T_{ref} .

It is convenient to define the equivalent thermal moment vector

$$\{M_t\} = - \int_z [G_e] \{\alpha_e\} (\overline{T} - T_{ref}) z dz$$

$$= - \int_{-t/2}^{+t/2} [G_e] \{\alpha_e\} (T_o' + T') z dz \tag{76}$$

$$= - [G_e] \{\alpha_e\} T' \frac{t^3}{12}$$

Substituting for t from equation (16a) and for T' from equation (68),

$$\{M_t\} = - \frac{1}{12} [G_e] \{\alpha_e\} T' \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \sum_{i_3=1}^{3} \sum_{j=1}^{3} c_{i_1} c_{i_2} c_{i_3}$$
$$x^{(r_{i_1}+r_{i_2}+r_{i_3}+p_j)} \; y^{(s_{i_1}+s_{i_2}+s_{i_3}+q_j)} \tag{77}$$

At the three vertices the value of $\{M_t\}$ will be given by

$$\{M_t\}_1 = - [G_e] \{\alpha_e\} \frac{t_1^3}{12} T_1' \tag{78}$$

$$\{M_t\}_3 = - [G_e] \{\alpha_e\} \frac{t_3^3}{12} T_3' \tag{79}$$

$$\{M_t\}_5 = - [G_e] \{\alpha_e\} \frac{t_5^3}{12} T_5' \tag{80}$$

where t_1 , t_3 , and t_5 are the thicknesses at the vertices G_1 , G_3 , and G_5 , respectively, of the element.

The "effective thermal gradient," T' , at the vertices is defined as

$$T_1' = \frac{1}{I_1} \int \overline{T}_1 \; z \; dz \tag{78a}$$

$$T_3' = \frac{1}{I_1} \int \overline{T}_3 \, z \, dz \tag{79a}$$

$$T_5' = \frac{1}{I_1} \int \overline{T}_5 \, z \, dz \tag{80a}$$

This capability of specifying the thermal gradients or the thermal moments at the three vertices of the element is not currently implemented. The theoretical derivations of the evaluation of the thermal load vector is, however, given for such linear variation of the thermal gradient values over the surface of the element.

Substituting equations (16), (53), (67), (68), and (69) into equation (66).

$$\{P^t_{gen}\} = - \frac{1}{12} \frac{\partial}{\partial \{a\}} \iint \Big(\{\chi_1\} + \{\chi_2\}\Big)^T [G_e] \{\alpha_e\}$$

$$\sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \sum_{i_3=1}^{3} \sum_{j=1}^{3} c_{i_1} c_{i_2} c_{i_3} d'_j \tag{81}$$

$$x^{(r_{i_1}+r_{i_2}+r_{i_3}+p_j)} \; y^{(s_{i_1}+s_{i_2}+s_{i_3}+q_j)} \; dxdy$$

As in the case of the derivation of the generalized stiffness matrix, the generalized thermal load vector will be evaluated in two stages, viz., the closed form expression $[P^t_{gen}]_1$ due to $[\chi_1]$, the vector of curvatures in the absence of transverse shear, and the numerically integrated expression $[P^t_{gen}]_2$ due to $[\chi_2]$, the contribution of transverse shear to the vector of curvatures. Using the following notations, viz.,

$$G'_{11} = G_{11}\alpha_{e_1} + G_{12}\alpha_{e_2} + G_{13}\alpha_{e_3} \tag{82}$$

$$G'_{22} = G_{12}\alpha_{e_1} + G_{22}\alpha_{e_2} + G_{23}\alpha_{e_3} \tag{83}$$

$$G'_{33} = G_{13}\alpha_{e_1} + G_{23}\alpha_{e_2} + G_{33}\alpha_{e_3} \tag{84}$$

the ith element of the generalized load vector $\{P^t_{gen}\}$ will be given by

$$
\begin{aligned}
\{P^t_{gen}\}_i = \frac{1}{12} \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \sum_{i_3=1}^{3} \sum_{j=1}^{3} & c_{i_1} c_{i_2} c_{i_3} d'_j \\
& \left[G'_{11} m_i (m_i - 1)\, F(m_i + r_{i_1} + r_{i_2} + r_{i_3} + p_j - 2, \right. \\
& n_i + s_{i_1} + s_{i_2} + s_{i_3} + q_j) + G'_{22} n_i (n_i - 1) \\
& F(m_i + r_{i_1} + r_{i_2} + r_{i_3} + p_j,\ n_i + s_{i_1} + s_{i_2} \\
& + s_{i_3} + q_j - 2) + G'_{33} m_i n_i\, F(m_i + r_{i_1} + r_{i_2} \\
& \left. + r_{i_3} + p_j - 1,\ n_i + s_{i_1} + s_{i_2} + s_{i_3} + q_j - 1) \right]
\end{aligned}
\qquad (85)
$$

The load vector $\{P^t_{gen}\}_2$ is evaluated using numerical integration and $[P^t_{gen}]$ is obtained as the sum of $[P^t_{gen}]_1$ and $[P^t_{gen}]_2$ For plates infinitely rigid in transverse shear, $[P^t_{gen}]_2$ is null. The equivalent thermal bending load in the local element coordinate system is obtained as, by virtue of equation (17),

$$\{P^t_e\} = [S]^T \{P^t_{gen}\} \qquad (86)$$

The load vector can then be transformed to the global system by

$$\{P^t_g\} = [T]^T [E] \{P^t_e\} \qquad (87)$$

RECOVERY OF INTERNAL FORCES

The bending moments and shear forces are recovered at the three vertices, the stresses are evaluated at the three vertices and at the centroid of the element. After the displacements of the element are transformed from the global system $[u_g]$ to the element coordinate system $[u_e]$, the generalized coordinates $\{a\}$ are evaluated from equation (12) The curvatures $\{\chi\}$ are evaluated from equation (53) with the nonzero elements of $[B_3]$ being as listed below.

$B_3(1,11) = -24A_{11}$

$B_3(1,12) = -6A_{31}$

$B_3(1,13) = -4A_{32}$

$B_3(1,14) = -6A_{15}$

$B_3(1,16) = -120A_{11}x$

$B_3(1,17) = -24(A_{31}x + A_{11}y)$

$B_3(1,18) = -12(A_{32}x + A_{31}y)$

$B_3(1,19) = -12(A_{15}x + A_{32}y)$

$B_3(1,20) = -24A_{15}y$

$B_3(2,12) = -6A_{21}$

$B_3(2,13) = -4A_{33}$

$B_3(2,14) = -6A_{34}$

$B_3(2,15) = -24A_{25}$

$B_3(2,17) = -24A_{21}x$

$B_3(2,18) = -12(A_{33}x + A_{21}y)$

$B_3(2,19) = -12(A_{34}x + A_{33}y)$

$B_3(2,20) = -24(A_{25}x + A_{34}y)$

$B_3(2,21) = -120A_{25}y$

$$B_3(3,11) = -24A_{21}$$

$$B_3(3,12) = -6(A_{11} + A_{33})$$

$$B_3(3,13) = -4(A_{31} + A_{34})$$

$$B_3(3,14) = -6(A_{32} + A_{25})$$

$$B_3(3,15) = -120A_{15}$$

$$B_3(3,16) = -120A_{21}x$$

$$B_3(3,17) = -24\left[(A_{11} + A_{33})x + A_{21}y\right]$$

$$B_3(3,18) = -12\left[(A_{34} + A_{31})x + (A_{34} + A_{11})y\right]$$

$$B_3(3,19) = -12\left[(A_{25} + A_{32})x + (A_{34} + A_{31})y\right]$$

$$B_3(3,20) = -24A_{15}x + (A_{32} + A_{25})y$$

$$B_3(3,21) = -120A_{15}y$$

where A_{11}, A_{12}, . . . , A_{34} are as given in equations (33) and (48) Moments at the vertices are then obtained from

$$\{M\}_1 = [D]_1 \{\chi\} - \{M_t\}_1 \tag{88}$$

$$\{M\}_3 = [D]_3 \{\chi\} - \{M_t\}_3 \tag{89}$$

$$\{M\}_5 = [D]_5 \{\chi\} - \{M_t\}_5 \tag{90}$$

The transverse shears are evaluated as follows:

$\{\gamma\}$ is evaluated from equations (28) and (47).

$\{V\}$ is then evaluated from equations (24) and (25).

The stresses at the three vertices are evaluated at distances Z11, Z21, Z13, Z23, Z15, and Z25 specified by the user. The stresses at the centroid are evaluated at the top and bottom fibers of the element.

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5.14 TRSHL, Higher Order Shallow Shell Element

This element was developed by Narayanaswami (ref. 1). The element has grid points at the vertices and at the midpoints of the sides of the triangle At each grid point, there are five degrees of freedom in the element coordinate system· viz., the membrane displacements u and v parallel to the x and y axes, the transverse displacement, w , in the z-direction normal to the x-y plane, with positive direction outward from the paper, and the rotations of the normal to the shell α and β , about the xz and yz planes, with positive directions following from the right-hand rule. The element, thus, has 30 degrees of freedom in the element coordinate system.

The membrane displacements u and v for the shell are expressed as quadratic polynomials and are the same as for the higher order membrane triangular element, TRIM6. The displacement function for the normal deflection, w , is taken as a quintic polynomial as for the higher order bending triangular element, TRPLT1. The geometry of the shell surface is approximated by a quadratic polynomial in base coordinates Shallow shell theory of Novozhilov (ref. 2) is used for including the membrane bending coupling effects. Thus, the element can strictly be used only in cases where the shell is shallow. However, reasonably good accuracy is seen even when the elements are used to analyze shells that are only marginally shallow The user is cautioned, however, to be careful while interpreting results obtained when the shell analyzed is very deep Due to the excessive computation time associated with such calculations, the transverse shear flexibility is not taken into account in the element formulation. The element can be used in the statics, normal modes and differential stiffness rigid formats.

Derivation of Element Properties

Element geometry Rectangular Cartesian coordinates are used in the formulation An arbitrary triangular element is shown in figure A4 X , Y , and Z are the basic coordinates, x , y , and z are the local coordinates The grid points of the element are numbered in counterclockwise direction as shown in the figure.

The lengths a , b , and c shown in figure A4 can be easily evaluated from the basic coordinates (X_1, Y_1, Z_1), (X_3, Y_3, Z_3) and (X_5, Y_5, Z_5) of the vertices of the triangle

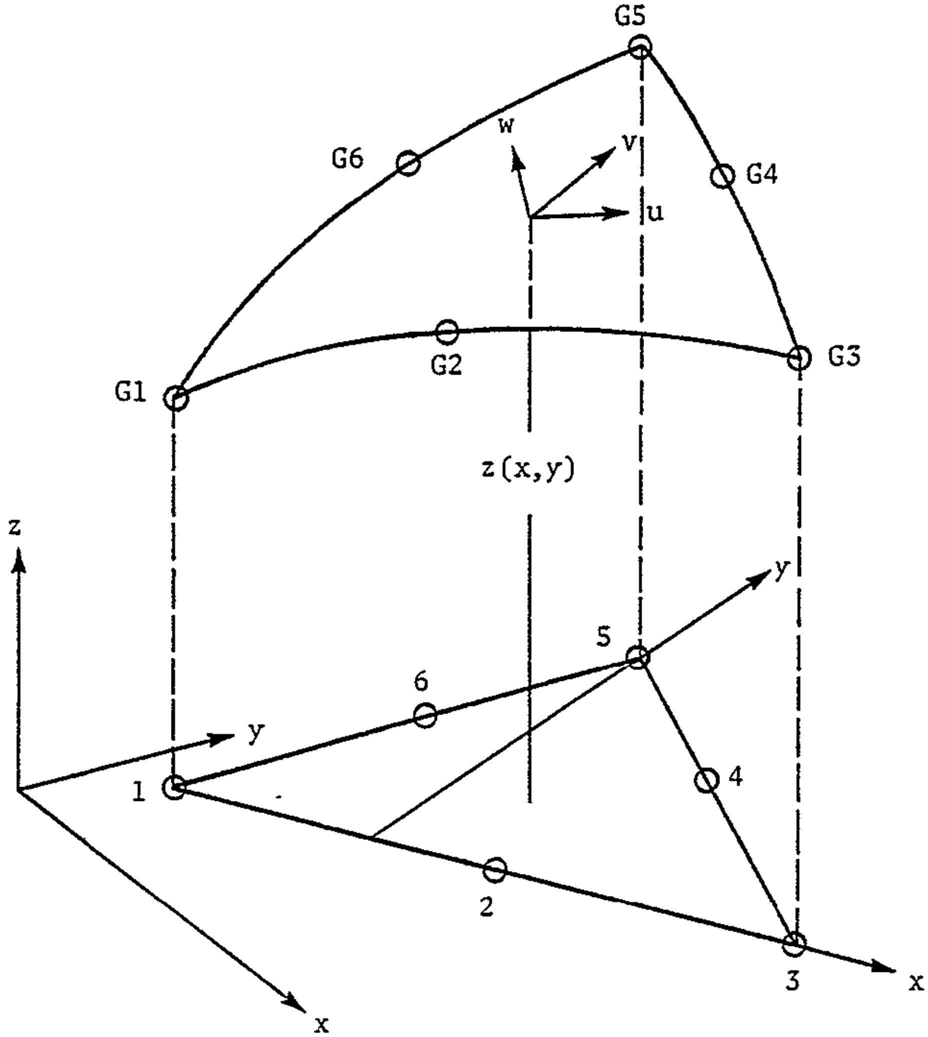


Figure A4. Triangular shell element geometry.

Displacement Field The $u(x,y)$ and $v(x,y)$ displacements are assumed to vary quadratically with position on the plane of the element, while displacement $w(x,y)$ within the triangular element is assumed to vary as a quintic polynomial in the local coordinates.

$$u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

$$v(x,y) = a_7 + a_8x + a_9y + a_{10}x^2 + a_{11}xy + a_{12}y^2$$

$$\begin{aligned} w(x,y) &= a_{13} + a_{14}x + a_{15}y + a_{16}x^2 + a_{17}xy + a_{18}y^2 \\ &+ a_{19}x^3 + a_{20}x^2y + a_{21}xy^2 + a_{22}y^3 + a_{24}x^3y \\ &+ a_{25}x^2y^2 + a_{26}xy^3 + a_{27}y^4 + a_{28}x^5 + a_{29}x^4y \\ &+ a_{30}x^3y^2 + a_{31}x^2y^3 + a_{32}xy^4 + a_{33}y^5 \end{aligned} \tag{1}$$

In concise form, u , v and w can be written as

$$u = \sum_{i=1}^{33} a_i x^{m_i} y^{n_i} \qquad a_i = m_i = n_i = 0;\ i = 7 \text{ to } 33 \tag{2}$$

$$v = \sum_{i=1}^{33} b_i x^{p_i} y^{q_i} \qquad b_i = p_i = q_i = 0,\ i = 1 \text{ to } 6,\ i = 13 \text{ to } 33 \tag{3}$$

$$w = \sum_{i=1}^{33} c_i x^{r_i} y^{s_i} \qquad c_i = r_i = s_i = 0,\ i = 1 \text{ to } 12 \tag{4}$$

The detailed derivation of the stiffness matrix for the triangular shell element follows closely that for the TRIM6 and TRPLT1 elements. Hence, only the salient features of the derivation are given in this section

The geometry of the shell surface is approximated by a quadratic polynomial in base coordinates

$$z(x,y) = h_1 + h_2x + h_3y + h_4x^2 + h_5xy + h_6y^2 \tag{5}$$

Hence the curvatures of the shell surface are

$$z,_{xx} = 2h_4 \tag{6}$$

$$z,_{xy} = h_5 \tag{7}$$

$$z,_{yy} = 2h_6 \tag{8}$$

The membrane thickness of the shell element is assumed to vary linearly over the surface of the element, i.e.,

$$t_m = \sum_{i=1}^{3} d_i \, x^{t_i} \, y^{u_i} \tag{9}$$

The bending thickness of the shell element is also assumed a similar linear variation

$$t_b = \sum_{i=1}^{3} d'_i \, x^{t'_i} \, y^{u'_i}$$

Following the shallow shell theory of Novozhilov (ref. 2), the membrane strains in the shell are given by

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} - z,_{xx} \, w \\ &= \sum_{i=1}^{33} \left(m_i \, a_i \, x^{m_i-1} \, y^{n_i} - 2h_4 \, c_i \, x^{v_i} \, y^{s_i} \right) \end{aligned} \tag{10}$$

$$\begin{aligned} \varepsilon_y &= \frac{\partial v}{\partial y} - z,_{yy} \, w \\ &= \sum_{i=1}^{33} \left(q_i \, b_i \, x^{p_i} \, y^{q_i-1} - 2h_6 \, c_i \, x^{v_i} \, y^{s_i} \right) \end{aligned} \tag{11}$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z,_{xy}\, w$$

$$= \sum_{i=1}^{33} \Big(n_i\, a_i\, x^{m_i}\, y^{n_i - 1} + p_i\, b_i\, x^{p_i - 1}\, y^{q_i} \tag{12}$$

$$-2h_5\, c_i\, x^{v_i}\, y^{s_i} \Big)$$

In the absence of transverse shear effects, the bending strains are given by

$$\chi_x = \frac{\partial^2 w}{\partial x^2} = \sum_{i=1}^{33} v_i\, (v_i - 1)\, c_i\, x^{v_i - 2}\, y^{s_i} \tag{13}$$

$$\chi_y = \frac{\partial^2 w}{\partial x^2} = \sum_{i=1}^{33} s_i\, (s_i - 1)\, c_i\, x^{v_i}\, y^{s_i - 2} \tag{14}$$

$$\chi_{xy} = 2\, \frac{\partial^2 w}{\partial x \partial y} = \sum_{i=1}^{33} 2\, v_i\, s_i\, c_i\, x^{v_i - 1}\, y^{s_i - 1} \tag{15}$$

Following the procedure outlined in sections 5.8.6 and 5.8.7, the jth column of the ith row of the generalized stiffness matrix is obtained as

$$K_{ij} = \sum_{k=1}^{3} \Big[G_{11} \Big(m_i\, m_j\, d_k\, F(m_i + m_j + t_k - 2,\ n_i + n_j + u_k)$$

$$- h_4\, m_i\, d_k\, F(m_i + r_j + t_k - 1,\ n_i + s_j + u_k)$$

$$- h_4\, m_j\, d_k\, F(m_j + r_i + t_k - 1,\ n_j + s_i + u_k)$$

$$+ h_4^2\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k) \Big)$$

$$+ G_{22} \Big(q_i\, q_j\, d_k\, F(p_i + p_j + t_k,\ q_i + q_j + u_k - 2) \tag{16}$$

$$- h_6\, q_i\, d_k\, F(p_i + r_j + t_k,\ q_i + s_j + u_k - 1)$$

$$- h_6\, q_j\, d_k\, F(r_i + p_j + t_k,\ s_i + q_j + u_k - 1)$$

$$+ h_6^2\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k) \Big)$$

(continued)

$$
\begin{aligned}
& + G_{33}\bigl(n_i\, n_j\, d_k\, F(m_i + m_j + t_k,\ n_i + n_j + u_k - 2) \\
& \quad + n_i\, p_j\, d_k\, F(m_i + p_j + t_k - 1,\ n_i + q_j + u_k - 1) \\
& \quad - h_5\, n_i\, d_k\, F(m_i + r_j + t_k,\ n_i + s_j + u_k - 1) \\
& \quad + p_i\, n_j\, d_k\, F(p_i + m_j + t_k - 1,\ q_i + n_j + u_k - 1) \\
& \quad + p_i\, p_j\, d_k\, F(p_i + p_j + t_k - 2,\ q_i + q_j + u_k) \\
& \quad - h_5\, p_i\, d_k\, F(p_i + r_j + t_k - 1,\ q_i + s_j + u_k) \\
& \quad - h_5\, n_j\, d_k\, F(r_i + m_j + t_k,\ s_i + n_j + u_k - 1) \\
& \quad - h_5\, p_j\, d_k\, F(r_i + p_j + t_k - 1,\ s_i + q_j + u_k) \\
& \quad + h_5^2\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k)\bigr) \\
& + G_{12}\bigl(m_i\, q_j\, d_k\, F(m_i + p_j + t_k - 1,\ n_i + q_j + u_k - 1) \\
& \quad - h_6\, m_i\, d_k\, F(m_i + r_j + t_k - 1,\ n_i + s_j + u_k) \qquad\qquad (16) \\
& \quad - h_4\, q_j\, d_k\, F(r_i + p_j + t_k,\ s_i + q_j + u_k - 1) \\
& \quad + 2h_4\, h_6\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k) \\
& \quad + q_i\, m_j\, d_k\, F(p_i + m_j + t_k - 1,\ q_i + n_j + u_k - 1) \\
& \quad - h_4\, q_i\, d_k\, F(p_i + r_j + t_k,\ q_i + s_j + u_k - 1) \\
& \quad - h_6\, m_j\, d_k\, F(r_i + m_j + t_k - 1,\ s_i + n_j + u_k)\bigr) \\
& + G_{13}\bigl(m_i\, n_j\, d_k\, F(m_i + m_j + t_k - 1,\ n_i + n_j + u_k - 1) \\
& \quad + m_i\, p_j\, d_k\, F(m_i + p_j + t_k - 2,\ n_i + q_j + u_k) \\
& \quad - h_5\, m_i\, d_k\, F(m_i + r_j + t_k - 1,\ n_i + s_j + u_k) \\
& \quad - h_4\, n_j\, d_k\, F(r_i + m_j + t_k,\ s_i + n_j + u_k - 1) \\
& \quad - h_4\, p_j\, d_k\, F(r_i + p_j + t_k - 1,\ s_i + q_j + u_k)
\end{aligned}
$$

(continued)

$$
\begin{aligned}
&+ 2h_4\, h_5\, d_k\, F(r_i + r_j + t_k, s_i + s_j + u_k) \\
&+ n_i\, m_j\, d_k\, F(m_i + m_j + t_k - 1, n_i + n_j + u_k - 1) \\
&- h_4\, n_i\, d_k\, F(m_i + r_j + t_k, n_i + s_j + u_k - 1) \\
&+ p_i\, m_j\, d_k\, F(p_i + m_j + t_k - 2, q_i + n_j + u_k) \\
&- h_4\, p_i\, d_k\, F(p_i + r_j + t_k - 1, q_i + s_j + u_k) \\
&- h_5\, m_j\, d_k\, F(r_i + m_j + t_k - 1, s_i + n_j + u_k)\Big) \\
+ G_{23}\Big(&q_i\, n_j\, d_k\, F(p_i + m_j + t_k, q_i + n_j + u_k - 2) \\
&+ q_i\, p_j\, d_k\, F(p_i + p_j + t_k - 1, q_i + q_j + u_k - 1) \\
&- h_5\, q_i\, d_k\, F(p_i + r_j + t_k, q_i + s_j + u_k - 1) \\
&- h_6\, n_j\, d_k\, F(r_i + m_j + t_k, s_i + n_j + u_k - 1) \qquad (16) \\
&- h_6\, p_j\, d_k\, F(r_i + p_j + t_k - 1, s_i + q_j + u_k) \\
&+ 2h_5\, h_6\, d_k\, F(r_i + r_j + t_k, s_i + s_j + u_k) \\
&+ n_i\, q_j\, d_k\, F(m_i + p_j + t_k, n_i + q_j + u_k - 2) \\
&- h_6\, n_i\, d_k\, F(m_i + r_j + t_k, n_i + s_j + u_k - 1) \\
&+ p_i\, q_j\, d_k\, F(p_i + p_j + t_k - 1, q_i + q_j + u_k - 1) \\
&- h_6\, p_i\, d_k\, F(p_i + r_j + t_k - 1, q_i + s_j + u_k) \\
&- h_5\, q_j\, d_k\, F(r_i + p_j + t_k, s_i + q_j + u_k - 1)\Big)\Big] \\
+ &\sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} \Big[\frac{1}{12}\; d'_{k_1} d'_{k_2} d'_{k_3}\, (G_{11}\, r_i\, r_j\, (r_i - 1)(r_j - 1) \\
&\cdot F(r_i + r_j + t'_{k_1} + t'_{k_2} + t'_{k_3} - 4, s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3})
\end{aligned}
$$

(continued)

$$+ G_{22}\, s_i\, s_j\, (s_i - 1)(s_j - 1)\, F(r_i + r_j + t'_{k_1} + t'_{k_2} + t'_{k_3},$$

$$+ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 4)$$

$$+ \Big(4G_{33}\, r_i\, r_j\, s_i\, s_j + G_{12}\{r_i\, s_j\, (r_i - 1)(s_j - 1)$$

$$+ r_j\, s_i\, (r_j - 1)(s_i - 1)\}\Big)\, F(r_i + r_j + t'_{k_1} + t'_{k_2} + t'_{k_3} - 2,$$

$$+ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 2)$$

$$+ 2G_{13}\{r_i\, r_j\, s_j\, (r_i - 1) + r_i\, r_j\, s_i\, (r_j - 1)\}\, F(r_i + r_j \qquad (16)\ \text{(concluded)}$$

$$+ t'_{k_1} + t'_{k_2} + t'_{k_3} - 3,\ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 1)$$

$$+ 2G_{23}\{r_j\, s_i\, s_j\, (s_i - 1) + r_i\, s_i\, s_j\, (s_j - 1)\}\, F(r_i + r_j$$

$$+ t'_{k_1} + t'_{k_2} + t'_{k_3} - 1,\ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 3)\Big]$$

The generalized stiffness matrix can be transformed to the element and global coordinates by transformations similar to that for TRIM6 and TRPLT1 elements.

Equivalent Thermal Load Vector:

The equivalent thermal load vector for the shallow shell triangular element consists of loads due to thermal expansion as well as due to thermal bending caused by variation of temperature with depth. The detailed derivation of the thermal load vector is similar to that used for TRIM6 and TRPLT1 elements; hence, only the essential steps are given here.

The vector of thermal strains is

$$\{\varepsilon_t\} = \begin{Bmatrix} \varepsilon_{xt} \\ \varepsilon_{yt} \\ \varepsilon_{xyt} \end{Bmatrix} = \begin{Bmatrix} \alpha_{e_1} \\ \alpha_{e_2} \\ \alpha_{e_{12}} \end{Bmatrix} (\overline{T} - T_{ref}) = \{\alpha_e\}\ (\overline{T} - T_{ref}) \qquad (17)$$

where $\{\alpha_e\} = [U]^{-1}\ \{\alpha_m\}$ is a vector of thermal expansion coefficients, $[U]$ is the strain transformation matrix given in equation (15) of page 5.8.4, $\{\alpha_m\}$

is the vector of thermal expansion coefficients in the material axis system, T_{ref} is the reference or stress-free temperature of the material and $\overline{T}$ is the temperature at any point (x,y) in the element

An applied stress vector which would produce the thermal strains is

$$\{\varepsilon_t\} = \left[G_e\right] \{\varepsilon_t\} = \left[G_e\right] \{\alpha_e\} (\overline{T} - T_{ref}) \tag{18}$$

The generalized equivalent thermal load vector $\{P^t_{gen}\}$ is obtained as

$$\{P^t_{gen}\} = \frac{\partial}{\partial\{a\}} \int_V \{\varepsilon\}^t \{\sigma_t\} \, dv \tag{19}$$

The strain vector $\{\varepsilon\}$ is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} - z_{xx} w - z \chi_x \\ \frac{\partial v}{\partial y} - z_{yy} w - z \chi_y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z_{xy} w - z \chi_{xy} \end{Bmatrix} \tag{20}$$

where z_{xx} , z_{yy} and z_{xy} are the curvatures of the shell surface and z is measured from the neutral surface of the plate.

The temperature at any point (x,y,z), $\overline{T}$, is given by

$$\overline{T} = T_o + T' z \tag{21}$$

where T_o is the mean temperature and T' is the thermal gradient.

The following derivation to obtain the equivalent thermal load vector is given for the case of linear variation of thermal gradient over the planar coordinates of the element; the values of the thermal gradient at the three vertices being defined as T'_1 , T'_3 and T'_5 . This capability is not operational in NASTRAN currently. The derivation, however, is valid for cases with the same thermal gradient at the three vertices by setting T'_3 and T'_5 equal to T'_1 . Thus, T_o and T' of equation (21) vary over the element as follows:

$$T_o = e_1 + e_2x + e_3y \tag{22}$$

$$T' = e'_1 + e'_2x + e'_3y \tag{23}$$

i.e.,

$$T_o = \sum_{i=1}^{3} e_i \, x^{v_i} \, y^{w_i} \tag{24}$$

$$T' = \sum_{i=1}^{3} e'_i \, x^{v'_i} \, y^{w'_i} \tag{25}$$

The constants e_1 , e_2 , e_3 and e'_1 , e'_2 and e'_3 can be evaluated from the user supplied values of the mean temperature and temperature gradient at the vertices of the element; however, as stated earlier, only the capability of specifying a temperature gradient for the element is currently available and hence e'_1 will be equal to the element temperature gradient and e'_2 and e'_3 will be equal to zero.

Substituting equations (10) through (15) into equation (20) and substituting for $\{\varepsilon\}$ and $\{\sigma_t\}$ in equation (19), the generalized equivalent thermal load vector $\{P^t_{gen}\}$ is obtained as

$$\{P^t_{gen}\} = \frac{\partial}{\partial\{a\}} \int_V \begin{Bmatrix} \sum_{i=1}^{33} (m_i\, a_i\, x^{m_i-1} y^{n_i} - 2h_4\, c_i\, x^{v_i} y^{s_i} \\ \qquad - z\, v_i\, (v_i - 1)\, c_i\, x^{v_i-2} y^{s_i}) \\ \sum_{i=1}^{33} (q_i\, b_i\, x^{p_i} y^{q_i-1} - 2h_6\, c_i\, x^{v_i} y^{s_i} \\ \qquad - z\, s_i\, (s_i - 1)\, c_i\, x^{v_i} y^{s_i-2}) \\ \sum_{i=1}^{33} (n_i\, a_i\, x^{m_i} y^{n_i-1} + p_i\, b_i\, x^{p_i-1} y^{q_i} \\ \qquad - 2h_5\, c_i\, x^{v_i} y^{s_i} - 2z\, v_i\, s_i\, c_i\, x^{v_i-1} y^{s_i-1}) \end{Bmatrix}^t$$

$$\cdot \left[G_e\right] \{\alpha_e\} \left[\sum_{j=1}^{3} (e_j\, x^{v_j} y^{w_j} + e'_j\, x^{v'_j} y^{w'_j} z) \right] dxdydz \tag{26}$$

Integrating over the thickness and noting that

$$\int_{-t/2}^{t/2} f(x,y)\; z\; dx\; dy\; dz = 0 \tag{27}$$

equation (26) reduces to

$$\{P^t_{gen}\} = \frac{\partial}{\partial\{a\}} \left(\iint \begin{Bmatrix} \sum_{i=1}^{33} m_i\, a_i\, x^{m_i-1} y^{n_i} - 2h_4\, c_i\, x^{v_i} y^{s_i} \\ \sum_{i=1}^{33} q_i\, b_i\, x^{p_i} y^{q_i-1} - 2h_6\, c_i\, x^{v_i} y^{s_i} \\ \sum_{i=1}^{33} n_i\, q_i\, x^{m_i} y^{n_i-1} + p_i\, b_i\, x^{p_i-1} y^{q_i} - 2h_5\, c_i\, x^{v_i} y^{s_i} \end{Bmatrix}^t \right.$$

$$\left. \cdot \left[G_e\right] \{\alpha_e\} \left(\sum_{j=1}^{3} e_j\, x^{v_j} y^{w_j} \right) \left(\sum_{k=1}^{3} d_k\, x^{t_k} y^{u_k} \right) dxdy \right) \tag{28}$$

(continued)

$$
-\frac{\partial}{\partial\{a\}}\left(\frac{1}{12}\iint\begin{Bmatrix} v_i\,(v_i-1)\,c_i\,x^{v_i-2}\,y^{s_i} \\ s_i\,(s_i-1)\,c_i\,x^{v_i}\,y^{s_i-2} \\ 2v_i\,s_i\,c_i\,x^{v_i-1}\,y^{s_i-1} \end{Bmatrix}^t [G_e]\,\{\alpha_e\}\right. \tag{28 (concluded)}
$$

$$
\cdot\left(\sum_{j=1}^{3} e'_j\,x^{v'_j}\,y^{w'_j}\right)\sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\sum_{k_3=1}^{3} d_{k_1}d_{k_2}d_{k_3}\,x^{t_{k_1}+t_{k_2}+t_{k_3}}
$$

$$
\left.\cdot\, y^{u_{k_1}+u_{k_2}+u_{k_3}}\,dxdy\right)
$$

The generalized equivalent thermal load vector will be obtained by performing the differentiation and integration operations of equation (28) and the final expression for $\{P^t_{gen}\}$ will be similar to those obtained for the TRIM6 and TRPLT1 elements, except that an additional expression involving the curvatures of the shell surface h_4 , h_5 and h_6 will be added now. The generalized thermal load vector $\{P^t_{gen}\}$ can then be transformed to the element and global coordinate system by the usual procedures.

7.3.6. Differential Stiffness Matrix for Triangular Shell Element TRSHL

The expression that is used for the energy of differential stiffness per unit area of the shell element consists of a part U'_b due to out-of-plane motions and a part U'_m due to in-plane motions. The expressions for U'_b and U'_m are the same as for plate elements and given in equations (18) and (19) of section 7.3.1.; the expressions for membrane strains will, however, involve the effects of coupling due to bending. Thus,

$$
U = U'_b + U'_m \tag{1}
$$

where

$$
U'_b = \frac{t}{2}\left\{\bar{\sigma}_x\left(\frac{\partial w}{\partial x}\right)^2 + \bar{\sigma}_y\left(\frac{\partial w}{\partial y}\right)^2 + 2\bar{\tau}_{xy}\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\right\} \tag{2}
$$

and

$$U'_m = \frac{t}{2} \left\{ \bar{\sigma}_x \, (\omega_z^2 + 2\omega_z \, \varepsilon_{xy}) + \bar{\sigma}_y \, (\omega_z^2 - 2\omega_z \, \varepsilon_{xy}) + 2\bar{\tau}_{xy} \, (\varepsilon_y - \varepsilon_x) \, \omega_z \right\} \tag{3}$$

The stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$ at any point within the element is assumed to vary linearly, the values at the three corner grid points being used to evaluate the coefficients in the linear variation.

$$\bar{\sigma}_x \, (x,y) = e_1 + e_2 x + e_3 y \tag{4}$$

$$\bar{\sigma}_y \, (x,y) = f_1 + f_2 x + f_3 y \tag{5}$$

$$\bar{\sigma}_{xy} \, (x,y) = g_1 + g_2 x + g_3 y \tag{6}$$

In condensed form

$$\bar{\sigma}_x = \sum_{i=1}^{3} e_i \, x^{R_i} \, y^{S_i} \tag{7}$$

$$\bar{\sigma}_y = \sum_{i=1}^{3} f_i \, x^{R_i} \, y^{S_i} \tag{8}$$

$$\bar{\sigma}_{xy} = \sum_{i=1}^{3} g_i \, x^{R_i} \, y^{S_i} \tag{9}$$

Also

$$\omega_x = \frac{\partial w}{\partial y} \tag{10}$$

$$\omega_y = - \frac{\partial w}{\partial x} \tag{11}$$

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial w}{\partial y}\right) \tag{12}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} - z_{,xx}\, w \tag{13}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} - z_{,yy}\, w \tag{14}$$

$$\varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} - 2z_{,xy}\, w \tag{15}$$

The thickness of the element t at any point

$$t(x,y) = \sum_{i=1}^{3} d_k\, x^{t_k}\, x^{u_k} \tag{16}$$

The jth column of the ith row of the generalized differential stiffness matrix is

$$\begin{aligned} K_{ij} = \sum_{k=1}^{3} \sum_{\ell=1}^{3} \Big[& d_k\, e_\ell\, r_i\, r_j\, F(r_i + r_j + t_k + R_\ell - 2,\ s_i + s_j + u_k + S_\ell) \\ & + d_k\, f_\ell\, s_i\, s_j\, F(r_i + r_j + t_k + R_\ell,\ s_i + s_j + u_k + S_\ell - 2) \\ & + d_k\, g_\ell\, s_i\, r_j\, F(r_i + r_j + t_k + R_\ell - 1,\ s_i + s_j + u_k + S_\ell - 1) \\ & + d_k\, g_\ell\, s_j\, r_i\, F(r_i + r_j + t_k + R_\ell - 1,\ s_i + s_j + u_k + S_\ell - 1) \\ & + 0.25\, d_k\, e_\ell\, p_i\, p_j\, F(p_i + p_j + t_k + R_\ell - 2,\ q_i + q_j + u_k + S_\ell) \\ & + 0.25\, d_k\, e_\ell\, n_i\, n_j\, F(m_i + m_j + t_k + R_\ell,\ n_i + n_j + u_k + S_\ell - 2) \\ & - 0.25\, d_k\, e_\ell\, p_i\, n_j\, F(p_i + m_j + t_k + R_\ell - 1,\ q_i + n_j + u_k + S_\ell - 1) \\ & - 0.25\, d_k\, e_\ell\, n_i\, p_j\, F(m_i + p_j + t_k + R_\ell - 1,\ n_i + q_j + u_k + S_\ell - 1) \\ & + d_k\, e_\ell\, p_i\, n_j\, F(p_i + m_j + t_k + R_\ell - 1,\ q_i + n_j + u_k + S_\ell - 1) \end{aligned} \tag{17}$$

(continued)

$$
\begin{aligned}
&+ d_k\, e_\ell\, p_i\, p_j\, F(p_i + p_j + t_k + R_\ell - 2,\ q_i + q_j + u_k + S_\ell) \\
&- d_k\, e_\ell\, p_i\, h5\, F(p_i + r_j + t_k + R_\ell - 1,\ q_i + s_j + u_k + S_\ell) \\
&- d_k\, e_\ell\, n_i\, n_j\, F(m_i + m_j + t_k + R_\ell,\ n_i + n_j + u_k + S_\ell - 2) \\
&- d_k\, e_\ell\, n_i\, p_j\, F(m_i + p_j + t_k + R_\ell - 1,\ n_i + q_j + u_k + S_\ell - 1) \\
&+ d_k\, e_\ell\, n_i\, h5\, F(m_i + r_j + t_k + R_\ell,\ n_i + s_j + u_k + S_\ell - 1) \\
&+ 0.25\, d_k\, f_\ell\, p_i\, p_j\, F(p_i + p_j + t_k + R_\ell - 2,\ q_i + q_j + u_k + S_\ell) \\
&- 0.25\, d_k\, f_\ell\, p_i\, n_j\, F(p_i + m_j + t_k + R_\ell - 1,\ q_i + n_j + u_k + S_\ell - 1) \\
&+ 0.25\, d_k\, f_\ell\, n_i\, n_j\, F(m_i + m_j + t_k + R_\ell,\ n_i + n_j + u_k + S_\ell - 2) \\
&- 0.25\, d_k\, f_\ell\, n_i\, p_j\, F(m_i + p_j + t_k + R_\ell - 1,\ n_i + q_j + u_k + S_\ell - 1) \\
&- d_k\, f_\ell\, p_i\, n_j\, F(p_i + m_j + t_k + R_\ell - 1,\ q_i + n_j + u_k + S_\ell - 1) \\
&- d_k\, f_\ell\, p_i\, p_j\, F(p_i + p_j + t_k + R_\ell - 2,\ q_i + q_j + u_k + S_\ell) \qquad (17) \\
&+ d_k\, f_\ell\, p_i\, h5\, F(p_i + r_j + t_k + R_\ell - 1,\ q_i + s_j + u_k + S_\ell) \\
&+ d_k\, f_\ell\, n_i\, n_j\, F(m_i + m_j + t_k + R_\ell,\ n_i + n_j + u_k + S_\ell - 2) \\
&+ d_k\, f_\ell\, n_i\, p_j\, F(m_i + p_j + t_k + R_\ell - 1,\ n_i + q_j + u_k + S_\ell - 1) \\
&- d_k\, f_\ell\, n_i\, h5\, F(m_i + r_j + t_k + R_\ell,\ n_i + s_j + u_k + S_\ell - 1) \\
&+ 0.5\, d_k\, g_\ell\, q_i\, p_j\, F(p_i + p_j + t_k + R_\ell - 1,\ q_i + q_j + u_k + S_\ell - 1) \\
&+ 0.5\, d_k\, g_\ell\, q_j\, p_i\, F(p_i + p_j + t_k + R_\ell - 1,\ q_i + q_j + u_k + S_\ell - 1) \\
&- 0.5\, d_k\, g_\ell\, q_i\, n_j\, F(p_i + m_j + t_k + R_\ell,\ q_i + n_j + u_k + S_\ell - 2) \\
&- 0.5\, d_k\, g_\ell\, q_j\, n_i\, F(p_j + m_i + t_k + R_\ell,\ q_j + n_i + u_k + S_\ell - 2) \\
&- 0.5\, d_k\, g_\ell\, p_j\, h6\, F(v_i + p_j + t_k + R_\ell - 1,\ s_i + q_j + u_k + S_\ell) \quad \text{(continued)}
\end{aligned}
$$

$$- 0.5\, d_k\, g_\ell\, p_i\, h_6\, F(v_j + p_i + t_k + R_\ell - 1,\ s_j + q_i + u_k + S_\ell)$$
$$+ 0.5\, d_k\, g_\ell\, n_j\, h_6\, F(r_i + m_j + t_k + R_\ell,\ s_i + n_j + u_k + S_\ell - 1)$$
$$+ 0.5\, d_k\, g_\ell\, n_i\, h_6\, F(r_j + m_i + t_k + R_\ell,\ s_j + n_i + u_k + S_\ell - 1)$$
$$- 0.5\, d_k\, g_\ell\, m_i\, p_j\, F(m_i + p_j + t_k + R_\ell - 2,\ n_i + q_j + u_k + S_\ell)$$
$$- 0.5\, d_k\, g_\ell\, m_j\, p_i\, F(m_j + p_i + t_k + R_\ell - 2,\ n_j + q_i + u_k + S_\ell) \qquad (17)\ \text{(concluded)}$$
$$+ 0.5\, d_k\, g_\ell\, m_i\, n_j\, F(m_i + m_j + t_k + R_\ell - 1,\ n_i + n_j + u_k + S_\ell - 1)$$
$$+ 0.5\, d_k\, g_\ell\, m_j\, n_i\, F(m_i + m_j + t_k + R_\ell - 1,\ n_i + n_j + u_k + S_\ell - 1)$$
$$+ 0.5\, d_k\, g_\ell\, p_j\, h_4\, F(r_i + p_j + t_k + R_\ell - 1,\ s_i + q_j + u_k + S_\ell)$$
$$+ 0.5\, d_k\, g_\ell\, p_i\, h_4\, F(r_j + p_i + t_k + R_\ell - 1,\ s_j + q_i + u_k + S_\ell)$$
$$- 0.5\, d_k\, g_\ell\, n_j\, h_4\, F(r_i + m_j + t_k + R_\ell,\ s_i + n_j + u_k + S_\ell - 1)$$
$$- 0.5\, d_k\, g_\ell\, n_i\, h_4\, F(r_j + m_i + t_k + R_\ell,\ s_j + n_i + u_k + S_\ell - 1)\Big]$$

The generalized differential stiffness matrix is transformed to the element coordinate system, basic coordinate system and the global coordinate system in the normal manner.

REFERENCES

1. Narayanaswami, R. "New Triangular Shallow Shell Finite Element." Technical Report 74-T7, School of Engineering, Old Dominion University, Norfolk, Virginia, November 1974.

2. Novozhilov, V. V. "The Theory of Thin Shells." 2nd edition, Noordhoff (1964).

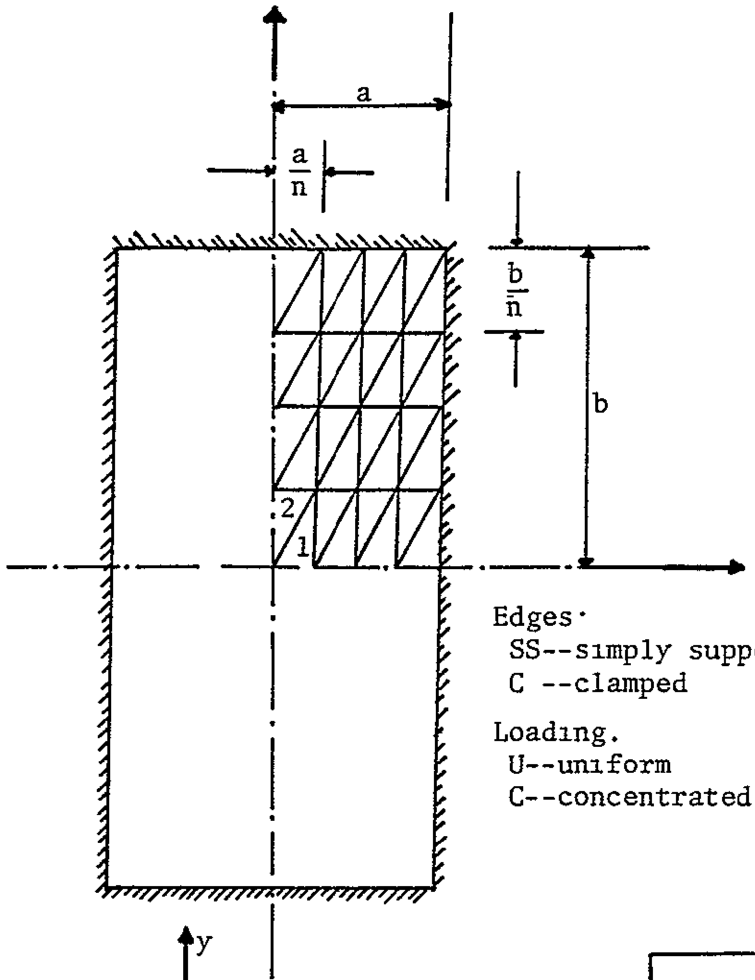
Modeling of Plate Structures Using TRPLT1 Elements

The Figure 1, shown on page 15.2-3, is modeled using higher order triangular bending element, (Figure 2, page 15.2-3(a)), CTRPLT1

Because of symmetry, the quarter section of the plate is discretized and detail of the discretization is given by the side of the modeled figure. Four different mesh sizes are used for each case.

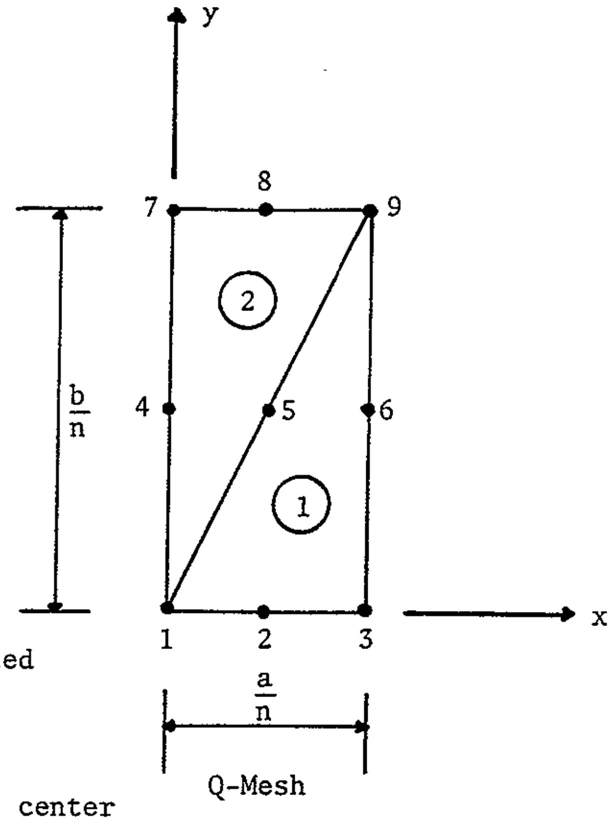
The central deflection is plotted in figures on pages 15.2-4 to 15.2-11 and also given in Tables 1 and 2 on 15.2-3(b) and 15 2-3(c).

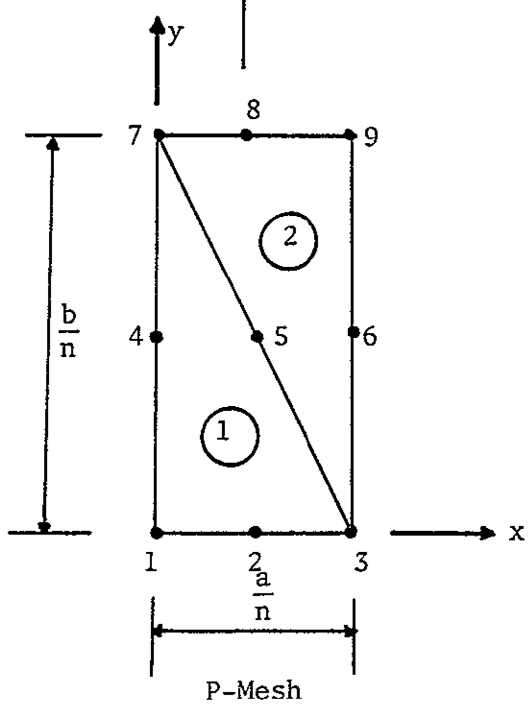
Such high accuracy is obtainable for other plate structure problems using the TRPLT1 element in view of the use of the quintic displacement field for the displacement pattern in the element.



Edges:
SS--simply supported
C --clamped

Loading.
U--uniform
C--concentrated at center





Case	b/a	Edges	Load
1	1	SS	U
2	2	SS	U
3	1	C	U
4	2	C	U
5	1	SS	C
6	2	SS	C
7	1	C	C
8	2	C	C

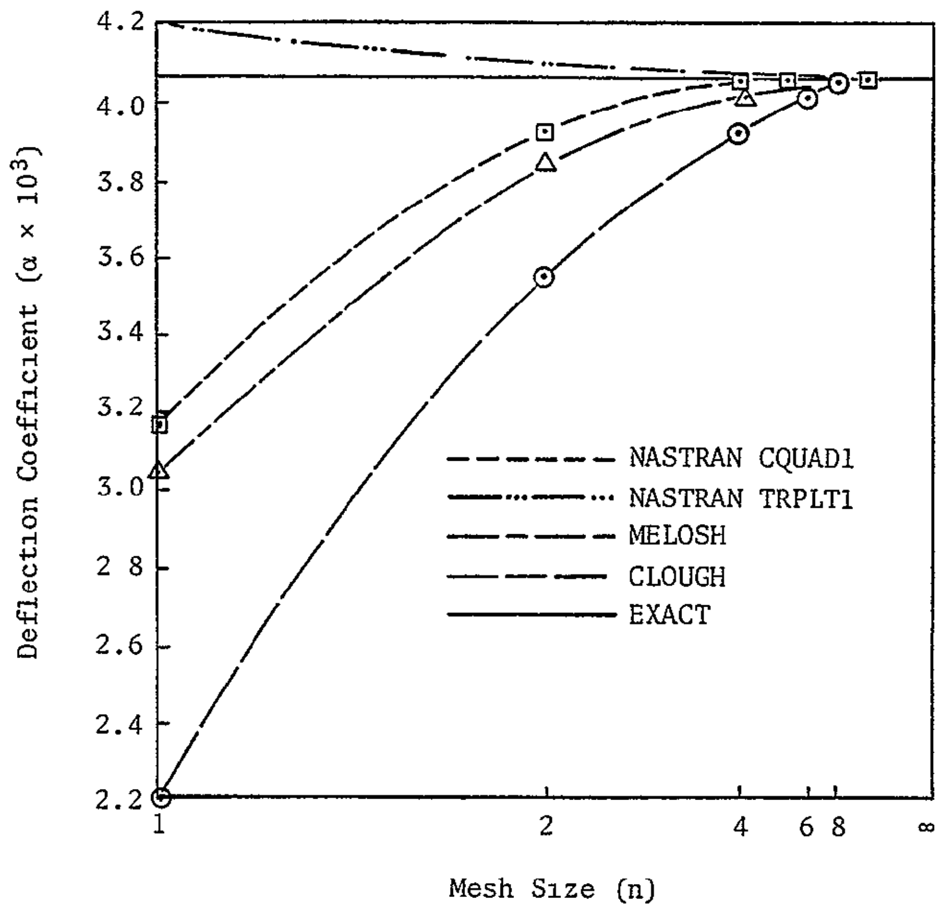
Figure 2. Discretization and schedule of rectangular plate
15.2-3(a)(1/1/77)

Table 1 Central deflection of simply supported rectangular plates $\frac{b}{a} = 2$.

Number of Elements per Side N	Concentrated Load at Center		Uniformly Distributed Load	
	Q-mesh	P-mesh	Q-mesh	P-mesh
2	21.3344	19.0681	10.9158	10.17408
4	17.7814	17.1537	10.1459	10.0245
8	16.9230	16.6984	10.0776	10.0548
12	16.7212	16.6073	10.1320	10.1230
Exact Solution	16.5		10.125	

Table 2. Central deflection of clamped rectangular plates $\frac{b}{a} = 2$.

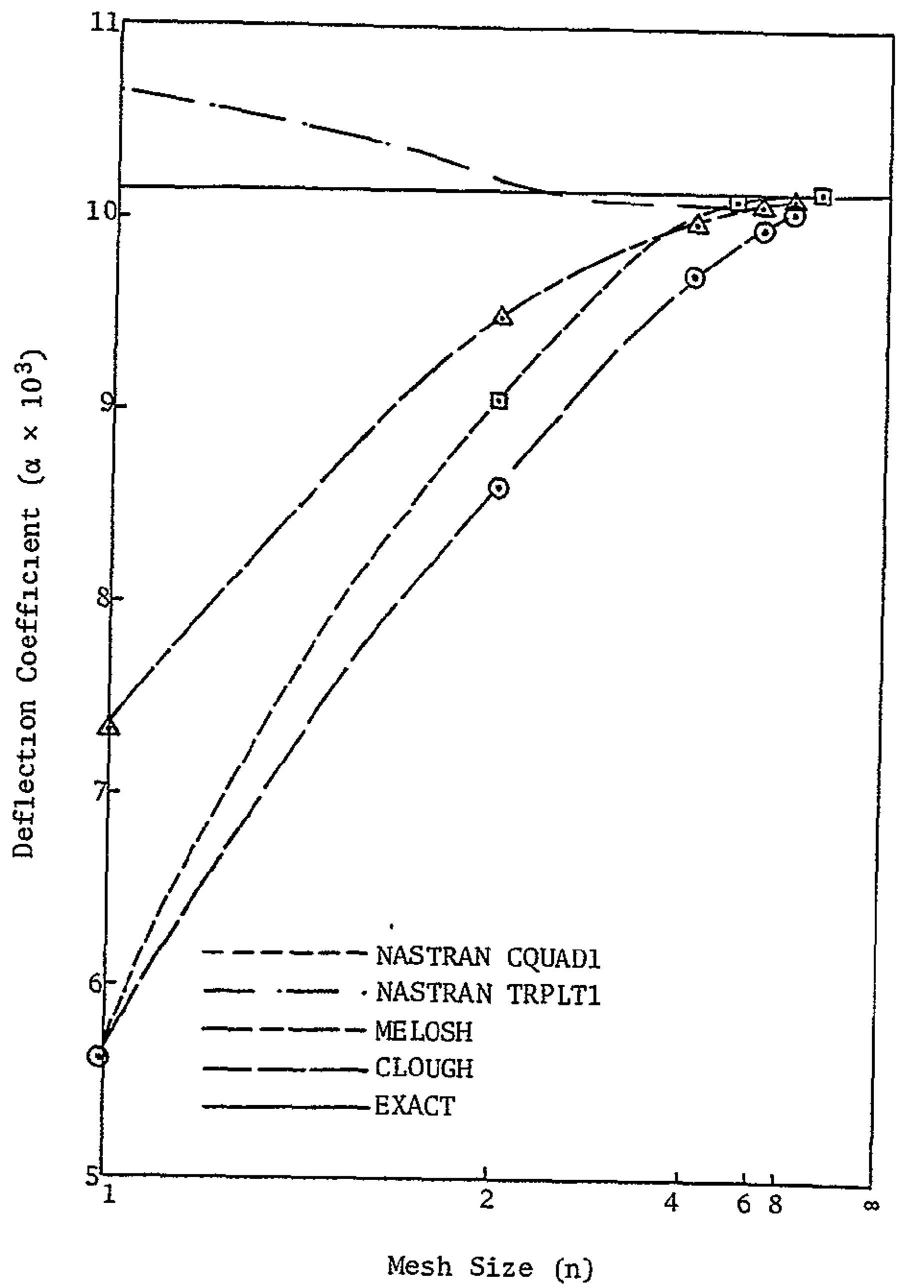
Number of Elements per Side N	Concentrated Load at Center		Uniformly Distributed Load	
	Q-mesh	P-mesh	Q-mesh	P-mesh
2	10.4294	10.8878	3.9168	3.870672
4	8.4193	8.0427	2.7757	2.7453
8	7.6242	7.4392	2.5791	2.5738
12	7.4282	7.3293	2.5603	2.5585
Exact Solution	7.22		2.54	



Central deflection of rectangular plate

Case 1 (1-SS-U)

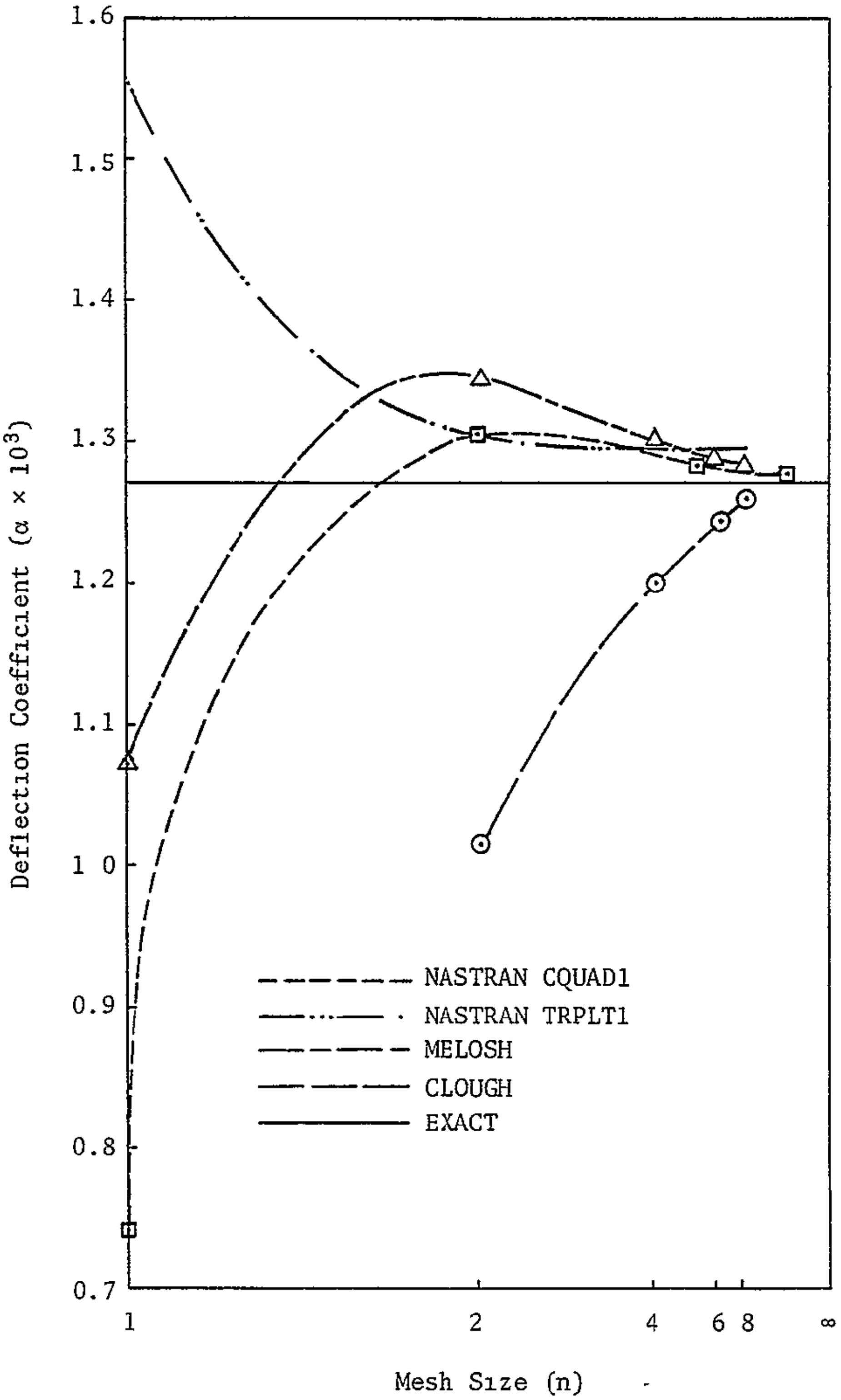
15.2-4 (1/1/77)



Central deflection of rectangular plate

Case 2 (2-SS-U)

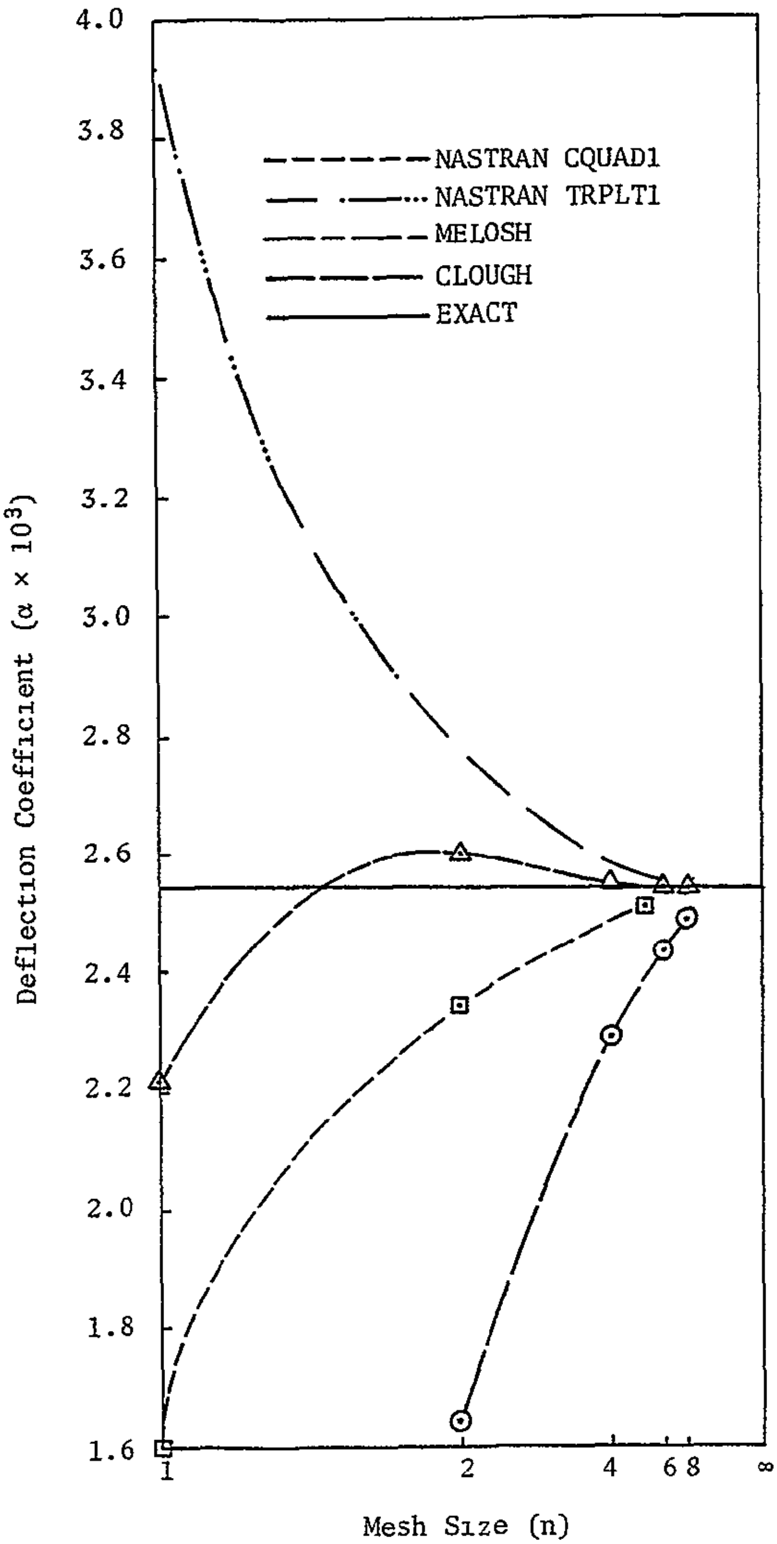
15.2-5 (1/1/77)



Central deflection of rectangular plate.

Case 3 (1-C-U)

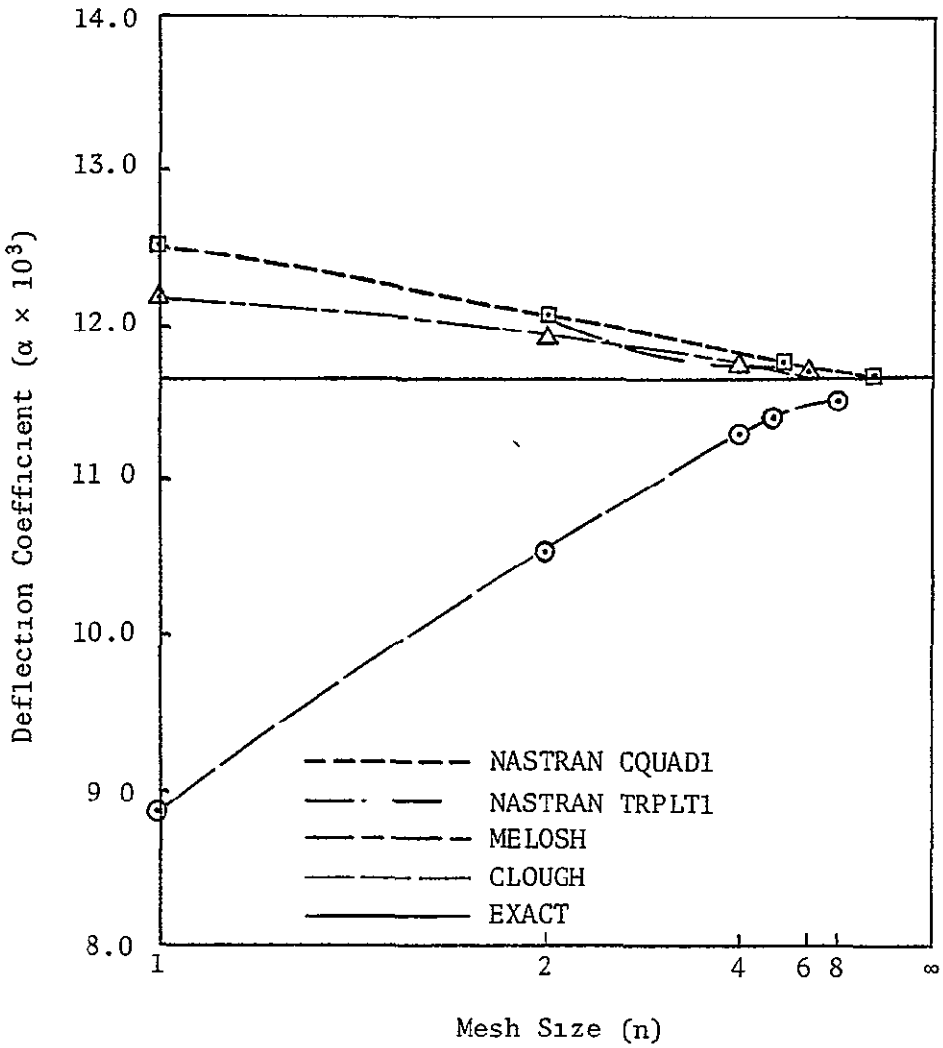
15 2-6 (1/1/77)



Central deflection of rectangular plate.

Case 4 (2-C-U)

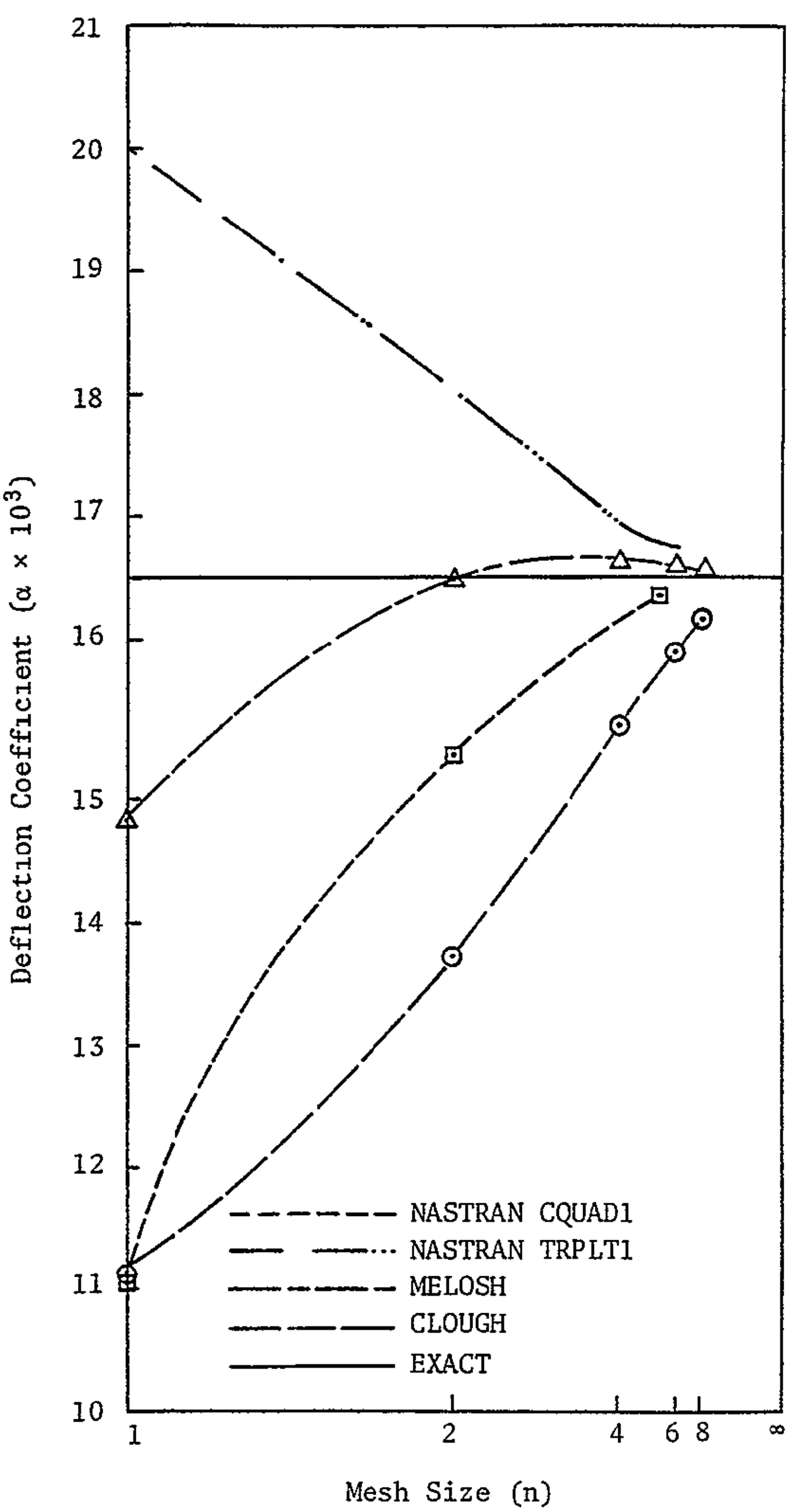
15.2-7 (1/1/77)



Central deflection of rectangular plate.

Case 5 (1-SS-C)

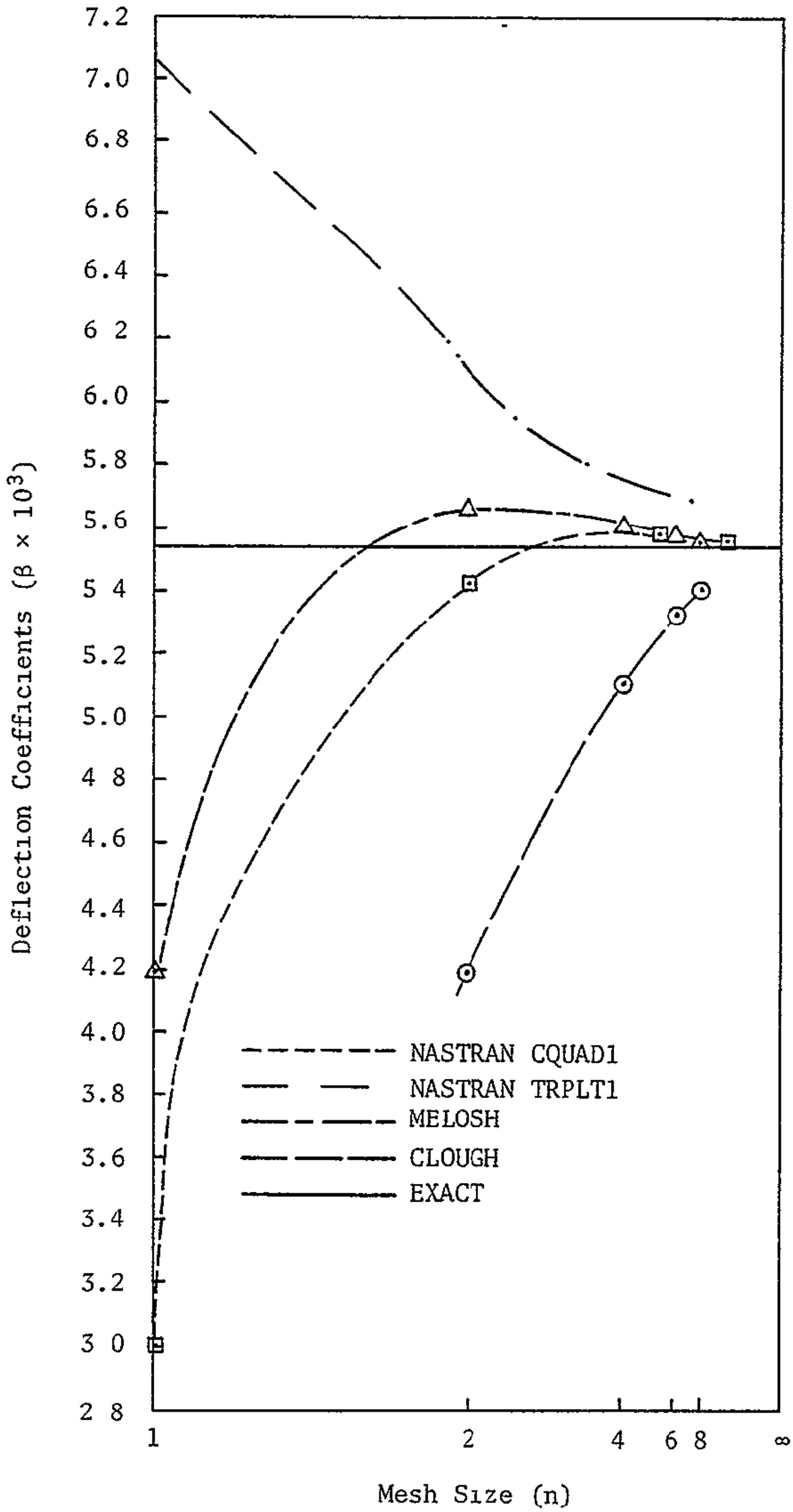
15 2-8 (1/1/77)



Central deflection of rectangular plate.

Case 6 (2-SS-C)

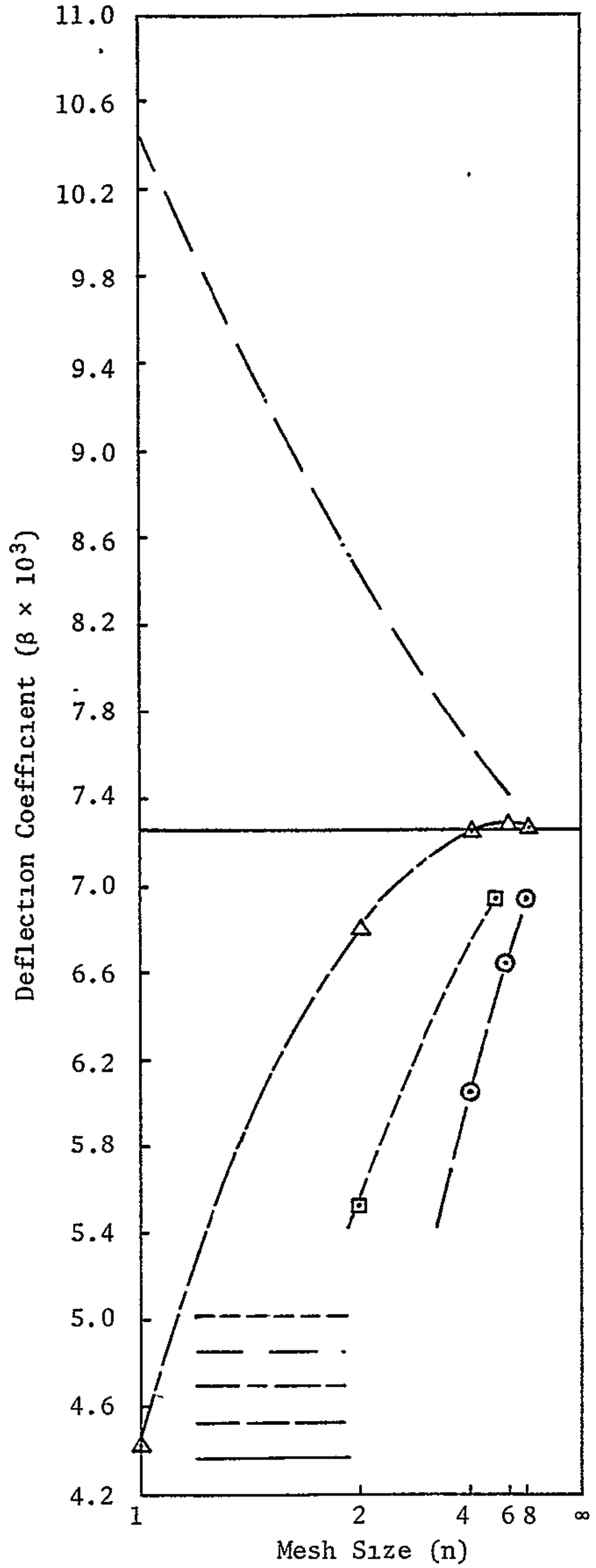
15 2-9 (1/1/77)



Central deflection of rectangular plate

Case 7 (1-C-C)

15 2-10 (1/1/77)



Central deflection of rectangular plate

Case 8 (2-C-C)

15.2-11 (1/1/77)

Modeling Membrane Plate Using TRIM6 Element

In the same figure 2, the cantilever beam is discretized using the linear strain triangular membrane element TRIM6. Discretization and result of the corresponding displacement is shown on page 15.3-3.

The cantilever beam shown on page 15.3-3 (figure 2) is divided into eight equal triangular (TRIM6) elements. The displacement pattern obtained using this mesh coincides with the exact one. Such high accuracy is obtainable for other membrane plate problems using the TRIM6 element in view of the quadratic displacement polynomial for the element.

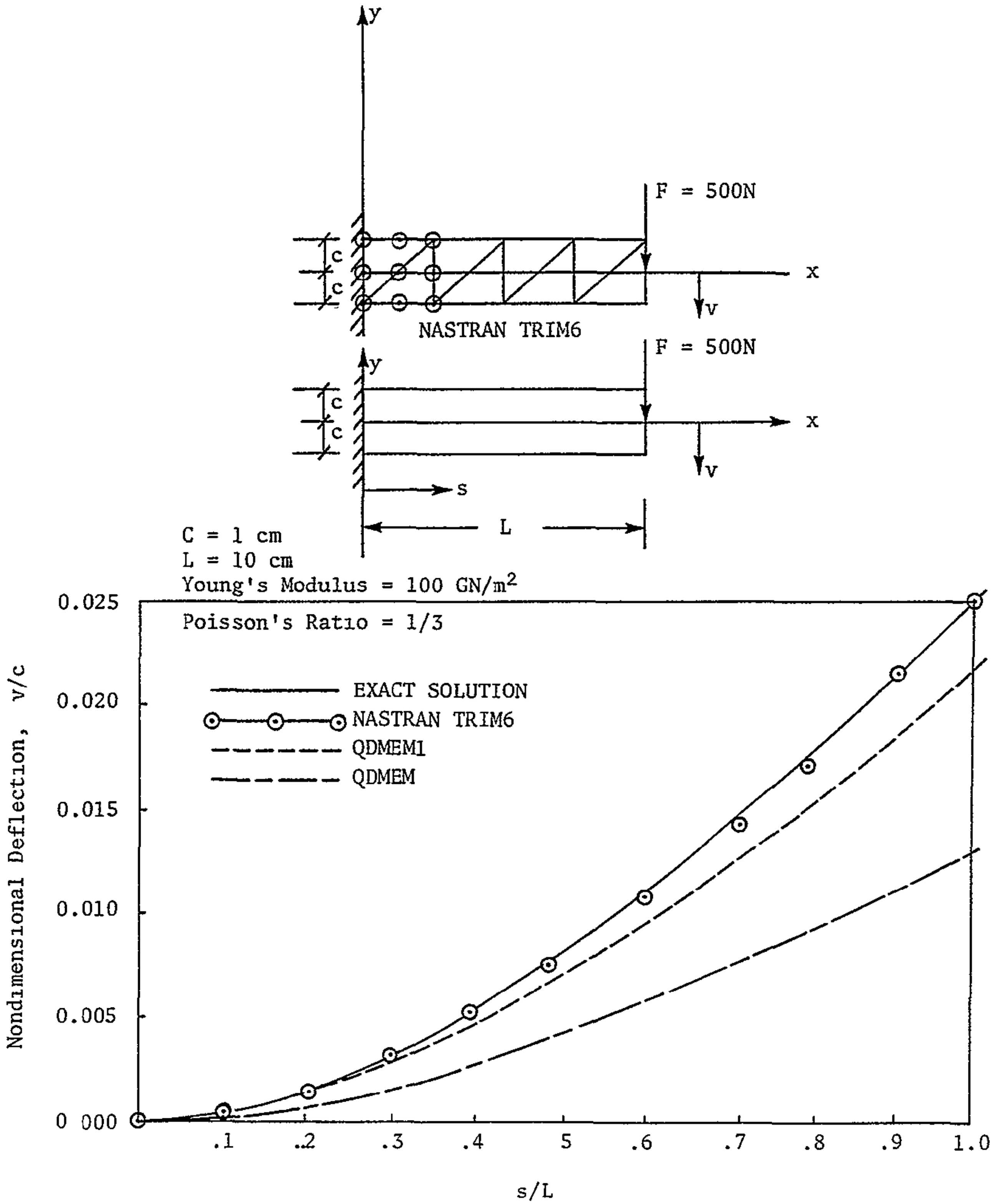


Figure 2. Deflection of cantilever beam idealized by QDMEM1 and TRIM6 elements.

APPENDIX B

Updates to the NASTRAN Users' Manual for the addition of TRIM6, TRPLT1 and TRSHL elements

1.3.5. Plate Elements

NASTRAN includes two different shapes of plate elements (triangular and quadrilateral) and two different stress systems (membrane and bending) which are uncoupled. There are in all a total of thirteen different forms of plate elements that are defined by connection cards as follows

1. CTRMEM - triangular element with finite in-plane stiffness and zero bending stiffness.

2. CTRIM6 - a triangular element with finite in-plane stiffness and zero bending stiffness.

3. CTRBSC - basic unit from which the bending properties of the other plate elements are formed.

4 CTRPLT - triangular element with zero in-plane stiffness and finite bending stiffness.

5. CTRPLT1 - higher order bending element--a triangular element with zero in-plane stiffness and finite bending stiffness.

6. CTRIA1 - triangular element with both in-plane and bending stiffness. It is designed for sandwich plates which can have different materials referenced for membrane, bending and transverse shear properties

7. CTRIA2 - triangular element with both in-plane and bending stiffness that assumes a solid homogeneous cross section.

8. CQDMEM - quadrilateral element consisting of four overlapping CTRMEM elements.

9. CQDMEM1 - an isoparametric quadrilateral membrane element.

10 CQDMEM2 - a quadrilateral membrane element consisting of four nonoverlapping CTRMEM elements.

11. CQDPLT - quadrilateral element with zero in-plane stiffness and finite bending stiffness

12. CQUAD1 - quadrilateral element with both in-plane and bending stiffness. It is designed for sandwich plates which can have different materials referenced for membrane, bending and transverse shear properties.

13. CQUAD2 - quadrilateral element with both in-plane and bending stiffness that assumes a solid homogeneous cross section.

Theoretical aspects of the plate elements are treated in Section 5.8 of the Theoretical Manual.

In addition, a shallow shell element, CTRSHL is also available. The elements and the coordinate systems are shown in figures 14(a), (b) and (c)

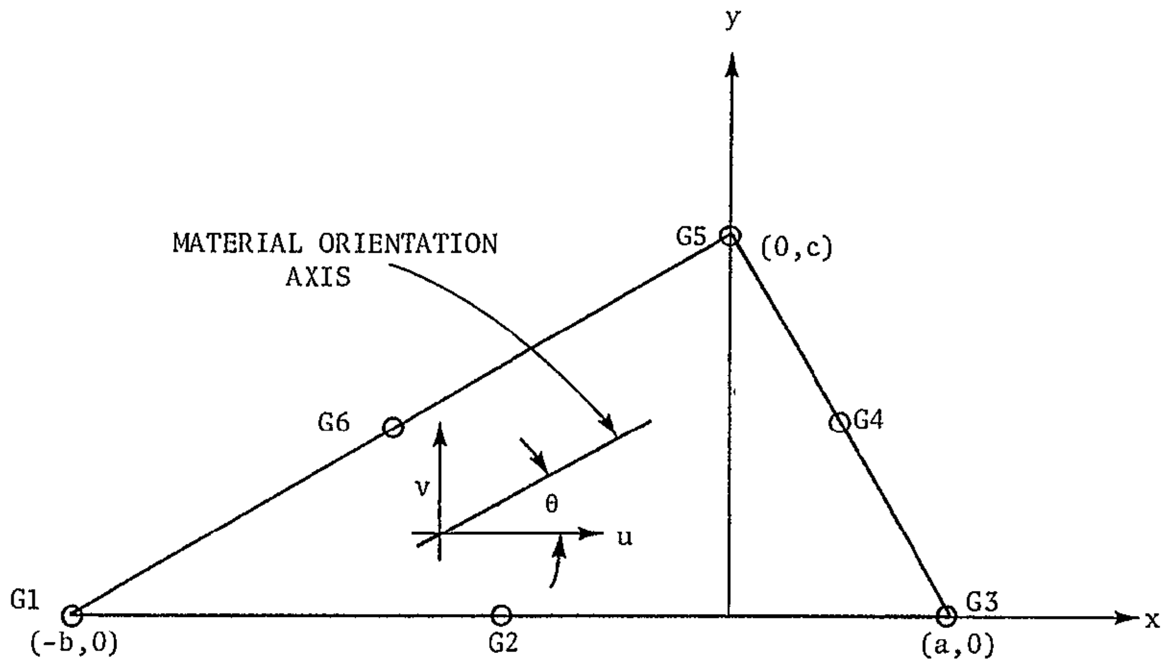


Figure 14(a). TRIM6 membrane element in element coordinate system

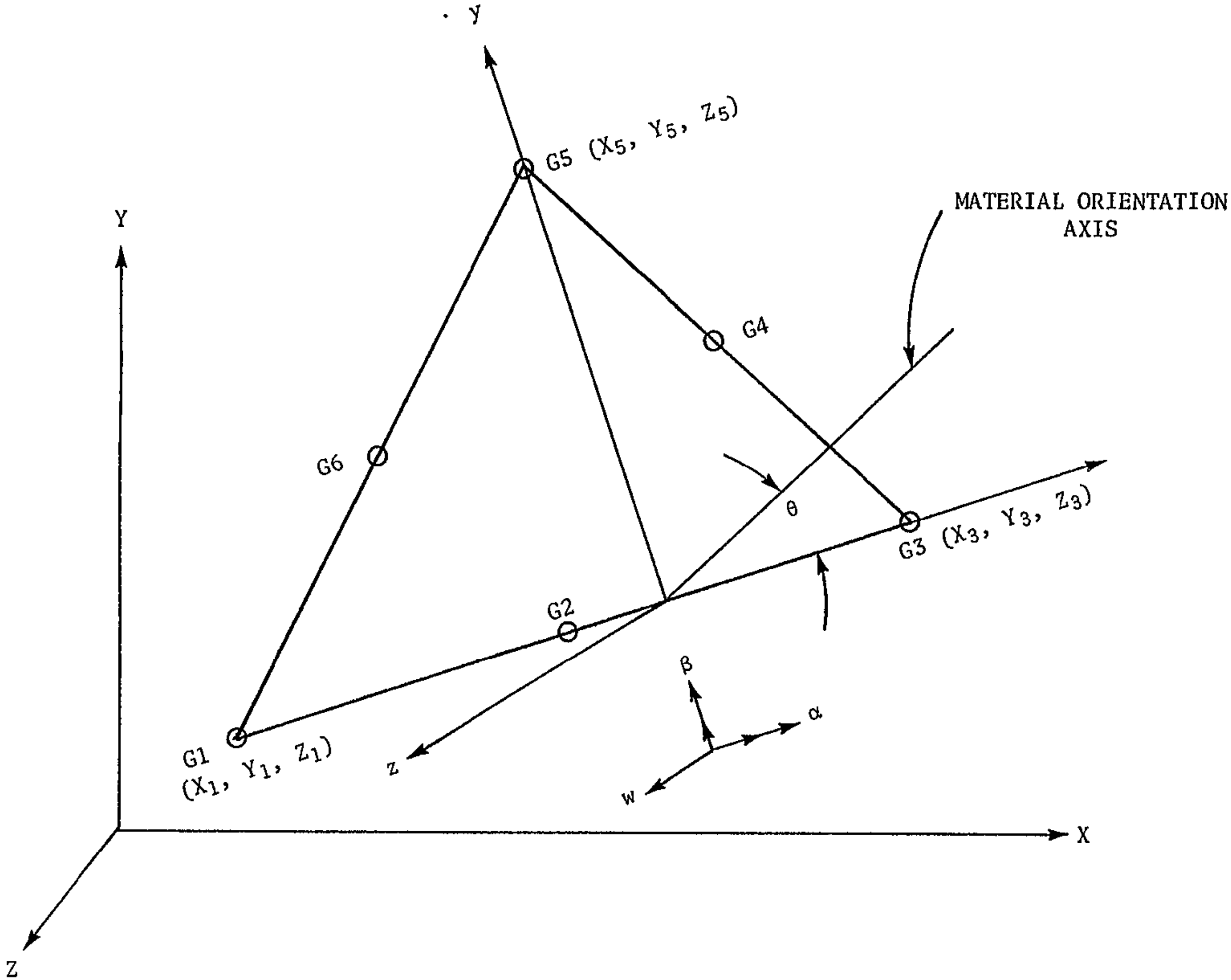


Figure 14(b). TRPLT1 triangular bending element geometry.

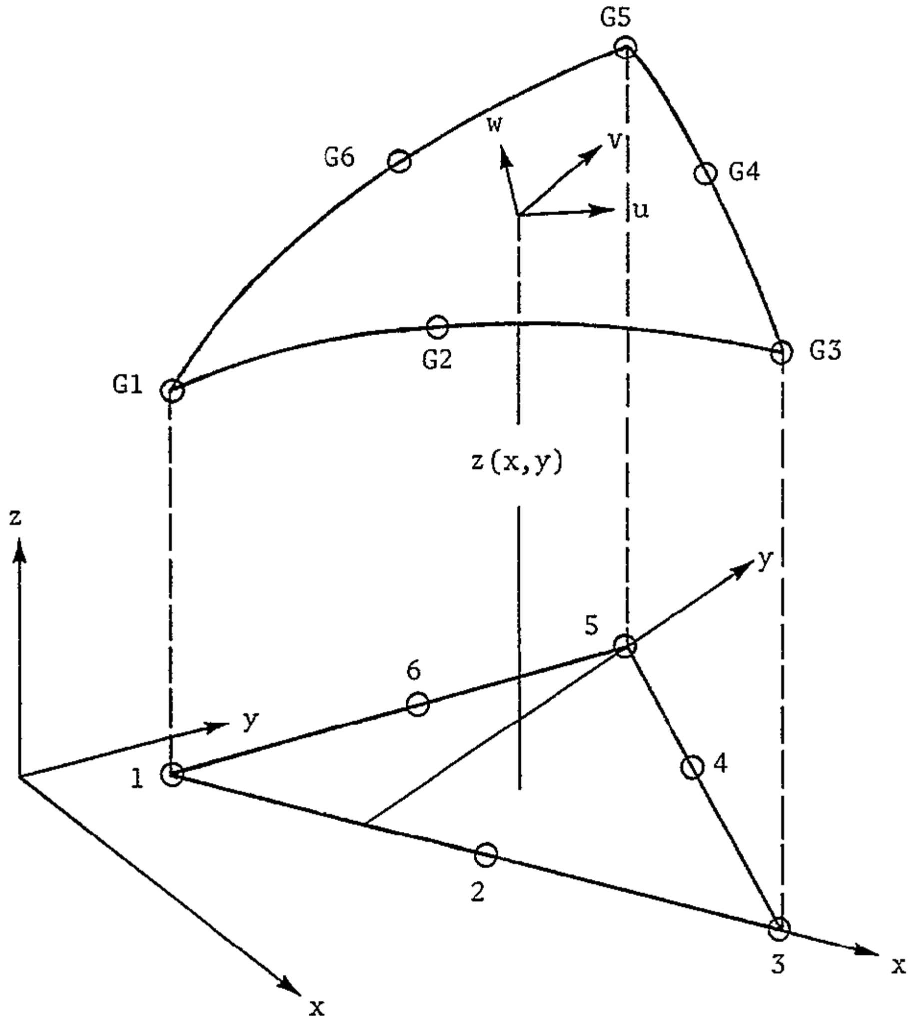


Figure 14(c) TRSHL shell element geometry and coordinate systems

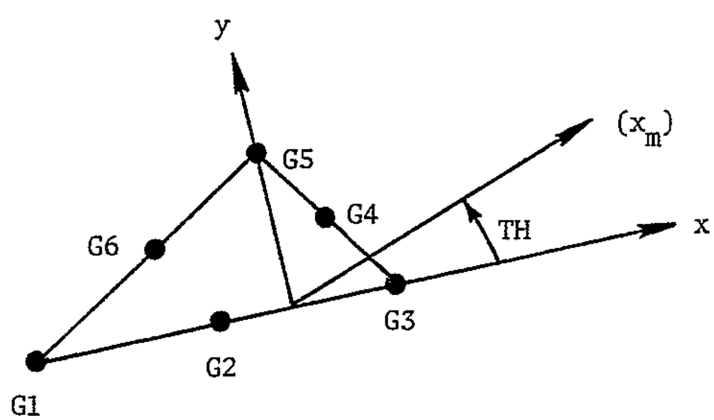
Input Data Card CTRIM6 Triangular Element Connection

Description: Defines a linear strain triangular membrane element (TRIM6) of the structural model.

Format and Example:

1	2	3	4	5	6	7	8	9	10
CTRIM6	EID	PID	G1	G2	G3	G4	G5	G6	A1
CTRIM6	220	666	100	110	120	210	220	320	+C2
+BC	TH								
+C22	9.0								

Field	Contents
EID	Element identification number (integer > 0).
PID	Property identification number (integer > 0).
GI thru G6	Grid point identification numbers of connected points (integers > 0; G1 ≠ G2 ≠ G3 ≠ G4 ≠ G5 ≠ G6).
TH	Material property orientation angle in degrees (Real). The sketch below gives sign convention for TH.



Remarks:

1. The grid points must be listed consecutively going around the perimeter in an anticlockwise direction and starting at a vertex.

2. Material properties (if MAT2) and stresses are given in the (x_m, y_m) coordinate system shown in the sketch.

3. G2, G4, and G6 are assumed to lie at the midpoints of the sides The locations of these grid points (on GRID Bulk Data cards) are used only for global coordinate system definition, GPWG (weight generator module), centrifugal forces, and deformed structure plotting.

4. Continuation card must be present.

5. Element identification numbers must be unique with respect to all other element identification numbers

Input Data Card PTRIM6 Linear Strain Triangular Element Property

Description: Defines the properties of a linear strain triangular membrane element TRIM6.

Format and Example:

1	2	3	4	5	6	7	8	9	10
PTRIM6	PID	MID	T1	T3	T5	NSM			
PTRIM6	666	999	1.17	2.52	3.84	8.3			

Field	Contents
PID	Property identification number (integer > 0).
MID	Material identification number (integer > 0).
T1, T3, T5	Thickness at the vertices of the element (Real).
NSM	Nonstructural mass per unit area (Real).

Remarks:

1. For structural problems, the material may be MAT1 or MAT2.
2. The thickness varies linearly over the triangle. If T3 or T5 is specified 0.0 or blank, it will be set equal to T1.
3. All PTRIM6 cards must have unique property identification numbers.

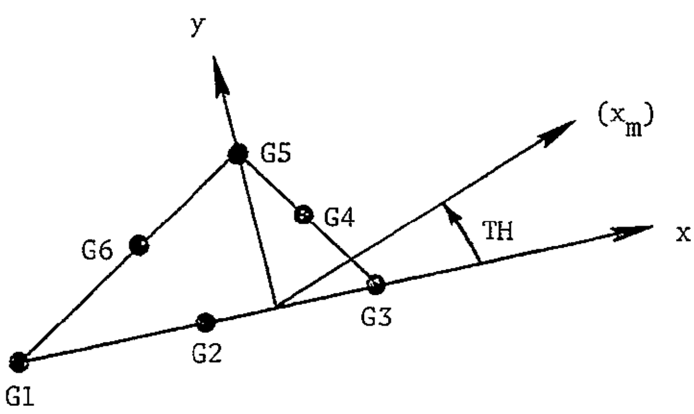
Input Data Card CTRPLT1 Triangular Element Connection

Description Defines a triangular bending element (TRPLT1) of the structural model.

Format and Example

1	2	3	4	5	6	7	8	9	10
CTRPLT	EID	PID	G1	G2	G3	G4	G5	G6	+abc
CTRPLT	160	20	120	10	30	40	70	110	+ABC
+abc	TH								
+ABC	16 2								

Field	Contents
EID	Element identification number (integer > 0)
PID	Identification number of a PTRPLT property card (Default is EID) (integer > 0).
G1, G2, G3, G4, G5, G6	Grid point identification numbers of connection points (integer > 0· G1 ≠ G2 ≠ G3 ≠ G4 ≠ G5 ≠ G6).
TH	Material property orientation angle in degrees (Real) - The sketch below gives the sign convention for TH.



Remarks:

1. Element identification numbers must be unique with respect to all other element identification numbers.

2. Interior angles must be less than 180°.

3. The grid points must be listed consecutively going around the perimeter in an anticlockwise direction and starting at a vertex.

4. Continuation card must be present.

Input Data Card PTRPLT1 Triangular Plate Property

Description: Used to define the bending properties of a triangular plate element. Referenced by CTRPLT1 card. No membrane properties are included.

Format and Example

1	2	3	4	5	6	7	8	9	10
PTRPLT1	PID	MID1	I1	I3	I5	MID2	TS1	TS3	+abc
PTRPLT1	15	25	20	30	40	35	3.0	1.15	+PQR
+abc	TS5	NSM	Z11	Z21	Z13	Z23	Z15	Z25	
+PQR	1.0	-1 0	1 5	-1.5	2.0	-2.0	2.5	-2.5	

Field	Contents
PID	Property identification number (integer > 0).
MID1	Material identification number for bending (integer > 0)
I1, I3, I5	Area moment of inertia of the element per unit width at the vertices 1, 3, 5 of the element (Real > 0.0) $I_1 = \frac{t_1^3}{12}, \quad I_3 = \frac{t_3^3}{12}, \quad I_5 = \frac{t_5^3}{12}$ where T_1, T_3, T_5 are the thickness of the element at the vertices 1, 3, 5.
MID2	Material identification number for transverse shear (integer > 0).
TS1, TS3, TS5	Transverse shear thickness (Real > 0 0) at the vertices 1, 3, 5 of the element.
NSM	Nonstructural mass per unit area (Real)
Z11, Z21, Z13, Z23, Z15, Z25	Fiber distances for stress computation at grid points G1, G3, G5, respectively, positive according to the right-hand sequence defined on the CTRPLT1 card (Real)

Remarks:

1. All PTRPLT1 cards must have unique property identification numbers.

2. If TS1 is zero, the element is assumed to be rigid in transverse shear.

3. If TS3 or TS5 is 0.0 or blank, it will be set equal to TS1.

4. If I3 or I5 is 0.0 or blank, it will be set equal to I1.

5. The stresses at the centroid will be computed at the top and bottom fibers.

C-2

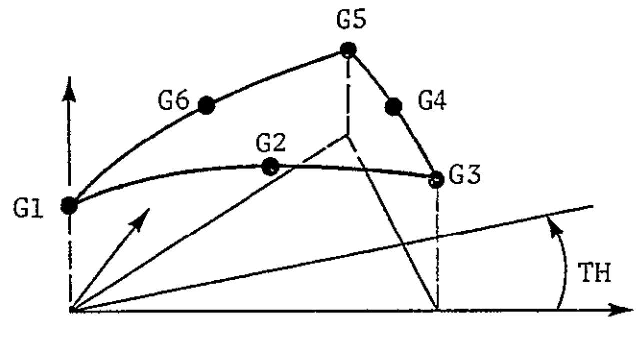
Input Data Card CTRSHL Triangular Shell Element Connection

Description: Defines a triangular thin shallow shell element (TRSHL) of the structural model.

Format and Example

1	2	3	4	5	6	7	8	9	10
CTRSHL	EID	PID	G1	G2	G3	G4	G5	G6	+abc
CTRPLT	160	20	120	10	30	40	70	110	+ABC
+abc	TH								
+ABC	16.2								

Field	Contents
EID	Element identification number (Integer > 0)
PID	Identification number of PTRSHL property card (Default is EID) (Integer > 0)
G1, G2, G3 G4, G5, G6	Grid point identification numbers of connection points (Integer > 0: G1 $\neq$ G2 $\neq$ G3 $\neq$ G4 $\neq$ G5 $\neq$ G6)
TH	Material property orientation angle in degrees (Real) - The sketch below gives the sign convention for TH.



Remarks

1. Element identification numbers must be unique with respect to all other element identification numbers

2. Interior angles must be less than 180°.

3. The grid points must be listed consecutively going around the perimeter in an anticlockwise direction and starting at a vertex.

4. Continuation card must be present.

Input Data Card PTRSHL Triangular Shell Property

Description Used to define the bending properties of a triangular shell element. Referenced by the CTRSHL card.

Format and Example

1	2	3	4	5	6	7	8	9	10
PTRSHL	PID	MID1	T1	T3	T5	MID2	I1	I3	+abc
PTRSHL	10	20	3.0	6.0	4.0	30	2.25	18.0	+PQR
+abc	I5	MID3	TS1	TS3	TS5	NSM	Z11	Z21	+def
+PQR	5.33	40	2.5	5.0	3.5	50	1.5	-1.5	+STU
+def	Z13	Z23	Z15	Z25					
+STU	3.0	-3.0	2.0	-2.0					

Field	Content
PID	Property Identification number (Integer > 0).
MID	Material identification number for membrane effect (Integer > 0).
T1, T3, T5	Thickness for membrane action at vertices 1, 3, 5 of the elements (Real > 0.0).
MID2	Material identification number for bending effects (Integer > 0).
I1, I3, I5	Area moments of inertia of the element at the vertices 1, 3, 5 of the element (Real > 0.0)
MID3	Material identification number for transverse shear (Integer > 0).
TS1, TS3, TS5	Transverse shear thickness (Real > 0.0) at the vertices 1, 3, 5 of the element.
NSM	Non-structural mass per unit area (Real).
Z11, Z12, Z13, Z23, Z15, Z25	Fiber distances for stress computation at grid points G1, G3, G5, respectively, positive according to the right-hand sequence defined on the CTRSHL card (Real > 0.0).

Remarks:

1. All PTRSHL cards must have unique property identification numbers.
2. If T3 or T5 equal to 0.0, or blank, they will be set equal to T1.
3. If I3 or I5 equal to 0.0, or blank, they will be set equal to I1.
4. If TS3 or TS5 equal to 0.0, or blank, they will be set equal to TS1.
5. If TS1 is 0.0, or blank, the element is assumed to be rigid in transverse shear.
6. The stresses at the centroid will be computed at the top and bottom fibers.

APPENDIX C

Updates to the NASTRAN Programmer's Manual
for the addition of TRIM6, TRPLT1 and TRSHL elements

4.87.21. TRIM6: Linear Strain Triangular Element

4.87.21.1 Input Data for TRIM6 Element

1. EST entries for TRIM6 are

Symbol	Description
EID	Element Identification Number
SIL_1, SIL_2, . . ., SIL_6	Scalar indices of connected grid points
θ	Anisotropic material orientation angle
Mat ID	Material Identification Number
T1, T3, T5	Thickness of corner grid points
μ	Nonstructural mass per unit area
N_i, X_i, Y_i, Z_i $i = 1, 6$	Local coordinate system numbers and location coordinates in the basic system for the connected grid points
T01, T02, T03, T04, T05, T06	Temperatures at the grid points

2. Coordinate system data

The numbers N_i, X_i, Y_i and Z_i are used to calculate 3 by 3 basic-to-global coordinate transformation matrices $[T_i]$ for points $i = 1, 2, 3, 4, 5$ and 6.

3. Material data

Symbol	Description
$[G]$	3 × 3 stress-strain matrix
ρ	Mass density
α_x, α_y, α_{xy}	Thermal expansion coefficients
T0	Reference temperature

g_e Structural damping coefficient

σ_t, σ_c, σ_s Stress limits for tension, compression, and shear

4.87.21.2 Basic Equations for TRIM6

1. The element coordinate system is defined by the following equations:

$$\{V_{13}\} = \begin{Bmatrix} X_3 - X_1 \\ Y_3 - Y_1 \\ Z_3 - Z_1 \end{Bmatrix} \tag{1}$$

$$\{V_{15}\} = \begin{Bmatrix} X_5 - X_1 \\ Y_5 - Y_1 \\ Z_5 - Z_1 \end{Bmatrix} \tag{2}$$

$$\{i\} = \frac{\{V_{13}\}}{|\{V_{13}\}|} \tag{3}$$

$$\{k\} = \frac{\{i\} \times \{V_{13}\}}{|\{i\} \times \{V_{13}\}|} \tag{4}$$

$$\{j\} = \{k\} \times \{i\} \tag{5}$$

2. The displacement transformation matrix from basic coordinates to in-plane coordinates is:

$$[E]^T = \begin{bmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{bmatrix} \tag{6}$$

3. The local (element) coordinate system of the element is as follows.

The x-axis is obtained by joining grid points 1 and 3 of the element.

The y-axis is the perpendicular from grid point 5 to the x-axis (line joining grid points 1 and 3).

Depending upon the location of grid point 5 relative to grid points 1 and 3, 3 cases of triangle orientation are possible: (refer to fig. 4.87.21.1)

Case I Acute angles at grid points 1 and 3

$$c = |\{i\} \times \{V_{15}\}| \tag{7}$$

$$b = \{i\} \cdot \{V_{15}\} \tag{8}$$

$$a = |\{V_{13}\}| - b \tag{9}$$

Coordinates of points are

$$x_1 = -b \ ; \quad x_2 = \frac{a - b}{2} \ , \quad x_3 = a \ , \quad x_4 = \frac{a}{2} \ ; \quad x_5 = 0 \ , \quad x_6 = -\frac{b}{2} \tag{10}$$

$$y_1 = 0 \ , \quad y_2 = 0 \ ; \quad y_3 = 0 \ , \quad y_4 = \frac{c}{2} \ , \quad y_5 = c \ , \quad y_6 = \frac{c}{2} \tag{11}$$

Case II. Obtuse angle at grid point 3

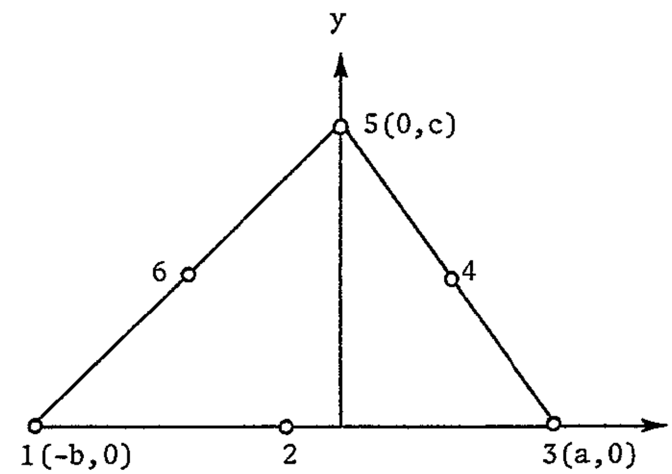
$$c = |\{i\} \times \{V_{15}\}| \tag{12}$$

$$b = \{i\} \cdot \{V_{15}\} \tag{13}$$

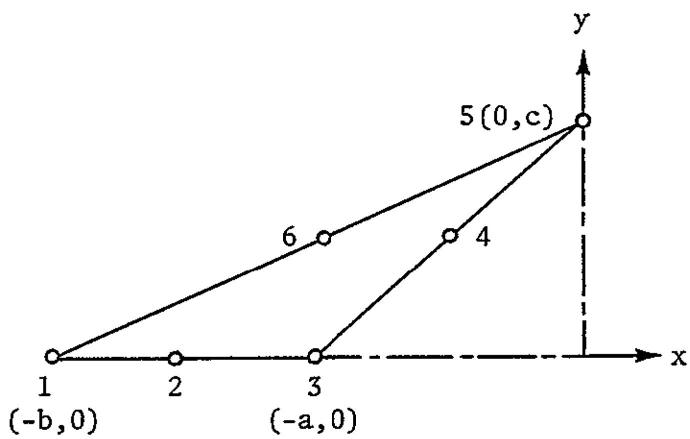
$$a = b - |\{V_{13}\}| \tag{14}$$

Coordinates of points are

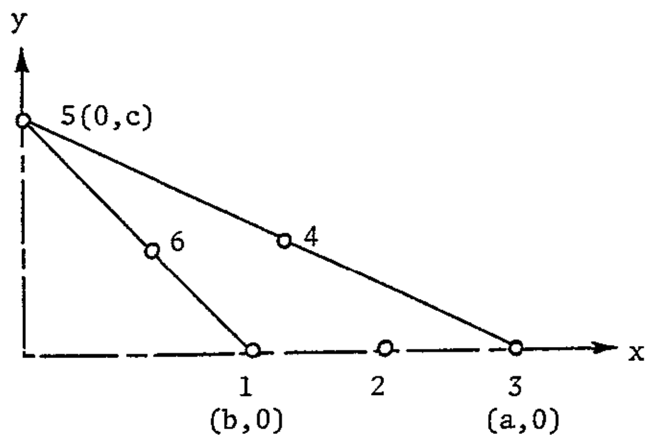
$$x_1 = -b \ , \quad x_2 = \frac{-a - b}{2} \ ; \quad x_3 = -a \ , \quad x_4 = -\frac{a}{2} \ , \quad x_5 = 0 \ , \quad x_6 = -\frac{b}{2} \tag{15}$$



Case I. Acute angles at grid points 1, 3, and 5.



Case II. Obtuse angle at grid point 3.



Case III. Obtuse angle at grid point 1

Figure 1 Triangular element shapes.
4.87-21.1

$$y_1 = 0 , \quad y_2 = 0 ; \quad y_3 = 0 ; \quad y_4 = \frac{c}{2} , \quad y_5 = c , \quad y_6 = \frac{c}{2} \tag{16}$$

Case III Obtuse angle at grid point 1

$$c = \left| \{i\} \times \{V_{15}\} \right| \tag{17}$$

$$b = \{i\} \cdot \{V_{15}\} \tag{18}$$

$$a = \left| \{V_{13}\} \right| + b \tag{19}$$

Coordinates of points are

$$x_1 = b , \quad x_2 = \frac{a + b}{2} ; \quad x_3 = a , \quad x_4 = \frac{a}{2} , \quad x_5 = 0 ; \quad x_6 = \frac{b}{2}$$

$$y_1 = 0 ; \quad y_2 = 0 ; \quad y_3 = 0 ; \quad y_4 = c_2 , \quad y_5 = c ; \quad y_6 = \frac{c}{2}$$

4. The matrix $\lfloor H \rfloor$ relating grid point displacements and the generalized coordinates (in the equation $\{u\} = H \{a\}$) is given by

$$\lfloor H \rceil = \begin{bmatrix} H_1 & 0 \\ 0 & H_1 \end{bmatrix} \tag{20}$$

where

$$[H_1] = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix} \tag{21}$$

5. The matrix $[B]$ relating strain vector to the generalized coordinates (in the equation $\{\varepsilon\} = [B]\{a\}$) is given by

$$B = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \quad (22)$$

4.87.21.3 <u>Stiffness Matrix Calculation for TRIM6 (Subroutine KTRM6S and KTRM6D)</u>

The polynomial expressions for variation of u, v and t within the element are

$$u = \sum_{i=1}^{12} a_i \, x^{m_i} \, y^{n_i} \quad (23)$$

$$v = \sum_{i=1}^{12} b_i \, x^{p_i} \, y^{q_i} \quad (24)$$

$$t = \sum_{i=1}^{3} c_i \, x^{r_i} \, y^{s_i} \quad (25)$$

The values of m_i, n_i, p_i, q_i, r_i, s_i are

$$m_1 = 0 \; ; \; m_2 = 1 \; , \; m_3 = 0 \; ; \; m_4 = 2 \; ; \; m_5 = 1 \; , \; m_6 = 0 \; , \quad m_7 \text{ to } m_{12} = 0 \quad (26)$$

$$n_1 = 0 \; ; \; n_2 = 0 \; ; \; n_3 = 1 \; ; \; n_4 = 0 \; ; \; n_5 = 1 \; , \; n_6 = 2 \; , \quad n_7 \text{ to } n_{12} = 0 \quad (27)$$

$$p_1 \text{ to } p_6 = 0 \; , \; p_7 = 0 \; , \; p_8 = 1 \; , \; p_9 = 0 \; , \; p_{10} = 2 \; , \quad p_{11} = 1 \; , \; p_{12} = 0 \quad (28)$$

$$q_1 \text{ to } q_6 = 0 \; ; \quad q_7 = 0 \, , \quad q_8 = 0 \; ; \quad q_9 = 1 \; ; \quad q_{10} = 0 \, , \quad q_{11} = 1 \, , \quad q_{12} = 2 \tag{29}$$

$$r_1 = 0 \; ; \quad r_2 = 1 \, , \quad r_3 = 0 \tag{30}$$

$$s_1 = 0 \; ; \quad s_2 = 0 \, , \quad s_3 = 1 \tag{31}$$

$$a_7 \text{ to } a_{12} = 0 \; ; \quad b_1 \text{ to } b_6 = 0 \tag{32}$$

The coefficients a_1 to a_6 and b_7 to b_{12} are generalized coordinates of the element and can be evaluated once the displacement vector is known.

The coefficients c_1 , c_2 and c_3 can be evaluated from the specified thicknesses t_1 , t_3 and t_5 of the 3 corner grid points and the geometric dimensions a , b and c of the element

$$c_1 = \frac{t_1 a + t_3 b}{(a + b)} \tag{33}$$

$$c_2 = \frac{t_3 - t_1}{(a + b)} \tag{34}$$

$$c_3 = \frac{1}{c} (t_5 - c_1) \tag{35}$$

The elements of the symmetric portion of the stress-strain matrix $[G_e]$ are denoted by G_{11} , G_{12} , G_{13} , G_{22} , G_{23} , G_{33} .

A formula for the integral of the type $x^m y^n$ taken over the area of the element is

$$\iint x^m y^n \, dx \, dy = F(m,n) = c^{n+1} \{a^{m+1} - (-b)^{m+1}\} \frac{m!n!}{(m + n + 2)!} \tag{36}$$

The equation used in the stiffness matrix generation in generalized coordinates is

$$
\begin{aligned}
(k_{ij})_{gen} = \sum_{k=1}^{3} c_k \Big[& G_{11} m_i m_j \; F\,(m_i + m_j + r_k - 2,\; n_i + n_j + s_k) \\
& + G_{22} q_i q_j \; F\,(p_i + p_j + r_k,\; q_i + q_j + s_k - 2) \\
& + G_{33} \{ n_i n_j \; F\,(m_i + m_j + r_k,\; n_i + n_j + s_k - 2) \\
& \qquad + p_i p_j \; F\,(p_i + p_j + r_k - 2,\; q_i + q_j + s_k) \} \\
& + (G_{33} n_i p_j + G_{12} m_i q_j) \; F\,(m_i + p_j + r_k - 1,\; n_i + q_j + s_k - 1) \\
& + (G_{33} n_j p_i + G_{12} m_j q_i) \; F\,(m_j + p_i + r_k - 1,\; n_j + q_i + s_k - 1) \\
& + G_{13} \{ (m_j n_i + m_i n_j) \; F\,(m_i + m_j + r_k - 1,\; n_i + n_j + s_k - 1) \\
& \qquad + m_j p_i \; F\,(m_j + p_i + r_k - 2,\; n_j + q_i + s_k) \\
& \qquad + m_i p_j \; F\,(m_i + p_j + r_k - 2,\; n_i + q_j + s_k) \} \\
& + G_{23} \{ (p_i q_j + p_j q_i) \; F\,(p_i + p_j + r_k - 1,\; q_i + q_j + s_k - 1) \\
& \qquad + n_i q_j \; F\,(m_i + p_j + r_k,\; n_i + q_j + s_k - 2) \\
& \qquad + n_j q_i \; F\,(m_j + p_i + r_k,\; r_j + q_i + s_k - 2) \} \Big]
\end{aligned}
\qquad (37)
$$

The stiffness matrix in global coordinates is

$$
[k] = [E]\;[T]^T\;[H^{-1}]^T\;[k]_{gen}\;[H^{-1}]\;[T]\;[E]^T \qquad (38)
$$

For use in the overall structural matrix, the 3 × 3 k_{ij} partition of the stiffness matrix $[k]$ corresponding to grid point i and connection point j is expanded to 6 × 6 to form

$$
k_{ij} = \left[\begin{array}{c|c} k_{ij} & 0 \\ \hline 0 & 0 \end{array}\right] \qquad (39)
$$

4.87.21.4 Mass Matrix Calculation for the TRIM6 Element (calculated in the stiffness subroutine KTRM6S and KTRM6D)

The mass is generated by the following algorithm

$$\{V_{13}\} = \begin{Bmatrix} X_3 - X_1 \\ Y_3 - Y_1 \\ Z_3 - Z_1 \end{Bmatrix} \tag{40}$$

$$\{V_{15}\} = \begin{Bmatrix} X_5 - X_1 \\ Y_5 - Y_1 \\ Z_5 - Z_1 \end{Bmatrix} \tag{41}$$

The area is

$$A = \frac{1}{2} \left|\{V_{13}\} \times \{V_{15}\}\right| \tag{42}$$

Volume

$$V = c_1 F(0,0) + c_2 F(1,0) + c_3 F(0,1) \tag{43}$$

where c_1, c_2, c_3 [see eq. (33), (34), (35)] are the constants in the thickness equation of the element [eq. (25)] and zero factorial has a value of 1. The mass at each point is

$$m = \frac{1}{6} (\rho V + A\mu) \tag{44}$$

which is $\frac{1}{6}$ the total mass.

For each point the diagonal mass matrix in element coordinate system at all the grid points is

$$[m_i] = \begin{bmatrix} m & & & & & 0 \\ & m & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ 0 & & & & & 0 \end{bmatrix} \quad i = 1, 2, \ldots, 6 \tag{45}$$

so that $[M_{ee}]$ the element mass matrix has $[m_i]$ matrices arranged diagonally.

The mass matrix in global coordinate system is obtained as

$$[M_{gg}] = [E]\,[T]^T\,[M_{ee}]\,[T]\,[E]^T \tag{46}$$

4.87.21.5 Element Load Calculations for TRIM6 (Subroutine TLØDM6)

The temperature within the element is assumed to vary bilinearly

$$T = \sum_{i=1}^{3} d_i x^{r_i} y^{s_i} \tag{47}$$

with

$$r_1 = 0\ ; \quad r_2 = 1\ , \quad r_3 = 0 \tag{48}$$

and

$$s_1 = 0\ , \quad s_2 = 0 \quad \text{and} \quad s_3 = 1 \tag{49}$$

The coefficients d_1 , d_2 and d_3 are evaluated from the specified temperatures T_{01} , T_{03} , T_{05} at the three corner grid points (obtained from the GPTT data block) and the reference temperature T_0 of the element

$$d_1 = \frac{T_{01}a + T_{03}b}{(a + b)} \tag{50}$$

$$d_2 = \frac{T_{03} - T_{01}}{(a + b)} \tag{51}$$

$$d_3 = \frac{1}{c} \left[T_{05} - d_1\right] \tag{52}$$

The constant d_1 is modified by the reference temperature, T_0, $d_1 = d_1 - T_0$. The ith element of the generalized load vector $\{P_{gen}\}$ is

$$\{P_i\}_{gen} = \sum_{k=1}^{3} \sum_{\ell=1}^{3} c_k d_\ell \Big[G_{11}^1 m_i F(m_i + r_k + t_\ell - 1, n_i + s_k + u_\ell)$$
$$+ G_{22}^1 q_i F(p_i + r_k + t_\ell, q_i + s_k + u_\ell - 1)$$
$$+ G_{33}^1 \{n_i F(m_i + r_k + t_\ell, n_i + s_k + u_\ell - 1)$$
$$+ p_i F(p_i + r_k + t_\ell - 1, q_i + s_k + u_\ell)\}\Big] \tag{53}$$

where

$$G_{11}^1 = G_{11}\,\alpha_1 + G_{12}\,\alpha_2 + G_{13}\,\alpha_{12}$$

$$G_{22}^1 = G_{12}\,\alpha_1 + G_{22}\,\alpha_2 + G_{23}\,\alpha_{12}$$

$$G_{33}^1 = G_{13}\,\alpha_1 + G_{23}\,\alpha_2 + G_{33}\,\alpha_{12}$$

The generalized equivalent load vector $\{P_{gen}\}$ is transformed to load vector $\{P_e\}$ in local element coordinates and to load vector $\{P_g\}$ in global grid-point coordinates by the following transformations

$$\{P_e\} = \left[H^{-1}\right]^T \{P_{gen}\} \tag{54}$$

$$\{P_g\} = [E]\,[T]^T\,\{P_e\} \tag{55}$$

$\{P_g\}$ is a 18 × 1 vector.

The forces are placed in the PG load vector data block.

4.87.21.6 Element Stress Calculations for TRIM6 Element (Subroutine STRM61 and STRM62 of module SDR2)

1. The relationship between strain and generalized coefficients is

$$\{\varepsilon\} = [B]\,\{a\} \tag{56}$$

where

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \tag{57}$$

The transformation from displacements to stress is·

$$[S_r] = [G_e]\,[B]\,[H^{-1}]\,[E]^T\,[T] \tag{58}$$

The temperature to stress relation is

$$\{S_t\} = -[G_e]\,\{\alpha\} \tag{59}$$

where

$$\{\alpha\} = \alpha \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{60}$$

for isotropic materials. $\{\alpha\}$ is input by the user for anisotropic materials and corrected for material angle by

$$\alpha = \left[V\right] \{\alpha_m\} \tag{61}$$

2. Calculations performed by STRM62 (Phase 2 calculations)

The equation for stress is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \left[\sum_{i=1}^{6} \left[S_i\right] \{u_g\} \right] + \{S_t\} (T_j - T_o) \tag{62}$$

where T_j is the loading temperature for the point where stress is evaluated (3 corner grid points and centroid) and is obtained from the GPTT data block. The temperature of the centroid is taken as the average of the grid point temperatures.

The principal stresses are

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} \tag{63}$$

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} \tag{64}$$

$$\Theta = \frac{1}{2} \arctan \left(\frac{2\sigma_{xy}}{\sigma_x - \sigma_y}\right) \quad \text{in degrees} \tag{65}$$

where Θ is limited to: $-90° \leq \Theta \leq 90°$

The maximum shear is

$$\tau = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} \tag{66}$$

The stresses are output for 4 points for every element. 3 corner grid points and the centroid.

4.87.22. TRPLT1 Higher Order Plate-bending Element

4.87.22.1 Input Data for TRPLT1 Element

1 EST entries for TRIBI are.

Symbol	Description
EID	Element Identification Number
$SIL_1, SIL_2, \ldots, SIL_6$	Scalar indices of connected grid points
Θ	Anisotropic material orientation angle
Mat ID_b	Material Identification Number for bending
Mat ID_5	Material Identification Number for shear
I1, I3, I5	Area moment of inertia per unit width at corner grid points $I1 = \frac{t_1^3}{12}$, $I3 = \frac{t_3^3}{12}$, $I5 = \frac{t_5^3}{12}$
TS1, TS3, TS5	Effective thickness for transverse shear at corner grid points
μ	Nonstructural mass per unit area
Z11, Z21, Z13, Z23, Z15, Z25	Distances Z1 and Z2 for stress calculation at 3 corner points
N_i, X_i, Y_i, Z_i ($i = 1, 6$)	Local coordinate system numbers and location coordinates in the basic system for the connected grid points
TEMP	Element temperature

2. Coordinate system data

The numbers N_i , X_i , and Z_i are used to calculate the 3 by 3 basic-to-global coordinate transformation matrices $[T_i]$ for points $i = 1, 2, 3, 4, 5, 6$, (via subroutines TRANSD or TRANSS).

3. Material data

Symbol	Description
$[G]$	3 × 3 stress-strain matrix
ρ	Mass density
For mat. ID_b { $\alpha_x, \alpha_y, \alpha_{xy}$	Thermal expansion coefficients
TO	Reference temperature
g_e	Structural damping coefficient
$\sigma_t, \sigma_c, \sigma_s$	Stress limits for tension, compression and shear
For mat. ID_s { G_s	Shear coefficient

4.87.22.2 Basic Equation for TRPLT1

1. The element coordinate system is defined by the following equations:

$$\{V_{13}\} = \begin{Bmatrix} X_3 - X_1 \\ Y_3 - Y_1 \\ Z_3 - Z_1 \end{Bmatrix} \tag{1}$$

$$\{V_{15}\} = \begin{Bmatrix} X_5 - X_1 \\ Y_5 - Y_1 \\ Z_5 - Z_1 \end{Bmatrix} \tag{2}$$

$$A = \frac{1}{2} \left| \{V_{13}\} \times \{V_{15}\} \right| \tag{3}$$

$$\{i\} = \frac{\{V_{13}\}}{|\{V_{13}\}|} \tag{4}$$

$$\{k\} = \frac{\{i\} \times \{V_{13}\}}{|\{i\} \times \{V_{13}\}|} \tag{5}$$

$$\{j\} = \{k\} \times \{i\} \tag{6}$$

2. The displacement transformation matrix from basic coordinates to in-plane coordinates is:

$$[E]^T = \begin{bmatrix} k_1 & k_2 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i_1 & i_2 & i_3 \\ 0 & 0 & 0 & j_1 & j_2 & j_3 \end{bmatrix} \tag{7}$$

3. The local (element) coordinate system of the element is as follows

The x-axis is obtained by joining grid points 1 and 3 of the element.

The y-axis is the perpendicular from grid point 5 to the x-axis (line joining grid points 1 and 3).

Depending upon the location of grid point 5 relative to grid points 1 and 3, three cases of triangle orientation are possible: (refer to fig. 4.87.21.1)

<u>Case I</u> Acute angles at grid points 1 and 3.

$$c = |\{i\} \times \{V_{15}\}| \tag{8}$$

$$b = \{i\} \cdot \{V_{15}\} \tag{9}$$

$$a = |\{V_{13}\}| - b \tag{10}$$

Coordinates of points are

$$x_1 = -b\ , \quad x_2 = \frac{a - b}{2}\ ; \quad x_3 = a\ , \quad x_4 = \frac{a}{2}\ , \quad x_5 = 0\ , \quad x_6 = -\frac{b}{2} \tag{11}$$

$$y_1 = 0 \ ; \quad y_2 = 0 \ ; \quad y_3 = 0 \ ; \quad y_4 = \frac{c}{2} \ ; \quad y_5 = c \ ; \quad y_6 = \frac{c}{2} \tag{12}$$

<u>Case II</u>: Obtuse angle at grid point 3:

$$c = |\{i\} \times \{V_{15}\}| \tag{13}$$

$$b = \{i\} \cdot \{V_{15}\} \tag{14}$$

$$a = b - |\{V_{13}\}| \tag{15}$$

Coordinates of points are

$$x_1 = -b \ ; \quad x_2 = -\frac{a - b}{2} \ ; \quad x_3 = 0 \ , \quad x_4 = -\frac{a}{2} \ , \quad x_5 = 0 \ , \quad x_6 = -\frac{b}{2} \tag{16}$$

$$y_1 = 0 \ ; \quad y_2 = 0 \ ; \quad y_3 = 0 \ ; \quad y_4 = \frac{c}{2} \ , \quad y_5 = c \ ; \quad y_6 = \frac{c}{2} \tag{17}$$

<u>Case III</u>: Obtuse angle at grid point 1.

$$c = |\{i\} \times \{V_{15}\}| \tag{18}$$

$$b = \{i\} \cdot \{V_{15}\} \tag{19}$$

$$a = |\{V_{13}\}| + b \tag{20}$$

Coordinates of points are:

$$x_1 = b \ , \quad x_2 = \frac{a + b}{2} \ ; \quad x_3 = a \ , \quad x_4 = \frac{a}{2} \ ; \quad x_5 = 0 \ , \quad x_6 = \frac{b}{2} \tag{21}$$

$$y_1 = 0 \ ; \quad y_2 = 0 \ , \quad y_3 = 0 \ ; \quad y_4 = \frac{c}{2} \ ; \quad y_5 = c \ , \quad y_6 = \frac{c}{2} \tag{22}$$

The matrix $[H]$ (for plates infinitely rigid in transverse shear) relating grid point displacements and the generalized coordinates (in the equation $\{u\} = [H]\{a\}$) is given by the matrix $[H]$, on the following page

5. The matrix $[B_2]$ relating curvatures (for plates infinitely rigid in transverse shear) to the generalized coordinates in the equation $\{\chi_1\} = [B_2]\{a\}$) is given by

$$[B_2] = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 12x^2 & bxy & 2y^2 & 0 & 0 & 20x^3 & 6xy^2 & 2y^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & by & 0 & 0 & 2x^2 & bxy & 12y^2 & 0 & 2x^3 & 6x^2y & 12xy^2 & 20y^3 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 0 & 6x^2 & 8xy & 6y^2 & 0 & 0 & 12x^2y & 12xy^2 & 8y^3 & 0 \end{bmatrix} \tag{24}$$

6. The matrix $[B_1]$ relating transverse shear strains $\{\gamma\}$ to the generalized coordinates (in the equation $\{\gamma\} = [B_1]\{a\}$) is a 2×20 matrix whose nonzero elements are as follows:

$$B_1(1,7) = 6A_{11} \tag{25a}$$

$$B_1(1,8) = 2A_{31} \tag{25b}$$

$$B_1(1,9) = 2A_{32} \tag{25c}$$

$$B_1(1,10) = 6A_{15} \tag{25d}$$

$$B_1(1,11) = 24A_{11}x \tag{25e}$$

$$B_1(1,12) = 6(A_{31}x + A_{11}y) \tag{25f}$$

$$B_1(1,13) = 4(A_{32}x + A_{31}y) \tag{25g}$$

$$B_1(1,14) = 6(A_{15}x + A_{32}y) \tag{25h}$$

$$B_1(1,15) = 24A_{15}y \tag{25i}$$

$$[H] = \begin{bmatrix}
1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & x_1^3 & x_1^2y_1 & x_1y_1^2 & y_1^3 & x_1^4 & x_1^3y_1 & x_1^2y_1^2 & x_1y_1^3 & y_1^4 & x_1^5 & x_1^3y_1^2 & x_1^2y_1^3 & x_1y_4^1 & y_1^5 \\
0 & 0 & 1 & 0 & x_1 & 2y_1 & 0 & x_1^2 & 2x_1y_1 & 3y_1^2 & 0 & x_1^3 & 2x_1^2y_1 & 3x_1y_1^2 & 4y_1^3 & 0 & 2x_1^3y_1 & 3x_1^2y_1^2 & 4x_1y_1^3 & 5y_1^4 \\
0 & -1 & 0 & -2x_1 & -y_1 & 0 & -3x_1^2 & -2x_1y_1 & -y_1^2 & 0 & -4x_1^3 & -3x_1^2y_1 & -2x_1y_1^2 & -y_1^3 & 0 & -5x_1^4 & -3x_1^2y_1^2 & -2x_1y_1^3 & -y_1^4 & 0 \\
1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & x_2^3 & x_2^2y_2 & x_2y_2^2 & y_2^3 & x_2^4 & x_2^3y_2 & x_2^2y_2^2 & x_2y_2^3 & y_2^4 & x_2^5 & x_2^3y_2^2 & x_2^2y_2^3 & x_2y_2^4 & y_2^5 \\
0 & 0 & 1 & 0 & x_2 & 2y_2 & 0 & x_2^2 & 2x_2y_2 & 3y_2^2 & 0 & x_2^3 & 2x_2^2y_2 & 3x_2y_2^2 & 4y_2^3 & 0 & 2x_2^3y_2 & 3x_2^2y_2^2 & 4x_2y_2^3 & 5y_2^4 \\
0 & -1 & 0 & -2x_2 & -y_2 & 0 & -3x_2^2 & -2x_2y_2 & -y_2^2 & 0 & -4x_2^3 & -3x_2^2y_2 & -2x_2y_2^2 & -y_2^3 & 0 & -5x_2^4 & -3x_2^2y_2^2 & -2x_2y_2^3 & -y_2^4 & 0 \\
1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 & x_3^3 & x_3^2y_3 & x_3y_3^2 & y_3^3 & x_3^4 & x_3^3y_3 & x_3^2y_3^2 & x_3y_3^2 & y_3^4 & x_3^5 & x_3^3y_3^2 & x_3^2y_3^3 & x_3y_3^4 & y_3^5 \\
0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & x_3^2 & 2x_3y_3 & 3y_3^2 & 0 & x_3^3 & 2x_3^2y_3 & 3x_3y_3^2 & 4y_3^3 & 0 & 2x_3^3y_3 & 3x_3^2y_3^2 & 4x_3y_3^3 & 5y_3^4 \\
0 & -1 & 0 & -2x_3 & -y_3 & 0 & -3x_3^2 & -2x_3y_3 & -y_3^2 & 0 & -4x_3^3 & -3x_3^2y_3 & -2x_3y_3^2 & -y_3^3 & 0 & -5x_3^4 & -3x_3^2y_3^2 & -2x_3y_3^3 & -y_3^4 & 0 \\
1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 & x_4^3 & x_4^2y_4 & x_4y_4^2 & y_4^3 & x_4^4 & x_4^3y_4 & x_4^2y_4^2 & x_4y_4^3 & y_4^4 & x_4^5 & x_4^3y_4^2 & x_4^2y_4^3 & x_4y_4^4 & y_4^5 \\
0 & 0 & 1 & 0 & x_4 & 2y_4 & 0 & x_4^2 & 2x_4y_4 & 3y_4^2 & 0 & x_4^3 & 2x_4^2y_4 & 3x_4y_4 & 4x_4^3 & 0 & 2x_4^3y_4 & 3x_4^2y_4^2 & 4x_4y_4^3 & 5y^4 \\
0 & 1 & 0 & -2x_4 & -y_4 & 0 & -3x_4^2 & -2x_4y_4 & -y_4 & 0 & -4x_4^3 & -3x_4^2y_4 & -2x_4y_4^2 & -y_4^3 & 0 & -5x_4^4 & -3x_4^2y_4^2 & -2x_4y_4^3 & -y_4 & 0 \\
1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 & x_5^3 & x_5^2y_5 & x_5y_5^2 & y_5^3 & x_5^4 & x_5^3y_5 & x_5^2y_5^2 & x_5y_5^3 & y_5^4 & x_5^5 & x_5^3y_5^2 & x_5^2y_5^3 & x_5y_5^4 & y_5^5 \\
0 & 0 & 1 & 0 & x_5 & 2y_5 & 0 & x_5^2 & 2x_5y_5 & 3y_5^2 & 0 & x_5^3 & 2x_5^2y_5 & 3x_5y_5^2 & 4y_5^3 & 0 & 2x_5^3y_5 & 3x_5^2y_5^2 & 4x_5y_5^3 & 5y_5^4 \\
0 & -1 & 0 & -2x_5 & -y_5 & 0 & -3x_5^2 & -2x_5y_5 & -y_5^2 & 0 & -4x_5^3 & -3x_5^2y_5 & -2x_5y_5^2 & -y_5^3 & 0 & -5x_5^4 & -3x_5^2y_5^2 & -2x_5y_5^3 & -y_5^4 & 0 \\
1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & x_6^3 & x_6^2y_6 & x_6y_6^2 & y_6^3 & x_6^4 & x_6^3y_6 & x_6^2y_6^2 & x_6y_6^3 & y_6^4 & x_6^5 & x_6^2y_6^2 & x_6^2y_6^3 & x_6y_6^4 & y_6^5 \\
0 & 0 & 1 & 0 & x_6 & 2y_6 & 0 & x_6^2 & 2x_6y_6 & 3y_6^2 & 0 & x_6^3 & 2x_6^2y_6 & 3x_6y_6^2 & 4y_6^3 & 0 & 2x_6^3y_6 & 3x_6^2y_6^2 & 4x_6y_6^3 & 5y_6^4 \\
0 & -1 & 0 & -2x_6 & -y_6 & 0 & -3x_6^2 & -2x_6y_6 & -y_6^2 & 0 & -4x_6^3 & -3x_6^2y_6 & -2x_6y_6^2 & -y_6^3 & 0 & -5x_6^4 & -3x_6^2y_6^2 & -2x_6y_6^3 & -y_6^4 & 0
\end{bmatrix}$$

$$B_1(1,16) = -120(A_{11}^2 + A_{13}A_{21} - 0.5A_{11}x^2) \quad (25j)$$

$$B_1(1,17) = -12(A_{11}A_{32} + A_{13}A_{34} + A_{38}A_{31} + A_{39}A_{33} + A_{11}A_{16} + A_{15}A_{21} - 0.5A_{32}x^2 - A_{31}xy - 0.5A_{11}y^2) \quad (25k)$$

$$B_1(1,18) = -12(A_{11}A_{15} + A_{13}A_{25} + A_{38}A_{32} + A_{39}A_{34} + A_{16}A_{31} + A_{15}A_{33} - 0.5A_{15}x^2 - A_{32}xy - 0.5A_{31}y^2) \quad (25l)$$

$$B_1(1,19) = -24(A_{15}A_{38} + A_{25}A_{39} + A_{16}A_{32} + A_{15}A_{34} - A_{15}xy - 0.5A_{32}y^2) \quad (25m)$$

$$B_1(1,20) = -120(A_{15}A_{16} + A_{15}A_{25} - 0.5A_{15}y^2) \quad (25n)$$

$$B_1(2,7) = 6A_{21} \quad (25o)$$

$$B_1(2,8) = 2A_{33} \quad (25p)$$

$$B_1(2,9) = 2A_{34} \quad (25q)$$

$$B_1(2,10) = 6A_{25} \quad (25r)$$

$$B_1(2,11) = 24A_{21}x \quad (25s)$$

$$B_1(2,12) = 6(A_{33}x + A_{21}y) \quad (25t)$$

$$B_1(2,13) = 4(A_{34}x + A_{33}y) \quad (25u)$$

$$B_1(2,14) = 6(A_{25}x + A_{34}y) \quad (25v)$$

$$B_1(2,15) = 24A_{25}y \quad (25w)$$

$$B_1(2,16) = -120(A_{11}A_{21} + A_{23}A_{21} - 0.5A_{21}x^2) \quad (25x)$$

$$B_1(2,17) = -12(A_{21}A_{32} + A_{23}A_{34} + A_{40}A_{31} + A_{41}A_{33} + A_{26}A_{11} + A_{25}A_{21} - 0.5A_{34}x^2 - A_{33}xy - 0.5A_{21}y^2) \quad (25y)$$

$$B_1(2,18) = -12(A_{21}A_{15} + A_{23}A_{25} + A_{40}A_{32} + A_{41}A_{34} + A_{26}A_{31} + A_{25}A_{33} - 0.5A_{25}x^2 - A_{34}xy - 0.5A_{33}y^2) \quad (25z)$$

$$B_1(2,19) = -24(A_{15}A_{40} + A_{25}A_{41} + A_{26}A_{32} + A_{25}A_{34} - A_{25}xy - 0.5A_{34}y^2) \quad (25aa)$$

$$B_1(2,20) = -120(A_{15}A_{26} + A_{25}^2 - 0.5A_{25}y^2) \quad (25bb)$$

where

$$\left.\begin{aligned}
A_{11} &= -(J_{11}D_{11} + J_{12}D_{13}) \\
A_{12} &= -(J_{11}D_{12} + J_{12}D_{23}) \\
A_{13} &= -(J_{11}D_{13} + J_{12}D_{33}) \\
A_{14} &= -(J_{11}D_{13} + J_{12}D_{12}) \\
A_{15} &= -(J_{11}D_{23} + J_{12}D_{22}) \\
A_{16} &= -(J_{11}D_{33} + J_{12}D_{23}) \\
A_{21} &= -(J_{12}D_{11} + J_{22}D_{13}) \\
A_{22} &= -(J_{12}D_{13} + J_{22}D_{23}) \\
A_{23} &= -(J_{12}D_{13} + J_{22}D_{33}) \\
A_{24} &= -(J_{12}D_{13} + J_{22}D_{12})
\end{aligned}\right\} \quad (25cc)$$

(continued)

$$A_{25} = -(J_{12}D_{23} + J_{22}D_{22})$$

$$A_{26} = -(J_{12}D_{33} + J_{22}D_{23})$$

$$A_{31} = A_{14} + 2A_{13}$$

$$A_{32} = A_{12} + 2A_{16}$$

$$A_{33} = A_{24} + 2A_{23}$$

$$A_{34} = A_{22} + 2A_{26}$$

$$A_{35} = A_{33} + A_{11}$$

$$A_{36} = A_{34} + A_{31} \qquad \text{(25cc) (concluded)}$$

$$A_{37} = A_{25} + A_{32}$$

$$A_{38} = A_{13} + A_{14}$$

$$A_{39} = A_{12} + A_{16}$$

$$A_{40} = A_{23} + A_{24}$$

$$A_{41} = A_{22} + A_{26}$$

7. The matrix $[B_3]$ relating $\{\chi_2\}$, the contribution of transverse shear to the vector of curvatures, to the generalized coordinates (in the equation $\{\chi_2\} = [B_3]\{a\}$) is given by

$$B_3(1,11) = -24A_{11} \qquad \text{(26a)}$$

$$B_3(1,12) = -6A_{31} \qquad \text{(26b)}$$

$$B_3(1,13) = -4A_{32} \qquad \text{(26c)}$$

$$B_3(1,14) = -6A_{15} \tag{26d}$$

$$B_3(1,16) = -120A_{11}x \tag{26e}$$

$$B_3(1,17) = -12(A_{32}x + A_{31}y) \tag{26f}$$

$$B_3(1,18) = -12(A_{15}x + A_{32}y) \tag{26g}$$

$$B_3(1,19) = -24A_{15}y \tag{26h}$$

$$B_3(2,12) = -6A_{21} \tag{26i}$$

$$B_3(2,13) = -4A_{33} \tag{26j}$$

$$B_3(2,14) = -6A_{34} \tag{26k}$$

$$B_3(2,15) = -24A_{25} \tag{26l}$$

$$B_3(2,17) = -12(A_{33}x + A_{21}y) \tag{26m}$$

$$B_3(2,18) = -12(A_{34}x + A_{33}y) \tag{26n}$$

$$B_3(2,19) = -24(A_{25}x + A_{34}y) \tag{26o}$$

$$B_3(2,20) = -120A_{25}y \tag{26p}$$

$$B_3(3,11) = -24A_{21} \tag{26q}$$

$$B_3(3,12) = -6(A_{11} + A_{33}) \tag{26r}$$

$$B_3(3,13) = -4(A_{31} + A_{34}) \tag{26s}$$

$$B_3(3,14) = -6(A_{32} + A_{25}) \tag{26t}$$

$$B_3(3,15) = -24A_{15} \tag{26u}$$

$$B_3(3,16) = -120A_{21}x \tag{26v}$$

$$B_3(3,17) = -12 \left[(A_{34} + A_{31})x + (A_{33} + A_{11})y\right] \tag{26w}$$

$$B_3(3,18) = -12 \left[(A_{25} + A_{32})x + (A_{34} + A_{31})y\right] \tag{26x}$$

$$B_3(3,19) = -24 \left[A_{15}x + (A_{32} + A_{25})y\right] \tag{26y}$$

$$B_3(3,20) = -120A_{15}y \tag{26z}$$

where A_{11} , A_{12} , . . . , A_{34} are as given in equations (25cc).

8. For plates with transverse shear flexibility the modified $[H]$ matrix, $[H']$, is given by subtracting the matrix $[B_1]$ for each of the six grid points from the respective rows of α and β of the grid points in the $[H]$ matrix.

9. For plates infinitely rigid in transverse shear,

$$[H^1] \equiv [H] \tag{27}$$

10. The two constraint equations involving the coefficeints a_{16} , a_{17} , a_{18} , a_{19} , and a_{20} of the quintic polynomial for transverse displacement so as to insure cubic edge rotation on the sloping edges of the triangular element are now entered as the 19th and 20th rows of $[H']$, i.e., the 19th and 20th rows are:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5b^4c & (3b^2c^3-2b^4c) & (2bc^4-3b^3c^2) & (c^5-4b^2c^3) & -5bc^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5a^4c & (3a^2c^3-2a^4c) & (-2ac^4+3a^3c^2) & (c^5-4a^2c^3) & 5ac^4 \end{bmatrix}$$

This is now added as the 19th and 20th row of the $[H']$ matrix to form $[H'']$ matrix. $[H'']$ is a 20 × 20 square matrix that is nonsingular [1]

[1] A numerical experiment to verify that $[H'']$ is nonsingular for all practical element sizes is described in "New Triangular and Quadrilateral Plate-bending Finite Elements" by R. Narayanaswami, NASA TN D-7407, Apr 74.

11. The $[H'']$ matrix is inverted. The first 18 columns of the $[H'']^{-1}$ matrix is denoted by the matrix $[H''']$ (Size 20 × 18), i.e.,

$$[H'''] = \text{The first 18 columns of } [H'']^{-1} \tag{28}$$

4.87.22.3 Stiffness Matrix Calculation for TRPLT1 (Subroutines KTRPLS and KTRPLD)

The polynomial expressions, for variation of w and t within the element, are

$$w = \sum_{i=1}^{20} a_i x^{m_i} y^{n_i} \tag{29}$$

$$t = \sum_{i=1}^{3} c_i x^{r_i} y^{s_i} \tag{30}$$

The values of m_i , n_i , p_i and q_i are

$$\begin{aligned} &m_1 = 0 ; \; m_2 = 1 , \; m_3 = 0 ; \; m_4 = 2 ; \; m_5 = 1 ; \; m_6 = 0 , \\ &m_7 = 3 ; \; m_8 = 2 ; \; m_9 = 1 ; \; m_{10} = 0 ; \; m_{11} = 4 ; \\ &m_{12} = 3 ; \; m_{13} = 2 , \; m_{14} = 1 ; \; m_{15} = 0 , \; m_{16} = 5 , \\ &m_{17} = 3 , \; m_{18} = 2 ; \; m_{19} = 1 ; \; m_{20} = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} &n_1 = 0 , \; n_2 = 0 ; \; n_3 = 1 , \; n_4 = 0 , \; n_5 = 1 , \; n_6 = 2 ; \\ &n_7 = 0 ; \; n_8 = 1 ; \; n_9 = 2 , \; n_{10} = 3 ; \; n_{11} = 0 , \\ &n_{12} = 1 ; \; n_{13} = 2 , \; n_{14} = 3 ; \; n_{15} = 4 , \; n_{16} = 0 ; \\ &n_{17} = 2 ; \; n_{18} = 3 , \; n_{19} = 4 ; \; n_{20} = 5 \end{aligned} \tag{32}$$

$$r_1 = 0 ; \; r_2 = 1 , \; r_3 = 0 \tag{33}$$

$$s_1 = 0 ; \; s_2 = 0 ; \; s_3 = 1 \tag{34}$$

The coefficients a_1 to a_{20} are generalized coordinates of the element and can be evaluated once the displacement vector is known.

The coefficients c_1, c_2, and c_3 can be evaluated from the specified thicknesses t_1, t_3, and t_5 of the 3 corner grid points and the geometric dimensions of the element

$$c_1 = \frac{t_1 a + t_3 b}{(a + b)} \tag{35}$$

$$c_2 = \frac{t_3 - t_1}{(a + b)} \tag{36}$$

$$c_3 = \frac{1}{c} (t_5 - c_1) \tag{37}$$

where t_1, t_3, t_5 are evaluated from the values I_1, I_3 and I_5 respectively.

The elements of the symmetric portion of the stress-strain matrix $[G_e]$ are denoted by G_{11}, G_{12}, G_{13}, G_{22}, G_{23}, and G_{33}.

A formula for the integral of the type $x^m y^n$ taken over the area of the element is

$$\iint x^m y^n \, dx \, dy = F(m,n) = c^{n+1}\{a^{m+1} - (-b)^{m+1}\} \frac{m!n!}{(m + n + 2)!} \tag{36}$$

The equation used is the stiffness matrix generation in generalized coordinates for plates infinitely rigid in transverse shear is given by

$$(k_{ij})_{gen} = \frac{1}{12} \sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} c_{k_1} c_{k_2} c_{k_3}$$

$$\Big[G_{11} m_i m_j (m_i - 1)(m_j - 1) \, F(m_i + m_j + r_{k_1} + r_{k_2} + r_{k_3} - 4,$$
$$n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3}) + G_{22} n_i n_j (n_i - 1)(n_j - 1) \, F(m_i + m_j \tag{39}$$
$$+ r_{k_1} + r_{k_2} + r_{k_3}, n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3} - 4)$$

(continued)

$$+ (4G_{33}m_i m_j n_i n_j + G_{12}\{m_i n_j (m_i - 1)(n_j - 1) + m_j n_i (m_j - 1)$$

$$\cdot (n_i - 1)\}) F(m_i + m_j + r_{k_1} + r_{k_2} + r_{k_3} - 2, n_i + n_j + s_{k_1}$$

$$+ s_{k_2} + s_{k_3} - 2) + 2G_{13}\{m_i m_j n_j (m_i - 1) + m_i n_i m_j (m_j - 1)\} F(m_i \qquad (39)\ \text{(concluded)}$$

$$+ m_j + r_{k_1} + r_{k_2} + r_{k_3} - 3, n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3} - 1)$$

$$+ 2G_{23}\{m_j n_i n_j (n_i - 1) + m_i n_i n_j (n_j - 1)\} F(m_i + m_j + r_{k_1} + r_{k_2}$$

$$+ r_{k_3} - 1, n_i + n_j + s_{k_1} + s_{k_2} + s_{k_3} - 3)\Big]$$

For plates with transverse shear flexibility, the expression for generalized stiffness matrix consists not only of the closed form expression but four additional integrals, given below, that are evaluated using numerical integration, i.e.,

$$[K_{gen}] = [K_{gen}]_{\substack{(\text{eq. } 39) \\ \text{closed form}}} + \iint [B_2]^T [D] [B_3] \, dx \, dy$$

$$+ \iint [B_3]^T [D] [B_2] \, dx \, dy + \iint [B_3]^T [D] [B_3] \, dx \, dy \qquad (40)$$

$$+ \iint [B_1]^T [G_s] [B_1] \, dx \, dy$$

$[D]$ matrix is obtained from the stress-strain matrix $[G_e]$ as

$$[D] = \frac{1}{12} [G_e] \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} c_i c_j c_k \, x^{r_i + r_j + r_k} \, y^{s_i + s_j + s_k} \right) \qquad (41)$$

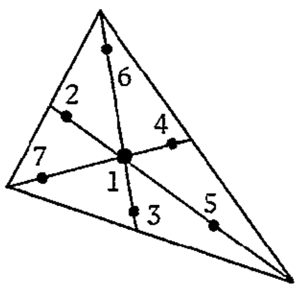
$$[G_s] = G \, t^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (42)$$

where t^* is the effective thickness for shear at the integration point and is evaluated from the user specified values TS1, TS3, and TS5. The numerical integration formulae used are the seven-point integration scheme listed in Zienkiewicz[1] and are given below.

For a triangle, the integrals of the form

$$I = \int_0^1 \int_0^{1-L} f(L_1 L_2 L_3)\, dL_1\, dL_2 = \sum_{k=1}^{7} w_k f_k (L_1 L_2 L_3)$$

where the points (L_1, L_2, L_3) and the weighting factors are as follows



Point	Triangular Coordinates L_1, L_2, L_3	Weight, $2W_k$
1	1/3, 1/3, 1/3	0.225
2	α_1 β_1 β_1	
3	β_1 α_1 β_1	0.13239415
4	β_1 β_1 α_1	
5	α_2 β_2 β_2	
6	β_2 α_2 β_2	0.12593928
7	β_2 β_2 α_2	

with

$\alpha_1 = 0.05971588$ $\qquad$ $\beta_1 = 0.47014206$

$\alpha_2 = 0.79742699$ $\qquad$ $\beta_2 = 0.101286505$

The stiffness matrix in global coordinates is

$$[\bar{k}] = [E]\,[T]^T\,[H''']^T\,[k]_{gen}\,[H''']\,[T]\,[E]^T \tag{43}$$

[1] O. C. Zienkiewicz, "Finite Element Method on Engineering Science," New York, London McGraw-Hill, 1971.

4.87.22.4 Mass Matrix Calculation for TRPLT1 (Calculated in Stiffness Subroutines KTRPLS and KTRPLD)

Two different mass matrices are used: the lumped mass and the consistent mass. The lumped mass matrix is calculated in the same manner as for TRIM6:

$$m = \frac{1}{6}\,(\rho V + A\mu) \tag{44}$$

where

$$\begin{aligned} V, \text{ the volume of the element} &= c_1 F(0,0) + c_2 F(1,0) \\ &\quad + c_3 F(0,1) \end{aligned} \tag{45}$$

For each point, the diagonal mass matrix in element coordinates at all the grid points is

$$\left[m_i\right] = \begin{bmatrix} 0 & & & & & \\ & 0 & & & 0 & \\ & & m & & & \\ & 0 & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \quad i = 1, 2, \ldots, 6 \tag{46}$$

so that $\left[M_e\right]$ the element mass matrix has $\left[m_i\right]$ matrices arranged diagonally.

The mass matrix in the global coordinate system is obtained as

$$\left[M_{gg}\right] = \left[E\right]\left[T\right]^T\left[M_{ee}\right]\left[T\right]\left[E\right]^T \tag{47}$$

If the parameter COUPMASS is set by the user, the consistent mass matrix will be formed. The jth element of the ith row of generalized mass matrix is given by

$$[M_{ij}]_{gen} = \rho \iint \sum_{k=1}^{3} c_k \, x^{m_i+m_j+r_k} \, y^{n_i+n_j+s_k} \, dx \, dy$$

$$+ \mu \iint x^{m_i+m_j} \, y^{n_i+n_j} \, dx \, dy \tag{48}$$

$$= \rho \sum_{k=1}^{3} \{c_k \, F(m_i + m_j + r_k, n_i + n_j + s_k)\}$$

$$+ \mu \, F(m_i + m_j, n_i + n_j) \tag{49}$$

The mass matrix in global coordinates is

$$[M] = [E] [T]^T [H'''] ^T [M_{gen}] [H'''] [T] [E]^T \tag{50}$$

4.87.22.5 Structural Damping Matrices for the TRPLT1 Element

The structural damping matrices are

$$[k^4_{ij}] = g_e [k^g_{ij}] \tag{51}$$

where g_e is the structural damping coefficient for the bending material referenced.

4.87 22.6 Stress and Element Force Calculations for the TRPLT1 Element (Subroutines STRP11 and STRP12 of Modules SDR2)

1. STRP11 is used to calculate the phase 1 stress-displacement relations.

Frequent reference will be made to the equations from sections 4.87 22.2 and 4 87.22.3

The following data are calculated

1. $[H''']$ - 20 × 18 Matrix relating generalized coordinates to grid point displacements.

2. $[B_2]$ - 3 × 20 Matrix relating bending curvatures and generalized coordinates.

3. $[B_3]$ - 2 × 20 Matrix relating curvature contribution of transverse shear strains and generalized coordinates

4. $[E]$ - element to basic coordinate transformation.

5. $[D]$ - 3 × 3 Matrix of elastic coefficients relating bending moments and curvatures.

6. $[G_s]$ - 2 × 2 Matrix relating transverse shear forces and shear strains.

7. $[T_i]$ - i = 1, 2, . . . , 6 - Global to basic transformations.

The following calculations are performed.

$$[S_M^*] = [D]\,[B_2]\,[H''] \tag{52}$$

$[S_M^*]$ is a 3 × 18 matrix; this is split into six 3 × 3 matrix partitions as follows:

$$[S_M^*] = \left[\, S_{M_1}^* \,\middle|\, S_{M_2}^* \,\middle|\, \cdots \,\middle|\, S_{M_6}^* \,\right] \tag{53}$$

Each of the six matrix partitions is multiplied as follows:

$$\left[S_{M_i}\right] = \left[S_{M_i}^*\right][E]^T\,[T_i] \qquad i = 1, 2, \ldots, 6 \tag{54}$$

$$[S_G^*] = [G_s]\,[B_3]\,[H''] \tag{55}$$

$[S_G^*]$ is a 2 × 18 matrix, this is split into six 2 × 3 matrix partitions as follows:

$$[S_G^*] = \left[\, S_{G_1}^* \,\middle|\, S_{G_2}^* \,\middle|\, \cdot \,\middle|\, S_{G_6}^* \,\right] \tag{56}$$

Each of the six matrix partitions is multiplied as follows:

$$\left[S_{G_i}\right] = \left[S^*_{G_i}\right] \left[E\right]^T \left[T_i\right] \qquad i = 1, 2, \ldots, 6 \tag{57}$$

The 5 × 6 matrix $\left[S_i\right]$ is obtained as

$$\left[S_i\right] = \begin{bmatrix} S_{M_i} \\ \hline S_{G_i} \end{bmatrix} \qquad i = 1, 2, \ldots, 6 \tag{58}$$

2. Phase 2

(a) The vector of forces is computed as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ V_x \\ V_y \end{Bmatrix} = \left(\sum_{i=1}^{6} \left[S_i\right] \{u_i\} \right) - \{M_t\} \tag{59}$$

where $\{M_t\}$ is the thermal moment vector If the thermal gradient is specified,

$$\{M_t\}_i = -\left[G_e\right]\{\alpha_e\} I_i T'_i \tag{60}$$

where I_i is the moment of inertia of the cross section and T'_i is the thermal gradient at vertex i of the element.

The stresses and forces are evaluated at the vertices of the element, in addition, the stresses are also evaluated at the centroid. The simplification is made that the thermal moment vector at the centroid is the average of that at the vertices.

(b) With no given temperatures at the stress points, the stresses are then calculated from the equations

$$\left.\begin{aligned} \sigma_x &= \frac{M_x z}{I} \\ \sigma_y &= \frac{M_y z}{I} \\ \sigma_{xy} &= \frac{M_{xy} z}{I} \end{aligned}\right\} \qquad (61)$$

M_x , M_y , M_{xy} and I for the appropriate points (vertices or centroid); z values for the corner grid points are as those given in the PTRPLT1 card and for the centroid, the z values are the top and bottom fibre distances.

If mean temperature T_0 and the gradient T' are specified at the three vertices,

$$\begin{Bmatrix} \sigma_{x_i} \\ \sigma_{y_i} \\ \sigma_{xy_i} \end{Bmatrix} = -\frac{z_i}{I}\left[\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} + [D]\{\alpha\}T' - \frac{1}{I}(T_0 - \overline{T})[D]\{\alpha\}\right] \qquad i = 1, 2 \qquad (62)$$

where $\overline{T}$ is the average temperature of the element.

The principal stresses and angles are calculated using the same formula as for the membrane element TRIM6 (section 4.87.21 6).

4.87.22.7 Thermal Load Calculations for the Bending Elements (Subroutine TLODT1, TLODT2 and TLODT3 of Module SSG1)

The variation over the surface of the element of the mean temperature, T_0 , and the thermal gradient at a cross section, T' , is assumed as a bilinear polynomial

$$T_0 = \sum_{i=1}^{3} d_i \, x^{p_i} y^{q_i} \qquad (63)$$

$$T' = \sum_{i=1}^{3} d_i^1 \, x^{p_i} y^{q_i} \tag{64}$$

so that the temperature at any point (x, y, z) is $T = T_0 + T'$.

The constants d_i and d'_i are evaluated from the values at the vertices and the reference temperature T_0 of the element

$$d_1 = \frac{T_{01}a + T_{03}b}{(a + b)} - T_0 \;;\quad d_2 = \frac{T_{03} - T_{01}}{(a + b)} \;;\quad d_3 = \frac{1}{c}\left[T_{05} - d_1\right] \tag{66}$$

$$d_1^1 = \frac{T_1^1 a + T_3^1 b}{(a + b)} \;,\quad d_2^1 = \frac{T_3^1 - T_1^1}{(a + b)} \;;\quad d_3^1 = \frac{1}{c}\left[T_5^1 - d_1^1\right] \tag{67}$$

It is convenient to define the elements of $\left[G_e\right]\{\alpha_e\}$ as

$$G'_{11} = G_{11}\alpha_{e_1} + G_{12}\alpha_{e_2} + G_{13}\alpha_{e_3} \tag{68}$$

$$G'_{22} = G_{12}\alpha_{e_1} + G_{22}\alpha_{e_2} + G_{23}\alpha_{e_3} \tag{69}$$

$$G'_{33} = G_{13}\alpha_{e_1} + G_{23}\alpha_{e_3} + G_{33}\alpha_{e_3} \tag{70}$$

The thermal load vector in generalized coordinates, $\{P^t_{gen}\}$, will be evaluated in two stages, viz., the closed form expression $\{P^t_{gen}\}_1$, due to the vector of curvatures in the absence of transverse shear and the numerically integrated expression $\{P^t_{gen}\}_2$ due to the contribution of transverse shear to the vector of curvatures. The ith element of $\{P^t_{gen}\}_1$ is given by

$$\left[\{P^t_{gen}\}_1\right]_i = \frac{1}{12} \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \sum_{i_3=1}^{3} \sum_{j=1}^{3} c_{i_1} c_{i_2} c_{i_3} d'_j \Big[G'_{11} m_i (m_i - 1)\, F(m_i + r_{i_1} + r_{i_2} + r_{i_3} + p_j - 2,\, n_i + s_{i_1} + s_{i_2} + s_{i_3} + q_j) + G'_{22} n_i (n_i - 1)\, F(m_i + r_{i_1} + r_{i_2} + r_{i_3} + p_j,\, n_i + s_{i_1} + s_{i_2} + s_{i_3} + q_j - 2) \tag{71}$$

(continued)

$$+ G'_{33} m_i n_l F(m_l + r_{l_1} + r_{k_2} + r_{k_3} + p_j - 1, n_l + s_{l_1} + s_{l_2} + s_{l_3} + q_j - 1)\Big] \quad (71) \text{ (concluded)}$$

The load vector $\{P^t_{gen}\}_2$ is evaluated using numerical integration of the following expression.

$$\{P^t_{gen}\}_2 = \frac{1}{12} \iint [B_3]^T [G_e] \{\alpha_e\} T' t^3 \, dx \, dy \quad (72)$$

The generalized thermal load vector is

$$\{P^t_{gen}\} = \{P^t_{gen}\}_1 + \{P^t_{gen}\}_2 \quad (73)$$

The thermal load vector in global coordinates is

$$\{P^t\}_g = [E] [T]^T [H'']^T \{P^t_{gen}\} \quad (74)$$

The forces are placed in the PG load vector data block.

4.87.23. TRSHL. Shallow Shell Triangular Element

4.87.23.1 Input Data for TRSHL Element

1. EST entries for TRSHL are

Symbol	Description
EID	Element Identification Number
SIL1, . . . SIL6	Scalar indices of connected grid points
Θ	Anisotropic material orientation angle
Mat ID_m	Material-Identification Number for membrane behavior
T_1, T_3, T_5	Membrane thickness at corner grid points
Mat ID_b	Material-Identification Number for bending
I_1, I_3, I_5	Area moments of inertia at corner grid points
Mat ID_s	Material-Identification Number for transverse shear
TS1, TS3, TS5	Thickness for transverse shear at corner grid points
μ	Nonstructural mass per unit area
Z11, Z21, Z13, Z23 Z15, Z25	Distances Z1 and Z2 for stress calculations at three corner grid points
N_i, X_i, Y_i, Z_i } $i = 1, \ldots 6$	Local coordinate system numbers and location of coordinates in the basic system for the connected grid points
TE1, TE2, . . TE6	Element temperature at the six grid points

2. Coordinate system data

The numbers N_i , X_i , and Z_i are used to calculate the 3 by 3 basic-to-global coordinate transformation matrices T_i for points $i = 1, 2, 3, 4, 5, 6$, (via subroutines TRANSD or TRANSS).

3. Material data

	Symbol	Description
For mat. ID_m	$[G]$	3 × 3 stress-strain matrix
	ρ	Mass density
	α_x, α_y, α_{xy}	Thermal expansion coefficients
	TO	Reference temperature
	g_e	Structural damping coefficient
	σ_t, σ_c, σ_s	Stress limits for tension, compression and shear
For mat. ID_b	D	3 × 3 bending stress-strain matrix
For mat. ID_s	G_s	Shear coefficient

4.87.23.2 Basic Equation for TRSHL

The calculations for the TRSHL element are very similar to those of TRIM6 and TRPLT1 (sections 4.87.21.2 and 4.87.22.2 respectively) that only the essential details are given here.

The displacement transformation matrix from basic coordinates to in-plane coordinates is

$$[E]^T = \begin{bmatrix} i_1 & i_2 & i_3 & 0 & 0 & 0 \\ j_1 & j_2 & j_3 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i_1 & i_2 & i_3 \\ 0 & 0 & 0 & j_1 & j_2 & j_3 \end{bmatrix} \qquad (1)$$

where i, j and k are as given by equations (4), (5) and (6) of section 4.87.22.2. No transverse shear effects are considered for the TRSHL element.

The matrix $[H]$ relating grid point displacements and the generalized coordinates (in the equation $\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [H]\{a\}$) is similar to that for TRIM6 and TRPLT1. The inverse matrix relating generalized coordinates to the grid-point displacement vector is a 32 × 30 matrix and is given by

$$[H]^{-1} = \left[\begin{array}{c|c|c} \begin{matrix} H_I^{-1} \\ 6\times 6 \end{matrix} & 0 & \\ \hline & \begin{matrix} H_I^{-1} \\ 6\times 6 \end{matrix} & \\ \hline & & \begin{matrix} H''' \\ 20\times 18 \end{matrix} \end{array}\right] \tag{2}$$

where H_I is a 6 × 6 matrix given in equation (21), section 4 87.21.2 and H''' is a 20 × 18 matrix given in equation (28) of section 4.87 22 2.

4.87.23.3 Stiffness Matrix Calculation for TRSHL (Subroutine KTSHLS and KTSHLD)

The polynomial expressions for variation of u, v, w and thickness t within the element are

$$u = \sum_{i=1}^{32} a_i \, x^{m_i} \, y^{n_i} \tag{3}$$

$$v = \sum_{i=1}^{32} b_i \, x^{p_i} \, y^{q_i} \tag{4}$$

$$w = \sum_{i=1}^{32} c_i \, x^{v_i} \, y^{s_i} \tag{5}$$

$$t_m = \sum_{i=1}^{3} d_i \, x^{t_i} \, y^{u_i} \tag{6}$$

$$t_b = \sum_{i=1}^{3} d'_i \, x^{t'_i} \, y^{u'_i}$$

The values of m_i , n_i , p_i , q_i , v_i , s_i , t_i , u_i , t'_i and u'_i are

$m_i(i = 1,32)$: 0, 1, 0, 2, 1, 0, 26*0

$n_i(i = 1,32)$: 0, 0, 1, 0, 1, 2, 26*0

$p_i(i = 1,32)$: 6*0, 0, 1, 0, 2, 1, 0, 20*0

$q_i(i = 1,32)$: 6*0, 0, 0, 1, 0, 1, 2, 20*0

$v_i(i = 1,32)$: 12*0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 0, 4, 3, 2, 1, 0, 5, 3, 2, 1, 0 (7)

$s_i(i = 1,32)$: 12*0, 0, 0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, 0, 2, 3, 4, 5

$t_i(i = 1,3)$: 0, 1, 0; t'_i:0, 1, 0, s_i:0, 0, 1; s'_i·0, 0,1

The coefficients a_i , b_i and c_i are undetermined parameters such that

$a_i = 0$ $i = 7$ to 32

$b_i = 0$ $i = 1$ to 6 and 13 to 32 (8)

$c_i = 0$ $i = 1$ to 12

a_i, $i = 1$ to 6; b_i, $i = 7$ to 12, and c_i, $i = 13$ to 32 can be determined once the element displacement vector is known.

The coefficients d_i and d'_i, $i = 1$ to 3 are evaluated from the user specified values of the membrane thickness and the area moments of inertia, respectively, by equations similar to those for TRIM6 and TRPLT1 elements.

The equation used in the stiffness matrix generation in generalized coordinates is (following the procedure outlined in sections 5.8.6 and 5.8.7), the jth column of the ith row of the generalized stiffness matrix is obtained as

$$
\begin{aligned}
K_{ij} = \sum_{k=1}^{3} \Big[& G_{11}\Big(m_i\, m_j\, d_k\, F(m_i + m_j + t_k - 2,\ n_i + n_j + u_k) \\
& \quad - h_4\, m_i\, d_k\, F(m_i + r_j + t_k - 1,\ n_i + s_j + u_k) \\
& \quad - h_4\, m_j\, d_k\, F(m_j + r_i + t_k - 1,\ n_j + s_i + u_k) \\
& \quad + h_4^2\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k)\Big) \\
& + G_{22}\Big(q_i\, q_j\, d_k\, F(p_i + p_j + t_k,\ q_i + q_j + u_k - 2) \\
& \quad - h_6\, q_i\, d_k\, F(p_i + r_j + t_k,\ q_i + s_j + u_k - 1) \\
& \quad - h_6\, q_j\, d_k\, F(r_i + p_j + t_k,\ s_i + q_j + u_k - 1) \\
& \quad + h_6^2\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k)\Big) \\
& + G_{33}\Big(n_i\, n_j\, d_k\, F(m_i + m_j + t_k,\ n_i + n_j + u_k - 2) \\
& \quad + n_i\, p_j\, d_k\, F(m_i + p_j + t_k - 1,\ n_i + q_j + u_k - 1) \\
& \quad - h_5\, n_i\, d_k\, F(m_i + r_j + t_k,\ n_i + s_j + u_k - 1) \\
& \quad + p_i\, n_j\, d_k\, F(p_i + m_j + t_k - 1,\ q_i + n_j + u_k - 1) \\
& \quad + p_i\, p_j\, d_k\, F(p_i + p_j + t_k - 2,\ q_i + q_j + u_k) \\
& \quad - h_5\, p_i\, d_k\, F(p_i + r_j + t_k - 1,\ q_i + s_j + u_k) \\
& \quad - h_5\, n_j\, d_k\, F(r_i + m_j + t_k,\ s_i + n_j + u_k - 1)
\end{aligned}
\tag{9}
$$

(continued)

$$
\begin{aligned}
&- h_5\, p_j\, d_k\, F(r_i + p_j + t_k - 1,\ s_i + q_j + u_k) \\
&+ h_5^2\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k)\Bigr) \\
+\, G_{12}\Bigl(&m_i\, q_j\, d_k\, F(m_i\ p_j + t_k - 1,\ n_i + q_j + u_k - 1) \\
&- h_6\, m_i\, d_k\, F(m_i + r_j + t_k - 1,\ n_i + s_j + u_k) \\
&- h_4\, q_j\, d_k\, F(r_i + p_j + t_k,\ s_i + q_j + u_k - 1) \\
&+ 2h_4\, h_6\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k) \\
&+ q_i\, m_j\, d_k\, F(p_i + m_j + t_k - 1,\ q_i + n_j + u_k - 1) \\
&- h_4\, q_i\, d_k\, F(p_i + r_j + t_k,\ q_i + s_j + u_k - 1) \\
&- h_6\, m_j\, d_k\, F(r_i + m_j + t_k - 1,\ s_i + n_j + u_k)\Bigr) \\
+\, G_{13}\Bigl(&m_i\, n_j\, d_k\, F(m_i + m_j + t_k - 1,\ n_i + n_j + u_k - 1) \\
&+ m_i\, p_j\, d_k\, F(m_i + p_j + t_k - 2,\ n_i + q_j + u_k) \qquad (9) \\
&- h_5\, m_i\, d_k\, F(m_i + r_j + t_k - 1,\ n_i + s_j + u_k) \\
&- h_4\, n_j\, d_k\, F(r_i + m_j + t_k,\ s_i + n_j + u_k - 1) \\
&- h_4\, p_j\, d_k\, F(r_i + p_j + t_k - 1,\ s_i + q_j + u_k) \\
&+ 2h_4\, h_5\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k) \\
&+ n_i\, m_j\, d_k\, F(m_i + m_j + t_k - 1,\ n_i + n_j + u_k - 1) \\
&- h_4\, n_i\, d_k\, F(m_i + r_j + t_k,\ n_i + s_j + u_k - 1) \\
&+ p_i\, m_j\, d_k\, F(p_i + m_j + t_k - 2,\ q_i + n_j + u_k) \\
&- h_4\, p_i\, d_k\, F(p_i + r_j + t_k - 1,\ q_i + s_j + u_k) \\
&- h_5\, m_j\, d_k\, F(r_i + m_j + t_k - 1,\ s_i + n_j + u_k)\Bigr) \\
+\, G_{23}\Bigl(&q_i\, n_j\, d_k\, F(p_i + m_j + t_k,\ q_i + n_j + u_k - 2) \\
&+ q_i\, p_j\, d_k\, F(p_i + p_j + t_k - 1,\ q_i + q_j + u_k - 1)
\end{aligned}
$$

(continued)

$$- h_5\, q_i\, d_k\, F(p_i + r_j + t_k,\ q_i + s_j + u_k - 1)$$

$$- h_6\, n_j\, d_k\, F(r_i + m_j + t_k,\ s_i + n_j + u_k - 1)$$

$$- h_6\, p_j\, d_k\, F(r_i + p_j + t_k - 1,\ s_i + q_j + u_k)$$

$$+ 2h_5\, h_6\, d_k\, F(r_i + r_j + t_k,\ s_i + s_j + u_k)$$

$$+ n_i\, q_j\, d_k\, F(m_i + p_j + t_k,\ n_i + q_j + u_k - 2)$$

$$- h_6\, n_i\, d_k\, F(m_i + r_j + t_k,\ n_i + s_j + u_k - 1)$$

$$+ p_i\, q_j\, d_k\, F(p_i + p_j + t_k - 1,\ q_i + q_j + u_k - 1)$$

$$- h_6\, p_i\, d_k\, F(p_i + r_j + t_k - 1,\ q_i + s_j + u_k)$$

$$- h_5\, q_j\, d_k\, F(r_i + p_j + t_k,\ s_i + q_j + u_k - 1)\Big)\Big]$$

$$+ \sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} \Big[\frac{1}{12}\ d'_{k_1} d'_{k_2} d'_{k_3}\ (G_{11}\, r_i\, r_j\, (r_i - 1)(r_j - 1)$$

$$\cdot F(r_i + r_j + t'_{k_1} + t'_{k_2} + t'_{k_3} - 4,\ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3}) \qquad (9)$$

$$+ G_{22}\, s_i\, s_j\, (s_i - 1)(s_j - 1)\ F(r_i + r_j + t'_{k_1} + t'_{k_2} + t'_{k_3}, \qquad \text{(concluded)}$$

$$+ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 4)$$

$$+ \Big(4G_{33}\, r_i\, r_j\, s_i\, s_j + G_{12}\{r_i\, s_j\, (r_i - 1)(s_j - 1)$$

$$+ r_j\, s_i\, (r_j - 1)(s_i - 1)\}\Big)\ F(r_i + r_j + t'_{k_1} + t'_{k_2} + t'_{k_3} - 2,$$

$$+ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 2)$$

$$+ 2G_{13}\{r_i\, r_j\, s_j\, (r_i - 1) + r_i\, r_j\, s_i\, (r_j - 1)\}\ F(r_i + r_j$$

$$+ t'_{k_1} + t'_{k_2} + t'_{k_3} - 3,\ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 1)$$

$$+ 2G_{23}\{r_j\, s_i\, s_j\, (s_i - 1) + r_i\, s_i\, s_j\, (s_j - 1)\}\ F(r_i + r_j$$

$$+ t'_{k_1} + t'_{k_2} + t'_{k_3} - 1,\ s_i + s_j + u'_{k_1} + u'_{k_2} + u'_{k_3} - 3)\Big]$$

The generalized stiffness matrix can be transformed to the element and global coordinates by transformations similar to those for TRIM6 and TRPLT1 elements.

4.87.23.4 Mass Matrix Calculation for the TRSHL Element (Calculated in the stiffness subroutine KTSHLS and KTSHLD)

Two different mass matrices are calculated: lumped mass and consistent mass. The calculations are the same as for TRIM6 and TRPLT1.

4.87.23.5 Structural Damping Matrices for the TRSHL Element

The calculations are similar to those for TRIM6 and TRPLT1 elements.

4.87.23.6 Stress and Element Force Calculations for TRSHL Element (Subroutines STRSL1, STRSLV and STRSL2 of module SDR2)

The calculations are similar to those of TRIM6 and TRPLT1 elements.

4.87.23.7 Thermal Load Calculations for the TRSHL Element (Subroutines TLØDSL of module SSG1)

The calculations are similar to those for TRIM6 and TRPLT1 elements.

4.87.23.8 Differential Stiffness Matrix Calculations for the TRSHL Element (Subroutine DTSHLD of module SSG1)

The steps leading to the calculations of the differential stiffness matrix are given in section 7.3.6 of the theoretical manual (pages 66 to 70 of this report).

APPENDIX D

Updates to the NASTRAN Demonstration Problem Manual for the addition of TRIM6, TRPLT1 and TRSHL elements

Demo Problem 1.3-4(a)

Analysis of a Free Rectangular Plate with thermal loading using higher order triangular membrane TRIM6 element The quarter section of the plate shown in figure 1, is discretized using TRIM6 element. Discretization is given on page 1.3-4(b), figure 3(a).

The graphs for the measured stresses σ_x and σ_y at $x = 1.5$ shown on pages 1.3-5 and 1.3-6. The results obtained by this analysis are not included in the same graph, since for the chosen mesh the stresses are evaluated at locations different from those shown in the graph. However, good agreement is seen for the stresses for the chosen mesh.

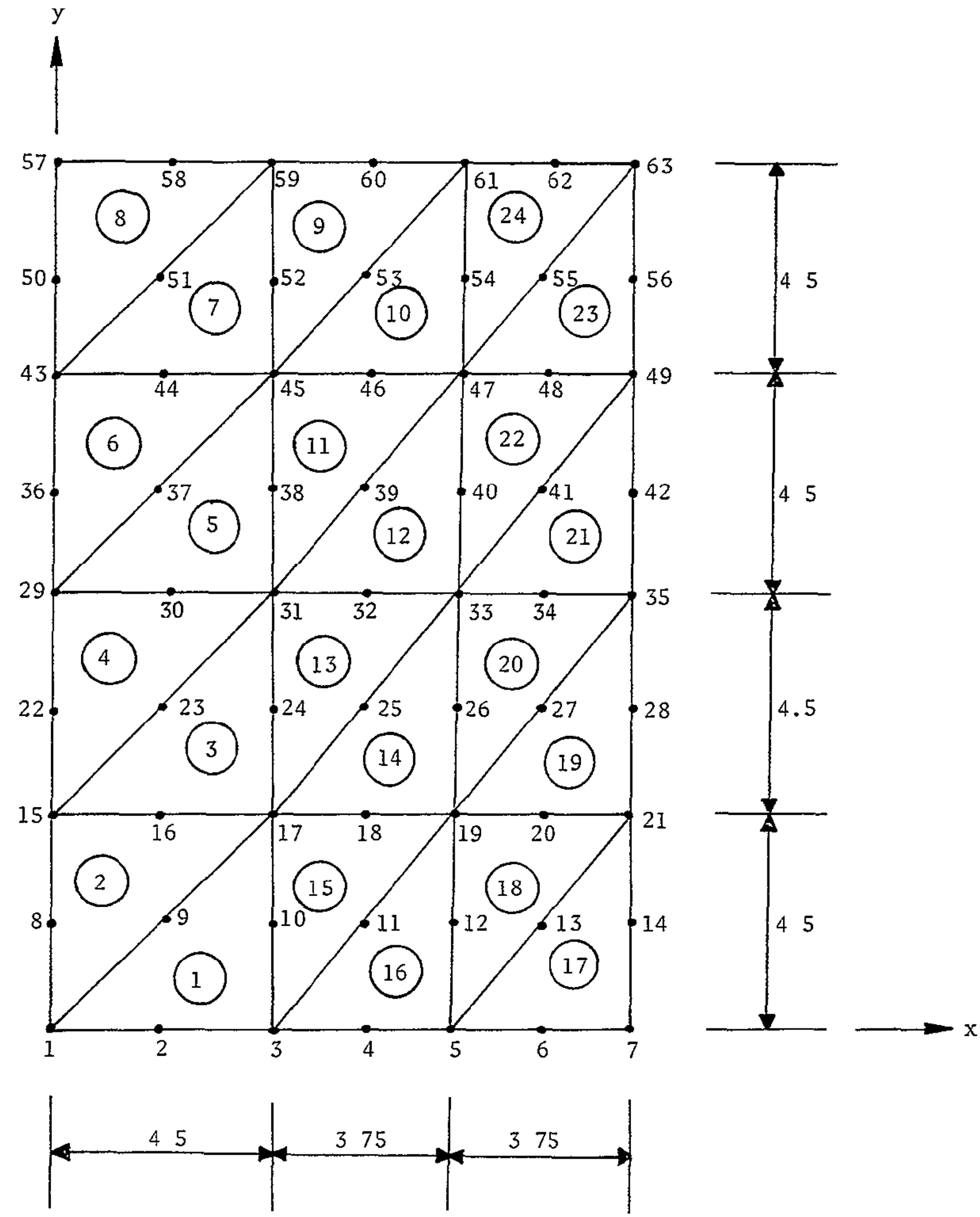


Figure 3(a). Model of free rectangular plate using TRIM6 element.
1 3-4(b) (1/1/76)

Demo Problem 1.7-5

Triangular Shallow Shell Element

Two problems, (1) that of a spherical cap, and (2) that of a cylindrical shell roof, are considered. These are the same two example problems analyzed in reference 23.

(1) The spherical cap, with finite element discretization is shown in figure 5. Due to symmetry, only one fourth (quarter) of the cap was analyzed.

Good agreement in deflections at the center of the cap is obtained even with relatively coarse mesh sizes as shown in Table 1. Even though the results appear to be oscillating about the exact value, the percentage error in the converged solution is very negligible.

(2) The shell is shown in figure 6 along with pertinent dimensions and associated material properties. The finite element discretization for the shell is shown in the same figure. Due to symmetry, only one fourth (quarter) of the shell was analyzed.

Results for the shell roof problem and the exact solution reported by Cowper et al (ref. 23) are given in Table 2. Reasonable agreement is seen between the finite element and the exact solutions.

The values given in Table 2 are obtained from a stand-alone program wherein the global stiffness matrix had 5 d.o.f. per grid-point, viz., u , v , w , α and β . This is consistent with shallow shell theory. In NASTRAN, the global stiffness matrix has 6 d.o.f. per grid point, viz., u , v , w , α , β and γ . It is necessary therefore to constrain the sixth degree of freedom at all grid points where all the elements connected to that grid point are in the same plane. This requirement is to ensure that the global stiffness matrix is nonsingular for a given sufficiently supported condition of the structure. Theoretically, however, the above requirement is equivalent to the introduction of additional constraints on the problem and hence the solution obtained from NASTRAN will be lower bounds to the actual values obtained from the stand alone program and given in table 1. The values obtained from NASTRAN and CTRSHL elements are given in table 2. For shells that are strictly shallow, the solution from NASTRAN will approach that obtainable from stand alone programs based on a strict application of shallow shell theory.

An alternative to solve problems where shell is only marginally shallow, as the example discussed herein, is to use combination of TRIM6 and TRPLT1 elements. The result of using a 2 × 4 and 3 × 3 mesh of CTRIM6 and CTRPLT1 elements from NASTRAN is given in table 4. Note that the values are very close to the exact values.

Table 1. Center deflections for spherical cap problem.

	Deflection $\delta_c = \frac{Etw_c}{P_o R^2}$	
Finite Element Grid	$\frac{Rt}{L^2} = 0.02$	$\frac{Rt}{L^2} = 0.005$
1 × 1	1.15107	1.13951
2 × 2	1.00774	0.99178
3 × 3	1.00452	1.00177
4 × 4	1.00437	1.00084
Exact	1.00978	1.00043

Table 2. Results for a cylindrical shell roof from stand-alone program (values are in element coordinates).

Finite Element Grids	$10u_A$ (in.)	w_B (in.)	$10v_B$ (in.)	$10w_C$ (in.)	$10^{-3}N_{xxB}$ (lb./in.)	$10^{-3}M_{yyC}$ (lb. in./in.)	$10^{-2}M_{xxC}$ (lb. in./in.)
1 × 1	-0.45168	-0.29100	-2.48424	-4.0700	2.4659	0.7685	2.8520
2 × 2	-0.7812	-12516	-4.77312	-2.1344	4.2801	-0.9395	-0.8896
3 × 3	-1.09590	-2.49876	-7.12872	-1.3606	5.4948	-2.0283	-1.1136
4 × 4	-1.2939	-3.4332	-8.57580	2.2224	6.0277	-2.3828	-1.7912
2 × 4	-0.9041	-2.2815	-6.15	0.888	5.1312	-1.415	-2.0196
3 × 6	-1.1244	-3.6227	-8.5968	3.1031	6.1862	-1.9414	-1.8912
4 × 8	-1.3845	-4.1526	-9.5295	3.9238	6.4839	-2.0459	-1.6724
5 × 5	-1.4160	-3.88152	-9.29000	2.8182	6.3279	-2.3538	-1.9770
6 × 6	-1.4733	-4.09176	-9.76992	3.0900	6.4444	-2.3242	-2.0638
Exact	-1.51325	-4.09916	-8.76147	5.2494	6.4124	-2.0562	-0.9272

Table 3. Results for cylindrical shell roof from NASTRAN using CTRSHL elements (values are in global coordinates).

Finite Element Grids	U_A (in.)	W_B (in.)	V_B (in.)	W_C (in.)
2 × 4	-0.0945	-1.6437	-0.5181	0.1938
3 × 3	-0.09054	-1.7309	-0.4801	0.3813
Exact	-0.151325	-3.70331	-1.96372	0.52494

Table 4. Results for cylindrical shell roof from NASTRAN using CTRIM6 and CTRPLT1 elements (values are in global coordinates).

Finite Element Grids	U_A (in.)	W_B (in.)	V_B (in.)	W_C (in.)
2 × 4	-0.1233	-3.4162	-1.7445	0.4287
3 × 3	-0.1335	-4.2560	-2.1226	1.1007
Exact	-0.151325	-3.70331	-1.96372	0.52494

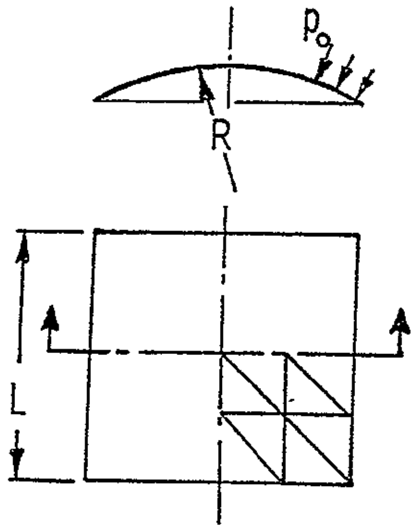


Figure 5.

I 7-11 (1/1/77)

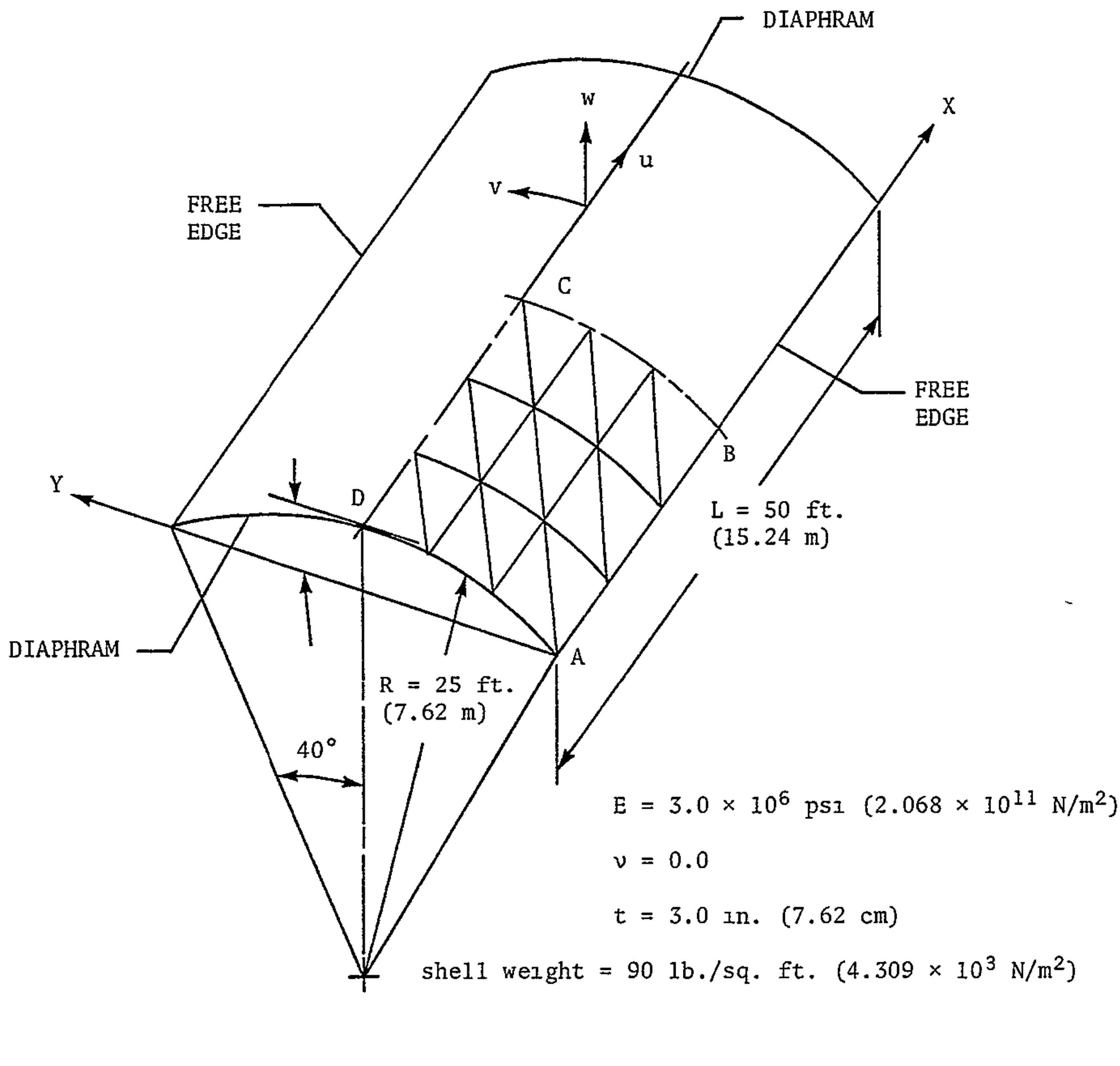


Figure 6. Geometry of cylindrical shell roof and finite element idealization

Demo Problem 1.8-5

Analysis of a Beam Using TRIM6

A. Description

The cantilever beam shown on page 1.8-4 is modeled with the NASTRAN TRIM6 element as shown in figure 2, page 1.8-7. This problem demonstrates the analysis of a beam subdivided into the six noded triangular membrane elements.

The loads were chosen to approximate the stress distribution due to a moment on one end of a beam. The other end is constrained to resist the moment. The plane of symmetry is not used.

B. Input

1. Parameters Similar to those listed on page 1 8-1.
2. Boundary Constraints. on $x = 0$ plane, $u_x = u_y = 0$.
3. Loads· total moment = $M_y = 2.048 \times 10^3$. This moment will produce bending about the z-axis. It is modeled by a set of axial loads at $x = \ell$ which, in turn, represents an axial stress distribution: $\sigma_x = 1.5y$.
4. Subcase - 1 Consistent loading.
5. Subcase - 2: Lumped loading.
   Considering strip near extreme fiber
   $F_x = \frac{1}{2} (1.5 \times 6 + 1.5 \times 8)\ 2 \quad 4 = 84$

C. Analysis and Results

Analysis. refer to pages 1 8-1 and 1.8-2

Results

Comparisons of Displacement

Grid Pts.	Theory (10^{-4}) $y = \frac{Mx^2}{2EI}$	Consistent Loading Subcase 1 (10^{-4})	Lumped Loading Subcase 2 (10^{-4})
3	0	0	0
13	.0625	.0515	.0523
23	.25	.2377	.2467
33	.5625	.549	.5744
43	1.0	.985	1.0172

Comparisons of Stress

Figures 3(a) and 3(b) show stresses obtained from the analysis.

Referring to figure 3a, subcase 1, we have stress at

Grid point 5 = 9.567 (C)

Grid point 3 = $\frac{1.64 + 1.31 + 1.31}{3}$ = 1.4 (C)

Grid point 1 = $\frac{11.7 + 15.46}{2}$ = 13.58 (T)

Referring to figure 3b, subcase 2, we have stress at

Grid point 5 = 9.87 (C)

Grid point 3 = $\frac{1.64 + 1.35 + 1.35}{3}$ = 1.48 (C)

Grid point 1 = $\frac{12.1 + 15.96}{2}$ = 14.03 (T)

If the cantilever beam is discretized with the same type of mesh and the same number of elements, but the diagonal oriented in the opposite direction to figure 2, i.e., as shown in figure 4, then the stresses for subcase 1, at grid points 5, 3, and 1 would be

Grid point 5 = 13.58 (C)

Grid point 3 = 1.4 (C)

Grid point 1 = 9.567 (T)

Therefore the stresses at grid points 5, 3, and 1 are taken as the average of the two types of meshes (i.e., figure 2, and figure 4).

Therefore, for subcase 1, we have stresses at

$$\text{Grid point 5} = \frac{9.567 + 13.58}{2} = 11.575 \text{ (C)}$$

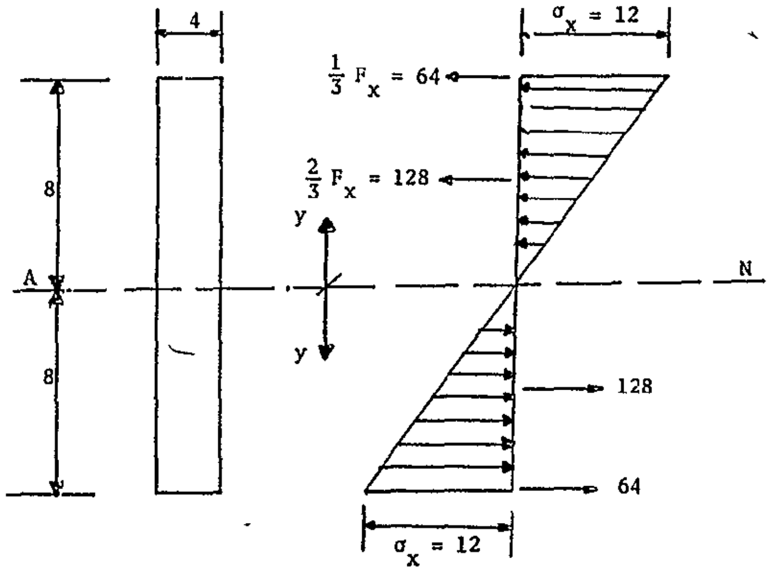
$$\text{Grid point 3} = \frac{1.4 + 1\ 4}{2} = 1.4 \text{ (C)}$$

$$\text{Grid point 1} = \frac{13.58 + 9.567}{2} = 11.575 \text{ (T)}$$

Conclusion

Stresses at Grid Point	Theory	NASTRAN TRIM6
5	12	11.575
3	0	1.4
1	12	11.575

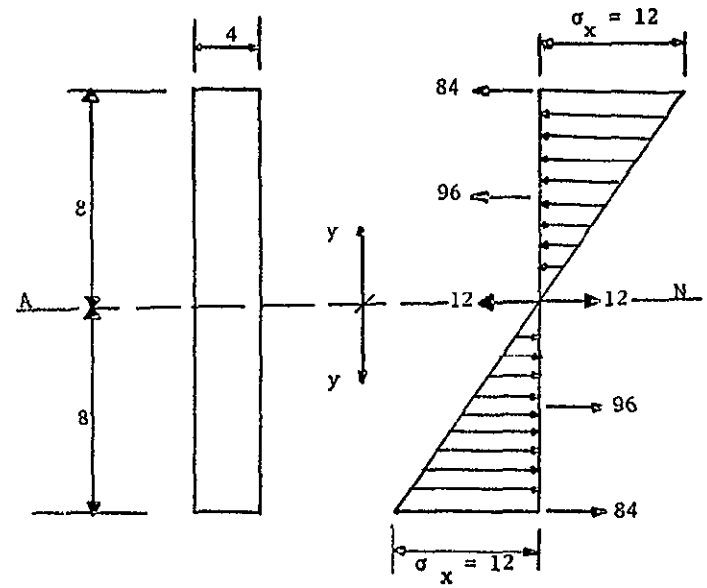
Subcase 1: Consistent Loading



at $y = 8$, $\sigma_x = 1.5y = 1.5 \times 8 = 12$

Total Axial Force $= F_x = \frac{1}{2} \times 8 \times 12 \times 4 = 192$

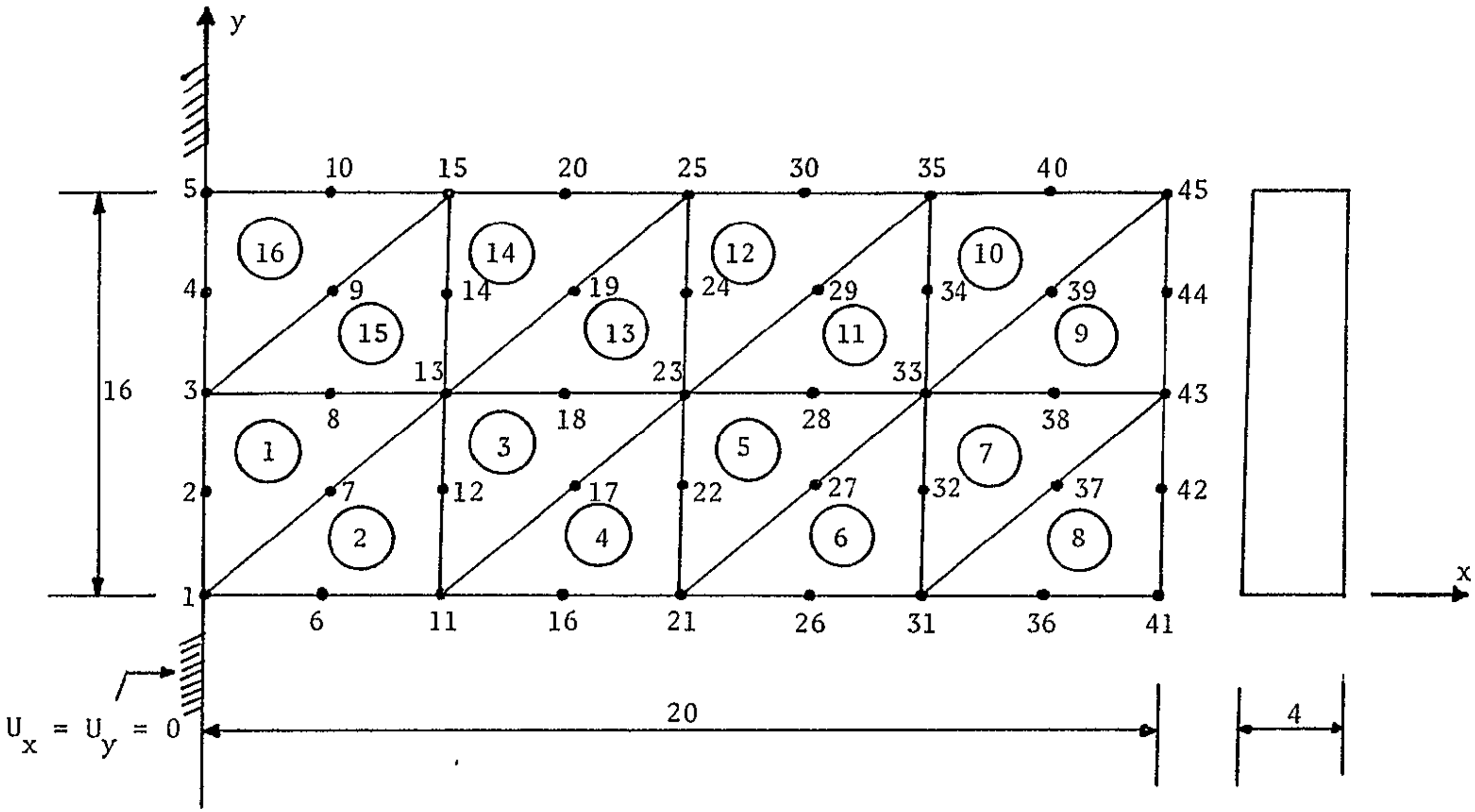
Subcase 2. Lumped Loading



Total Axial Force $= F_x = \frac{1}{2} \times 8 \times 12 \times 4 = 192$

Considering strip near N.A., $F_x = \frac{1}{2} \times 2 \times (1.5 \times 2)4 = 12$

Considering strip in between N.A. and extreme fiber $F_x = \frac{1}{2}(1.5 \times 2 + 1.5 \times 6)4 \times 4 = 96$


y
x
16
20
4
$U_x = U_y = 0$
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2
3
4
5
6
7
8
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14
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16
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Figure 2

Subcase 1

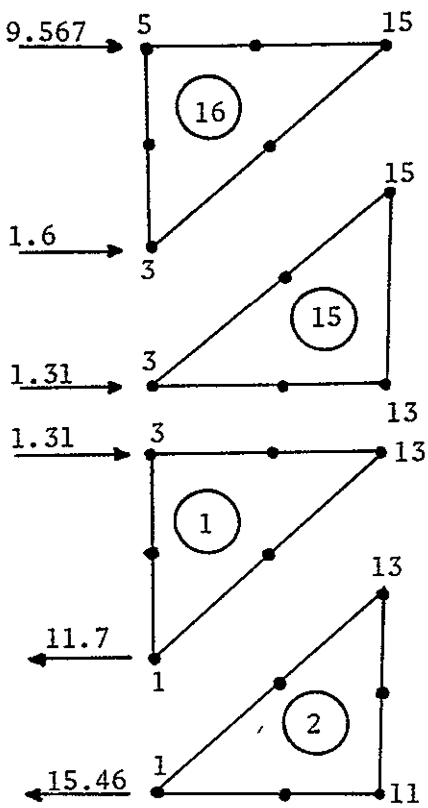


Figure 3a

Subcase 2

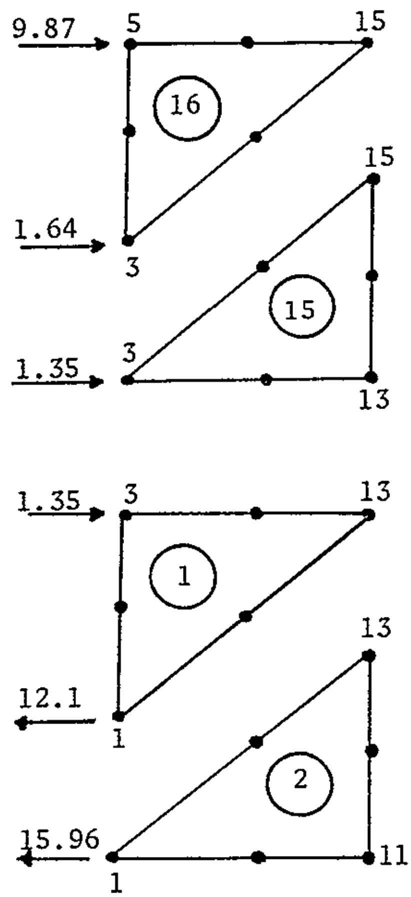
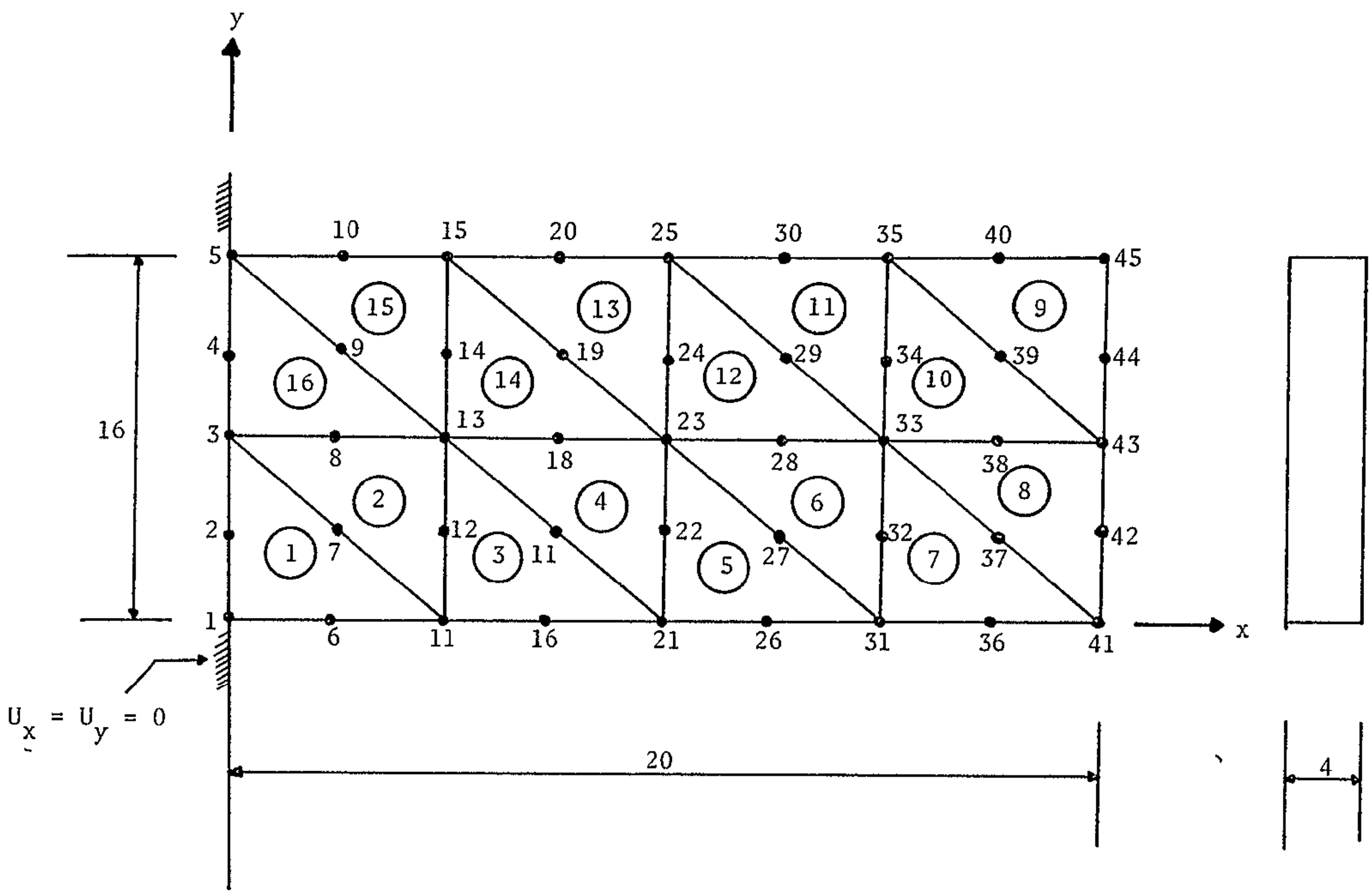


Figure 3b


y
x
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20
4
$U_x = U_y = 0$
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Figure 4

Demo Problem 1.9-4

Thermal and Applied Loads on TRIM6 Elements

A. Description

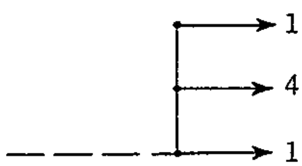
This problem demonstrates the use of the TRIM6 elements. Ten triangular membrane elements are used to model a 2 × 1 × 10 beam. The dimensions and boundary conditions are shown in figure 2. Two loading conditions are applied: axial stress and thermal expansion. Symmetry boundary conditions are used.

B. Input

1. Parameters. similar to those listed on page 1.9-1.
2. Boundary Constraints: $u_x = u_y = 0$ at $x = 0$, $u_y = 0$ at $y = 0$.
3. Loads: Subcase 1, consistent loading

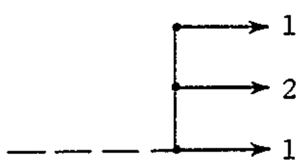
   $F_x = 24 \times 10^3$ (total axial force)

   Total force on symmetric part $= \frac{24}{2} = 12$



Force divided into the ratio of 1:4 1, i.e., $\frac{1 \times 12}{6}$, $\frac{4 \times 12}{6}$, $\frac{1 \times 12}{6}$

Subcase 3, Lumped loading



Force divided into the ratio of 1 2.1, i.e., $\frac{1 \times 12}{4}$, $\frac{2 \times 12}{4}$, $\frac{1 \times 12}{4}$

Subcase 2, Thermal loading

$T = 60°$ (Uniform temperature field)

$T_o = 10°$ (Reference temperature)

C. Analysis and Results

Analysis: refer to page 1 9-2.

Results:

x		TRIM6 Sol. (10^{-3})		Subcase 2	
	Exact Sol. (10^{-3})	Subcase 1	Subcase 3	Exact Sol.	TRIM6 Sol.
0	0	0	0	0	0
2	1	0.98	0.98	0.1	0.109
4	2	1.98	1.98	0.2	0.2093
6	3	2.98	2.981	0.3	0.3093
8	4	3.98	3.98	0.4	0 4093
10	5	4.98	4.981	0.5	0.5093
12	6	5.98	5.981	0.6	0.6093
14	7	6.98	6 98	0.7	0.7093
16	8	7.98	7.98	0.8	0.8093
18	9	8.98	8.99	0.9	0.9093
20	10	9.98	10.026	1 0	1.00937

Displacement (U_x)

Graph is given on pages 1.9-7 and 1 9-8

Conclusion

The results of all three subcases are exact to the single precision limits.

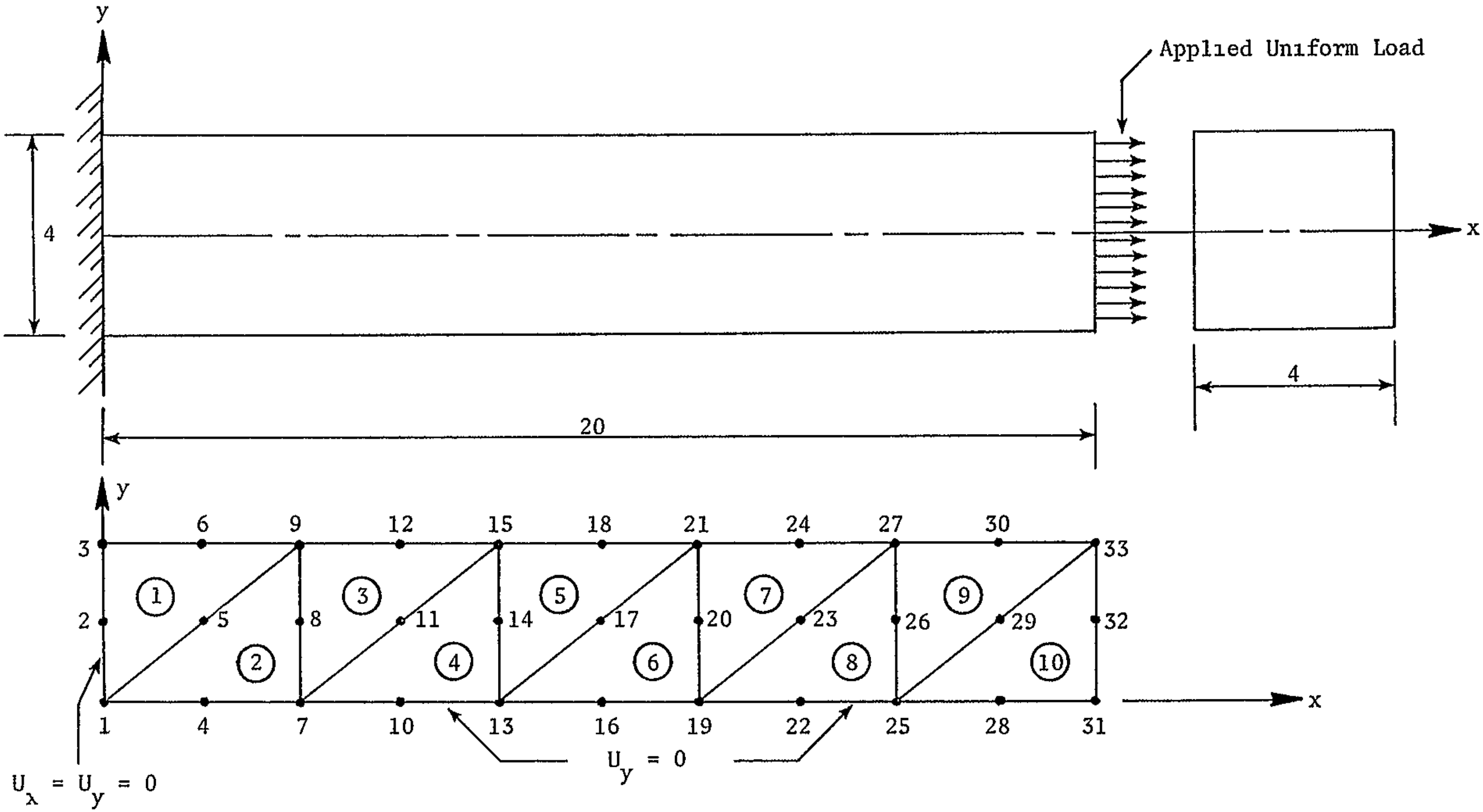
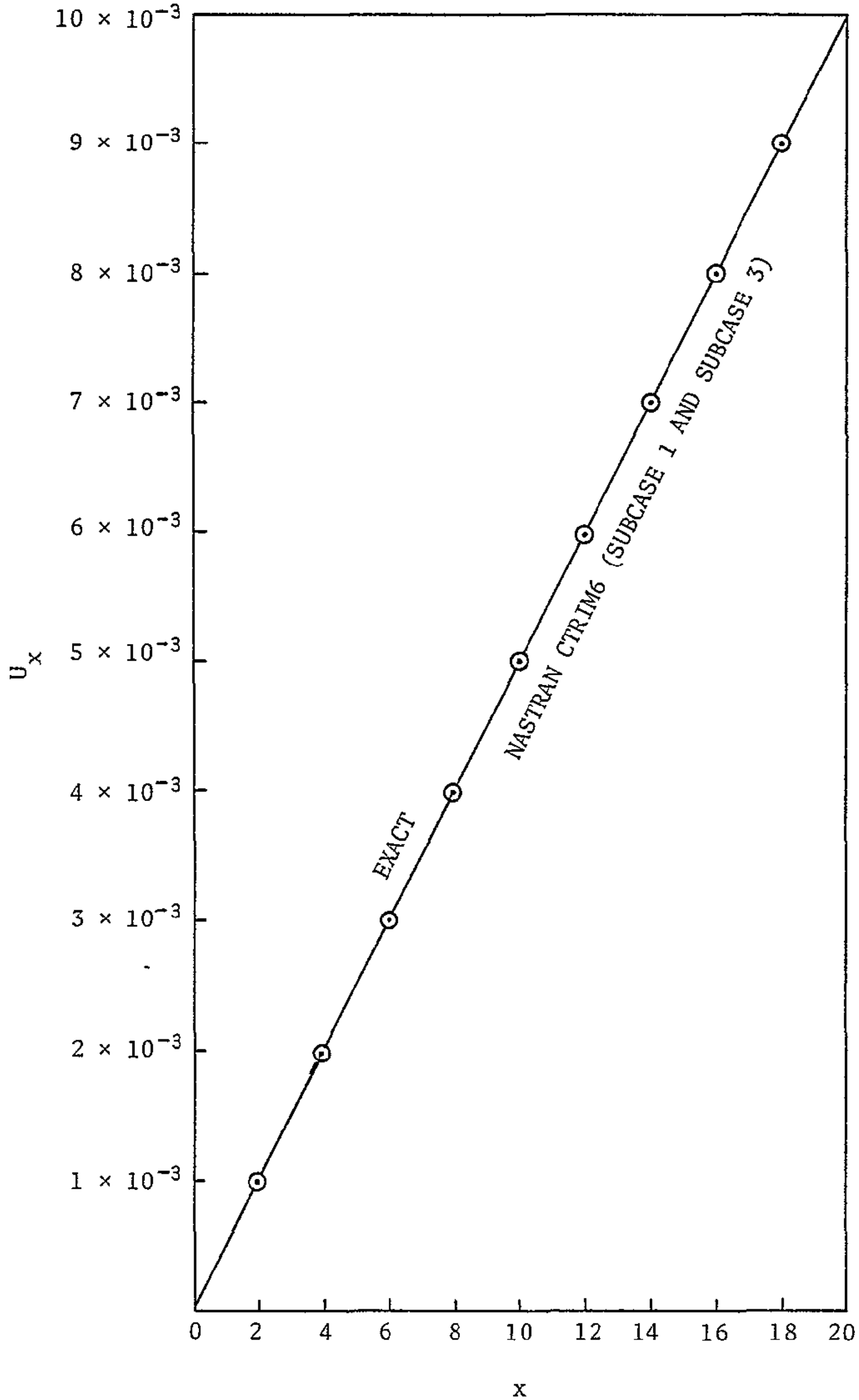
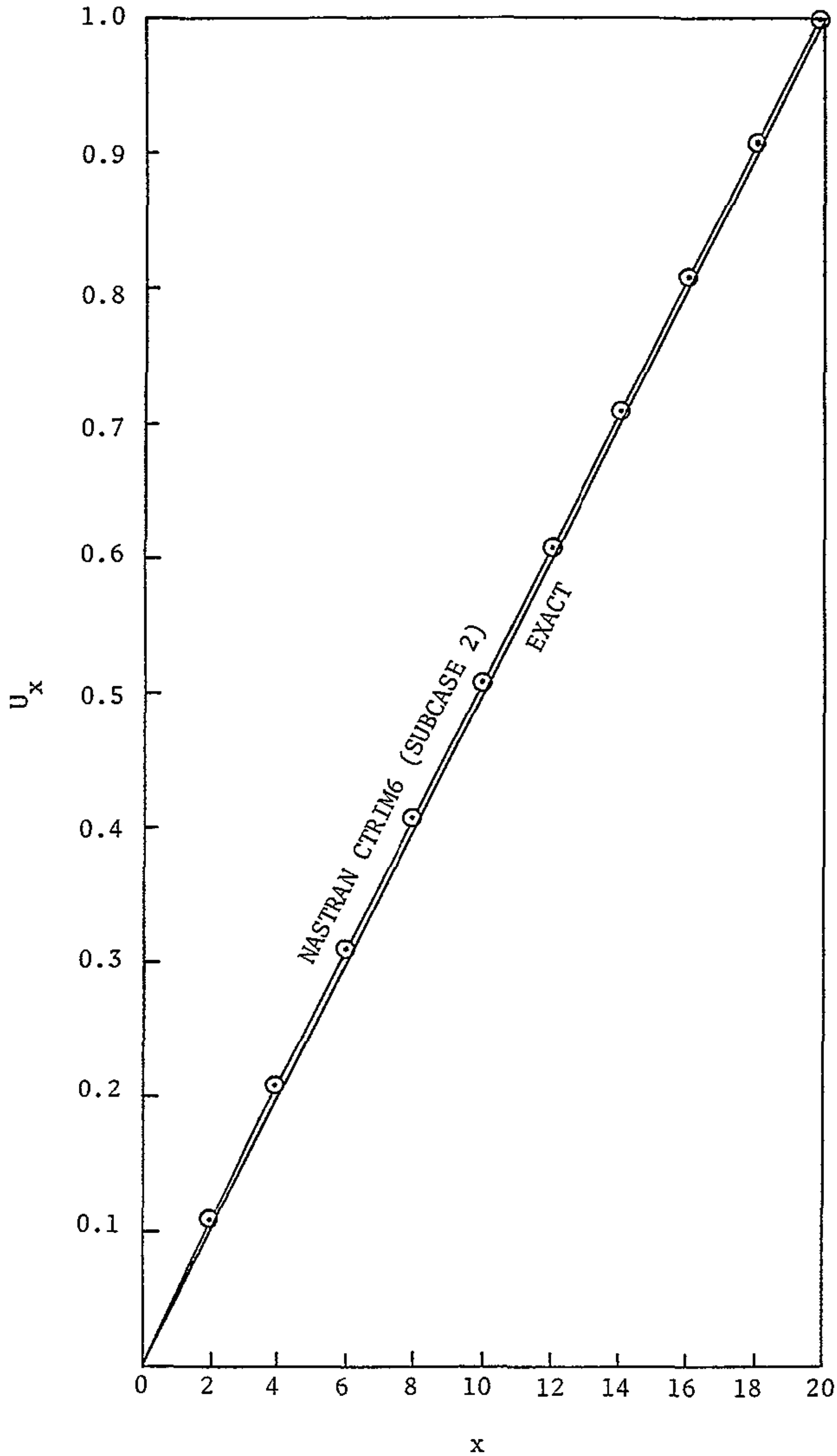


Figure 2. Model of Cantilever beam using TRIM6 element.



Deflection U_x for subcase-1 and subcase-3.

1 9-7 (1/1/77)



Deflection U_x for subcase 2.

1 9-8 (1/1/77)

Demo Problem 1.11-4(a)

Analysis of a simply supported rectangular plate with a thermal gradient using higher order triangular bending TRPLT1 elements: The quarter section of the plate is discretized using TRPLT1 element. Discretization is given on page 1.11-4(b), figure 2.

For input and theory, refer to pages 1.11-1, 1.11-2, and 1.11-3.

Result:

The maximum displacement obtained was 0.5935, a difference of about 5 percent from the analytical value. A more refined mesh is likely to yield closer values to the exact ones.

The graph for the moments M_x, M_y and M_{xy} obtained by analysis at $x = 0.5$ is shown on page 1.11-6. The results obtained by this analysis is not included in the same graph, since for the chosen mesh the moments are evaluated at locations different from those shown in the graph. However, good agreement is seen for the moments for the chosen mesh.

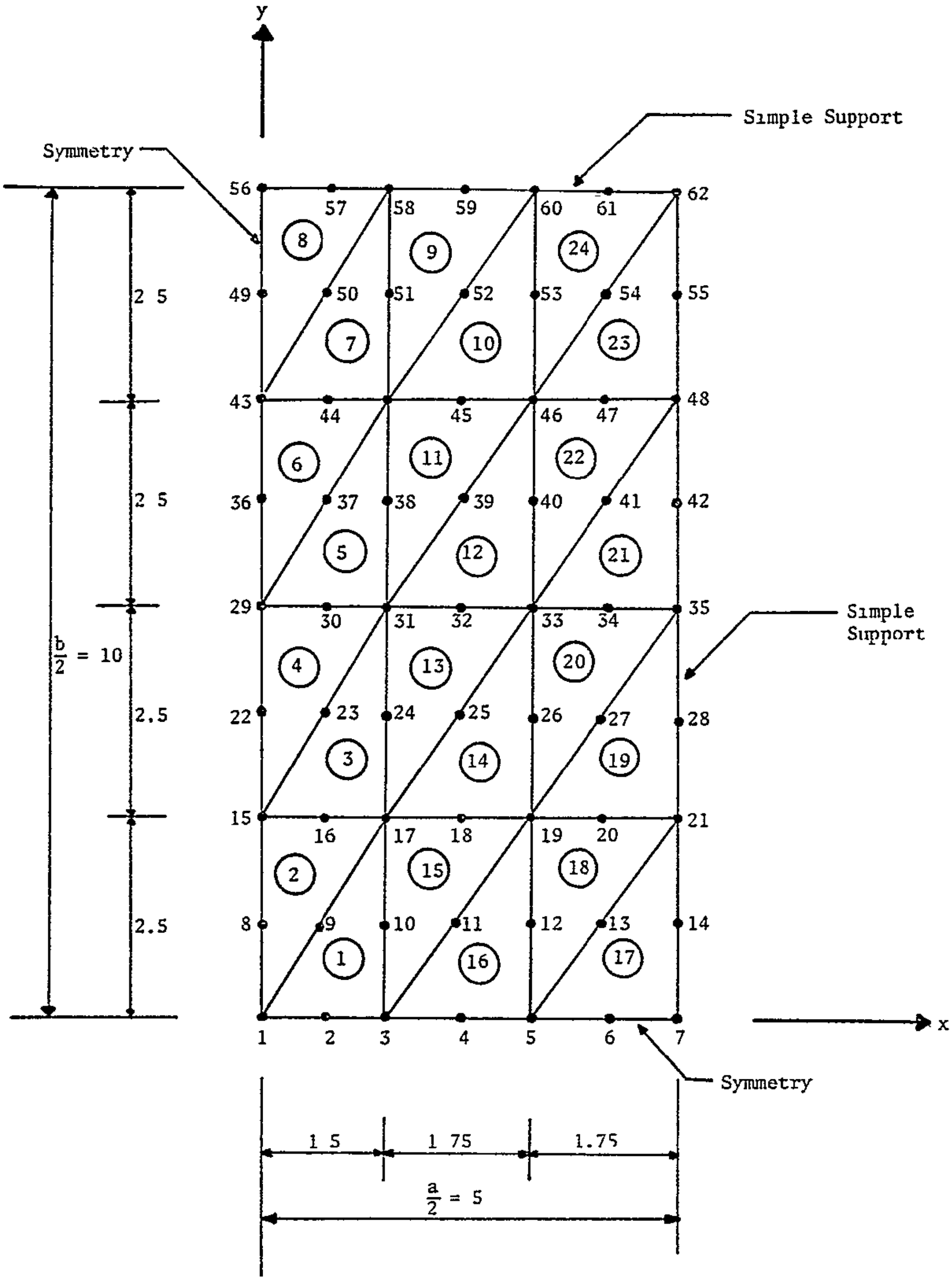


Figure 2. Model of simply supported rectangular plate using TRPLT1 element.

1.11-4(b) (1/1/77)

Demo Problem 1.11-8

Deflection of Thick Rectangular Plate Using TRPLT1 Element

The TRPLT1 element is used for solving moderately thick to thick plate where the effects of transverse shear are present.

The rectangular plate in figure 1, shown on page 1 11-11, is discretized by using higher order triangular bending element TRPLT1. Because of symmetry, the quarter section of the plate is discretized and details of the discretization are given in the same figure.

Four different mesh sizes are used for Q-mesh and P-mesh The result is tabulated for the simply supported and clamped edge with two different $\frac{a}{t}$ ratios, on page 1.11-9 and 1.11-10 respectively.

Input:

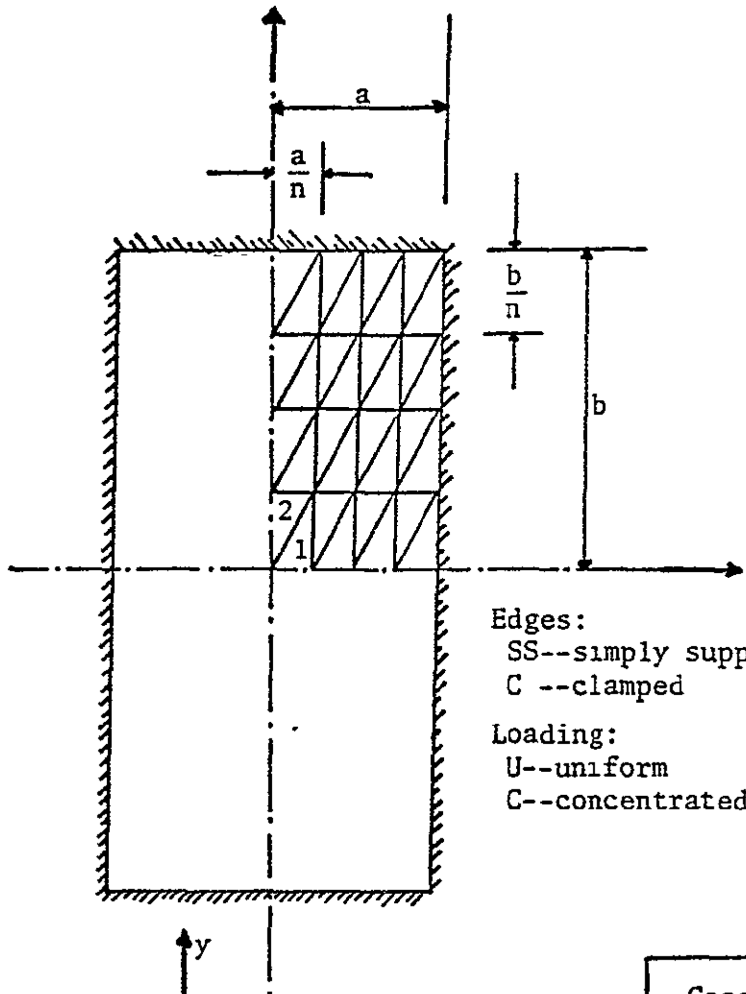
$E = 3.0 \times 10^7$ lbs/in.^{2}	(Young's modulus)
$\nu = 0.3$	(Poisson's ratio)
$q = 1000$ lbs/in.^{2}	(Uniform distributed load)
$q = 1000$ lbs.	(Concentrated load at center)
$\frac{b}{a} = 2$	(Length/width)
$t = 1.0$ in.	(Thickness of the plate)

Table 1. Central deflection of simply supported rectangular plates (Q-mesh) including the effects of transverse shear.

Number of Elements per Side N	Concentrated Load at Center		Uniformly Distributed Load	
	$\frac{a}{t} = 100$	$\frac{a}{t} = 4$	$\frac{a}{t} = 100$	$\frac{a}{t} = 4$
2	21.3373	22.22	10.983	11.2651
4	17.7854	20.0133	10.2084	10.5612
8	16.9276	19.4859	10.1396	10.4831
12	16.73	19.5322	10.1330	10.5468

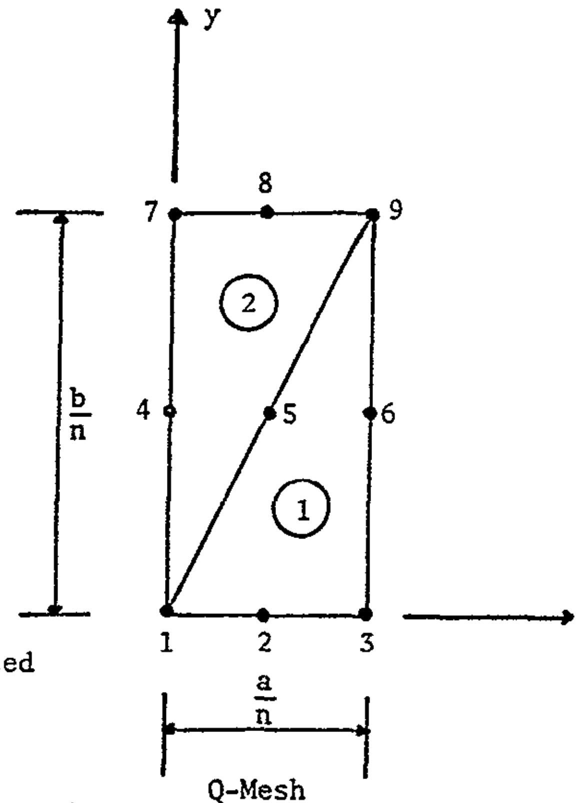
Table 2. Central deflection of clamped rectangular plates (Q-mesh) including the effects of transverse shear.

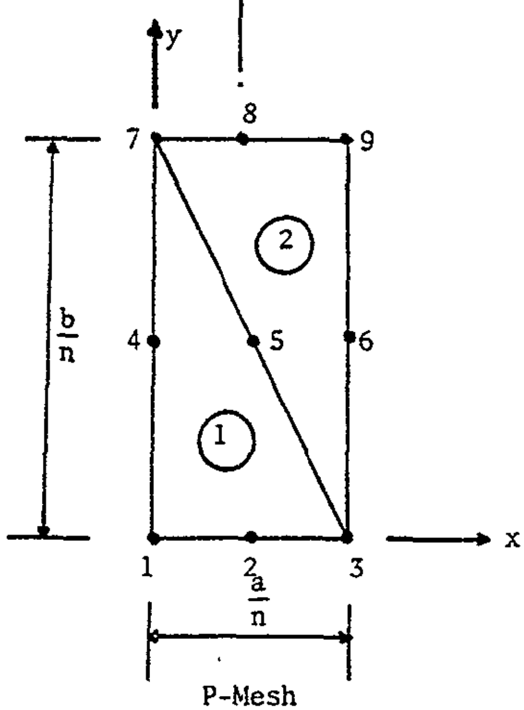
Number of Elements per Side N	Concentrated Load at Center		Uniformly Distributed Load	
	$\frac{a}{t} = 100$	$\frac{a}{t} = 4$	$\frac{a}{t} = 100$	$\frac{a}{t} = 4$
2	10.4330	12.5582	3.942	4.5086
4	8.4230	10.565	2.7932	3.133
8	7.6283	10 0845	2.5953	2.9375
12	7.4336	10.1791	2.561	2 9475



Edges:
SS--simply supported
C --clamped

Loading:
U--uniform
C--concentrated at center





Case	b/a	Edges	Load
1	1	SS	U
2	2	SS	U
3	1	C	U
4	2	C	U
5	1	SS	C
6	2	SS	C
7	1	C	C
8	2	C	C

Figure 1. Discretization and schedule of rectangular plate.

1.11-11 (1/1/77)

Demo Problem 1.11-12

Analysis of Rectangular Plate Using TRPLT1 Element

This problem demonstrates the accuracy of TRPLT1 element in the evaluation of moments in rectangular plate.

The rectangular plate with discretization, shown on page 1 11-9 (figure 1), is analyzed.

Two different mesh arrangements are used, i.e., Q-mesh and P-mesh. The result is tabulated for the points marked on x- and y-axis, as shown in figure 2, page 1.11-17.

Input:

$E = 3.0 \times 10^7$ lbs/in.^{2}	(Young's modulus)
$\nu = 0.3$	(Poisson's ratio)
$q = 1000$ lbs/in.^{2}	(Uniform distributed load)
$\frac{b}{a} = 2$	(Length/width)
$t = 1.0$ in.	(Thickness of the plate)
$N = 12$	(No. of elements per side)

Table 1. Numerical factor β' in the equation $M_x = \beta' qa^2$ for bending moments of simply supported rectangular plate under uniformly distributed load (along $y = 0$).

x	NASTRAN-TRPLT1			Exact β'
	β' ($y = 0$)			
	Q-mesh	P-mesh	Average	
0.5a (center)	0.1001332	0.1022486	0.1012	0.1017
0.416667a	0.105326	0.1070887	0.10620733	0.098875
0.33333a	0.099048	0.10247159	0.100759796	0.09075
0.25a	0.087942	0.0923552	0.0901486	0.076875
0.166667a	0.0710574	0.0751297	0.07309355	0.058
0.083333a	0.04853	0.0393438	0.0439369	0.032
0a (end)	0.022253	0.0254971	0.02387506	0.0

Table 2. Numerical factor β_1' in the equation $M_y = \beta_1' q a^2$ for bending moments of simply supported rectangular plate under uniformly distributed load (along y = 0).

x	NASTRAN-TRPLT1			Exact
	β_1' (y = 0)			
	Q-mesh	P-mesh	Average	β_1'
0.5a (center)	0.046605	0.0470838	0.0468444	0.0464
0.416667a	0.047194	0.0474771	0 0473355	0.045
0.33333a	0.0432597	0.0441237	0.04369171	0.0410625
0.25a	0.0369394	0.0379362	0.037438	0.0344375
0.166667a	0.0280123	0.02855153	0.028282	0.0252
0.08333a	0.01630765	0.0100907	0.0132	0.013875
0a (end)	0.006676	0.01902541	0.0128507	0.0

Table 3. Numerical factor β'' in the equation $M_x = \beta'' qa^2$ for bending moments of simply supported rectangular plate under uniformly distributed load (along $x = 0$).

y	NASTRAN-TRPLT1			Exact β''
	β'' (x = 0)			
	Q-mesh	P-mesh	Average	
0.5b (center)	0.1001332	0.1022486	0.1012	0.1017
0.416667b	0.103584	0.1087446	0.1061643	0.099625
0.33333b	0.0969304567	0.1131157037	0.1050231	0.09025
0.25b	0.0850907	0.1218893745	0.10349	0.080625
0.1666667b	0.0674407	0.14173259	0.1045866	0.06175
0.08333b	0.04365204	0.1811511029	0.1124016	0.033625
0.0b (end)	0.01275513	0.037393	0.025074	0.0

Table 4. Numerical factor β_1'' in the equation $M_y = \beta_1'' q a^2$ for bending moments of simply supported rectangular plate under uniformly distributed load (along x = 0).

y	NASTRAN-TRPLT1			Exact β_1''
	β_1'' (x = 0)			
	Q-mesh	P-mesh	Average	
0.5b (center)	0.046605	0.0470838	0.0468444	0.0464
0.416667b	0.0486921	0.05105566	0.049874	0.0465625
0.33333b	0.049534	0.057433	0.0534835	0.0465625
0.25b	0.049873	0.0679783	0 05892565	0.0460625
0.166667b	0.04794	0.0850651	0.06650255	0.041125
0.083333b	0.03933845	0.109444	0.0743913	0.0288125
0.0b (end)	0.001104162	0 01272338	0.0069137754	0.0

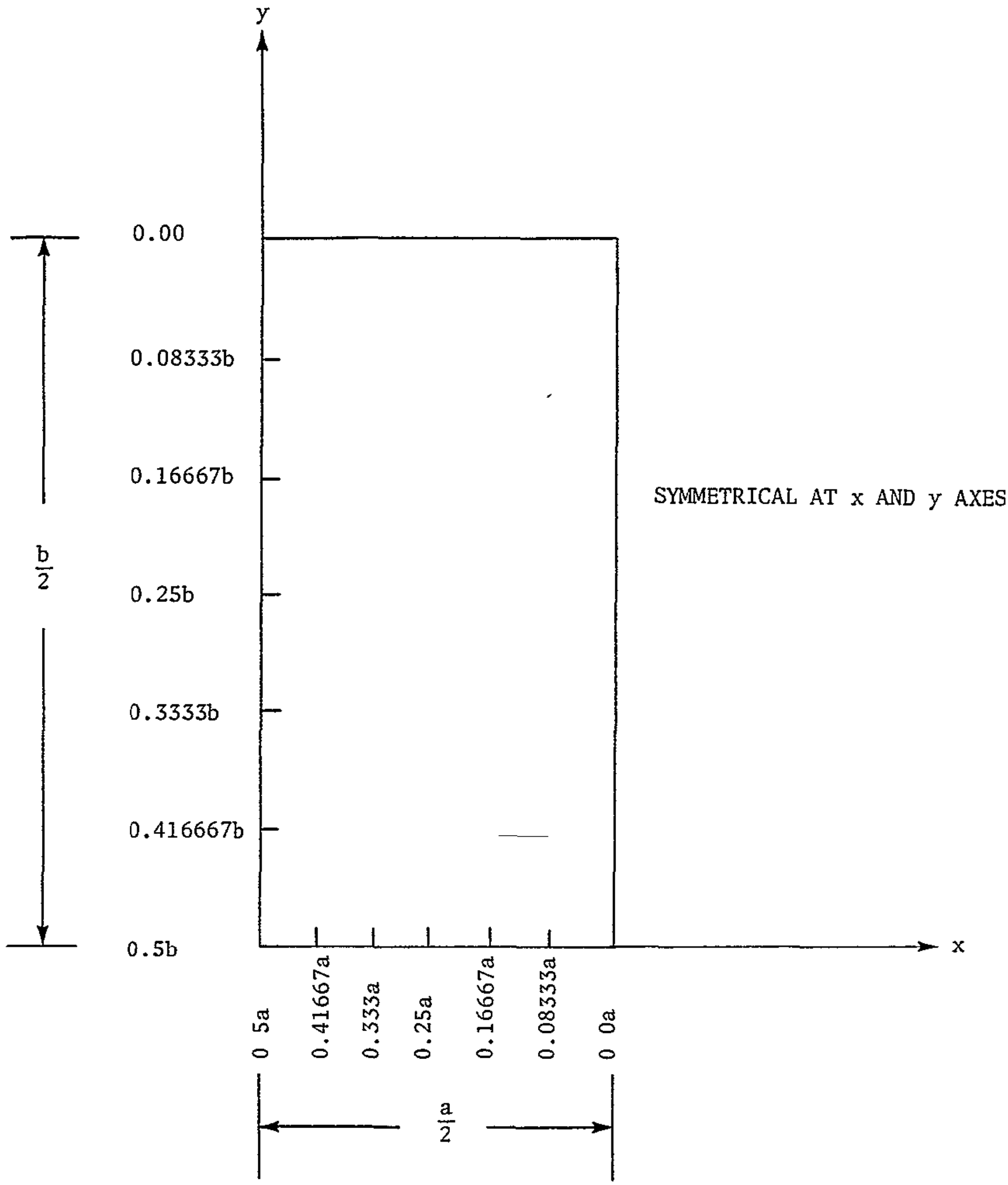


Figure 2.

$\underline{y = 0}$

$M_x = \beta' qa^2$, therefore $\beta' = \dfrac{M_x}{qa^2}$

$M_y = \beta_1' qa^2$, therefore $\beta_1' = \dfrac{M_y}{qa^2}$

$\underline{x = 0}$

$M_x = \beta'' qa^2$, therefore $\beta'' = \dfrac{M_x}{qa^2}$

$M_y = \beta_1'' qa^2$, therefore $\beta_1'' = \dfrac{M_y}{qa^2}$

Demo Problem 3.1-6

Vibration of Tapered Rectangular Plates

This problem demonstrates the use of the higher order triangular bending element CTRPLT1 to solve problems in vibration of thin isotropic plates.

The structural problem consists of a linearly tapered rectangular plate with two different support conditions, namely (i) simply supported and (ii) cantilever.

(i) Linearly tapered simply supported rectangular plate The model as shown in figure 4a uses only half of the plate due to symmetry. The plate thickness is given by:

$$t = t_o \left(1 + \kappa \frac{x}{a}\right) \tag{1}$$

where κ is a constant determining the rate of taper. Two different mesh sizes of the finite element model, 1 × 2 and 2 × 4, are used. Nondimensional fundamental frequencies for rectangular plates for three different aspect ratios $\frac{a}{b}$ and $\kappa = 0.5$ and 0.8 are presented in table 2.

The frequency parameter is defined as·

$$\Omega = \omega\, a^2 \sqrt{\frac{\rho t_o}{D_o}} \tag{2}$$

where ω is the circular frequency, a is the length, ρ is the mass density, t_o is thickness and D_o is the bending rigidity. Analytical results from reference 20 are also shown for comparison.

(ii) Linearly tapered cantilever rectangular plate The plate is idealized with a mesh size of 2 × 4 or 16 elements, as shown in figure 4b. Results of frequency parameters Ω_{mn} as defined in equation (2), where m and n represent the number of nodal lines perpendicular and parallel to the support, respectively, using TRIA2 and TRPLT1 are shown in table 3

Constant thicknesses of 0.0405 in. (0.1029 cm) and 0.1215 in. (0.0386 cm) were used when modeling with TRIA2 element. Experimental data obtained by Plunkett in reference 21 are also given.

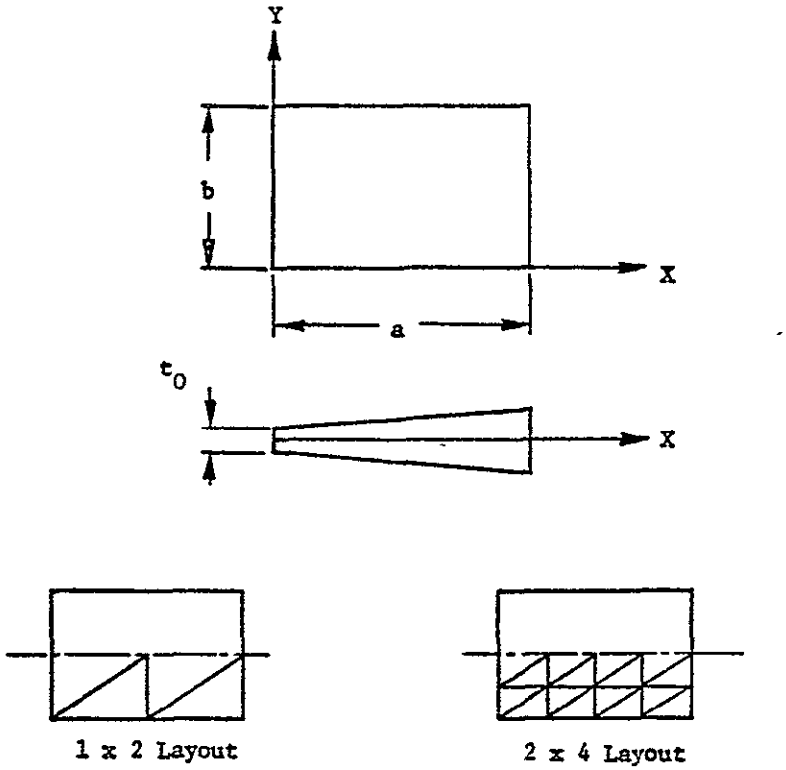
Tables 2 and 3 show that very good results have been obtained using the higher order plate element.

Table 2. Fundamental frequency for linearly tapered rectangular plates simply supported on all edges· $\nu = 0.3$.

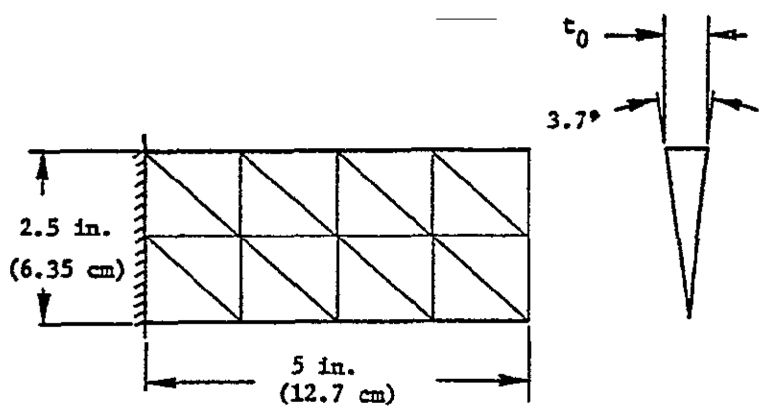
Aspect Ratio $\frac{a}{b}$	NASTRAN TRPLT1 Finite Element Layout	Frequency Parameter $\Omega = \omega a^2 \left(\frac{\rho t_0}{D_0}\right)^{1/2}$	
		Taper Rate $\kappa = 0.5$	Taper Rate $\kappa = 0.8$
0.5	1 × 2	14.662	16.242
	Theory	15.304	16.994
1 0	1 × 2	24.171	26.901
	2 × 4	24.454	---
	Theory	24.556	27 354
2.0	1 × 2	58.560	64.770
	2 × 4	60.346	---
	Theory	60.982	67.500

Table 3. Frequency parameters for a linearly tapered rectangular cantilever plate; $\nu = 0.3$.

Mode		Frequency Parameter $\Omega_{mn} = \omega_{mn} a^2 \left(\frac{\rho t_0}{D_0}\right)^{1/2}$		
		NASTRAN		Experiment
m	n	TRIA2	TRPLT1	
0	0	2.28	2.25	2.47
1	0	9.8	10.0	10.6
0	1	14.5	13.6	14.5
1	1	23.8	27.0	28.7
0	2	35.9	32.8	34.4
0	3	51.5	47.3	47.4
2	0	31.0	53.3	52.5
1	2	64.0	57.7	54.0



(a) Simply supported plate.



(b) Cantilever plate

Figure 4. Plate geometry and finite elment idealization for the TRPLT1 element test problems.

Demo Problem 5.1-5

Buckling of Columns and Plates

The out-of-plane buckling of plate elements is evaluated from the differential stiffness matrix of bending plate element TRPLT1 due to membrane prestress effects obtained from a membrane analysis using TRIM6 elements. To solve out-of-plane buckling of plates, a membrane-bending combination element is necessary. TRSHL is such a combination element with the added feature of membrane bending coupling for shell problems; where the curvature is zero, there is no coupling between membrane, and bending effects, and TRSHL for such cases reduces to a combination element. The results for problems in this section have been obtained using TRSHL elements.

Three buckling problems were investigated using the triangular plate and membrane elements. Following are the three different problems: (i) Buckling of a tapered column fixed at the base is shown in figure 3a. The area moment of inertia at any cross section can be expressed in the form

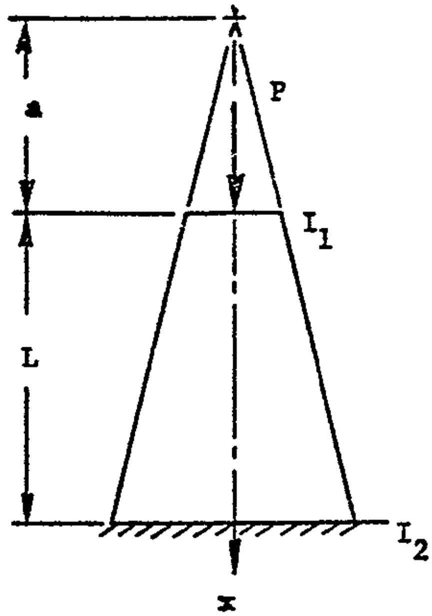
$$I_x = I_1 \left(\frac{x}{a} \right)^4 \qquad (1)$$

where I_1 is the moment of inertia at the top of the column (x = a). Results for the buckling factor for a tapered column of $I_1/I_2 = 0.2$ have been obtained from NASTRAN using TRIA2 and TRSHL, and an analytical solution from reference 22 is given in table 1 for comparison. (ii) Buckling of a simply supported square plate subjected to uniform compression in one direction. Owing to symmetry, only one quarter of the plate (modeled with 2 × 2 mesh size) is used as shown in figure 3b. Results of the buckling factor from NASTRAN TRIA2 and TRSHL elements and the exact solution are shown in table 2.

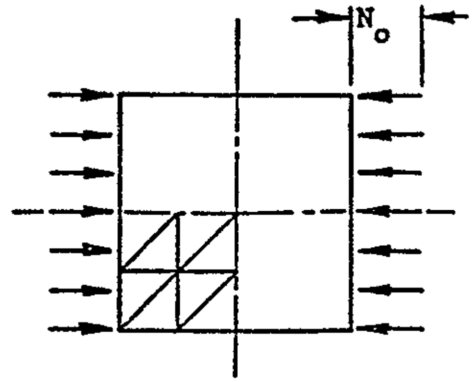
The nondimensional buckling factor λ is represented by the formula:

$$N_{cr} = \frac{\pi^2 D}{b^2} \qquad (4)$$

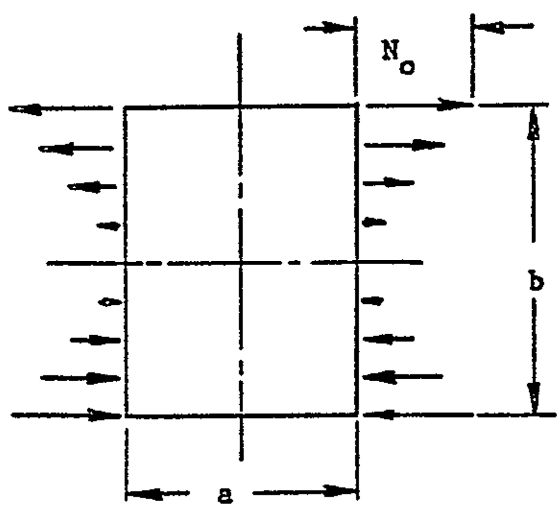
(iii) The third problem considered is buckling of simply supported rectangular plate of aspect ratio a/b - 0.8 under in-plane bending loading shown in figure 3c. Due to symmetry, only half of the plate is used in the analysis. NASTRAN results using TRIA2 and TRSHL with different mesh sizes are shown in table 3, along with analytical results from reference 22. Table 3 clearly shows that the TRSHL elements gave a much better prediction of the critical buckling load than the TRIA2 elements.



(a) Tapered column.



(b) Simply supported square plate under uniform compression.



(c) Simply supported plate under in-plane bending.

Figure 3. Column and plate geometry for TRSHL element buckling test problems.

Table 1. Buckling factor for a tapered column.

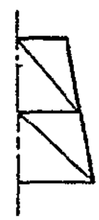
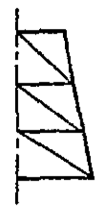
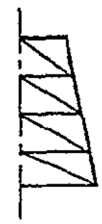
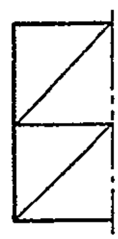
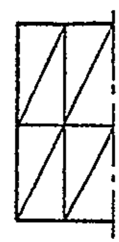
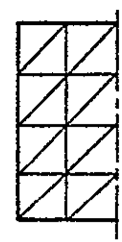
	Buckling Factor $\lambda = \frac{P_{cr}L^2}{EI_2}$		
	Finite Element Layout		
			
TRIA2	1.4242	1.3618	1.3420
TRSHL	1.6437	1.6050	1.5853
Theory		1.505	

Table 2. Buckling factor for simply supported square plate uniformly compressed in one direction; $\nu = 0.3$.

	Buckling Factor $\lambda = \frac{N_{cr}b^2}{\pi^2 D}$
TRIA2	4.0356
TRSHL	3.9779
Exact	4.0000

Table 3. Buckling factor for a simply supported rectangular plate of aspect ratio 0.8 under in-plane bending, $\nu = 0.3$.

	Buckling Factor $\lambda = \frac{(N_o)_{cr} b^2}{\pi^2 D}$		
	Finite Element Layout		
			
TRIA2	29.7815	35.3289	23.8702
TRSHL	24.5507	24.1103	24.1708
Theory	24.4		

20. Leissa, A. W. "Vibration of Plates," NASA SP-160 pp. 285-297, 1967.

21. Plunkett, R.· "Natural Frequencies of Uniform and Non-Uniform Rectangular Cantilever Plates," J. Mech. Engr Sci., Vol. 5, pp. 146-156, 1963.

22. Timoshenko, S. P. and Gere, J. M.. "Theory of Elastic Stability," McGraw Hill, pp. 125-132, and 372-379, 1961.

23. Cowper, G. R., Lindberg, G. M. and Olson, M. D.. "A Shallow Shell Finite Element of Triangular Shape," Int. J. Solids and Structures, Vol. 6, pp. 1133-1156, 1970.

APPENDIX E

The NASTRAN source code subroutines that
need to be modified or added for
inclusion of TRIM6, TRPLT1 and TRSHL elements

APPENDIX E

The NASTRAN Subroutines that are modified to add the TRIM6, TRPLT1 and TRSHL elements are

1. DS1
2. DS1A
3. EDTLZZZ
4. ELELBL
5. EMGPRØ
6. GPTABD
7. IFP
8. IFS1P
9. IFX1BD
10. IFX2BD
11. IFX3BD
12. IFX4BD
13. IFX5BD
14. IFX6BD
15. IFX7BD
16. LDØ1
17. LDØ2
18. LDØ3
19. LDØ4
20. LDØ5
21. LDØ6
22. LDØ7
23. LDØ8
24. LDØ9
25. LD10
26. LD11
27. LD12
28. LD13
29. LD14
30. LD15
31. LD21
32. LD22
33. LD23
34. LD34
35. LINEL
36. ØFP1A
37. ØFP1BD
38. ØFP5BD
39. ØF1PBD
40. ØF5PBD
41. SDR2B
42. SDR2E

New Subroutines Added

1. KTRM6S: Stiffness and mass matrix generation subroutine, single precision version, for element CTRIM6.

2. KTRM6D: Stiffness and mass matrix generation subroutine, double precision version, for element CTRIM6.

3. TLØDM6: Thermal load vector calculation for element CTRIM6.

4. STRM61 Stress data recovery, Phase I for element CTRIM6.

5. STRM62: Stress data recovery, Phase II for element CTRIM6.

6. KTRPLS Stiffness and mass matrix calculations, single precision version (without the effects of transverse shear), for element CTRPLT1.

7. KTRPLD Stiffness and mass matrix calculations, double precision version (without the effects of transverse shear), for element CTRPLT1.

C-3

8. TSPL1S. Transverse shear calculations, single precision version, for element CTRPLT1. This subroutine performs the numerical integration to obtain the contribution to the generalized stiffness matrix due to transverse shear effects.

9. TSPL1D. Same as TSPL1S, double precision version.

10. TSPL2S Calculations, single precision version, to obtain the matrix, $[B_2]$, relating curvatures to generalized coordinates (in the equation $\{x_1\} = [B_2]\ \{a\}$).

11. TSPL2D: Same as TSPL2S, double precision version.

12. TSPL3S Calculations, single precision version, to obtain the matrix, $[B_1]$, relating transverse shear strains to the generalized coordinates (in the equation $\{\gamma_1\} = [B_1]\ \{a\}$) for use in the TSPL1S subroutine.

13. TSPL3D· Calculations, double precision version, to obtain the matrix $[B_1]$, as in TSPL3S, for use in the TSPL1D subroutine.

14. TLØDT1: Thermal load vector calculations in the absence of transverse shear effects for element CTRPLT1.

15. TLØDT2: Numerically integrated contribution to the thermal load vector for element CTRPLT1 due to transverse shear effects.

16. TLØDT3. Calculations to obtain the matrix, $[B_1]$, relating transverse shear strains to the generalized coordinates (in the equation $\{\gamma_1\} = [B_1]\ \{a\}$) for use in the TLØDT1 subroutine.

17. STRP11: Stress data recovery, Phase I, for CTRPLT1 element.

18. STRPTS: Calculations to evaluate matrices for recovery of shear forces for CTRPLT1 element.

19. STRP12 Stress data recovery, Phase II, for CTRPLT1 element.

20. KTSHLS Stiffness and mass matrix calculations, single precision version, for triangular shallow shell element CTRSHL.

21. KTSHLD: Stiffness and mass matrix calculations, double precision version, for triangular shallow shell element CTRSHL.

22. TLØDSL Thermal laod vector calculations for element CTRSHL.

23. STRSL1· Stress Data Recovery, Phase I, for element CTRSHL.

24. STRSLV Calculations to evaluate matrices for recovery of shear forces for CTRSHL element.

25. STRSL2· Stress Data Recovery, Phase II, for element CTRSHL.

26. DTSHLD. Differential Stiffness Matrix generation, double precision version, for element CTRSHL.