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THE DYNAMICS AND OPTIMAL CONTROL OF SPINNING SPACECRAFT WITH MOVABLE TELESCOPING APPENDAGES

Part C

EFFECT OF FLEXIBILITY DURING BOOM DEPLOYMENT

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ABSTRACT

The dynamics of a spinning symmetrical spacecraft system during the deployment (or retraction) of flexible boom-type appendages is the subject of this investigation. The effect of flexibility during boom deployment is treated by modelling the deployable members as compound spherical pendula of varying length (according to a control law). The orientation of the flexible booms with respect to the hub, is described by a sequence of two Euler angles. The boom members contain a flexural stiffness which can be related to an assumed effective restoring linear spring constant, and structural damping which effects the entire system. It is seen that the linearized equations of motion for this system, when the boom length is constant, involve periodic coefficients with the frequency of the hub spin. A bounded transformation is found which converts this system into a kinematically equivalent one involving only constant coefficients. According to the Lyapunov reducibility theorem the stability of the reduced system can be analyzed using the standard techniques - i.e. an application of the Kelvin-Tait-Chetaev theorem. With the presence of structural boom damping only (and no damping on the hub), criteria for stability in the Lyapunov sense may be developed and analyzed before or immediately after each deployment maneuver. The

simulation of the first-order nonlinear equations may be used to predict the dynamics of the system before, during, and after each extension or retraction.

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NOMENCLATURE

A	Linearized system state matrix
a_i	Flexural coordinate in the transverse x,y plane
b_i	Flexural coordinate out of the x,y plane
C	Equivalent (structural) linear viscous rate damping of the booms
c_i	Offset of booms from the principal x axes
D	Symmetric positive semidefinite damping matrix
G	Skew-symmetric matrix involving gyroscopic terms
I_1, I_3	Composite principal moments of inertia ($I_1 = I_2$)
I_{H_i}	Hub principal moment of inertia ($i = 1,2,3$)
K	Symmetric stiffness matrix
k	Equivalent linear restoring spring constant of the booms
ℓ	Instantaneous boom length
M, M_1, M_2, M_3	System composite parameters in the equations of motion
M	Symmetric mass inertia matrix
m_H	Hub mass
m_t	Tip mass
dm(t)	Arbitrary differential mass within the system

$dm_i(t)$	Stored mass
Q_{a_i}	Generalized force associated with the flexural coordinate in the transverse x,y plane ($i = 1 \rightarrow 4$)
Q_{b_i}	Generalized force associated with the flexural coordinate out of the x,y plane ($i = 1 \rightarrow 4$)
q	Position of an arbitrary volume element along the boom (wire)
$\bar{R}_{HV}$	Position vector of arbitrary volume element with respect to the hub center of mass
$\bar{R}_{OH}$	Inertial position vector of the hub center of mass
$\bar{R}_{OV}$	Inertial position vector of an arbitrary volume element within the system
r	Offset of the booms from the system center of mass
S(t)	Bounded nonsingular transformation matrix used to transform nonautonomous system of equations to an autonomous set
T	System kinetic energy
t	Time
dV	Differential volume within the system
$W_{0,1,2,3}$	Indicates the order of magnitude of the angular velocity vector components
w_0	Nominal value of hub spin rate
X,Y,Z	Inertial coordinate system
x,y,z	Hub coordinate system
X(t)	State vector

δ	Small perturbation associated with rotation about the spin axes
ϵ_1, ϵ_2	Transformed form of the coordinates θ_1, θ_2
$\theta_1,\theta_2,\theta_3$	Euler angles which define the orientation of the hub with respect to the inertial system
ρ	Boom (wire) mass per unit length
ψ	Angle swept out in the transverse plane by the hub spin vector
$\bar{\omega}$	Inertial angular velocity vector of the hub
$\mathcal{F}$	Rayleigh dissipation function
Superscripts	
Dot	Indicates time differentiation
Superscripts	
f	Associated with the final length of the boom
H	Hub reference frame
I	Inertial reference frame
i	Refers to boom number ($i = 1 \rightarrow 4$)
j	Index number of Lagrangian equation associated with a flexural coordinate ($j = 1,2,3,4$)

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CHAPTER I

Introduction

The effect of flexibility during boom deployment on the dynamics of a spinning spacecraft system is treated by modelling the deployable members as compound spherical pendula of varying length. The Euler angles, a_i and b_i, describe flexural coordinates within the transverse plane and normal to this plane (Fig. 1). This model utilizing four deployable transverse booms (Fig. 1 showing one boom) nominally lying in a plane orthogonal to the spin axes will be considered. The booms are assumed to emanate from points a distance, r, from the center of mass of the hub and, in their undeflected state, are not necessarily aligned with the hub principal axes ($c_i \neq 0$, $i = 1 \rightarrow 4$).

The development by Longman and Fedor[1] for treating the dynamics of a system with constant length flexible wires will be modified to include the effect of varying the length. A quadratic approximation to the kinetic energy is developed which involves a total of 14 generalized coordinates. The degree of flexibility is treated by introducing a restoring potential energy function proportional to the square of the flexural coordinates, where the proportionality (spring) constants can be varied according to the in-plane and out-of-plane resistance to bending. Boom bending in the transverse plane and also out-of-plane can be simulated by varying the magnitude of restoring linear spring constants, which can be related to the stiffness (EI) of the boom material.

A related recent paper by Janssens,[2] considered the flexibility of fixed length appendages attached to a hub which was assumed to spin at a uniform rate with no transverse components of angular velocity. The motion of the appendages was simulated as spherical pendula in a centrifugal force field having two degrees of freedom, thus neglecting the perturbing effects of the wires on the motion of the central body. The author[2] has examined this simpler problem to obtain the oscillatory behavior of the pendula, which describe the nature of non-linearities in the equations of motion, and gives a description of the asymmetry between forward and backward swings in the presense of out-of-plane oscillations.[2]

Two other related problems have also been examined. The first considers stability boundaries on the extension of a pair of axial antennas whose undeformed state lies along the nominal spin axis.[3] It was assumed that the rate of extension was sufficiently small so that Coriolis effects due to the rate of change of length could be neglected. The author[3] determines how far the antennas can be extended before a stability boundary is approached but does not simulate the actual dynamics during deployment.[3] In the second problem, the author's[4] concern is to determine the maximum nutation angles expected to remain after the deployment, the maximum bending moments, and deflections of long flexible booms. This analysis neglects boom extension by considering boom length (ℓ) to be constant in the formulation. The spin rate (ω_3) is also assumed constant.[4]

The objective of the present investigation is to consider both the stability as well as the development of the deployment dynamics of spinning spacecraft with movable flexible appendages. The final objective will be to reach conclusions about the effect of first order appendage flexibility during extension or retraction maneuvers.

CHAPTER II

Equations of Motion

The first order equations of motion will be developed using the Lagrangian formulation, treating flexible boom (wire) members as spherical pendula of varying length. It is assumed that the amplitudes of the flexural coordinates as well as the amplitudes of the variational coordinates that describe the motion of the rigid part are small. It is sufficient to use a quadratic approximation to the Lagrangian to obtain the first order equations of motion.[1] We will first examine the development of the quadratic approximation to the kinetic energy.

Fig. 1 illustrates an inertially fixed coordinate system: X, Y, Z, and a coordinate system: x, y, z, fixed to the symmetric hub of the spacecraft and centered about its center of mass. Also from Fig. 1, we let $\bar{R}_{OV}$ be the vector from the cen[illegible] of the inertially fixed coordinates to an arbitrary volume element, dV, and $\bar{R}_{OH}$, the vector from the center of the inertial coordinate system to the origin of the hub fixed axes. $\bar{R}_{HV}$, is the vector from the origin of the hub fixed axes to the arbitrary volume element dV. The quantity, $d\bar{R}_{OV}/dt|_I$, is defined as the velocity vector of the volume element, dV, relative to inertial space.

A. Kinetic Energy

The kinetic energy of each element of volume can be expressed as,

$$dT = dm\left|d\bar{R}_{OV}/dt\right|^2/2 \tag{1}$$

The total kinetic energy of the system is obtained by integrating Eq. (1) over the total volume in the system:

$$K.E. = (1/2)\iiint \rho_V \left(\frac{d\bar{R}_{OV}}{dt}\bigg|_I \cdot \frac{d\bar{R}_{OV}}{dt}\bigg|_I \right) dV \tag{2}$$

where ρ_V is the density (mass per unit volume). The hub and wires (booms) are assumed to be continuous, whereas the tip mass and the stored mass are considered discrete particles. Thus, the kinetic energy for the entire system can be expressed as:

$$\begin{aligned} T = {} & (1/2)\iiint_{hub} \rho_V \left(\frac{d\bar{R}_{OV}}{dt}\bigg|_I \cdot \frac{d\bar{R}_{OV}}{dt}\bigg|_I \right) dV \\ & + (1/2)\iiint_{wires} \rho_V \left(\frac{d\bar{R}_{OV}}{dt}\bigg|_I \cdot \frac{d\bar{R}_{OV}}{dt}\bigg|_I \right) dV \\ & + (1/2)\sum_{tip\ mass} m_t \left(\frac{d\bar{R}_{OV}}{dt}\bigg|_I \cdot \frac{d\bar{R}_{OV}}{dt}\bigg|_I \right) \\ & + (1/2)\sum_{i=1}^{4} dm_i \left(\frac{d\bar{R}_{OV}}{dt}\bigg|_I \cdot \frac{d\bar{R}_{OV}}{dt}\bigg|_I \right)_{stored\ mass} \end{aligned} \tag{3}$$

It should be noted that the last three terms in Eq. (3) include the effect of the varying length of the wires.

B. Angular Velocity

The orientation of the hub axes relative to inertial space can be expressed by using an Euler angle transformation, where θ_1, θ_2, and θ_3 are the Euler angles. The rotations follow the right hand sense, and are taken in the following order[1]: (1) a rotation through θ_1 about the inertial X axes, followed by (2) a rotation θ_2 about the (intermediate) Y' axis, and then (3) a rotation θ_3 about the resulting Z" axis. After appropriate combination of the three transformation matrices, the following over-all transformation describes the orientation of the hub system unit vectors relative to the inertial system unit vectors:

$$\begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} = \begin{bmatrix} c\theta_2 c\theta_3 & s\theta_3 c\theta_1 - s\theta_1 s\theta_2 s\theta_3 & s\theta_1 s\theta_3 + c\theta_1 s\theta_2 c\theta_3 \\ -c\theta_2 s\theta_3 & c\theta_1 c\theta_3 + s\theta_1 s\theta_2 s\theta_3 & -s\theta_1 c\theta_3 - c\theta_1 s\theta_2 s\theta_3 \\ s\theta_2 & -s\theta_1 c\theta_3 & c\theta_1 c\theta_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{4}$$

where c, s indicate the cosine and sine functions, respectively. From consideration of this specific Euler angle sequence,

$$\bar{\omega} = \dot{\theta}_1 \hat{X} + \dot{\theta}_2 \hat{Y}' + \dot{\theta}_3 \hat{Z}'' \tag{5}$$

Eq. (5) can be expanded by using Eq. (4) and the intermediate individual transformation matrices to yield the components of angular velocity, $\bar{\omega}$, in the hub system as,

$$\bar{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 c\theta_2 c\theta_3 + \dot{\theta}_2 s\theta_3 \\ -\dot{\theta}_1 c\theta_2 s\theta_3 + \dot{\theta}_2 c\theta_3 \\ \dot{\theta}_1 s\theta_2 + \dot{\theta}_3 \end{bmatrix} . \quad (6)$$

C. Development of First Order Expression for the Kinetic Energy

In order to develop a first order expression for the kinetic energy, T, we can write:

$$\begin{aligned} \dot{\theta}_3 &= w_0(t) + \delta\dot{\theta}_3 \\ \theta_3 &= \psi(t) + \delta\theta_3 \end{aligned} \quad (7)$$

where $\delta\dot{\theta}_3/w_0$ and $\delta\theta_3 << 1$. In addition we assume $|\theta_1|$, $|\theta_2|$, $|a_i|$, and $|b_i|$, $<< 1$. For a rigid satellite in or close to its nominal state of spin (i.e. not during a deliberate spin-up maneuver),

$$\begin{aligned} w_0(t) &= \omega_0 = \text{constant (about spin axis)} \\ \psi(t) &= \omega_0 t \; . \end{aligned} \quad (8)$$

Since small angles have been assumed, we can replace, e.g. - cos θ_2 by 1, sin θ_2 by θ_2, etc. Then, with the aid of Eq. (7), Eq. (6) can be approximated by:

$$\bar{\omega} \cong \underbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_0 \end{bmatrix}}_{W_0} + \underbrace{\begin{bmatrix} \dot{\theta}_1 c\psi + \dot{\theta}_2 s\psi \\ -\dot{\theta}_1 s\psi + \dot{\theta}_2 c\psi \\ \delta\dot{\theta}_3 \end{bmatrix}}_{W_1} + \underbrace{\begin{bmatrix} -\dot{\theta}_1 \delta\theta_3 s\psi + \dot{\theta}_2 \delta\theta_3 c\psi \\ -\dot{\theta}_1 \delta\theta_3 c\psi - \dot{\theta}_2 \delta\theta_3 s\psi \\ \dot{\theta}_1 \theta_2 \end{bmatrix}}_{W_2} + \ldots, \qquad (9)$$

where the three dots indicate terms of higher order than quadratic.

The generalized coordinates for the center of mass of the hub are the cartesian components of $\bar{R}_{OH}$ in inertial space. $\dot{X}$, $\dot{Y}$, $\dot{Z}$ represent components in the coordinate system (I) of $d\bar{R}_{OH}/dt|_I$, an inertial derivative. The quantities $\dot{x}_H, \dot{y}_H$, and $\dot{z}_H$ are defined as components of this vector projected into the hub (H), coordinate system (x_H y_H z_H) as seen from the inertial reference frame.

The first term of Eq. (3), the kinetic energy of the hub, contains contributions due to the translational kinetic energy of the center of mass and the rotational kinetic energy about the center of mass.

$$T_{hub} = (1/2)\, m_H(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + (1/2)\, I_{H_1}(\omega_1^2 + \omega_2^2) + (1/2)\, I_{H_3}\, \omega_3^2 \qquad (10)$$

It is assumed in the development of Eq. (10) that the hub is symmetric and homogenous (i.e. $I_{H_1} = I_{H_2}$) . With the aid of Eq. (9), Eq. (10) may be expressed as,

$$T_{hub} = (1/2)\, m_H(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + (1/2)\, I_{H_1}\, [(W_0 + W_1 + W_2)_1^2$$

$$+ (W_0 + W_1 + W_2)_2^2]$$

$$+ (1/2)\, I_{H_3}\, (W_0 + W_1 + W_2)_3^2 \;. \qquad (11)$$

As an example, $W_0+W_1+W_2$ may be expressed in terms of the Euler angles and rates as follows:

$$(W_0+W_1+W_2)_1 = \dot{\theta}_1 \cos\psi + \dot{\theta}_2 \sin\psi + (-\dot{\theta}_1 \delta\theta_3 \sin\psi + \dot{\theta}_2 \delta\theta_3 \cos\psi) \qquad (12)$$

W_3 is higher order than quadratic; thus it can be omitted, along with all other higher order terms.

To calculate T_{wire} and $T_{tip\ mass}$ it is necessary to determine the inertial velocity, $d\bar{R}_{OV}/dt|_I$. From Fig. 1 , and with the application of Coriolis' law,

$$d\bar{R}_{OV}/dt|_I = d\bar{R}_{OH}/dt|_I + d\bar{R}_{HV}/dt|_I$$

$$= d\bar{R}_{OH}/dt|_I + d\bar{R}_{HV}/dt|_H + \bar{\omega} \times \bar{R}_{HV} \qquad (13)$$

(It should be noted that in the development of T_{hub} this expression simplifies to:

$$d\bar{R}_{OV}/dt|_I = d\bar{R}_{OH}/dt|_I + \bar{\omega} \times \bar{R}_{HV} \tag{14}$$

since $d\bar{R}_{HV}/dt|_H = 0$ for a differential volume element inside the hub.)

The quadratic approximation of $\bar{R}_{HV}$, written in matrix form, following Fig. 2, can be expressed as:

$$\left[\bar{R}_{HV}\right]_H = (q+r)\begin{bmatrix}\cos c_i\\ \sin c_i\\ 0\end{bmatrix} + q\begin{bmatrix}-a_i \sin c_i\\ a_i \cos c_i\\ b_i\end{bmatrix} - (1/2)q(b_i^2 + a_i^2)\begin{bmatrix}\cos c_i\\ \sin c_i\\ 0\end{bmatrix}$$

$$+ \ldots + (\dot{q} + \dot{r})\begin{bmatrix}\cos c_i\\ \sin c_i\\ 0\end{bmatrix} + \dot{q}\begin{bmatrix}-a_i \sin c_i\\ a_i \cos c_i\\ b_i\end{bmatrix}$$

$$+ (1/2)\,\dot{q}(b_i^2 + a_i^2)\begin{bmatrix}\cos c_i\\ \sin c_i\\ 0\end{bmatrix} + \ldots \tag{15}$$

where $q = q(t)$

In Eq. (15), the components of $\bar{R}_{HV}$ are given in the hub coordinate system, (H).

For the case of dV along the wire at an instantaneous length, q(t),

$$[d\bar{R}_{HV}/dt\ _H]_H = \dot{q} \begin{bmatrix} \cos c_i \\ \sin c_i \\ 0 \end{bmatrix} + q \begin{bmatrix} -\dot{a}_i \sin c_i \\ \dot{a}_i \cos c_i \\ \dot{b}_i \end{bmatrix} + \dot{q} \begin{bmatrix} -a_i \sin c_i \\ a_i \cos c_i \\ b_i \end{bmatrix}$$

$$- q\ (b_i\dot{b}_i + a_i\dot{a}_i) \begin{bmatrix} \cos c_i \\ \sin c_i \\ 0 \end{bmatrix} - (1/2)\ \dot{q}\ (b_i^2 + a_i^2) \times$$

$$\begin{bmatrix} \cos c_i \\ \sin c_i \\ 0 \end{bmatrix} + \ldots \qquad (16)$$

The quadratic approximation to the kinetic energy per unit mass of the volume element can be obtained by calculating,

$$(1/2)\ [d\bar{R}_{OV}/dt|_I]_H^T \cdot [d\bar{R}_{OV}/dt|_I]_H \qquad (17)$$

through second order.

If dV is along the wire, Eq. (13) may be expanded with the aid of Eqs. (15) and (16) to yield:

$$d\bar{R}_{OV}/dt|_I = [\dot{x}_H - q\dot{a}_i \sin c_i - q(b_i\dot{b}_i + a_i\dot{a}_i) \cos c_i]\bar{i}$$

$$+ [\dot{y}_H + q\dot{a}_i \cos c_i - q(b_i\dot{b}_i + a_i\dot{a}_i) \sin c_i]\bar{j}$$

$$+ [q\ \dot{b}_i + \dot{z}_H]\bar{k} + \bar{\omega} \times \bar{R}_{HV}\ . \qquad (18)$$

In order to include the effect of the wire on the composite system moments of inertia we consider, for example, the contribution of the wire to I_y ,

$$I_{y_{wire}} = I_{uu}\alpha^2 + I_{vv}\beta^2 + I_{ww}\gamma^2 + md^2 \qquad (19)$$

where I_{uu}, I_{vv} and I_{ww} represent the principal moments of inertia of the wire, (u is the longitudinal wire axis, v and w are transverse axes). The term md^2 represents the moment of inertia of the wire due to the offset of its center of mass from the hub principal "y_H" axis. α, β, γ are the direction cosines: $(\hat{u} \cdot \hat{y})$, $(\hat{v} \cdot \hat{y})$, and $(\hat{w} \cdot \hat{y})$, respectively. Assuming the wire is thin and noting that for small displacements of a_i, b_i, $\gamma \ll 1$, Eq. (19) may be approximated by:

$$I_{y_{wire}} \cong I_{vv}\beta^2 + md^2 \qquad (20)$$

where $\beta^2 = \cos^2 a_i$. The moment of inertia of the wire about the v axes (Fig. 2), can be expressed as:

$$I_{vv} = 2\int_0^{\ell/2} q^2\rho dq = m\ell^2/12 \qquad (21)$$

where ρ is the mass per unit length. After substitution of Eq. (21) into Eq. (20),

$$I_{y_{wire}} = (m\ell^2/12)\cos^2 a_i + m\,[r+(\ell/2)\cos a_i]^2 \quad (22)$$

which can be simplified to,

$$I_{y\ wire} = (m\ell^2/3)\cos^2 a_i + mr^2 + mr\ell\cos a_i \quad (23)$$

The contribution of the moment of inertia of the tip mass can be developed in a similar manner as:

$$I_{y_{tip\ mass}} = m_t[(r+\ell\cos a_i)^2 + (\ell\sin a_i)^2] \quad (24)$$

The combined effect of the appendages can be obtained by adding Eqs. (23) and (24) to yield:

$$I_{y_{appendages}} \cong (m/3+m_t)\ell^2+2(m/2+m_t)r\ell + (m+m_t)r^2 \quad (25)$$

for the case where $a_i \ll 1$. From symmetry, the wire and tip mass in the $(\pm x, \pm y)$ quadrant also contribute to I_y. Eq. (25) now becomes,

$$I_y = [2(m/3+m_t)\ell^2 + 4(m/2+m_t)\ell r + 2(m_t+m)r^2] = I_x \quad (26)$$

Assuming the hub is symmetrical, $(I_{H_1} = I_{H_2})$ then the composite principal moments of inertia can be expressed as:

$$I_1 = I_2 = I_{H_1} + I_y \quad (27)$$

The composite moment of inertia about the third principal axis of the hub can be written as,

$$I_3 = I_{H_3} + I_{3\ wire} + I_{3\ tip\ mass} \quad (28)$$

The second term in Eq. (28) can be approximated as:

$$I_{3_{wire}} \cong I_{wire} + md^2 \tag{29}$$

With the aid of Figure 2, Eq. (29) can be expanded to yield,

$$I_{3_{wire}} = (m\ell^2/3) + mr^2 + m\ell r \cos a_i \tag{30}$$

The third term of Eq. (28) can be written as:

$$I_{3_{tip\ mass}} = m_t r^2 + 2m_t \ell r \cos a_i + \ell^2 m_t \tag{31}$$

After substitution of Eqs. (30) and (31) into Eq. (29),

$$I_3 = I_{H_3} + (m_t + m/3)\ell^2 + 2(m_t + m/2)\ell r + (m_t + m)r^2 \tag{32}$$

If we consider the four appendages, then Eq. (32) can be written as:

$$I_3 = I_{H_3} + [4(m_t + m/3)\ell^2 + 8(m_t + m/2)\ell r + 4(m_t + m)r^2] \tag{33}$$

When Eq. (17) is expanded in the calculation of T_{wires}, terms which are periodic such as $\sin(\psi + c_i)$ and $\cos(\psi + c_i)$ appear. If we consider the four appendages, then the terms can be related using the following transformations:

Let $\alpha = \psi + c_1$ thus,

$$\begin{aligned}
\sin\alpha &= \sin\alpha \\
\cos\alpha &= \cos\alpha \\
\sin(\alpha+\pi/2) &= \cos\alpha \\
\cos(\alpha+\pi/2) &= -\sin\alpha \\
\sin(\alpha+\pi) &= -\sin\alpha \\
\cos(\alpha+\pi) &= -\cos\alpha \\
\sin(\alpha+3\pi/2) &= -\cos\alpha \\
\cos(\alpha+3\pi/2) &= \sin\alpha
\end{aligned} \tag{34}$$

Now, volume integration over each wire can be performed, with the result that the kinetic energy of the wires for constant length is calculated as:

$$T_{wires} = (1/2)[2(\dot{\theta}_1^2 + \dot{\theta}_2^2)m\ell^2/3 + 2(\dot{\theta}_1^2 + \dot{\theta}_2^2)rm\ell + 2(\dot{\theta}_1^2 + \dot{\theta}_2^2)r^2m] \tag{35}$$

If dV were at the $i^{\underline{th}}$ tip mass, then expansion of Eq. (17), multiplication by m_t, and substitution of ℓ for q (for constant length), would result in the kinetic energy for the $i^{\underline{th}}$ tip mass.

$$T_{tip\ mass} = (1/2)(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2)\, m_t + \dot{z}_H\, \dot{b}_i m_t \ell + (1/2)(\dot{a}_i^2 + \dot{b}_i^2) m_t \ell^2$$

$$+ (1/2) w_0^2\, m_t[\ell^2 + r^2 + r\ell - \ell^2(b_i^2 + a_i^2) - r\ell(b_i^2 + a_i^2)]$$

$$+ (1/2)(\delta\dot{\theta}_3)^2\, m_t(r^2 + \ell^2 + r\ell) + w_0\, \dot{\theta}_1 \theta_2\, m_t(r^2 + \ell^2 + 2r\ell)$$

$$+ w_0\, \delta\dot{\theta}_3\, m_t(r^2 + \ell^2 + 2r\ell) + (1/2)(a_i^2\, w_0^2\, \ell^2\, m_t)$$

$$- m_t \ell(\dot{a}_i \dot{x}_H + \dot{y}_H\, a_i w_0) \sin c_i + m_t \ell(\dot{y}_H\, \dot{a}_i - \dot{x}_H\, a_i w_0) \cos c_i$$

$$- m_t \ell[b_i w_0\, \dot{\theta}_1(r+\ell) + \dot{b}_i\, \dot{\theta}_2(r+\ell)] \cos(\psi + c_i)$$

$$- m_t \ell[b_i w_0\, \dot{\theta}_2(r+\ell) + \dot{b}_i\, \dot{\theta}_1(r+\ell)] \sin(\psi + c_i)$$

$$+ m_t(\dot{\theta}_1^2 + \dot{\theta}_2^2)(r^2 + 2r\ell + \ell^2) \qquad (36)$$

If we consider variable length appendages, the formulation of Ref. 1 can still be used, except length is now a function of time and the position of an arbitrary volume element along the wire, (q), is a function of time. The additional terms which result for $d\bar{R}_{0V}/dt|_{deployment}$ can be expressed as follows:

$$d\bar{R}_{0V}/dt|_{deployment} = [\dot{q} \cos c_i - \dot{q}\, a_i \sin c_i - (1/2)\, \dot{q}\, (b_i^2 + a_i^2) \cos c_i]\, \bar{I} + [\dot{q} \sin c_i + \dot{q}\, a_i \cos c_i - (1/2)\, \dot{q}(b_i^2 + a_i^2) \sin c_i]\, \bar{J} + \dot{q}\, b_i\, \bar{K} \qquad (37)$$

When Eq. (37) is substituted into Eq. (17), the resulting additional terms in the kinetic energy per unit mass are:

$$
\begin{aligned}
dT_{deployment} = {} & (1/2)\,\dot{q}^2 - \dot{q}q(\dot{b}_i b_i + \dot{a}_i a_i) - \dot{q}\, w_0\, q\, a_i - \dot{q}\, \delta\dot{\theta}_3\, q\, a_i \\
& - \dot{q}\, \dot{\theta}_1 \theta_2 q\, a_i + q \cdot \dot{q}\, \dot{a}_i\, a_i + \dot{q}\, w_0\, a_i(q+r) + \dot{q}\, a_i\, \delta\dot{\theta}_3(q+r) \\
& + \dot{q}\, a_i\, \dot{\theta}_1 \theta_2 (q+r) + \dot{x}_H \dot{q} \cos c_i - \dot{x}_H \dot{q}\, a_i \sin c_i \\
& - (1/2)\dot{x}_H\, \dot{q}(b_i^2 + a_i^2) \cos c_i + \dot{y}_H\, \dot{q} \sin c_i + \dot{y}_H \dot{q}\, a_i \cos c_i \\
& - (1/2)\dot{y}_H \dot{q}(b_i^2 + a_i^2) \sin c_i + \dot{q}\, \dot{z}_H\, b_i + \dot{q}\, q\, \dot{b}_i\, b_i \\
& + r\dot{q}b_i\dot{\theta}_2\, \delta\theta_3 \sin(\psi + c_i) + r\dot{q}\, \dot{\theta}_1 b_i \sin(\psi + c_i) \\
& + r\dot{q}\, \dot{\theta}_1\, \delta\theta_3\, b_i \cos(\psi + c_i) - r\dot{\theta}_2 \dot{q}\, b_i \cos(\psi + c_i)
\end{aligned}
\tag{38}
$$

The limits of integration over the differential volume in the expression for T_{wires} can be formulated as follows:

$q(t) = \dot{q}t$, where $\dot{q}$ is assumed to be constant (for an assumed uniform extension rate). Then, $dq = \dot{q}\,dt$, and

$$\int_0^{\ell} (\ldots)dq \rightarrow \int_0^{\ell/\dot{q}} (\ldots)\dot{q}\,dt, \quad 0 \leq t \leq t_f$$

when

$$t = 0,\ q = 0;\ t = \ell/\dot{\ell},\ q = \ell(t)$$

when

$$\ell \leq \ell_f,\ t_f = \ell_f/\dot{\ell} \quad \text{when} \quad t(0) = 0;\ \dot{q} = \dot{\ell}$$

where ℓ_f and t_f represent final length and time, respectively.

The following additional terms result for T_{wires}.

$$T_{wires_{depl.}} = (2/3)m\dot{\ell}^2 + (1/2)\sum_{i=1}^{4} w_0 a_i r m_i \dot{\ell}^2 + (1/2)\sum_{i=1}^{4} a_i \, \delta\dot{\theta}_3 r m_i \dot{\ell}^2$$

$$+ (1/2)\sum_{i=1}^{4} a_i \dot{\theta}_1 \theta_2 r m_i \dot{\ell}^2 + (1/2)\sum_{i=1}^{4} \dot{z}_H b_i m_i \dot{\ell} \qquad (39)$$

If we consider $T_{tip\ mass}$, where the expansion of Eq. (17) is similar to that for T_{wires}, then multiplication by m_t and substituting $\dot{\ell}$ for $\dot{q}$ yields:

$$T_{tip\ mass_{depl.}} = 2m_t\dot{\ell}^2 + \sum_{i=1}^{4} \ddot{\ell} m_t w_0 a_i r + \sum_{i=1}^{4} \dot{\ell} m_t a_i \delta\dot{\theta}_3 r + \sum_{i=1}^{4} \ddot{\ell} z_H b_i m_t \qquad (40)$$

To consider the kinetic energy due to the stored mass of the rods inside the hub, the following assumptions are made:

(1) at full extension there is no stored mass;

(2) the mass is stored at a point inside the hub, a distance r from the hub center;

(3) the extension rate is the same for all rods;

(4) the wires are homogenous.

The extended mass per unit volume along the wire can be defined as:

$$m(t) = \rho \ell(t) \tag{41}$$

where $0 \leq t \leq t_f$, and ρ is the mass per unit length. The stored mass per unit volume can be defined as:

$$dm_{i\ \text{stored}} = (\ell_f - \ell(t))\rho \tag{42}$$

where $\ell(t)$ resprsents the instantaneous length. Since the stored mass is inside the hub, then Eq. (14) can be used to express $d\overline{R}_{OV}/dt|_I$, where;

$$\overline{R}_{HV} = r(\cos c_i\ \overline{i} + \sin c_i\ \overline{j}) \tag{43}$$

If we substitute Eqs. (14) and (43) into Eq. (17), and apply the same procedure, the terms which result for the combined effect of the four locations, yield:

$$T_{\text{stored mass}} = (1/2) \sum_{i=1}^{4} \rho_i(\ell_f - \ell(t))\ [4(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + 4r^2 \{w_0^2 + (\dot{\theta}_1\theta_2)^2 + (\delta\dot{\theta}_3)^2\} + 8r^2(w_0\ \delta\dot{\theta}_3 + w_0\ \dot{\theta}_1\theta_2 + \delta\dot{\theta}_3\dot{\theta}_1\theta_2) + 8r^2\ \dot{\theta}_1\ \theta_2\ \delta\theta_3 + 2r^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + 2r^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)(\delta\theta_3)^2] \tag{44}$$

Eqs. (11), (35), (36), (39), (40) and (44) can now be combined to provide the quadratic approximation of the kinetic energy for the entire system, which can be written as:

$$T = (1/2)M(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + I_3\, w_0(\delta\dot{\theta}_3 + \dot{\theta}_1\theta_2) + (1/2)\, I_3\, w_0^{\,2}$$

$$+ (1/2)\, I_1(\dot{\theta}_1^2 + \dot{\theta}_2^2) + (1/2)\, I_3(\delta\dot{\theta}_3)^2 + M_1(w_0 + \delta\dot{\theta}_3) \sum_{i=1}^{4} a_i$$

$$+ (1/2)\, M_3\, w_0^2 \sum_{i=1}^{4} a_i^2 + M_2\, \dot{z}_H \sum_{i=1}^{4} \ddot{b}_i - (1/2)\, M_1\, w_0^{\,2} \sum_{i=1}^{4} (a_i^2 + b_i^2)$$

$$+ (1/2)\, M_3 \sum_{i=1}^{4} (\dot{a}_i^{\,2} + \dot{b}_i^2)$$

$$+ M_2 \sum_{i=1}^{4} [- \dot{x}_H(\dot{a}_i \sin c_i + w_0\, a_i \cos c_i) + \dot{y}_H\, (\dot{a}_i \cos c_i$$

$$- w_0 a_i \sin c_i)] + M_1 \sum_{i=1}^{4} \{\dot{\theta}_1\, [b_i \sin (\psi + c_i) - w_0 b_i \cos(\psi + c_i)]$$

$$- \dot{\theta}_2[\dot{b}_i \cos (\psi + c_i) + w_0\, b_i \sin (\psi + c_i)]\} + (2/3)m\dot{\ell}^2$$

$$+ (1/2) \sum_{i=1}^{4} w_0\, a_i r m_i\, \dot{\ell}^2 + (1/2) \sum_{i=1}^{4} a_i\, \delta\dot{\theta}_3\, r m_i \dot{\ell}^2$$

$$+ (1/2) \sum_{i=1}^{4} a_i \dot{\theta}_1 \theta_2\, r m_i\, \dot{\ell}^2 + (1/2) \sum_{i=1}^{4} \dot{z}_H b_i m_i \dot{\ell} + 2 m_t \dot{\ell}^2$$

$$+ \sum_{i=1}^{4} w_0 a_i r m_t \dot{\ell} + \sum_{i=1}^{4} a_i\, \delta\dot{\theta}_3\, r m_t \dot{\ell} + \sum_{i=1}^{4} a_i \dot{\theta}_1 \theta_2\, r m_t \dot{\ell}$$

$$+ \sum_{i=1}^{4} \dot{z}_H b_i m_t \dot{\ell} + (1/2) \sum_{i=1}^{4} \rho_i(\ell_f - \ell(t))\, [4(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + 4r^2\, [w_0^{\,2}$$

$$+ (\dot{\theta}_1\theta_2)^2 + (\delta\dot{\theta}_3)^2] + 8r^2(w_0\, \delta\dot{\theta}_3 + w_0\, \dot{\theta}_1\theta_2 + \delta\dot{\theta}_3\theta_1\theta_2)$$

$$+ 8r^2\dot{\theta}_1\dot{\theta}_2\delta\dot{\theta}_3 + 2r^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + 2r^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)\, (\delta\theta_3)^2] \tag{45}$$

where the varying length terms (containing $\dot{\ell}$) and the stored mass terms are additions to the constant length terms developed in Ref. 1. In connection with Eq. (45) the following parameters have been defined:

$$M = m_H + 4m + 4m_t$$

$$M_1 = (m_t + m\ell/3 + m_t r + mr/2)\ell = M_3 + rM_2$$

$$M_2 = (m_t + m/2)\ell$$

$$M_3 = (m_t + m/3)\ell^2$$

where m = m(ℓ), and are defined similarly to those given in Ref. 1

Terms associated with the stored mass in Eq. (45) can be included with the composite moments of inertia expressed in Eqs. (27) and (33), and also with M_1, M_2, and M_3. These quantities can now be redefined in modified form as:

$$M = m_H + 4\rho\ell(t) + 4m_{stored} + 4m_t = \text{constant} \qquad (46a)$$

where $m_{stored} = \rho(\ell_f - \ell(t))$

$$M_1 = (m_t\ \ell(t) + (1/3)\rho[\ell\ (t)]^2 + rm_t + (1/2)r\,\ell\,(t))\ \ell(t) \qquad (46b)$$

$$M_2 = (m_t + (1/2)\rho\,\ell\,(t))\,\ell(t) \qquad (46c)$$

$$M_3 = (m_t + (1/3)\ \rho\ell(t)[\ell(t)]^2 \qquad (46d)$$

$$I_1 = I_{H_1} + [2(m_t + (1/2)\rho\ell(t))[\ell(t)]^2$$

$$+ 4(m_t + (1/2)\rho\ell(t))r\,\ell(t) + 2(m_t + \rho\ell_f)r^2 \qquad (46e)$$

$$I_3 = I_{H_3} + [4(m_t + (1/3)\rho\ell(t)[\ell(t)]^2$$

$$+ 8(m_t\ (1/2)\rho\ell(t))r\ell(t) + 4(m_t + \rho\ell_f)r^2] \qquad (46f)$$

For varying length, m has been defined in Eq. (41), assuming the mass per unit length, (ρ), of each appendage is the same.

The corresponding time rates of change of the (new) composite parameters and of the (new) composite moments of inertia can be written as:

$$\dot{M} = 0 \qquad (46g)$$

$$\dot{M}_1 = 2m_t\,\dot{\ell}\ell + \rho\dot{\ell}\ell^2 + rm_t\dot{\ell} + r\rho\dot{\ell}\ell \qquad (46h)$$

$$\dot{M}_2 = m_t\dot{\ell} + \rho\dot{\ell}\ell \qquad (46i)$$

$$\dot{M}_3 = 2m_t\dot{\ell}\ell + \rho\dot{\ell}\ell^2 \qquad (46j)$$

$$\dot{I}_1 = 4m_t\dot{\ell}\ell + 2\rho\dot{\ell}\ell^2 + 4m_t\dot{\ell} + 4r\rho\dot{\ell}\ell \qquad (46k)$$

$$\dot{I}_3 = 8m_t\dot{\ell}\ell + 4\rho\dot{\ell}\ell^2 + 8m_tr\dot{\ell} + 8r\rho\dot{\ell}\ell \qquad (46l)$$

where $\ell = \ell(t)$ and $\dot{\ell} = \dot{\ell}(t)$

D. Development of the Equations of Motion

The fourteen generalized coordinates selected are; X, Y, Z, θ_1, θ_2, θ_3, a_1, a_2, a_3, a_4, b_1, b_2. b_3 and b_4. We note that X, Y and Z, are inertially fixed coordinates and do not appear in Eq. (45), thus they are cyclic. Lagrange's equations of motion:

$$d/dt(\partial T/\partial \dot{q}_i) - \partial T/\partial q_i = Q_i \quad i = 1,2,\ldots 14 \qquad (47)$$

where Q_i for the wires (booms) is the generalized force derivable from a potential energy function which is assumed to be proportional to the square of the flexural coordinates, a_i and b_i, and involves a linear structural restoring spring constant, k; also Q is derivable from a Rayleigh dissapative function, $\mathcal{F}$ (i.e. $\mathcal{F} \propto \dot{a}_i^2$ and $\mathcal{F} \propto \dot{b}_i^2$, i=1 - 4 booms), involving assumed linear viscous structural rate damping, C. Eq. (47) may be developed in the coordinates, q_i, to yield:

$$d/dt(\partial T/\partial \dot{X}) = C_1 = p_1;\ d/dt(\partial T/\partial \dot{Y}) = C_2 = p_2;$$

$$d/dt(\partial T/\partial \dot{Z}) = C_3 = p_3 \qquad (47a)$$

where p_1, p_2, p_3 are the constant generalized momenta associated with X, Y and Z, respectively. If we choose the inertial axes such that there is no momentum of the spacecraft relative to the inertial axes at time t = 0, then the constant momenta are zero, therefore $C_1 = C_2 = C_3 = 0$. Eq. (47) can be developed for this system as a set of fourteen equations of motion. Furthermore if we keep $\dot{x}_H$, $\dot{y}_H$ and $\dot{z}_H$ as coordinates, then solving the linear differential equations with variable coefficients (when ℓ is constant), is made easier.[1] The equations of motion can be expressed as follows:

$$M\ddot{x}_H - (m_t\dot{\ell} + \rho\ell\dot{\ell}) \sum_{i=1}^{4} (\dot{a}_i \sin c_i + w_0 \cos c_i) - M_2 \sum_{i=1}^{4} (\ddot{a} \sin c_i + w_0 \dot{a}_i \cos c_i + \dot{w}_0 a_i \cos c_i) = 0 \tag{48a}$$

$$M\ddot{y}_H + \dot{M}_2 \sum_{i=1}^{4} (\dot{a}_i \cos c_i - w_0 a_i \sin c_i) + M_2 \sum_{i=1}^{4} (\ddot{a}_i \cos c_i - w_0 \dot{a}_i \sin c_i - \dot{w}_0 a_i \sin c_i) = 0 \tag{48b}$$

$$M\ddot{z}_H + \dot{M}_2 \sum_{i=1}^{4} \dot{b}_i + M_2 \sum_{i=1}^{4} \ddot{b}_i + \sum_{i=1}^{4} \rho_i [\ddot{b}_i \ell\dot{\ell} + b_i(\dot{\ell}^2 + \ell\ddot{\ell})]/2 + \sum_{i=1}^{4} m_{t_i} (\dot{b}_i \dot{\ell} + \ddot{\ell} b_i) = 0 \tag{48c}$$

$$\dot{I}_3 w_0\theta_2 + I_3(\dot{w}_0\theta_2 + w_0 \dot{\theta}_2) + \dot{I}_1\dot{\theta}_1 + I_1\ddot{\theta}_1 + \dot{M}_1 \sum_{i=1}^{4} [\dot{b}_i \sin (\psi + c_i) - w_0 b_i \cos(\psi + c_i)] + M_1 \sum_{i=1}^{4} [(\ddot{b}_i + w_0^2 b_i) \sin (\psi + c_i) - \dot{w}_0 b_i \cos (\psi + c_i)] + \sum_{i=1}^{4} r\rho_i \{\dot{a}_i\theta_2\ell\dot{\ell}^2 + a_i [\dot{\theta}_2 \ell\dot{\ell}^2 + \theta_2 (\dot{\ell}^3 + 2\ell \dot{\ell}\ddot{\ell})]\}/2 + \sum_{i=1}^{4} rm_{t_i} [\dot{a}_i\theta_2\dot{\ell} + a_i(\dot{\theta}_2 \dot{\ell} + \theta_2\ddot{\ell})]$$

$$+ 4 \sum_{i=1}^{4} \rho_i r^2 \{\ell_f(\ddot{\theta}_2 \delta\theta_3 + \dot{\theta}_2 \delta\dot{\theta}_3) - [\dot{\ell}\dot{\theta}_2 \delta\theta_3 + \ell(\ddot{\theta}_2 \delta\theta_3 + \dot{\theta}_2 \delta\dot{\theta}_3)]\} = 0 \qquad (48d)$$

$$\dot{I}_1 \dot{\theta}_2 + I_1 \ddot{\theta}_2 - \dot{M}_1 \sum_{i=1}^{4} [\dot{b}_i \cos(\psi + c_i) + w_0 b_i \sin(\psi + c_i)]$$

$$- M_1 \sum_{i=1}^{4} [(\ddot{b}_i + w_0^2 b_i) \cos(\psi + c_i) + \dot{w}_0 b_i \sin(\psi + c_i)]$$

$$+ 4 \sum_{i=1}^{4} \rho_i r^2 \{\ell_f(\ddot{\theta}_1 \delta\theta_3 + \dot{\theta}_1 \delta\dot{\theta}_3) - [\dot{\ell}\dot{\theta}_1 \delta\theta_3 + \ell(\ddot{\theta}_1 \delta\theta_3 + \dot{\theta}_1 \delta\dot{\theta}_3)]\}$$

$$- I_3 w_0 \dot{\theta}_1 - (1/2) \sum_{i=1}^{4} a_i \dot{\theta}_1 r\ell\rho \dot{\ell}^2 = 0 \qquad (48e)$$

$$\dot{I}_3 (w_0 + \delta\dot{\theta}_3) + I_3(\dot{w}_0 + \delta\ddot{\theta}_3) + \dot{M}_1 \sum_{i=1}^{4} \dot{a}_i + M_1 \sum_{i=1}^{4} \ddot{a}_i$$

$$+ (1/2) \sum_{i=1}^{4} \rho_i r(\dot{a}_i \ell\dot{\ell}^2 + a_i [\dot{\ell}^3 + 2\ell\dot{\ell}\ddot{\ell}]) + \sum_{i=1}^{4} r m_{t_i} (\dot{a}_i \dot{\ell} + a_i \ddot{\ell})$$

$$- 4 \sum_{i=1}^{4} \rho_i (\ell_f - \ell) r^2 \dot{\theta}_1 \dot{\theta}_2 = 0 \qquad (48f)$$

$$M_1(w_0^2 a_j + \dot{w}_0 + \delta\ddot{\theta}_3) + \dot{M}_1(w_0 + \delta\dot{\theta}_3) + M_3(\ddot{a}_j - w_0^2 a_j)$$

$$+ \dot{M}_3 \dot{a}_j + M_2 \sum_{j=1}^{4} [(-\ddot{x}_H + \dot{y}_H w_0) \sin c_i + (\ddot{y}_H + \dot{x}_H w_0) \cos c_i]$$

$$+ \dot{M}_2 \sum_{i=1}^{4} (\dot{x}_H \sin c_i + \dot{y}_H \cos c_i) - (1/2) \sum_{i=1}^{4} \rho_i r \ell \dot{\ell}^2 (w_0$$

$$+ \delta\dot{\theta}_3 + \dot{\theta}_1 \theta_2) - \dot{\theta}_1 \theta_2 r \dot{\ell} \sum_{i=1}^{4} m_{t_i} + ka_j + C\dot{a}_j = 0 \qquad (48g)$$

$$M_2 \ddot{z}_H + \dot{M}_2 \dot{z}_H + M_3 \ddot{b}_j + \dot{M}_3 \dot{b}_j + M_1 \sum_{i=1}^{4} [(\ddot{\theta}_1 + 2w_0 \dot{\theta}_2) \sin(\psi + c_i)$$

$$+ (-\ddot{\theta}_2 + 2w_0 \dot{\theta}_1) \cos(\psi + c_i)] + \dot{M}_1 \sum_{i=1}^{4} [\dot{\theta}_1 \sin(\psi + c_i)$$

$$- \dot{\theta}_2 \cos(\psi + c_i)] + M_1 w_0^2 b_j + kb_j + C\dot{b}_j = 0 \qquad (48h)$$

where $\dot{M}_1 = 2m_t \dot{\ell}\ell + \rho\dot{\ell}\ell^2 + r\rho\dot{\ell}\ell$ in Eq. (48g);

$\dot{M}_2 = (\rho\dot{\ell}\ell/2)$ in Eq. (48h).

In Eqs. (48g) and (48h), $j = 1,2,3,4$; i.e. these equations represent a set of eight Lagrangian equations corresponding to the coordinates $a_1 - a_4$ and $b_1 - b_4$, respectively. The terms ka_j, kb_j, $C\dot{a}_j$ and $C\dot{b}_j$ in these equations, are associated with the generalized force Q_i for the wires (booms) due to the effect of a linear (structural) restoring spring constant, k, and structural damping represented by C, (a linear viscous rate damping constant), which is assumed to exist in the four booms. Both k and C are assumed to be the same for each boom, and associated with a potential energy function and a Raleigh dissapative function, respectively.

Since Eqs. (48a), (48b) and (48c) yield first integrals where $\dot{x}_H$, $\dot{y}_H$ and $\dot{z}_H$ can be related to the constant generalized momenta p_1, p_2, and p_3, respectively, then $\ddot{x}_H$, $\ddot{y}_H$ and $\ddot{z}_H$ are zero and can be eliminated from these equations. Eq. (48) can be reduced to a set of eleven equations by solving Eqs. (48a), (48b) and (48c) for $\dot{x}_H$, $\dot{y}_H$ and $\dot{z}_H$ respectively and substituting them into Eqs. (48g) and (48h). In Eqs. (48), we will now assume that the tip mass, m_{t_i}, and mass per unit length, ρ_i, are the same for the four appendages.

CHAPTER III

Stability

To examine stability before and immediately after deployment, we assume $\dot{\ell} = \ddot{\ell} = 0$. Eq. (48) for constant length[1], can be reduced to the following set of ten equations by solving for $\delta\ddot{\theta}_3$ in Eq. (48f) and substituting the results into Eq. (48g).

$$I_1\ddot{\theta}_1 + I_3 w_0 \dot{\theta}_2 + M_1[(\ddot{b}_1 + w_0^2\, b_1) \sin(\psi + c_1)$$

$$+ (\ddot{b}_2 + w_0^2\, b_2) \cos(\psi + c_1) - (\ddot{b}_3 + w_0^2\, b_3) \sin(\psi + c_1)$$

$$- (\ddot{b}_4 + w_0^2\, b_4) \cos(\psi + c_1)] = 0 \tag{49a}$$

$$I_1\ddot{\theta}_2 - I_3 w_0 \dot{\theta}_1 - M_1[(\ddot{b}_1 + w_0^2\, b_2) \cos(\psi + c_1)$$

$$+ (\ddot{b}_2 + w_0^2\, b_2) \sin(\psi + c_1) + (\ddot{b}_3 + w_0^2\, b_3) \cos(\psi + c_1)$$

$$- (\ddot{b}_4 + w_0^2\, b_4) \sin(\psi + c_1) = 0 \tag{49b}$$

$$(M_3 - M_1^2/I_3)\ddot{a}_j + (M_1 - M_3) w_0^2\, a_j + M_2[(-M_2 \sum_{i=1}^{4} (\dot{a}_j \cos c_i$$

$$- w_0 a_i \sin c_i) w_0 \sin c_j + M_2 \sum_{i=1}^{4} (\dot{a}_i \sin c_i$$

$$+ w_0 a_i \cos c_i) w_0 \cos c_j]/M + ka_j + C\dot{a}_j = 0$$

j= 1,2,3,4 equations (49c)

$$M_3\ddot{b}_j + M_1 w_0^2 b_j + M_1[(\ddot{\theta}_1 + 2w_0\dot{\theta}_2) \sin(\psi + c_j)$$

$$+ (-\ddot{\theta}_2 + 2w_0 \dot{\theta}_1) \cos(\psi + c_j) + kb_j + C\dot{b}_j = 0 \qquad (49d)$$

j = 1,2,3,4 equations

The homogenous system of linear differential equations can be expressed in state vector form:[5]

$$\dot{X}(t) = A(t + T)X(t) \qquad (50)$$

where X(t) is a (20 x 1) column matrix of the coordinates and velocities and A(t + T) represents a (20 x 20) matrix. Typical periodic coefficients in A(t + T) are:

$$\sin(\psi + c_i) \quad \text{and} \quad \cos(\psi + c_i)$$

which occur in Eqs. (49a), (49b) and (49d) for constant length.

A. Application of the Lyapunov Reducibility Theorem

The Lyapunov reducibility theorem[5] may be applied to reduce the linear system of differential equations with periodic coefficients to a kinematically similar system of differential equations. (A recent application of this theorem in studying the stability of a dual-spin spacecraft with a flexible momentum wheel is given in

Ref. 6.) The theorem states that there is a bounded nonsingular transformation matrix S(t) with a bounded inverse $S^{-1}(t)$ such that a matrix D, defined by:

$$D = S^{-1} AS - S^{-1} \dot{S} \tag{51}$$

is a constant matrix. We can then relate

$$X = S(t)Y \tag{52}$$

and obtain the system

$$\dot{Y} = DY \tag{53}$$

as the kinematically equivalent of Eq. (50). The bounded transformation of the form of Eq. (52) for the present system can be expressed:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & & & & & & & & \\ \sin\psi & \cos\psi & & & & & & \bigcirc & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & \bigcirc & & & & 1 & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{54}$$

In Eq. (54) if $Z = [\varepsilon_1 \varepsilon_2 \ldots b_4]^T$, then

$$Y = \begin{pmatrix} Z \\ \dot{Z} \end{pmatrix}$$

Thus, the kinematically equivalent system can be expressed in terms of the Z matrix and the constant coefficient matrices, M, G, D, K, as:

$$M\ddot{Z} + G\dot{Z} + D\dot{Z} + KZ = 0 \qquad (55)$$

M is the symmetric mass inertia matrix, G is a skew symmetric matrix involving gyroscopic terms, D is a symmetric positive semidefinite damping matrix and K is a symmetric stiffness matrix. These matrices can be expressed as follows:

$$
M = \left[\begin{array}{cccccc|c|c|c|c}
I_1 & 0 & 0 & 0 & 0 & 0 & M_1 \sin c_1 & M_1 \cos c_1 & -M_1 \sin c_1 & -M_1 \cos c_1 \\
0 & I_2 & 0 & 0 & 0 & 0 & -M_1 \cos c_1 & M_1 \sin c_1 & M_1 \cos c_1 & -M_1 \sin c_1 \\
0 & 0 & M_3 - M_1^2/I_3 & -M_1^2/I_3 & -M_1^2/I_3 & -M_1^2/I^3 & 0 & 0 & 0 & 0 \\
0 & 0 & -M_1^2/I_3 & M_3 - M_1^2/I_3 & -M_1^2/I_3 & -M_1^2/I^3 & 0 & 0 & 0 & 0 \\
0 & 0 & -M_1^2/I_3 & -M_1^2/I_3 & M_3 - M_1^2/I_3 & -M_1^2/I_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -M_1^2/I_3 & -M_1^2/I_3 & -M_1^2/I_3 & M_3 - M_1^2/I_3 & 0 & 0 & 0 & 0 \\
M_1 \sin c_1 & -M_1 \cos c_1 & 0 & 0 & 0 & 0 & M_3 & 0 & 0 & 0 \\
M_1 \cos c_1 & M_1 \sin c_1 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 & 0 \\
-M_1 \sin c_1 & M_1 \cos c_1 & 0 & 0 & 0 & 0 & 0 & 0 & M_3 & 0 \\
-M_1 \cos c_1 & -M_1 \sin c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3
\end{array}\right] \tag{56}
$$

$$
G = \begin{bmatrix}
0 & w_0(I_3-2I_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
w_0(2I_1-I_3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_2^2 w_0/M & 0 & -M_2^2 w_0/M & 0 & 0 & 0 & 0 \\
0 & 0 & -M_2^2 w_0/M & 0 & M_2^2 w_0/M & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -M_2^2 w_0/M & 0 & M_2^2 w_0/M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -M_2^2 w_0/M & 0 & 0 & 0 & 0 & 0 \\
 & & & & \bigcirc & & & & &
\end{bmatrix}
\tag{57}
$$

$$D = \begin{bmatrix} 0 & & & & & & & \bigcirc & & \\ & 0 & & & & & & & & \\ & & C & & & & & & & \\ & & & C & & & & & & \\ & & & & C & & & & & \\ & & & & & C & & & & \\ & \bigcirc & & & & & C & & & \\ & & & & & & & C & & \\ & & & & & & & & C & \\ & & & & & & & & & C \end{bmatrix} \tag{58}$$

$$K = \begin{bmatrix} G & 0 & 0 & 0 & 0 & 0 & D & E & -D & -E \\ 0 & G & 0 & 0 & 0 & 0 & -E & D & E & -D \\ 0 & 0 & A & B & -C & -B & 0 & 0 & 0 & 0 \\ 0 & 0 & B & A & -B & C & 0 & 0 & 0 & 0 \\ 0 & 0 & -C & -B & A & B & 0 & 0 & 0 & 0 \\ 0 & 0 & -B & C & B & A & 0 & 0 & 0 & 0 \\ D & -E & 0 & 0 & 0 & 0 & F & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 & F & 0 & 0 \\ -D & E & 0 & 0 & 0 & 0 & 0 & 0 & F & 0 \\ -E & -D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F \end{bmatrix} \tag{59}$$

where, in Eq. (59),

$$A = k + r M_2 w_0^2 [1 - M_2 (\sin^2 c_1 - \cos^2 c_1)/rM]$$

$$B = 2(M_2^2 w_0^2/M) \sin c_1 \cos c_1$$

$$C = M_2^2 w_0^2/M (\sin^2 c_1 - \cos^2 c_1)$$

$$D = M_1 w_0^2 \sin c_1$$

$$E = M_1 w_0^2 \cos c_1$$

$$F = M_1 w_0^2 + k$$

$$G = I_3 w_0 - I_1 w_0^2$$

B. Application of the Kelvin-Tait-Chetaev Theorem

In the equation:

$$M\ddot{Z} + DZ + GZ + KZ = 0 \tag{60}$$

we assume that M and D are symmetric positive definite, G is skew symmetric, and that none of the eigenvalues of the symmetric matrix, K, is zero. Then the stability of the solutions of Eq. (60) is the same as the stability of the solutions of the truncated equation[7]

$$M\ddot{Z} + KZ = 0.$$

If M, K, and D are positive definite, the zero solution of Eq. (55) is asymptotically stable; if M and K are positive definite and D is positive semidefinite with ($D \neq 0$), the solution of Eq. (55) is stable in the sense of Lyapunov but not asymptotically stable.[7] The Kelvin-Tait-Chetaev Theorem can be applied to this particular system, where D is positive semidefinite in the absence of damping on the main hub, by examining the M and K matrices for positive definiteness. If, for a particular set of system parameters, any of the principal minor subdeterminants of either the M or K matrices, Eq. (56) or Eq. (59), is negative, then the system will be unstable in the Lyapunov sense.

C. Extension of Stability

A recent extension of the Kelvin-Tait-Chetaev theorem to establish asymptotic stability of linear systems where the damping matrix is only positive semidefinite, was presented by Müller[8] and is summarized briefly, here.

The mechanical system,

$$M\ddot{Z} + (D + G)\dot{Z} + KZ = 0$$

with the fxf matrices

$$M = M^T > 0,\ D = D^T \geq 0,\ G = -G^T,\ K = K^T$$

is then asymptotically stable if the fxf matrix K is; (a) positive definite, and (b), that the rank of an augmented matrix, S be:

$$\text{Rank of } S = \text{Rank } [\Gamma \vdots F\Gamma \vdots F^2\Gamma \vdots \ldots \vdots F^{n-1}\Gamma] = n \quad (n = 2f)$$

with

$$F = \left[\begin{array}{c|c} 0 & E_f \\ \hline -M^{-1}K & -M^{-1}G \end{array}\right], \quad \Gamma = \begin{bmatrix} 0 \\ M^{-1} D \end{bmatrix}$$

where F is a (2f x 2f) matrix and Γ is a (2f x f) matrix. Condition (a) is a result of applying Sylvester's theorem when the principal minors of matrix K are positive definite.[8,9] This extension of the Kelvin-Tait-Chetaev theorem which tests the system for asymptotic stability can be applied to this particular system. K can be tested for positive definiteness for a particular set of system parameters and the augmented matrix, S, can be constructed as previously indicated and tested for a rank equal to n (twenty, for this system). If the rank of S is n then the system will be asymptotically stable in the Lyapunov sense, even when there is no damping on the main (hub) part of the satellite.

CHAPTER IV

Concluding Comments

The rotational equations of motion for a spinning spacecraft system with deployable appendages and the criteria for the stability of such a system when the appendages are at a fixed length, have been developed. The future analysis of this system will include the following:

1. an attempt to identify mode shapes for constant extension rates.
2. an application of the Kelvin-Tait-Chetaev theorem and its extension to predict the stability of this system before and after deployment maneuvers for different values of system parameters.
3. a computer simulation of the equations of motion with first order flexibility present and then absent. ($a_i = b_i = \dot{a}_i = \dot{b}_i = 0$)
4. a computer simulation of the equations of motion with first order flexibility, in-plane bending only.
5. an attempt to relate the EI (flexural rigidity) of the wires to an effective restoring linear spring constant, k.

Conclusions regarding the effect of flexibility on the system during extension or retraction as well as on the stability of the system before or immediately after extension or retraction, (numerical results) will be forthcoming.

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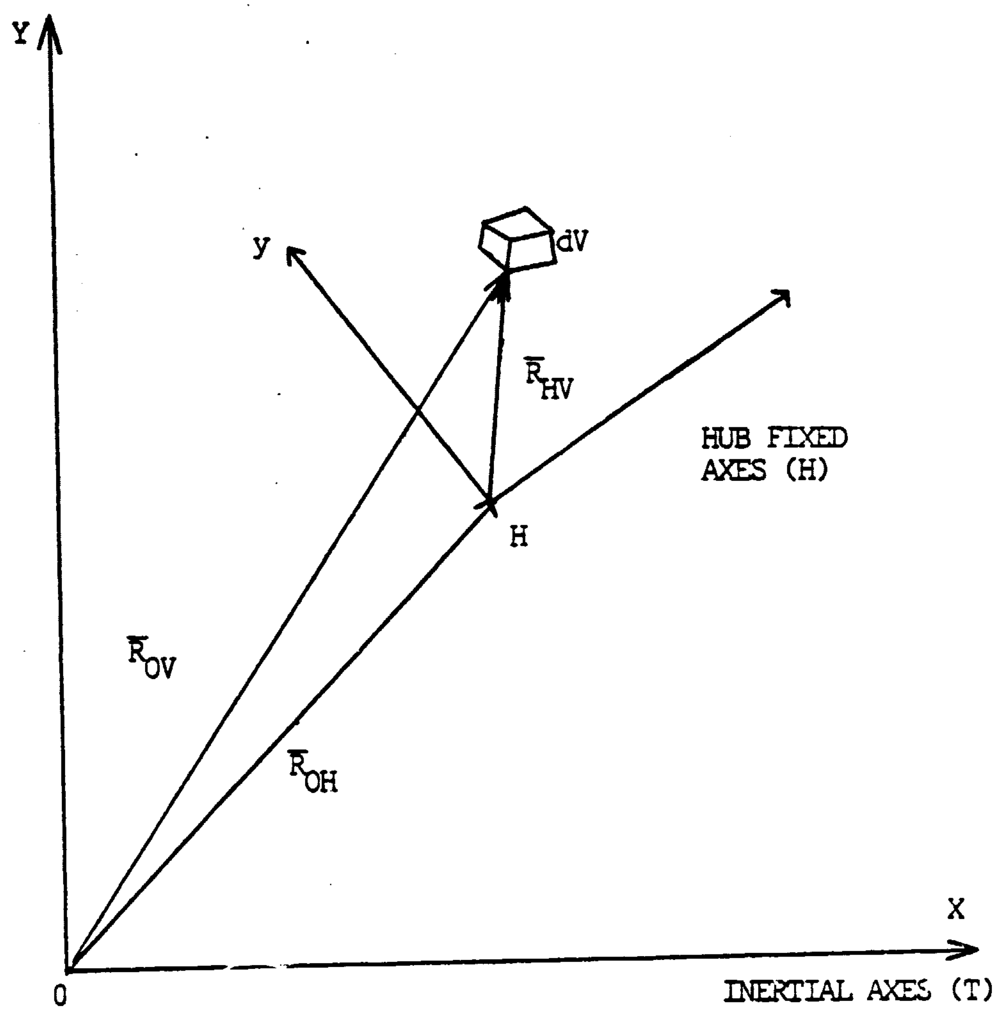

Fig. 1. Inertial and hub fixed coordinate systems.

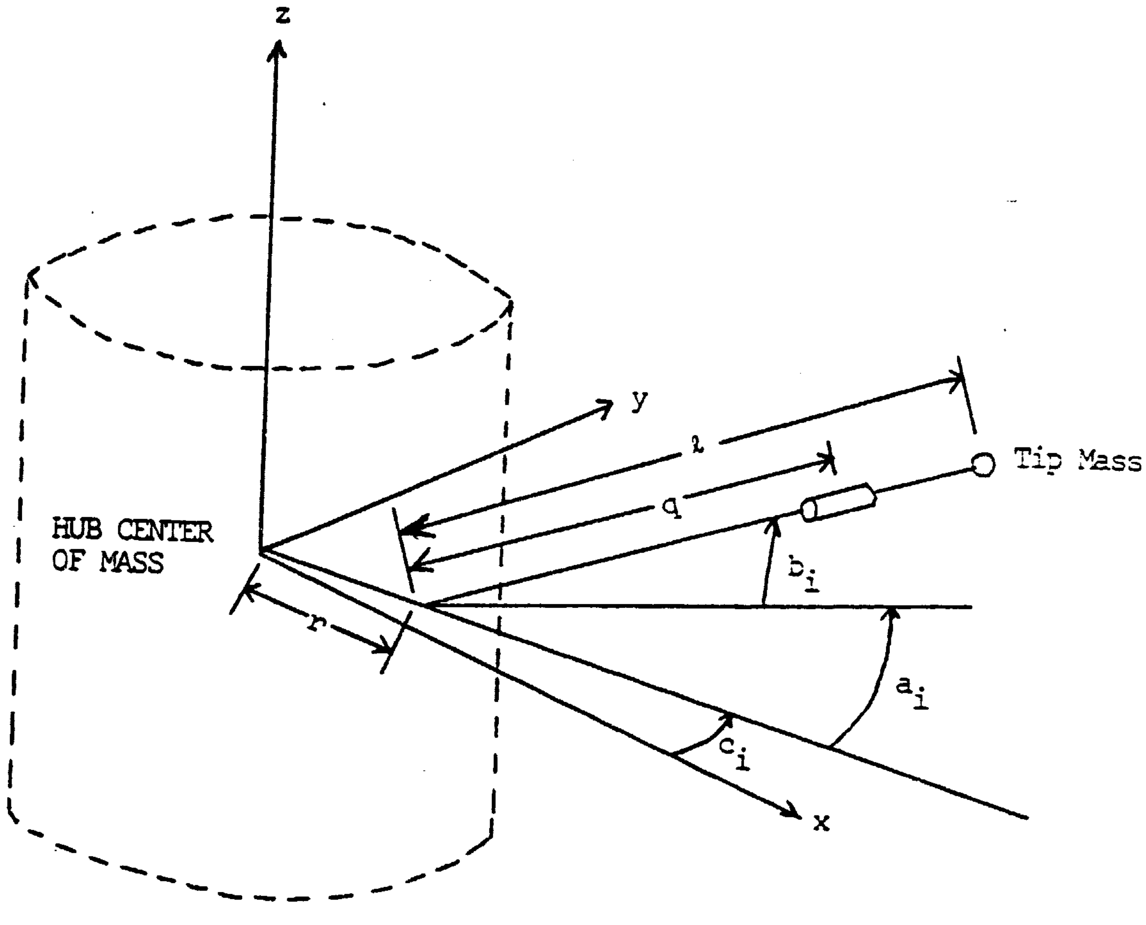

Fig. 2. Variables specifying position of wire element.