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NASA TECHNICAL MEMORANDUM

A GREEN'S FUNCTION FORMULATION FOR A NONLINEAR
POTENTIAL FLOW SOLUTION APPLICABLE TO TRANSONIC FLOW

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16. Abstract

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SUMMARY

Routine determination of inviscid subsonic flow fields about wing-body-tail configurations is state-of-the-art employing a Green's function approach for numerical solution of the perturbation velocity potential equation. This approach has been successfully extended into the high subsonic subcritical flow regime and into the shock-free supersonic flow regime. However, it has not been successfully extended into the transonic flow regime. The present study develops a modified Green's function formulation, valid throughout a range of Mach numbers including transonic, that takes an explicit accounting of the intrinsic non-linearity in the parent governing partial differential equations. Some considerations pertinent to flow field predictions in the transonic flow regime are discussed.

INTRODUCTION

Potential flow aerodynamics has traditionally examined the solution of the Laplacian of the incompressible flow perturbation potential function. This solution has found wide application in a variety of problems. Of particular interest here is its successful application at subsonic and supersonic speeds by means of the Goethert transformation. However, attempts to achieve a method of utilizing this solution at transonic speeds have proved unsuccessful. It is felt that this failure is due to shortcomings in the traditional formulation of the approximate potential equation at transonic speeds. The purpose of the present study is to evaluate the more comprehensive tensor form of the governing equation for irrotational potential flow, so as to extract a potentially useful Green's function solution form, valid throughout a Mach number range, specifically including transonic flow.

SYMBOLS

A	function; coefficient matrix
c	isentropic sound speed
e	alternating tensor
f	function; load vector
F	function
g	function; Green's function
G	Green's tensor function
i,j,k	unit vector triad
L	differential operator on R
m	scalar component of Mach tensor
M	Mach number; tensor function
n	normal vector
p	pressure
r	geodesic distance
R	solution domain; tensor geodesic distance
t	time

u	perturbation velocity vector
U	velocity vector
x_i	Cartesian coordinate system
x,y,z	rectangular Cartesian coordinate system
β	$1 - M_\infty^2$
γ	ratio of specific heats
δ	Dirac delta function; Kronecker delta
∂R	closure of R
ε	small parameter
ρ	density
σ	singularity strength
Σ	summation operator
ϕ	perturbation velocity potential
Φ	total velocity potential
ψ	function

Subscripts

∞	freestream reference
i,j,k	tensor indices

n   normal to closure

o   stagnation reference

t   partial derivative by time

,   gradient vector operator

–   not a tensor index

Superscripts

$\sim$ source point position vector

$\rightarrow$ vector

$\wedge$ unit vector

REVIEW OF THE LINEARIZED PROBLEM FORMULATION

Computational solutions for three-dimensional, inviscid irrotational flows about practical aircraft configurations have achieved a high state of capability. The theoretical formulation is a Green's function solution of the Laplacian differential equation governing the distribution of velocity perturbation potential function. For incompressible flow, the continuity equation states

$$\vec{\nabla} \cdot \vec{U} = U_{k;k} = 0 \tag{1}$$

The second form in Eq. (1) introduces the preferred tensor indicial notation to be eventually required. The differential constraint of irrotationality is

$$\vec{\nabla} \times \vec{U} = e_{ijk} U_{k,j} \equiv 0 \tag{2}$$

Equations (1) - (2) completely describe the incompressible flow velocity vector. Furthermore, from Eq. (2), since the curl of the gradient vanishes identically, we can express $\vec{U}$ as

$$\vec{U} = -\vec{\nabla}\Phi = -\Phi_{,k} \tag{3}$$

Hence, the velocity field may be completely determined independent of the pressure, and neglecting momentum considerations (see reference 1, page 6), by solution of

$$\nabla^2 \phi = \phi_{,kk} = 0 \qquad (4)$$

on the solution domain R, as determined by combining Eqs. (1) - (3). Eq. (4) is an elliptic boundary value description; the boundary conditions are typically Neumann, obtained by forming the inner product of Eq. (3) with a unit normal vector $\hat{n}$ on the closure, ∂R, of R as

$$-\vec{\nabla} \phi \cdot \hat{n} = U_n \qquad (5)$$

In Eq. (5), U_n is the local velocity efflux through ∂R. Furthermore, from Eq. (3), the level of ϕ is arbitrary and may be specified at a convenient value for some location on ∂R.

For small disturbance theory (see reference 2, chapter 14), the local flow velocity vector can be represented as

$$\vec{U}(x_i) = \vec{U}_\infty + \vec{u}(x_i) \qquad (6)$$

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where $\vec{u}(x_i)$ is the amount by which the local irrotational velocity differs from the reference freestream value, $\vec{U}_\infty$. Assuming $\vec{U}_\infty$ aligned parallel to the x axis, and identifying the perturbation velocity potential, ϕ

$$\vec{u} = -\vec{\nabla}\phi \tag{7}$$

we have

$$\vec{U} = -\vec{\nabla}\Phi = U_\infty \hat{i} - \vec{\nabla}\phi \tag{8}$$

Since $\vec{u}$ describes an irrotational, inviscid flow, we obtain as before

$$\nabla^2 \phi = 0 \tag{9}$$

The Neumann boundary condition becomes, from Eq. (8)

$$-\vec{\nabla}\phi \cdot \hat{n} = (\vec{U} - U_\infty \hat{i}) \cdot \hat{n} \tag{10}$$

The theory of Green's functions and potential theory are based upon the divergence theorem

$$\int_R \vec{\nabla} \cdot \vec{F} d\tau = \oint_{\partial R} \vec{F} \cdot \hat{n} d\sigma \qquad (11)$$

If we let $\vec{F} = \phi\vec{\nabla}\psi$ and $\psi\vec{\nabla}\phi$ respectively, insert each into Eq. (11) and subtract, we obtain Green's third identity.

$$\int_R [\psi\nabla^2\phi - \phi\nabla^2\psi] d\tau = \oint_{\partial R} [\psi\vec{\nabla}\phi - \phi\vec{\nabla}\psi] \cdot \hat{n} d\sigma \qquad (12)$$

Eq. (12) is the foundation for a Green's function solution to Eqs. (9) - (10). Let ψ take the specific form

$$\psi \equiv g = \frac{1}{|\vec{r}|} + A \qquad (13)$$

where

$$\vec{r} \equiv \vec{x} - \vec{x}' \qquad (14)$$

We assume $\nabla^2 A = 0$, and that A has no singularities in R. The function A can be used to adjust boundary values of g, and $|\vec{r}|^{-1}$ is singular at $\vec{x} = \vec{x}'$, where $\vec{x}$ is the position vector of a field point and $\vec{x}'$ is the position vector to a source point. From vector field theory, (see

reference 3, section 1.7), we readily establish in three-dimensional space,

$$\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi\delta(\vec{x} - \vec{x}') \qquad (15)$$

where δ is the Dirac delta function. (In two-dimensions, the multiplier in Eq. (15) becomes 2π.) Furthermore, we know

$$\int_R f(\vec{x})\delta(\vec{x} - \vec{x}')d\tau = f(\vec{x}') \qquad (16)$$

where f is an arbitrary function. Substituting Eqs. (13) - (16) into (12), we obtain

$$4\pi\phi(\vec{x}) = -\int_R g\nabla^2\phi d\vec{x}' + \oint_{\partial R}[g\vec{\nabla}\phi - \phi\vec{\nabla}g] \cdot \hat{n}d\sigma' \qquad (17)$$

For our perturbation potential flow problem, the first right side term in Eq. (17) vanishes identically, due to Eq. (9). This is perhaps the single key feature of this approach in practical aerodynamics analysis, since only integrals on the closure ∂R need be evaluated, and only surface velocities and pressures are of practical usefulness.

(Were the potential Eq. (9) to be the inhomogeneous Poisson equation, then quadrature throughout R would be required in Eq. (17) and the method loses considerable computational appeal.) The lead term in the surface integral can be evaluated by a limiting operation, using the normal velocity boundary condition, Eq. (10), and the fact that $|\vec{\nabla}\phi| \to 0$ as $|\vec{x}| \to \infty$. Eq. (17) then becomes, for incompressible perturbation potential flow, using Eq. (13) (See reference 1, page 14)

$$2\pi\phi(\vec{x}) - \oint_{\partial R} \phi(\vec{x}')\vec{\nabla}' \frac{1}{|\vec{r}(x,x')|} \cdot \hat{n}d\sigma' = -\hat{n}(\vec{x}) \cdot [U_{\infty}\hat{1} - \vec{U}] \quad (18)$$

The last term in Eq. (18) represents the specified velocity efflux boundary condition statement.

Using various potential flow singularity functions (e.g. line source, vortex, doublet, see reference 4), the integrand in Eq. (18) can be formed and evaluated. This yields a full, large order, linear algebraic equation system to be solved for the strength distribution, $\sigma(\vec{x})$, of the elementary singularities, as

$$[A]\{\sigma\} = \{r\} \quad (19)$$

These procedures can be extended into the high subsonic subcritical, and the linearized small disturbance purely supersonic flow regimes,

using a coordinate transformation (see reference 5). Now including momentum considerations (developed in detail in the next section), it is well known that the small disturbance linearized perturbation velocity potential flow equation becomes

$$[1 - M_\infty^2]\phi_{,xx} + \phi_{,yy} + \phi_{,zz} = 0 \qquad (20)$$

The boundary condition remains Eq. (10), and M_∞ is the reference Mach number associated with $U_\infty \hat{i}$. Since $\beta^2 \equiv [1 - M_\infty^2]$ is a constant, Eq. (20) can be returned to the form of the Laplacian, Eq. (9), by the independent variable coordinate transformation

$$\bar{x} = x$$
$$\bar{y} = \beta y \qquad (21)$$
$$\bar{z} = \beta z$$

Hence, incompressible flow results can be extended to appropriate Mach number ranges using Eq. (21) and solution to Eq. (9). For supersonic flows, β becomes complex and physical velocity components are extracted from the real part of the computational solution. No transonic flow results are available using this approach.

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FORMULATION OF THE NONLINEAR PROBLEM

In a way, it is remarkable the extent to which the solution of the Laplacian of the incompressible flow perturbation potential function has been successfully examined in potential flow aerodynamics. We seek to evaluate the more comprehensive tensor form of the governing equation for irrotational potential flow, so as to extract a potentially useful Green's function solution form, valid throughout a Mach number range, specifically including transonic flow. For the time-varying, isoenergetic flow of an inviscid perfect gas, conservation of mass, momentum and energy is expressed as the differential constraints.

$$\rho_{,t} + (\rho U_i)_{,i} = 0$$

$$\rho U_{i,t} + \rho U_j U_{i,j} + p_{,i} = 0 \tag{22}$$

$$c^2 \rho_{,i} - p_{,i} = 0$$

In the last equation, c is the local speed of sound, defined in terms of a stagnation condition as

$$c^2 = c_o{}^2 - \frac{\gamma - 1}{2} U_k U_k \tag{23}$$

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The five equations (22) can be conveniently combined into a single equation, by expanding the divergence in the first, substituting the density gradient by pressure from the fifth, and combining into the remainder. In expanded form, this yields

$$c^2 U_{j,j} - U_i U_j U_{i,j} = \frac{1}{2} (U_i^2)_{,t} + \ln \rho_{,t} \qquad (24)$$

In Eqs. (22) - (23), the subscript comma j denotes the gradient operator on x_j, and the subscript comma t denotes the partial derivative on time. For the additional constraint of irrotationality, using Eq. (3), Eq. (24) takes the compact form

$$[c^2 \delta_{ij} - \Phi_{,i} \Phi_{,j}] \Phi_{,ij} = \frac{1}{2} (\Phi_{,i})^2_{,t} + \ln \rho_{,t} \qquad (25)$$

The boundary condition for Eq. (25) remains Eq. (5). For steady flow, the right side vanishes identically. Dividing through by c^2 yields the familiar form

$$[\delta_{ij} - c^{-2} \Phi_{,i} \Phi_{,j}] \Phi_{,ij} \equiv M_{ij} \Phi_{,ij} = 0 \qquad (26)$$

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Eq. (26) is the fundamental form for which we seek to evaluate existence of solution forms. We shall also have occasion to explore use of Eq. (25), as a consequence of difficulties associated with Eq. (26). We cannot proceed directly to a Green's function formulation with Eq. (26) since, a) it does not exist in divergence form, and/or, b) the coefficients of the Mach tensor, M_{ij}, are neither constants nor functions of the spatial coordinates, but instead highly non-linear functions of the dependent variable.

Eq. (26) explicitly displays the complete mixed elliptic-parabolic-hyperbolic character of potential flow. For example, for $i = j$, the corresponding term in M_{ij} is of the form $[1 - m^2_{(i)}]$, where $m_{(i)}$ is the local Mach number associated with the velocity component parallel to the ith coordinate. i.e., x_i. Hence, for subsonic flows, M_{ij} is non-negative throughout R and Eq. (26) is elliptic. We therefore require knowledge of Φ or $\vec{\nabla}\Phi \cdot \hat{n}$ on the complete closure ∂R for a well-posed problem. For supersonic flows, at least one of the diagonal entries in M_{ij} is negative, and Eq. (26) displays a hyperbolic character. The well-posed solution requires knowledge of Φ and $\vec{\nabla}\Phi \cdot \hat{n}$ on a non-characteristic closure segment. In the intermediate range of primary interest (i.e., transonic), a diagonal coefficient in M_{ij} will be bounded around zero, Eq. (26) will display a globally elliptic character with a priori unknown interior regions of hyperbolic and/or parabolic character. In this instance of major interest, we may seek to employ the transient description of Eq. (25), which can bring in an additional hyperbolic character to uniformly superimpose on the solution.

Eqs. (26) and (25) are essentially intractible, but similarly describe a much more general flow field than is of aerodynamic interest. Therefore, we seek to employ the simplifying small disturbance approximation, but to retain the intrinsic tensor character while doing so. Hence, assume again that the external reference flow is parallel to the x axis (i.e., x_1), and express the local velocity vector as a perturbation on U_∞ of the form

$$U_k = -\phi_{,k} = U_\infty \delta_{k1} - \phi_{,k} \qquad (27)$$

In Eq. (27), δ_{k1} is the Kronecker delta which is zero except when k = 1. The reference state for the local sound speed can be efficiently modified to U_∞, yielding for Eq. (23)

$$c^2 = c_o^2 - \frac{\gamma - 1}{2} U_k U_k$$

$$= c_\infty^2 - \frac{\gamma - 1}{2} [\phi_{,k}\phi_{,k} - 2U_\infty \phi_{,1}] \qquad (28)$$

As a first step towards tractibility, under the small disturbance approximation for all subsonic and moderate supersonic (but explicitly not transonic) flows, Eq. (26) using Eqs. (27) - (28) will again yield Eq. (20). Rewritten in tensor form, Eq. (20) is formally identical to Eq. (26)

with the definition

$$M_{ij} = \begin{bmatrix} (1 - M_\infty^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = m_{(i)}\delta_{ij} \tag{29}$$

The last form emphasizes that M_{ij} is only a scalar modification of the Kronecker delta in principal coordinates. With this identification, Eq. (26) remains a linear partial differential equation but is no longer self-adjoint. To obtain the Green's function form, analogous to Eq. (12), we must identify the adjoint operator M operating on the Green's function g as (see reference 7, section 10 and reference 8, appendix I)

$$Mg \equiv (M_{ij}g)_{,ij} \tag{30}$$

Denoting the differential operator on ϕ, Eq. (20), as L, form $gL\phi$, i.e.,

$$gL\phi \equiv gM_{ij}\phi_{,ij} = 0 \tag{31}$$

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Inserting Eq. (31) into the divergence theorem, Eq. (11), and noting that M_{ij} is constant, obtain

$$\int_R \mathcal{E} M_{ij} \phi_{,ij} d\tau = \oint_{\partial R} \mathcal{E} M_{ij} \phi_{,i} n_j d\sigma - \int_R (\mathcal{E} M_{ij})_{,j} \phi_{,i} d\tau$$

$$= \oint_{\partial R} \mathcal{E} M_{ij} \phi_{,i} n_j d\sigma - \oint_{\partial R} (\mathcal{E} M_{ij})_{,j} \phi n_i d\sigma$$

$$+ \int_R (\mathcal{E} M_{ij})_{,ij} \phi d\tau \qquad (32)$$

Noting that the last term in Eq. (32) is Eq. (30), and rearranging, we obtain the tensor form of Green's third identity as

$$\int_R [\mathcal{E} L\phi - \phi Mg] d\tau = \oint_{\partial R} [\mathcal{E} M_{ij} \phi_{,i} - (\mathcal{E} M_{ij})_{,i} \phi] n_j d\sigma \qquad (33)$$

In writing Eq. (33), advantage was taken of rearrangement of dummy (repeated) subscripts. The first left term of Eq. (33) vanishes everywhere within R, see Eq. (26). For boundary conditions,

$$M_{ij}\phi_{,i}n_j = m_{(i)}\delta_{ij}\phi_{,i}n_j$$

$$= m_{(i)}\phi_{,i}n_i$$

$$= -m_{(i)}[U_i - U_\infty\delta_{i1}]n_i \qquad (34)$$

using Eq. (10). Noting that $M_{ij,i}$ vanishes, Eq. (33) can be rearranged to the final form, analogous to the scalar Green's theorem, Eq. (17), as

$$\int_R \phi M_{ij}g_{,ij}d\tau = -\oint_{\partial R}g m_{(i)}[U_i - U_\infty\delta_{i1}]n_i d\sigma$$

$$-\oint_{\partial R}\phi M_{ij}g_{,i}n_j d\sigma \qquad (35)$$

Eq. (35) is the final form, valid for both subsonic and purely supersonic flows under the small disturbance approximation. A few additional comments are warranted, however. For the supersonic flow case, use of Eq. (35) rather than the Goethert rule transformation of the incompressible flow solution, Eq. (9) might yield improved solution accuracies since the boundary conditions are applied to the physical

aerodynamic shape (rather than the transformed shape, see reference 6, page 8). Caution is required, however, in employing singularity methods to evaluate terms in Eq. (35). For the supersonic linearized case under consideration, Eq. (26) is hyperbolic, and g is more properly called a characteristic function (see reference 7, page 53).

A hyperbolic equation does not admit to isolated singularities, but instead every singularity is continued along a characteristic. Hence, theoretical evaluation of the character of g, and the all-important zones of silence ahead of a Mach cone, is required to assure proper application of Eq. (35) for $M_\infty > 1$. No such problems are encountered for the high subsonic subcritical case, and g takes on the familiar character of the Green's function. We defer this presentation pending establishment of the transonic flow equation.

The development leading to Eq. (35) required simplification of the basic equation using the small disturbance supersonic approximation. These linearizing assumptions are mutually exclusive to the transonic flow region of interest, and it is necessary to develop an alternative approach. As the first step, we seek to establish Eq. (26) in divergence form so as to employ the divergence theorem. Hence, using the chain rule

$$M_{ij}\phi,_{ij} = [M_{ij}\phi,_i],_j - \phi,_i M_{ij},_j \qquad (36)$$

Multiplying M_{ij} by c^2 for convenience, the last term in Eq. (36) is

$$\phi_{,i} M_{ij,j} = \phi_{,i}[\delta_{ij} - c^{-2}\phi_{,i}\phi_{,j}]_{,j}$$

$$= \phi_{,i}\left[\frac{1}{c^2}(c^2\delta_{ij} - \phi_{,i}\phi_{,j})\right]_{,j} \tag{37}$$

Inserting Eq. (28) for c^2 and Eq. (27) for $\phi_{,k}$ into Eq. (37), we obtain equivalently

$$\phi_{,i} M_{ij,j} = [U_\infty\delta_{i1} - \phi_{,i}]\left[\frac{1}{c^2}\{c_\infty^2\delta_{ij} - \frac{\gamma-1}{2}(\phi_{,k}\phi_{,k} - 2U_\infty\phi_{,1})\delta_{ij}\right.$$

$$\left. - (U_\infty\delta_{i1} - \phi_{,i})(U_\infty\delta_{j1} - \phi_{,j})\}\right]_{,j}$$

$$= [U_\infty\delta_{i1} - \phi_{,i}]\left[\frac{c_\infty^2}{c^2}\{\delta_{ij} - M_\infty^2\delta_{i1}\delta_{j1} + M_\infty\left[\frac{\phi_{,1}}{c_\infty}\delta_{i1} + \frac{\phi_{,i}}{c_\infty}\delta_{j1}\right]\right.$$

$$\left. - \frac{\gamma-1}{2}\left[\frac{\phi_{,k}\phi_{,k}}{c_\infty^2} - 2M_\infty\frac{\phi_{,1}}{c_\infty}\right]\delta_{ij} - \frac{\phi_{,i}\phi_{,i}}{c_\infty^2}\}\right]_{,j} \tag{38}$$

The last form in Eq. (38) utilizes the definition of $M_\infty \equiv U_\infty/c_\infty$. From Eq. (28), we have

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$$\left(\frac{c_\infty^2}{c^2}\right)^{-1} = 1 - \frac{\gamma-1}{2c_\infty^2}\left[\phi_{,k}\phi_{,k} - 2U_\infty\phi_{,1}\right] \qquad (39)$$

We now wish to evaluate Eqs. (38) - (39) under the small distur-
bance supersonic approximation, whereupon we assume

$$\frac{|u_k|}{U_\infty} = \frac{|\phi_{,k}|}{U_\infty} << 1 \qquad (40)$$

Since $M_\infty \approx 1$ in the transonic and low supersonic range, this implies

$c_\infty \approx U_\infty$. Substituting U_∞ for c_∞ in Eqs. (33) - (39), we observe that

$(c_\infty^2/c^2)^{-1} \approx 1$, and that all the non-constant terms within the outer

square bracket in Eq. (38) vanish identically. Since the derivative

of $[\delta_{ij} - M_\infty^2\delta_{i1}\delta_{j1}]$ also vanishes, we have established the conditions

appropriate for Eq. (38) to vanish entirely. Hence, under the assump-

tion of Eq. (40), which restricts validity to a region around $M_\infty = 1$, the

fundamental potential flow equation, Eq. (26), can be equivalently

written in divergence form as

$$[M_{ij}\phi_{,i}]_{,j} = 0 \qquad (41)$$

This is the desired result. It is of crucial importance, for practical computational usefulness of the Green's function method, that the governing partial differential equation be homogeneous, so as to avoid integrations over the entire solution domain R, see Eq. (17). Note that M_{ij} remains the complete tensor including off-diagonal entries, and thus encompasses the linearized compressible form, Eq. (29).

Using Eq. (41), and identifying $G(x_i,x_i')$ as the new tensor Green's function to be established, we can obtain the desired solution form using the divergence theorem and Eq. (12) as

$$\int_R [G(M_{ij}\Phi,_i),_j - \Phi(M_{ij}G),_{ij}]d\tau$$

$$= \oint_{\partial R} [GM_{ij}\Phi,_i n_j - \Phi(M_{ij}G),_j n_i]d\sigma$$

$$= \oint_{\partial R} [GM_{ij}\Phi,_i - \Phi M_{ij}G,_i]n_j d\sigma \qquad (42)$$

The last form in Eq. (42) is achieved recognizing that M_{ij} is symmetric. The first left side term vanishes identically throughout R due to Eq. (41). From the second left side term, we wish to extract the field point value of potential, i.e., $\Phi(x_i)$. The form is again given by Eq. (13); we seek to transform the integral as given by Eq. (16). The desired form of the Green's function is (ref. 9)

$$G(x_i, x_i') = [M^{ij}]^{1/2} \frac{1}{R} + A \qquad (43)$$

where,

$$R^2 = \sum_{i,j=1}^{3} M^{ij}(x_i - x_i')(x_j - x_j') \qquad (44)$$

M^{ij} is the inverse (matrix) of the Mach tensor M_{ij}, Eq. (26), and $\nabla^2 A = A_{,ii} = 0$. Eq. (44) presents the necessary generalization of the geodesic distance r, Eq. (14), associated with the scalar formulation. For the term of interest in Eq. (42), we obtain using Eqs. (43) - (44)

$$\int_R \phi[M_{ij}G]_{,ij}d\tau' = \int_R \phi \left[M_{ij}[M^{ij}]^{1/2} \frac{1}{\sqrt{\sum\limits_{i,j} M^{ij}(x_i - x_i')(x_j - x_j')}} \right]_{,ij} d\tau'$$

$$(45)$$

Since, to the order of approximation of this development, derivatives on M_{ij} vanish, Eqs. (38) - (40), so will they vanish on M^{ij}. Performing the indicated differentiation on x_i and x_j in Eq. (45) will bring M^{ij}

into the numerator while leaving $[M^{ij}]^{1/2}$ in the denominator. Cancellation yields the coefficient $M_{ij}M^{ij} = \delta_{ij}$, the Kronecker delta. The indices of differentiation then become contracted over i and j yielding the familiar evaluation given in Eq. (15). Using Eq. (16), we then obtain

$$\int_R \phi(x_i') \, [M_{ij}G]_{,ij} d\tau' = -4\pi\phi(x_i) \qquad (46)$$

The Green's function solution form for the non-linear transonic flow potential equation then becomes

$$4\pi\phi(x^i) = \oint_{\partial R} [GM_{ij}\phi_{,i}(x_i') - \phi(x_i')M_{ij}G_{,i}]n_j d\sigma' \qquad (47)$$

with the Green's function, $G(x_i, x_i')$ given by Eqs. (43) – (44).

Equation (47) is the terminal form we seek for transonic flow. As before, the boundary condition statement, Eq. (5) or (10), on velocity efflux over δR will be used to evaluate the first surface integral term. For transonic flow, it is highly improbable that M_{ij} will introduce an additional singularity into the integral, since non-zero velocity

boundary conditions are specified well away from the transonic region. The second integrand in Eq. (47) is again employed to fill out the influence coefficient matrix. The singularity that arises as $\bar{x}_i \rightarrow \bar{x}'_i$ (where the superscript bar constrains x_i to δR) may be isolated by encircling $\bar{x}'$ with a small hemispherical surface of radius ϵ. Proceeding to the limit as ϵ vanishes, a useful formula is

$$\lim_{\epsilon \rightarrow 0} \int_{\partial R_\epsilon} M_{ij} G_{,i}(\bar{x}_i, \bar{x}'_i) n_j d\sigma' = \frac{-1}{2} \tag{48}$$

Singularity distribution functions that are not constant over panels of the discretization, Eq. (47), will require individual evaluation of limiting behavior of the form of Eq. (48).

While Eq. (47) represents the Green's function solution form for transonic flow in completeness, several major application-oriented features remain to be resolved that will undoubtedly require some computational evaluation. First, when the flow goes locally sonic, M_{11} will vanish identically. If M_{ij} were a diagonal tensor, the determinant would also vanish, and the inverse tensor M^{ij} required for the Green's function would become singular. However, in the present formulation, M_{ij} is not diagonal but possesses the off diagonal elements, $\Phi_{,i} \Phi_{,j}$. In particular, $-(\Phi_{,1})^2$ must be sufficiently large at the sonic point to keep $\det|M_{ij}|$ non-zero for the algorithm to remain

stable. Of course, this is a function of the particular geometry, and probably requires a numerical evaluation. Downstream of this critical location for some distance, the flow will be locally super- sonic on the aerodynamic surface. In this region, M_{11} will be negative and Eq. (47) will display the mixed hyperbolic character for which the tensor Green's function formulation is unproven, as discussed for Eq. (35). Not all source points can exchange information with a field point in this region, as a manifestation of the zone of silence. Analyses of the transonic problem in this region using finite difference (ref. 10) and finite element (ref. 11) solutions to the field problem description have altered their difference kernel, and discarded negative "diffusion" coefficients, respectively to accurately model the locally supersonic flow regions. Whether this can be accomplished using a pure domain- closure procedure remains to be verified. Finally, for moderately high subsonic incident flow Mach numbers, the supersonic flow region will typically return to subsonic flow through a shock. The shock can be assumed to intersect the surface normally; since the flow parallel to each side of the shock remains uniform, the normal efflux boundary con- dition statement, Eq. (5), remains unaltered. However, some means will be required to accurately register the location of the shock, and to therein employ the Rankine-Hugoniot relations rather than Eq. (47), (see reference 12, page 13). For solution of the field form of the transonic flow, artificial viscosity is typically employed to smear the shock over several mesh points and thereby eliminate the discontinuity.

An alternative to the presented formulation exists which might alleviate these critical flow recognition problems. Since the coefficients of the Mach tensor are non-linear functions of the potential function, an iteration procedure will be required to establish solution of Eq. (47), in a manner corresponding to [A] becoming [A(σ)], Eq. (19). The alternative would be to reformulate the description in terms of the transient flow, Eq. (25). The addition of time as an independent variable can cast a uniform parabolic or hyperbolic character onto the solution. Using a four-vector approach (see reference 3, chapter 6), a time-retarded Green's function could possibly be developed to account for a finite speed for signal propagation. For a scalar differential equation, the retarded Green's function form is

$$g(x_i, t; x_i', t') = \frac{1}{|x_i - x_i'|} \ \delta(t' + \frac{x_i - x_i'}{c} - t) + A \qquad (49)$$

where c is the scalar signal propagation speed, the scalar equivalent of $|M_{ij}|$. The causal description in Eq. (49) indicates a signal is recorded at (x_i, t) due to a disturbance originating earlier at (x_i', t') where

$$t' = t - \frac{|x_i - x_i'|}{c} \qquad (50)$$

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For a parabolic form of the tensor equation, Eq. (25), a time-retarded Green's function is given as (see reference 9)

$$G(x_i, t; x_i', t') = \frac{[M^{ij}]^{1/2}}{(t - t')^{3/2}} \exp\,[-R^2/4(t - t')]$$ (51)

where R^2 remains the geodesic distance, Eq. (44). The equivalent tensor form for a hyperbolic equation is unknown. Its usefulness appears limited however, since in four vector form (see reference 13 for the completely linearized form), Eq. (25) is a generalized Poisson equation. The nonhomogeneity is $(\ln \rho)_{,t}$; on several occasions we have noted the practical usefulness requirement that the parent partial differential equation description be homogeneous. In the final approach to steady-state, this term of course will vanish. However, should its retention be required to get a transient solution started, then in Green's third identity, Eq. (42), we would have the four-vector equivalent

$$\int_R G[M_{ij} \Phi_{,i}]_{,j} \; d\tau = \int_R G \ln \rho_{,t} d\tau$$ (52)

Therefore, the formulation requires evaluation of an integral over the entire domain R, and the key feature of only surface paneling requirements is lost. From a user standpoint, as well as computer program design, this appears totally unacceptable.

CONCLUSIONS

The existence of Green's function solution formulations for non-linear potential flow has been determined, for the tensor form of the governing differential equation system. For the small disturbance, completely-linearized approximation, and for subsonic and purely supersonic flow, the Green's function solution form is given in Eq. (35). For the transonic Mach number range, the general tensor equation is cast into divergence form, Eq. (41), using the small disturbance approximation. Subsequently, the Green's function is established for this explicitly non-linear tensor differential equation, Eqs. (43) - (44), and the Green's function solution form established for transonic flow predictions, Eq. (47). Summary comments are made regarding anticipated problems with application of Eq. (47) to critical flow configurations. A plausible alternative solution procedure involving time-retarded Green's functions is briefly described, although practical programming considerations contraindicate its utility.

REFERENCES

1. Hess, J. L. & Smith, A. M. O., "Calculation of Potential Flow About Arbitrary Bodies," in _Prog. Aero. Sci._, V. 8, 1966, pp. 1-138.

2. Shames, I. H., _Mechanics of Fluids_, McGraw-Hill, New York, 1962.

3. Jackson, J. D., _Classical Electrodynamics_, Wiley, New York, 1963.

4. Woodward, F. A., "An Improved Method for the Aerodynamic Analysis of Wing-Body-Tail Configurations in Subsonic and Supersonic Flow. Part I - Theory and Application," NASA CR-2228, 1973.

5. Gothert, B., "Plane and Three-Dimensional Flow at High Subsonic Speeds," NACA TM 1105, 1946.

6. Woodward, F. A., Dvorak, F. A., Geller, E. W., "A Computer Program for Three-Dimensional Lifting Bodies in Subsonic Inviscid Flow," Report USAAMRDL-TR-74-18, 1974.

7. Sommerfeld, A., _Partial Differential Equations in Physics_, Academic Press, New York, 1949.

8. Jaunzemis, W., _Continuum Mechanics_, Appendix I, MacMillan, New York, 1967.

9. Chang, Y. P., Kang, C. S., and Chen, D. J., "The Use of Fundamental Green's Functions for the Solution of Problems of Heat Conduction in Anisotropic Media," Int. J. Heat Mass Transfer, V. 16, pp. 1905-1918, 1973.

10. Jameson, A., "Numerical Calculation of the Three Dimension Transonic Flow Over a Yawed Wing," Proceedings AIAA Computational Fluid Dynamics Conference, 1973, pp. 18-26.

11. Chan, S. T. K., Braeshears, M. R., and Young, V. Y. C., "Finite Element Analysis of Transonic Flow by the Method of Weighted Residuals," AIAA Paper 75-79, 1975.

12. Gullstrand, T. R., "The Flow Over Symmetrical Aerofoils Without Incidence at Sonic Speed," Technical Note KTH-AERO TN 24, Royal Institute of Technology, Division of Aeronautics, Sweden, 1952.

13. Morino, L. & Kuo, C. C., "Subsonic Potential Aerodynamics for Complex Configurations: A General Theory," AIAA J., V. 12, No. 2, pp. 191-197, Feb. 1974.