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ALGORITHMS FOR ADAPTIVE STOCHASTIC CONTROL FOR A

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SUMMARY

This report is concerned with the control of linear, discrete-time, stochastic systems with unknown control gain parameters. Two suboptimal adaptive control schemes are derived: one is based on underestimating future control and the other is based on overestimating future control. Both schemes require little on-line computation and incorporate in their control laws some information on estimation errors. The performance of these laws is studied by Monte Carlo simulations on a computer. Two single-input, third-order systems are considered, one stable and the other unstable, and the performance of the two adaptive control schemes is compared with that of the scheme based on enforced certainty equivalence and the scheme where the control gain parameters are known.

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I. Introduction

Problems of controlling systems under uncertainty have long attracted the attention of many control theorists and engineers because of their importance in practical control systems. Since the work of Bellman [1], the stochastic adaptive control approach has been useful for treating such problems ([2]; also see [3] for a survey). For state space models, the optimization approach for stochastic adaptive control has been studied extensively. However, explicit solutions have been obtained for only a limited class of problems, for example, the well known certainty equivalence solution of the standard linear quadratic gaussian problem. Although more general problems have been conceptually solved (i.e., requiring formal solutions of functional equations), explicit forms of the optimal control laws (if they exist) have yet to be obtained. In order to overcome the difficulties in solving the functional equations, many suboptimal schemes have been proposed [3]. Most of them incorporate approximations for some features of adaptive control. However, except for the ad hoc scheme where the certainty equivalence principle is enforced (this scheme will be called the CE law), they usually require a considerable amount of on-line computation, which can often be prohibitive. For example, the control law based on the dual control approach in [4], which exhibits an active learning property, requires extensive on-line computation to evaluate future observation programs. The open loop optimal feedback control law (OLOF) ignores future measurements but incorporates some information concerning the uncertainty (covariances of estimation errors) in its control algorithm [5 - 7]. In this sense, this scheme was called "cautious" in [3]. The OLOF law still requires numerical optimization techniques on-line.

The purpose of this study is to investigate two suboptimal schemes which require little on-line computation but incorporate the effects of estimation errors in their control laws, and to study the performance of these laws by Monte Carlo simulations on a computer. We consider discrete-time linear stochastic systems with unknown control gain parameters (essentially the same class of problems as that treated in [6]). Admittedly, this class of systems is small in practice. However, we believe that because of their conceptual simplicity and computational efficiency, the two laws derived in this report may provide a suitable framework for treating the more general problem, i.e., when the system state and control gain matrices are both unknown.

One of the two control laws is based on underestimating future control, hence called the UEFC law, and the other is based on overestimating future control, the OEFC law. Two single-input, third order systems (one stable and the other unstable) are simulated, and the performance of the UEFC and OEFC laws is compared with that of the CE law and the law where the control gain parameters are known. The sensitivity of the performance of the four laws is studied for various levels of initial uncertainties in the states and the control gain parameters.

This report is organized as follows: Section II defines the notations. A precise definition of the problem is given in Section III. Section IV presents the results of the application of Kalman filter theory for the optimal estimation problem. We derive the UEFC and OEFC laws in Section V, and Section VI shows the results of the Monte Carlo simulations. Section VII concludes with remarks on this study.

II. Notations

The transpose of a matrix X (vector x) is denoted by X^T (x^T). The trace of a square matrix X is denoted by $tr(X)$. The matrices I_n and $0_{m,n}$ denote the n-dimensional identity matrix and the $m \times n$ null matrix, respectively; the subscripts will be dropped when there is no ambiguity. The notation $X > 0$ ($X \geq 0$) denotes a positive definite (semidefinite) matrix X, and $X > Y$ ($X \geq Y$) implies $X - Y > 0$ ($X - Y \geq 0$). The Kronecker product of matrices X and Y is denoted by $X \otimes Y$. The mn dimensional row and column string vectors of an $m \times n$ matrix X are denoted by $rs(X)$ and $cs(X)$; i.e.,

$$[rs(X)]^T \equiv (x_{R1}^T \, x_{R2}^T \cdots x_{Rm}^T)$$

$$[cs(X)]^T \equiv (x_{C1}^T \, x_{C2}^T \cdots x_{Cn}^T)$$

where x_{Ri}^T (x_{Ci}) is the i-th row (column) vector of X.

The (conditional) expectation of a random vector x (given Y) is denoted by $E[x]$ ($E[x|Y]$). The notation $x \sim N(\bar{x}, X)$ means that a random vector x has Gaussian distribution with mean $\bar{x}$ and covariance X. Statements with "a.s." imply that they hold with probability 1.

Symbols with subscript or superscript "U" ("O") pertain to algorithms for UEFC (OEFC).

III. Problem Statement

We consider a standard finite-stage discrete-time linear stochastic control problem with a quadratic performance index. The system dynamics and measurement relations are described by

$$x(k+1) = A\,x(k) + B\,u(k) + D\,\xi(k) \tag{1}$$

$$y(k+1) = C\,x(k+1) + \eta(k+1), \qquad k = 0, 1, \cdots, N-1 \tag{2}$$

where the state $x(k)$, the control $u(k)$, the measurement $y(k)$ and the plant noise $\xi(k)$ are vectors of dimensions n, m, ℓ and r, respectively. The matrices A, C and D are of appropriate dimensions and are assumed to be known. The $n \times m$ control matrix B is a random matrix[1] with

$$b \sim N(\bar{b}, P_b), \qquad b \equiv rs(B)$$

The other primary random variables are

$$x(0) \sim N(\bar{x}_0, P_0)$$

$$\xi(k) \sim N(0, Q(k))$$

$$\eta(k) \sim N(0, R(k)), \qquad R(k) > 0$$

$\xi(k)$ and $\eta(k)$ are mutually independent white noise sequences, and both are independent of b and $x(0)$; b and $x(0)$ are also mutually independent.

[1]For simplicity of derivation, we assume that B is a constant matrix. The extension of our results to the case with linearly varying B as in [6] is straightforward.

The performance measure we wish to minimize is given by

$$J \equiv E\left[\sum_{k=0}^{N-1} J(k)\right] \equiv E\left[\sum_{k=0}^{N-1}\left\{x(k+1)^T S(k+1)\, x(k+1) + u(k)^T \Lambda(k)\, u(k)\right\}\right] \tag{3}$$

where $S(k+1) \geq 0$ and $\Lambda(k) > 0$. Admissible control laws are causal; i.e.,

$$u(k) \equiv u(k, Y(k), U(k-1))$$

where $Y(k) \equiv \{y(1), \cdots, y(k)\}$ and $U(k-1) \equiv \{u(0), \cdots, u(k-1)\}$. $u(0)$ must be a function of prior information on the system.

IV. Estimation

Since the system equations (1) and (2) are linear in the random vector $x(k)$ and random matrix B, Kalman filter theory can be applied to modified system equations to obtain the optimal minimum variance estimates.

Applying Lemma A.1 in the Appendix, we get

$$B\,u(k) = I_n B\,u(k) = [I_n \otimes u(k)^T]\, b \tag{4}$$

We can write the following system equations for the augmented state vector $z(k)^T \equiv (x(k)^T\, b^T)$

$$z(k+1) = F(k)\, z(k) + G\,\xi(k) \tag{5}$$

$$y(k+1) = H\, z(k+1) + \eta(k+1) \tag{6}$$

where[2]

$$F(k) \equiv \begin{bmatrix} A & I_n \otimes u(k)^T \\ 0_{nm,n} & I_{nm} \end{bmatrix}, \qquad G \equiv \begin{bmatrix} D \\ 0_{nm,r} \end{bmatrix} \tag{7}$$

$$H \equiv \begin{bmatrix} C & 0_{\ell,nm} \end{bmatrix} \tag{8}$$

Application of Kalman filter theory to the linear equations (5) and (6) yields the following optimal minimum variance estimate:

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[2] If we arrange the vectors of B columnwise we obtain augmented system equations of the same form as (5) - (8), except that $F(k)$ is given by

$$F(k) \equiv \begin{bmatrix} A & u(k)^T \otimes I_n \\ 0_{nm,n} & I_{nm} \end{bmatrix}$$

The augmented state vector for this case is $z(k)^T \equiv (x(k)^T\, b_c^T)$, where $b_c \equiv cs(B)$. The row string arrangement in (5) - (8) is preferred in order to facilitate backward optimization (see Section V).

$$\hat{z}(k+1|k+1) = F(k)\,\hat{z}(k|k) + K(k+1)\,[y(k+1) - H\,F(k)\,\hat{z}(k|k)] \tag{9}$$

$$K(k+1) = P(k+1|k+1)\,H^T[H\,P(k+1|k)\,H^T + R(k)]^{-1} \tag{10}$$

$$P(k+1|k) = F(k)\,P(k|k)\,F^T(k) + G\,Q(k)\,G^T \tag{11}$$

$$P(k+1|k+1) = [I_n - K(k+1)\,H]\,P(k+1|k) \tag{12}$$

$$\hat{z}(0|0) \equiv \begin{bmatrix} \bar{x}_0 \\ \bar{b} \end{bmatrix}, \qquad P(0|0) \equiv \begin{bmatrix} P_0 & 0_{n,nm} \\ 0_{nm,n} & P_b \end{bmatrix}$$

where $\hat{z}(k|k) \equiv E[z(k)|Y(k)]$, $\hat{z}(k+1|k) \equiv E[z(k+1)|Y(k)] = F(k)\,\hat{z}(k|k)$ and

$$P(k|k) = E[\{z(k) - \hat{z}(k|k)\}\{z(k) - \hat{z}(k|k)\}^T|Y(k)] \tag{13}$$

$$P(k+1|k) = E[\{z(k+1) - \hat{z}(k+1|k)\}\{z(k+1) - \hat{z}(k+1|k)\}^T|Y(k)] \tag{14}$$

We partition $\hat{z}(i|k)$ and $P(i|k)$ as

$$\hat{z}(i|k) \equiv \begin{bmatrix} \hat{x}(i|k) \\ \hat{b}(i|k) \end{bmatrix}, \qquad P(i|k) \equiv \begin{bmatrix} \pi_1(i|k) & \pi_3(i|k)^T \\ \pi_3(i|k) & \pi_2(i|k) \end{bmatrix} \tag{15}$$

where $\hat{x}(i|k)$ is an n-dimensional vector, and $\pi_1(i|k)$ and $\pi_2(i|k)$ are $n \times n$ and $nm \times nm$ matrices, respectively.

V. Feedback Control Laws

It is well known that the control laws which solve the optimization problem are the formal solutions of the functional equation [2]

$$J_k^* = \min_{u(k)} J_k, \qquad k = N-1, \cdots, 0 \tag{16}$$

where

$$J_k \equiv E[J(k) + J_{k+1}^* |Y(k)], \qquad J_N^* \equiv 0 .$$

However, closed form solutions of the backward optimization are not available, and various suboptimal schemes have been proposed (see, for example, [3] for a survey of such schemes). Some of the schemes [4, 6] require a considerable amount of on-line computation at each stage k. We derive here two feedback laws which do not require lengthy on-line computations. The two laws are obtained by carrying out the backward optimization (16) approximately. In the following derivations of the control laws, the time indices will be dropped for brevity when there is no ambiguity in notation.

V.1 Control Law Based on Underestimating Future Control Efforts (UEFC)

This control law is derived by underestimating the effects of future control. The backward "sub-optimization" proceeds as follows:

Last Stage: $k = N-1$

Since $J_N \equiv 0$, it is easy to obtain the quadratic cost-to-go functional

$$J_{N-1} = E[x(N)^T S(N)\, x(N) + u(N-1)^T \Lambda(N-1)\, u(N-1) \,|\, Y(N-1)]$$
$$= u(N-1)^T \Lambda(N-1)\, u(N-1) + \mathrm{tr}\{S(N)\, E[B\, u(N-1)\, u(N-1)^T B^T | Y(N-1)]\}$$
$$+ 2\,\mathrm{tr}\{A^T S(N)\, E[B\, u(N-1)\, x(N-1)^T | Y(N-1)]\} + \alpha(N-1) + \beta(N-1) \quad (17)$$

where

$$\alpha(N-1) \equiv \mathrm{tr}\{A^T S(N)\, A\, E[x(N-1)\, x(N-1)^T | Y(N-1)]\} \quad (18)$$

$$\beta(N-1) \equiv \mathrm{tr}[D^T S(N)\, D\, Q(N-1)] \quad (19)$$

are independent of $u(N-1)$.

Recalling (4), we can rewrite the second and third terms as

$$\mathrm{tr}\{S E[B\, u u^T B^T | Y] = \mathrm{tr}\{S(I_n \otimes u^T)\, E[b\, b^T | Y]\, (I_n \otimes u)\}$$
$$= \mathrm{tr}\{S(N)\, [I_n \otimes u(N-1)^T]\, M_2(N-1|N-1)\, [I_n \otimes u(N-1)]\} \quad (20)$$

$$\mathrm{tr}\{A^T S\, E[B\, u\, x^T | Y]\} = \mathrm{tr}\{A^T S [I_n \otimes u^T]\, E[b\, x^T | Y]\}$$
$$= \mathrm{tr}\{A^T S(N)\, [I_n \otimes u(N-1)^T]\, M_3(N-1|N-1)\} \quad (21)$$

where the M_j's are defined by

$$M(i|k) \equiv \begin{bmatrix} M_1(i|k) & M_3(i|k)^T \\ M_3(i|k) & M_2(i|k) \end{bmatrix} \equiv E\left[\begin{matrix} x(i)\, x(i)^T & x(i)\, b^T \\ b\, x(i)^T & b\, b^T \end{matrix} \middle| Y(k)\right]$$
$$= \begin{bmatrix} \pi_1(i|k) + \hat{x}(i|k)\, \hat{x}(i|k)^T & \pi_3(i|k)^T + \hat{x}(i|k)\, \hat{b}(i|k)^T \\ \pi_3(i|k) + \hat{b}(i|k)\, \hat{x}(i|k)^T & \pi_2(i|k) + \hat{b}(i|k)\hat{b}(i|k)^T \end{bmatrix} \quad (22)$$

Applying Lemma A.2 to (20) and (21), we have

$$\mathrm{tr}\{S(I_n \otimes u^T)\, M_2 (I_n \otimes u)\} = cs(I_n \otimes u)^T (S \otimes M_2)\, cs(I_n \otimes u)$$
$$= u(N-1)^T [\Gamma^T (S(N) \otimes M_2(N-1|N-1)\Gamma]\, u(N-1) \quad (23)$$

$$\mathrm{tr}\{A^T S(I_n \otimes u^T)\, M_3\} = \mathrm{tr}\{M_3 A^T S (I_n \otimes u^T)\}$$
$$= [cs(M_3 A^T S)]^T cs(I_n \otimes u^T)$$
$$= \{\Gamma^T cs[M_3(N-1|N-1)\, A^T S(N)]\}^T u(N-1) \quad (24)$$

where the following identity was used to obtain the final expressions:

$$cs(I_n \otimes u) = \Gamma\, u$$

$$\Gamma^T \equiv [I_m\, 0_{m,nm}\, I_m\, 0_{m,nm}\, I_m, \cdots, 0_{m,nm}\, I_m]$$

Note that Γ is an $n^2 m \times m$ matrix.

Thus, (17), (23) and (24) yield

$$J_{N-1} = u(N-1)^T [\Lambda(N-1) + \Theta(N-1)]\, u(N-1) + 2w(N-1)^T u(N-1) + \alpha(N-1) + \beta(N-1) \tag{26}$$

where

$$\Theta(N-1) \equiv \Gamma^T [S(N) \otimes M_2(N-1|N-1)]\, \Gamma \tag{27}$$

$$w(N-1) \equiv \Gamma^T cs[M_3(N-1|N-1)\, A^T S(N)] \tag{28}$$

Therefore, the optimal control law $u^*(N-1)$ and the associated cost-to-go are given by

$$u^*(N-1) = -[\Lambda(N-1) + \Theta(N-1)]^{-1} w(N-1) \tag{29}$$

$$J^*_{N-1} = -w(N-1)^T [\Lambda(N-1) + \Theta(N-1)]^{-1} w(N-1) + \alpha(N-1) + \beta(N-1) \tag{30}$$

Note that $\Theta(N-1) \geq 0$ a.s., since $S(N) \geq 0$ and $M_2(N-1|N-1) > 0$ a.s. (see Lemma A.3 in the Appendix). Hence $\Lambda(N-1) + \Theta(N-1) > 0$ and invertible a.s., since $\Lambda(N-1) > 0$.

Stage $k = N-2$

The functional relation (16) yields

$$\begin{aligned} J_{N-2} &= E[J(N-2) + J^*_{N-1} | Y(N-2)] \\ &= E[-w(N-1)^T \{\Lambda(N-1) + \Theta(N-1)\}^{-1} w(N-1) | Y(N-2)] \\ &\quad + E[J(N-2) + \alpha(N-1) | Y(N-2)] + \beta(N-1) \end{aligned} \tag{31}$$

Since $Y(N-1) = \{Y(N-2), y(N-1)\}$, from (18)

$$\begin{aligned} E[\alpha(N-1)|Y(N-2)] &= E[E\{x(N-1)^T A^T S(N)\, A\, x(N-1) | Y(N-1)\} | Y(N-2)] \\ &= E[x(N-1)^T A^T S(N)\, A\, x(N-1) | Y(N-2)] \end{aligned}$$

Therefore, it is straightforward to obtain

$$\begin{aligned} J^U_{N-2} &\equiv E[J(N-2) + \alpha(N-1) | Y(N-2)] + \beta(N-1) \\ &= u(N-2)^T [\Lambda(N-2) + \Theta_U(N-2)]\, u(N-2) \\ &\quad + 2\, w_U(N-2)^T u(N-2) + \alpha_U(N-2) + \beta_U(N-2) \end{aligned} \tag{32}$$

where

$$\Theta_U(N-2) \equiv \Gamma^T [V_U(N-2) \otimes M_2(N-2|N-2)]\, \Gamma \tag{33}$$

$$w_U(N-2) \equiv \Gamma^T cs[M_3(N-2|N-2)\, A^T V_U(N-2)] \tag{34}$$

$$\alpha_U(N-2) \equiv tr\{A^T V_U(N-2)\, A\, E[x(N-2)\, x(N-2)^T | Y(N-2)]\} \tag{35}$$

$$\beta_U(N-2) \equiv \beta(N-1) + \mathrm{tr}[D^T S(N-1)\, D\, Q(N-2)] \tag{36}$$

$$V_U(N-2) \equiv S(N-1) + A^T S(N)\, A \tag{37}$$

The difficulty in optimization lies in evaluating the first term in (31), since $\Theta(N-1)$ and $w(N-1)$ are complicated random matrix and vector, respectively, depending on $u(N-1)$. In this control law the term is neglected in order to simplify the backward optimization. Note that the term is nonpositive a.s., since $\Lambda(N-1) + \Theta(N-1) > 0$ a.s. This term originates from the first two terms in (26) (with the optimal law $u^*(N-1)$ in (29)), and accounts for the amount of reduced cost due to the control at stage $N-1$. Hence the omission of this term means that the control law at $N-2$ is designed by neglecting the control effect at $N-1$ ($E[\alpha(N-1)|Y(N-2)]$ accounts for the cost due to the free motion from $N-1$ to N). Although this approximation may seem somewhat ad hoc, the resulting control law requires little on-line computation and shows good performance in the simulated examples, as will be observed in Section VI.

With the above simplification, we have the control law $u_U(N-2)$ which minimizes ( ) and the associated cost-to-go functional J^U_{N-2}

$$u_U(N-2) = -[\Lambda(N-2) + \Theta_U(N-2)]^{-1} w_U(N-2) \tag{38}$$

$$J^*_{N-2} \le J^U_{N-2} = -w_U(N-2)^T [\Lambda(N-2) + \Theta_U(N-2)]^{-1} w_U(N-2) + \alpha_U(N-2) + \beta_U(N-2) \tag{39}$$

Algorithm for UEFC

By proceeding with the simplification described for stage N-2, we obtain the control law for a general stage k:

$$u_U(k) = -[\Lambda(k) + \Theta_U(k)]^{-1} w_U(k) \tag{40}$$

where

$$\Theta_U(k) \equiv \Gamma^T [V_U(k) \circledast M_2(k|k)]\, \Gamma \ge 0 \qquad \text{a.s.} \tag{41}$$

$$w_U(k) \equiv \Gamma^T \mathrm{cs}[M_3(k|k)\, A^T V_U(k)] \tag{42}$$

$$V_U(k) = S(k+1) + A^T V_U(k+1)\, A, \qquad k = N-1, \cdots, 0 \tag{43}$$

$$V_U(N) \equiv 0$$

and Γ and $M_i(k|k)$ are defined by (25) and (22), respectively.

Remarks:

1. Since $V_U(k)$ can be computed off-line by (43), this control law requires no on-line recursive computation, but computation of only $\Theta_U(k)$ and $w_U(k)$ to obtain $u_U(k)$.

2. Note that $\Theta_U(k)$ and $w_U(k)$ are functions of $\pi_2(k|k)$ and $\pi_3(k|k)$, measures of estimation error, as well as $\hat{x}(k|k)$ and $\hat{b}(k|k)$ (see equation (22)). In this sense UEFC is cautious like OLOF [3].
3. As mentioned above for stage $N-2$, $\Lambda(k) + \Theta_U(k) > 0$ and invertible a.s., hence (40) provides a well-defined control law a.s.

V.2 Control Law Based on Overestimating Future Control Efforts (OEFC)

Stage $k = N-2$

The UEFC law was obtained by neglecting the term due to the control efforts at stage $N-1$ because of the difficulty in approximating the term in a simple manner. Here we bound the term, the first (negative) term in (31), from below, thereby obtaining a control law (OEFC) by overestimating the control efforts at stage $N-1$.

Lemma

The first term in (31) can be bounded as

$$\begin{aligned} &-w_U(N-1)^T[\Lambda(N-1)+\Theta(N-1)]^{-1}w(N-1) \\ &\qquad \geq -\mathrm{tr}\{[\Lambda(N-1)+\Theta(N-1)]^{-1}\,\Theta(N-1)\}\,\alpha(N-1) \end{aligned} \tag{44}$$

Proof:

Using $S(N) \geq 0$ and $M(N-1|N-1) > 0$ in Lemma A.4 in the Appendix, we have

$$\begin{bmatrix} S(N) \circledast M_1(N-1|N-1) & S(N) \circledast M_3(N-1|N-1)^T \\ S(N) \circledast M_3(N-1|N-1) & S(N) \circledast M_2(N-1|N-1) \end{bmatrix} \geq 0 \tag{45}$$

We define

$$\Psi \equiv \begin{bmatrix} \mathrm{tr}\{A^T S(N)\, A\, M_1(N-1|N-1)\} & [\mathrm{cs}\{M_3(N-1|N-1)\, A^T S(N)\}]^T \\ \mathrm{cs}\{M_3(N-1|N-1)\, A^T S(N)\} & S(N) \circledast M_2(N-1|N-1) \end{bmatrix} \tag{46}$$

then

$$\begin{aligned} \alpha(N-1) &= \mathrm{tr}[A^T S(N)\, A\, M_1(N-1|N-1)] = \mathrm{tr}(M_1 A^T S A) \\ &= \mathrm{tr}(S A M_1 A^T) = [\mathrm{cs}(A^T)]^T (S \circledast M_1)\, \mathrm{cs}(A^T) \end{aligned}$$

where Lemma A.2 was used to obtain the last equality. Also from Lemma A.1

$$\mathrm{cs}[M_3(N-1|N-1)\, A^T S(N)] = (S \circledast M_3)\, \mathrm{cs}(A^T)$$

Therefore,

$$\Psi = \begin{bmatrix} \{cs(A^T)\}^T (S \otimes M_1)\, cs(A^T) & \{(S \otimes M_3)\, cs(A^T)\}^T \\ (S \otimes M_3)\, cs(A^T) & S \otimes M_2 \end{bmatrix}$$

$$= \begin{bmatrix} \{cs(A^T)\}^T & 0_{1,n^2m} \\ 0_{n^2m,n^2} & I_{n^2m} \end{bmatrix} \begin{bmatrix} S \otimes M_1 & S \otimes M_3{}^T \\ S \otimes M_3 & S \otimes M_2 \end{bmatrix} \begin{bmatrix} cs(A^T) & 0_{n^2,n^2m} \\ 0_{n^2m,1} & I_{n^2m} \end{bmatrix}$$

Hence, on noting (45), we have

$$\Psi \geq 0$$

and an application of Lemma A.4 to (46) yields

$$cs[M_3(N-1|N-1)\, A^T \Gamma(N)]\{cs[M_3(N-1|N-1)\, A^T S(N)]\}^T \leq \alpha(N-1)[S(N) \otimes M_2(N-1|N-1)] \quad (47)$$

Thus, from (28)

$$\begin{aligned} & w(N-1)^T[\Lambda(N-1)+\Theta(N-1)]^{-1} w(N-1) \\ & \quad = tr\{(\Lambda+\Theta)^{-1} \Gamma^T cs(M_3 A^T S)[cs(M_3 A^T S)]^T \Gamma\} \\ & \quad \leq tr\{(\Lambda+\Theta)^{-1} \Gamma^T \alpha(S \otimes M_2)\, \Gamma\} \\ & \quad = tr\{[\Lambda(N-1)+\Theta(N-1)]^{-1}\, \Theta(N-1)\}\, \alpha(N-1) \end{aligned}$$

where (47), $\Lambda + \Theta > 0$ a.s. and Lemma A.5 were used to obtain the inequality. This completes the proof.

Using the above Lemma and (31), we have a lower bound for J_{N-2}

$$J_{N-2} \geq J^U_{N-2} - E[tr\{[\Lambda(N-1)+\Theta(N-1)]^{-1}\, \Theta(N-1)\}\, \alpha(N-1)\,|Y(N-2)] \quad (48)$$

where $\Theta(N-1)$ and $\alpha(N-1)$ are random matrix and variable, respectively, given $Y(N-2)$, and no simple expression is available for the second term. As can be observed in (27), $\Theta(N-1)$ is a function of $M_2(N-1|N-1)$, the estimate of bb^T (a constant random matrix) at $N-1$. In order to proceed with the analysis in a simple manner, $\Theta(N-1)$ is replaced by its estimate

$$\hat{\Theta}(N-1|N-2) \equiv E[\Theta(N-1)\,|Y(N-2)] \approx \Gamma^T[S(N) \otimes M_2(N-2|N-2)]\,\Gamma \quad (49)$$

which is a function of $Y(N-2)$. Therefore, (48) is approximated by

$$J^O_{N-2} \equiv J^U_{N-2} - tr\{[\Lambda(N-1)+\hat{\Theta}(N-1|N-2)]^{-1}\, \hat{\Theta}(N-1|N-2)\, E[\alpha(N-1)\,|Y(N-2)]\} \quad (50)$$

For (32) - (37) and (50), we have the following cost-to-go expression for OLFC:

$$\begin{aligned} J^O_{N-2} = {} & u(N-2)^T[\Lambda(N-2)+\Theta_O(N-2)]\, u(N-2) \\ & + 2\, w_O(N-2)^T u(N-2) + \alpha_O(N-2) + \beta_O(N-2) \end{aligned} \quad (51)$$

where

$$\Theta_0(N-2) \equiv \Gamma^T[V_0(N-2) \circledast M_2(N-2|N-2)]\,\Gamma \tag{52}$$

$$w_0(N-2) \equiv \Gamma^T\, cs[M_3(N-2|N-2)\,A^T\,V_0(N-2)] \tag{53}$$

$$\alpha_0(N-2) \equiv tr[A^T\,V_0(N-2)\,A\,M_1(N-2|N-2)] \tag{54}$$

$$\beta_0(N-2) \equiv \beta_U(N-2)$$

$$V_0(N-2) \equiv S(N-1) + \epsilon(N-1|N-2)\,A^T\,S(N)\,A \tag{55}$$

$$\epsilon(N-1|N-2) \equiv 1 - tr\{[\Lambda(N-1) + \hat{\Theta}(N-1|N-2)]^{-1}\,\hat{\Theta}(N-1|N-2)\} \tag{56}$$

Therefore, the control law OEFC which minimizes J^0_{N-2} is given by

$$u_0(N-2) = -[\Lambda(N-2) + \Theta_0(N-2)]^{-1}\,w_0(N-2) \tag{57}$$

$$J^0_{N-2} = -w_0(N-2)^T[\Lambda(N-2) + \Theta_0(N-2)]^{-1}\,w_0(N-2)$$
$$+ \alpha_0(N-2) + \beta_0(N-2) \tag{58}$$

A'...ithm for OEFC

Since the expression (58) for J^0_{N-2} has the same quadratic form as (39) for J^U_{N-2}, it is easy to obtain the OEFC control law for a general stage k:

$$u_0(k) = -[\Lambda(k) + \Theta_0(k)]^{-1}\,w_0(k) \tag{59}$$

where

$$\Theta_0(k) \equiv \Gamma^T[V_0(k) \circledast M_2(k|k)]\,\Gamma \tag{60}$$

$$w_0(k) \equiv \Gamma^T\, cs[M_3(k|k)\,A^T\,V_0(k)] \tag{61}$$

The matrix $V_0(k)$ is computed by the following (backward) recursive formula

$$V(i|k) = S(i+1) + \epsilon(i+1|k)\,A^T\,V(i+1|k)\,A \qquad i = N-1,\ N-2,\ \cdots,\ k \tag{62}$$

$$V_0(k) \equiv V(k|k),\ V(N|k) \equiv 0_{n,n} \tag{63}$$

$$\epsilon(i+1|k) \equiv 1 - tr\{[\Lambda(i+1) + \hat{\Theta}(i+1|k)]^{-1}\,\hat{\Theta}(i+1|k)\} \tag{64}$$

$$\hat{\Theta}(i+1|k) \equiv \Gamma^T[V(i+1|k) \circledast M_2(k|k)]\,\Gamma \tag{65}$$

Remarks:

1. The OEFC algorithm has the same structure as the UEFC law given by (40) - (43), where $\epsilon(i+1|k) \equiv 1$ (compare (43) with (62)).
2. The OEFC law requires more on-line computation than the UEFC law, since $\hat{\Theta}(i+1|k)$ depends on $M_2(k|k) = E[bb^T|Y(k)]$ and (62) must be recursively computed for each stage k.

VI. Examples

A computer simulation study was performed to evaluate the performance of the UEFC and OEFC control laws. The two systems selected are single-input third order systems, and are essentially the same as those in [6]; one is a stable system and the other is an unstable system. The performance of the laws for Monte Carlo runs is statistically compared with the certainty equivalence law (CE) and the optimal control law when B is known (called the LQG algorithm—the solution of the standard LQG problem). The sensitivity of performance of the four algorithms is studied for various levels of initial uncertainties (P_b and P_0).

The system matrices common to the two systems are

$$C = [1 \quad 0 \quad 0], \quad D^T = [0.2 \quad 0.4 \quad 0.6]$$

$$\bar{x}_0^T = [1 \quad 1 \quad 1], \quad Q(k) = 0.01, \quad R(k) = 0.09$$

$$S(k+1) = I_3, \quad \Lambda(k) = 1$$

We simulate 20 stage ($N = 19$) processes, and compute the sample mean M_J and standard deviation S_J of the performance measure $\sum_{k=0}^{N-1} J(k)$ for 20 Monte Carlo runs.

VI.1. Stable System

The system matrices are given by

$$A = \begin{bmatrix} 1 & 0.2 & 0.0 \\ 0 & 1.0 & 0.2 \\ -1 & -1.4 & 0.4 \end{bmatrix}, \quad \bar{B} = \bar{b} = \begin{bmatrix} 0.0 \\ 0.0 \\ -0.4 \end{bmatrix}$$

where A has eigenvalues 0.8 and $0.8 \pm 0.4j$. The performance of the four algorithms (UEFC, OEFC, CE and LQG) for the Monte Carlo runs is plotted in Figure 1 (sample mean M_J) and Figure 2 (sample standard deviation S_J) for $P_0 = 4\,I_3$ and different P_b's. The abscissa in the figures is σ_b, where $P_b \equiv \sigma_b^2\, I_3$. For each of the 20 runs, $B = b$ and $x(0)$ are randomly generated by the distributions $b \sim N(\bar{b}, P_b)$ and $x(0) \sim N(\bar{x}_0, P_0)$. Similarly, Figures 3 and 4 show the dependence of M_J and S_J on various σ_0's, where $P_0 \equiv \sigma_0^2\, I_3$, for $P_b = 4\,I_3$. In order to see the normalized performance of the suboptimal laws, the ratio

$$r_J \equiv \frac{M_J \text{ for a suboptimal law}}{M_J \text{ for the LQG law}}$$

is plotted in Figures 5 and 6 for various σ_b's and σ_0's, respectively.

Observations:

1. The performance of UEFC and OEFC remains almost the same as σ_b increases, whereas the CE performance becomes considerably worse (Figures 1, 2 and 5). This is to be expected, since both UEFC and OEFC take the errors of estimates into consideration and are cautious in implementing control, while CE does not consider such uncertainty (see Remark 2 following equation (43)).
2. The normalized performance of the three suboptimal laws is rather insensitive to variations in P_0; however, r_J decreases slightly as P_0 increases (Figure 6). This is because the uncertainty in $\bar{x}_0$ (P_0), which is common to the four laws (including the LQG law), becomes comparatively more dominant than the uncertainty in $\bar{b}(P_b = 4\,I_3)$ as σ_0 increases, and as a result the performance degradation due to unknown B tends to decrease.
3. Considering that the performance of the LQG law is impossible to attain and that the optimal law with unknown B is worse than the LQG law (the optimal law with known B), the performance of the UEFC and OEFC laws ($r_J \doteq 1.5 - 3$, Figures 5 and 6) is good, especially since little on-line computation is required.

In order to study further the characteristics of the UEFC and OEFC laws, the time histories of the four laws for a representative run are plotted in

Figure 7:	Control $u(k)$
Figure 8:	Estimate $\hat{b}(k\|k) \equiv [\hat{b}_1 \;\; \hat{b}_2 \;\; \hat{b}_3]^T$
Figure 9 - 12:	Estimate $\hat{x}(k\|k) \equiv [\hat{x}_1 \;\; \hat{x}_2 \;\; \hat{x}_3]^T$
Figure 13:	Instantaneous cost $J(k)$

For this run $P_b = P_0 = 4\,I_3$, the true values of B and $x(0)$ are

$$B^T = [0.54 \quad -2.07 \quad -3.42], \quad x(0)^T = [1.19 \quad 3.65 \quad 5.56]$$

and the performance measure $\sum_{k=0}^{N-1} J(k)$ is 404, 787, 880, and 4301 for the LQG, UEFC, OEFC, and CE laws, respectively.

Observations:

The characteristics of the three suboptimal laws are clearly shown in these figures. The CE law erroneously exerts large control in the beginning ($k = 0 - 5$ in Figure 7), thereby incurring large costs (Figure 13). The large control accidentally results in fast learning of B (Figure 8), and less cost $J(k)$ than the UEFC and OEFC laws at later stages ($k \geq 7$). Both UEFC and OEFC are cautious and very little control energy is implemented in the beginning ($k \leq 7$ in Figure 7), when larger estimation errors are expected (see Remark 2 following equation (43)). Since UEFC underestimates future control efforts, it is less cautious than OEFC and exerts more control at $k = 8 - 14$ than OEFC, thereby attaining better cost (Figure 13) and better estimate $\hat{b}(k|k)$ (Figure 8). Note that the estimation of $x(k)$ for UEFC and OEFC is very good (compare Figures 10 and 11 with Figures 9 and 12), although the estimate $\hat{b}(k|k)$ is not as good as CE.

VI.2. Unstable System

The system matrices are given by

$$A = \begin{bmatrix} 1 & 0.2 & 0.0 \\ 0 & 1.0 & 0.2 \\ 1 & -0.6 & 0.8 \end{bmatrix}, \quad \bar{B} = \bar{b} = \begin{bmatrix} 0.0 \\ 0.0 \\ -0.2 \end{bmatrix}$$

where A has eigenvalues 1.2 and $0.8 \pm 0.4\,j$. As for the stable system, the performance of the four algorithms for 20 Monte Carlo runs is plotted in

Figure 14: Sample mean M_J for various P_b's

Figure 15: Sample standard deviation S_J for various P_b's

Figure 16: Sample mean M_J for various P_0's

Figure 17: Sample standard deviation S_J for various P_0's

Figure 18: Normalized sample mean r_J for various P_b's

Figure 19: Normalized sample mean r_J for various P_0's

The time histories for a representative run are plotted in

Figure 20: Control $u(k)$

Figure 21: Estimate $\hat{b}(k|k)$

Figure 22: Instantaneous cost $J(k)$

where $P_b = P_0 = 4\,I_3$, the true values of B and $x(0)$ are $B^T = [-1.90 \;\; 1.50 \;\; -2.07]$ and $x(0)^T = [0.19 \;\; 1.76 \;\; 0.37]$; and the performance measure $\sum_{k=0}^{N-1} J(k)$ is 64, 471, 708, and 4565 for the LQG, UEFC, OEFC, and CE laws, respectively.

Observations:

1. The characteristics of the three suboptimal laws are very similar to those observed for the stable system.
2. The performance of the OEFC law is somewhat worse than that in the stable case, whereas the UEFC law performs consistently well (Figures 14 - 19). The CE law performs better than the cautious OEFC and UEFC laws for small P_b (σ_b = 0.1 and 0.3; i.e., when there is little uncertainty in $\bar{b}$).
3. Figures 20 - 22 illustrate the characteristics of the three laws more clearly than the stable case (see Observations for Figures 7 - 13); the large control efforts at early stages for the CE law cause large cost $J(k)$ and accidentally fast learning of B (Figure 21), which results in small cost at later stages (Figure 22). The UEFC law is less cautious than the OEFC law and its peak control efforts are implemented earlier (k = 7 - 10 in Figure 20) than the OEFC law (k = 10 - 16), resulting in better overall cost and estimate $\hat{b}(k|k)$. Note that the peak of $J(k)$ is also earlier for the UEFC law (k = 9 - 12 in Figure 22) than for the OEFC law (k = 13 - 18).

VII. Conclusions

We have considered a discrete-time linear stochastic adaptive control system with unknown control gain matrix (B). Two suboptimal control laws have been derived: the UEFC law based on the underestimation of future control and the OEFC law based on the overestimation on future control. These laws require little on-line computation and at the same time incorporate some information on the estimation errors, hence they are in the category of "cautious" controls as classified by Wittenmark [3]. Two single-input third order systems have been simulated to compare the Monte Carlo performance of the laws with that of the CE and LQG laws. The dependence of the performance of the four laws on P_b and P_0 (the initial uncertainties on the state x and the control gain B) has been studied. The results indicate that the UEFC and OEFC laws perform much better than the CE law with only a little extra computation being required.

Admittedly, the class of systems considered in this study is small. However, the UEFC and OEFC laws derived for this class are conceptually simple and computationally efficient, and may provide a suitable framework for treating the more general class, where the system matrix (A) as well as the control gain matrix (B) are unknown. Further research is envisaged in this direction.

Appendix

The identities and inequalities used to derive the estimation and control laws in the preceding sections are collected and proved where necessary. The matrices involved in the following lemmas are assumed to be conformable.

Lemma A.1

$$cs(ABC) = (C^T \otimes A)\, cs(B) \tag{A1}$$

$$rs(ABC) = (A \otimes C)\, rs(B) \tag{A2}$$

Lemma A.2

$$tr(AB) = cs(A)\, cs(B) \tag{A3}$$

$$tr(A\, C^T B\, C) = cs(C)^T (A \otimes B^T)\, cs(C) \tag{A4}$$

For the proofs of (A1), (A3), and (A4), the reader is referred to [8]. The identity (A1) is due to Nissen [9]. The proof of (A2) is straightforward and is omitted.

Lemma A.3

$$\text{If} \quad A \geq 0 \quad \text{and} \quad B \geq 0, \text{ then} \quad A \otimes B \geq 0 \tag{A5}$$

$$\text{If} \quad A > 0 \quad \text{and} \quad B > 0, \text{ then} \quad A \otimes B > 0 \tag{A6}$$

Proof:

Since A and B are symmetric, $A \otimes B$ is symmetric. The eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$, where λ_i and μ_j are the eigenvalues of A and B, respectively [10, p. 235]. Since $A \geq 0$ and $B \geq 0$, $\lambda_i \geq 0$ and $\mu_j \geq 0$, hence

$$\lambda_i \mu_j \geq 0 \qquad \forall i, j$$

This implies that $A \otimes B \geq 0$. The proof of (A6) is similar.

Lemma A.4

If $A \geq 0$ and $B \equiv \begin{bmatrix} B_1 & B_3^T \\ B_3 & B_2 \end{bmatrix} > 0$, where B_1 and B_2 are square matrices of dimensions m and ℓ, respectively, then

$$B_2 - B_3 B_1^{-1} B_3^T > 0 \tag{A7}$$

and

$$C \equiv \begin{bmatrix} A \otimes B_1 & A \otimes B_3^T \\ A \otimes B_3 & A \otimes B_2 \end{bmatrix} \geq 0 \tag{A8}$$

If $B \geq 0$ and B_1 is a scalar, then

$$B_1 B_2 \geq B_3 B_3^T \tag{A9}$$

Proof:

Since $B > 0$, $B_1 > 0$ and invertible,

$$B = \begin{bmatrix} I_m & 0_{m,\ell} \\ B_3 B_1^{-1} & I_\ell \end{bmatrix} \begin{bmatrix} B_1 & 0_{m,\ell} \\ 0_{\ell,m} & B_2 - B_3 B_1^{-1} B_3^T \end{bmatrix} \begin{bmatrix} I_m & B_1^{-1} B_3^T \\ 0_{\ell,m} & I_\ell \end{bmatrix} > 0$$

which implies that $B_2 - B_3 B_1^{-1} B_3^T > 0$.

For the case $B \geq 0$, (A9) clearly holds if $B_1 = 0$. If $B_1 > 0$, we obtain $B_2 - B_3 B_1^{-1} B_3^T \geq 0$, which implies (A9).

To prove (A8) we assume that A is an n-dimensional matrix and let $\hat{A} = A + \epsilon I_n$; then from (A6) $\hat{A} \otimes B_1 > 0$ and is invertible, since $\hat{A} > 0$ and $B_1 > 0$. Therefore,

$$\hat{C} \equiv \begin{bmatrix} \hat{A} \otimes B_1 & \hat{A} \otimes B_3^T \\ \hat{A} \otimes B_3 & \hat{A} \otimes B_2 \end{bmatrix}$$

$$= D \begin{bmatrix} \hat{A} \otimes B_1 & 0_{nm,n\ell} \\ 0_{n\ell,nm} & \hat{A} \otimes B_2 - (\hat{A} \otimes B_3)(\hat{A} \otimes B_1)^{-1}(\hat{A} \otimes B_3^T) \end{bmatrix} D^T \tag{A10}$$

where

$$D \equiv \begin{bmatrix} I_{nm} & 0_{nm,n\ell} \\ (\hat{A} \otimes B_3)(\hat{A} \otimes B_1)^{-1} & I_{n\ell} \end{bmatrix}$$

Using identities for inverses and products of Kronecker products [8], we can easily write

$$(\hat{A} \otimes B_3)(\hat{A} \otimes B_1)^{-1} = I_n \otimes B_3 B_1^{-1}$$

$$\hat{A} \otimes B_2 - (\hat{A} \otimes B_3)(\hat{A} \otimes B_1)^{-1}(\hat{A} \otimes B_3^T) = \hat{A} \otimes (B_2 - B_3 B_1^{-1} B_3^T)$$

Therefore, from (A10)

$$C = \lim_{\epsilon \to 0} \hat{C} = \begin{bmatrix} I_{nm} & 0_{nm,n\ell} \\ I_n \otimes B_3 B_1^{-1} & I_{n\ell} \end{bmatrix} \begin{bmatrix} A \otimes B_1 & 0_{nm,n\ell} \\ 0_{n\ell,nm} & A \otimes (B_2 - B_3 B_1^{-1} B_3^T) \end{bmatrix} \begin{bmatrix} I_{nm} & I_n \otimes B_1^{-1} B_3^T \\ 0_{n\ell,nm} & I_{n\ell} \end{bmatrix} \tag{A11}$$

From (A5) and (A7), $A \otimes B_1 \geq 0$ and $A \otimes (B_2 - B_3 B_1^{-1} B_3^T) \geq 0$, hence (A11) implies $C \geq 0$.

Lemma A.5

If $A > 0$, $B \geq C \geq 0$, then

$$\mathrm{tr}[(A+B)^{-1} B] \geq \mathrm{tr}[(A+C)^{-1} C] \tag{A12}$$

Proof:

$$\begin{aligned} tr[(A+B)^{-1}B] &= tr\{(A+B)^{-1}[(A+B)-A]\} \\ &= tr(I_n) - tr[(A+B)^{-1}A] \end{aligned}$$

Now $tr[(A+B)^{-1}A] = tr[A^{\frac{1}{2}}(A+B)^{-1}A^{\frac{1}{2}}]$ and since $B \geq C \geq 0$, $(A+B)^{-1} \leq (A+C)^{-1}$ and

$$A^{\frac{1}{2}}(A+B)^{-1}A^{\frac{1}{2}} \leq A^{\frac{1}{2}}(A+C)^{-1}A^{\frac{1}{2}}$$

Therefore, $tr[(A+B)^{-1}A] \leq tr[(A+C)^{-1}A]$. Consequently,

$$tr[(A+B)^{-1}B] = tr(I_n) - tr[(A+B)^{-1}A] \geq tr(I_n) - tr[(A+C)^{-1}A] = tr[(A+C)^{-1}C]$$

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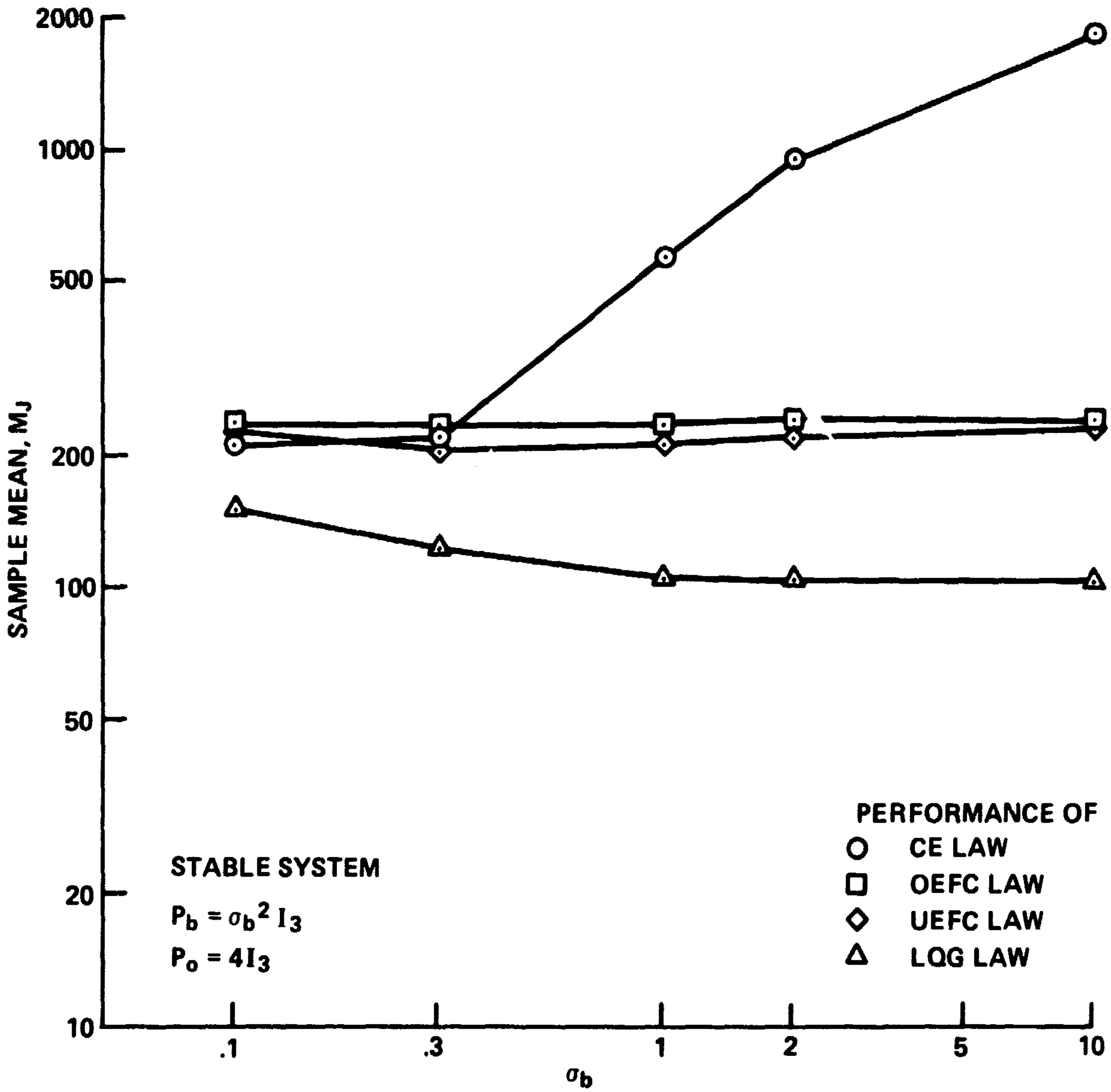

Figure 1. - Dependence of Monte Carlo performance on P_b (sample mean M_J).

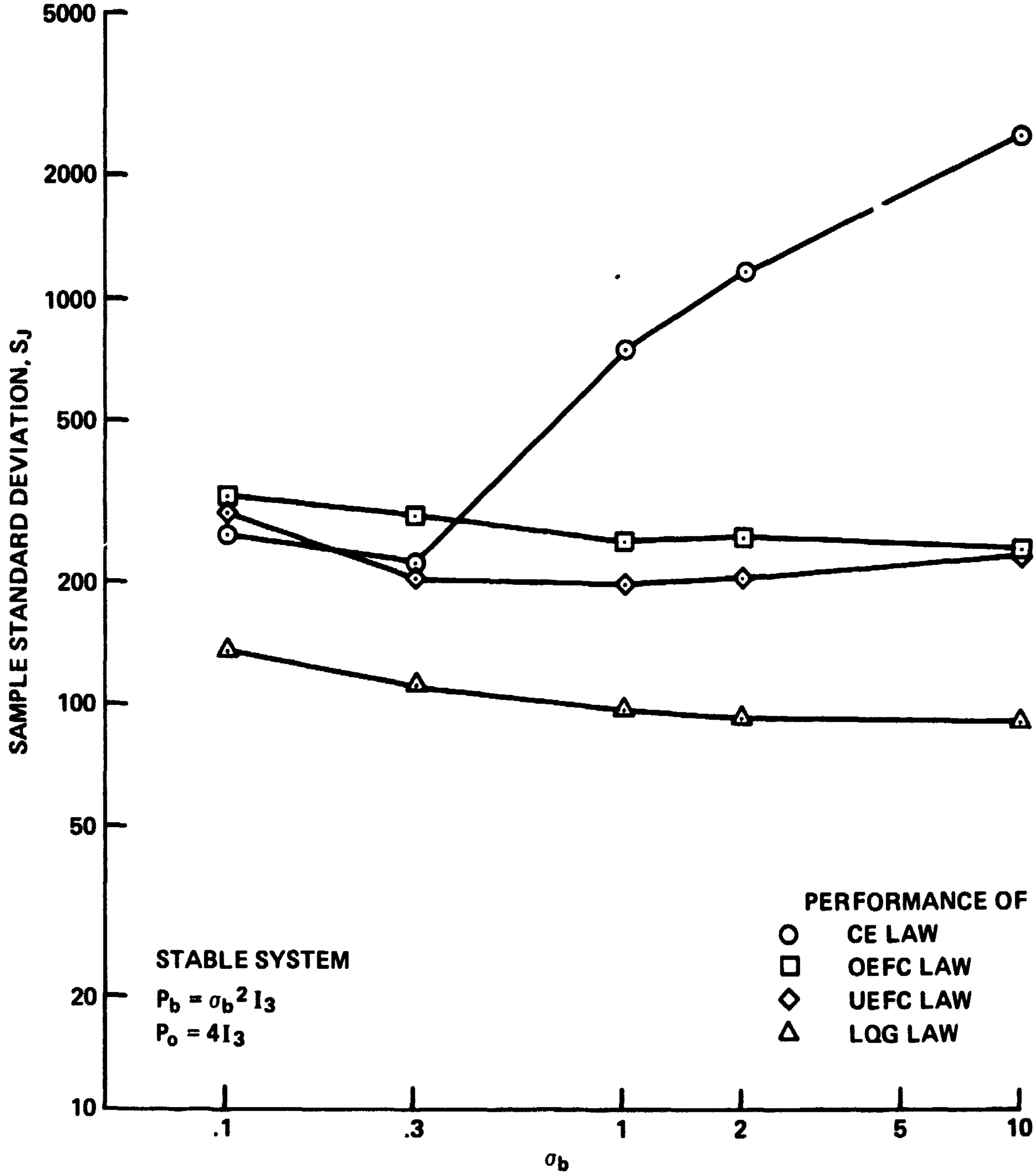

Figure 2. - Dependence of Monte Carlo performance on P_b (sample standard deviation S_J).

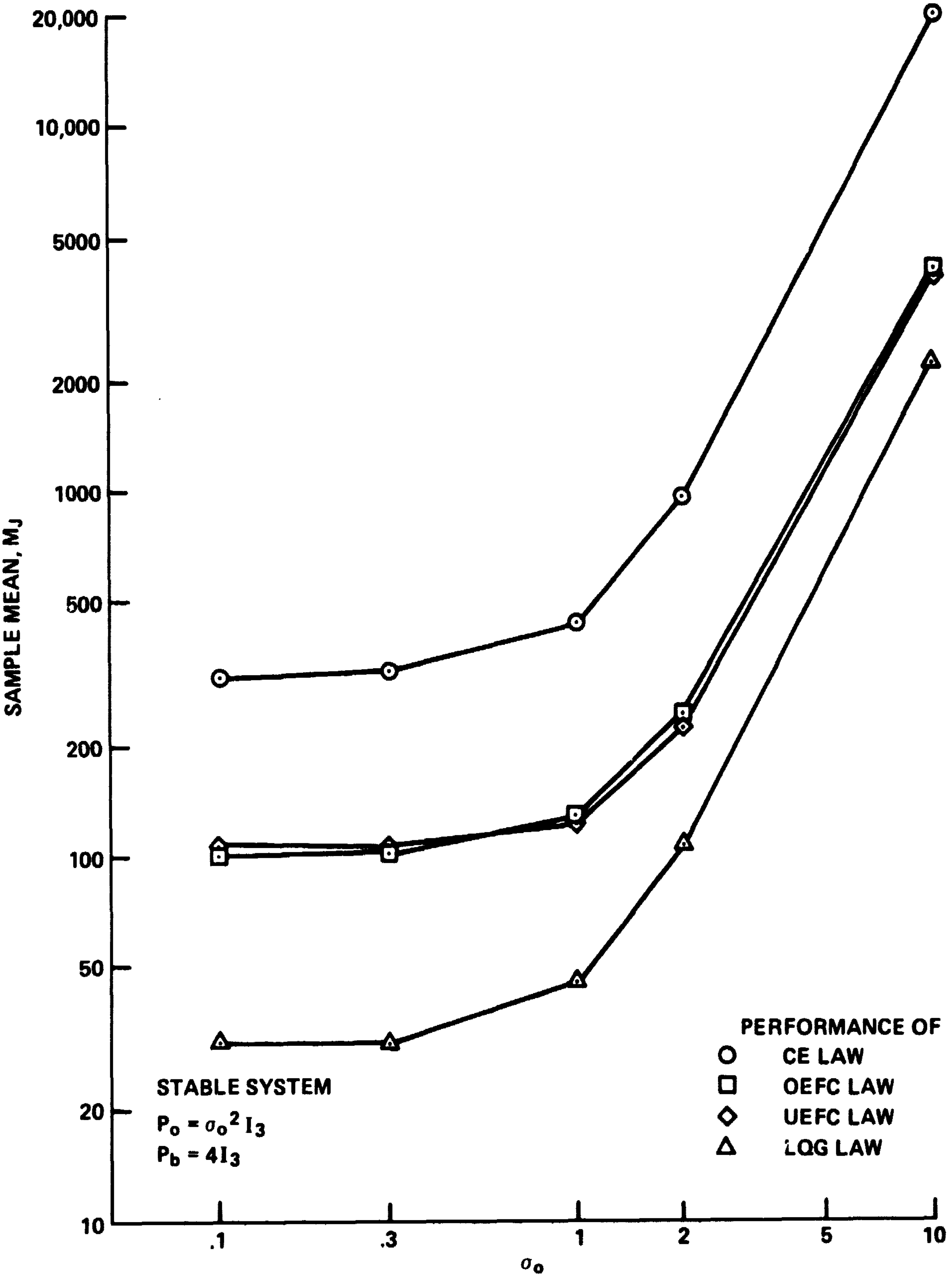

Figure 3. - Dependence of Monte Carlo performance on P_0 (sample mean M_J).

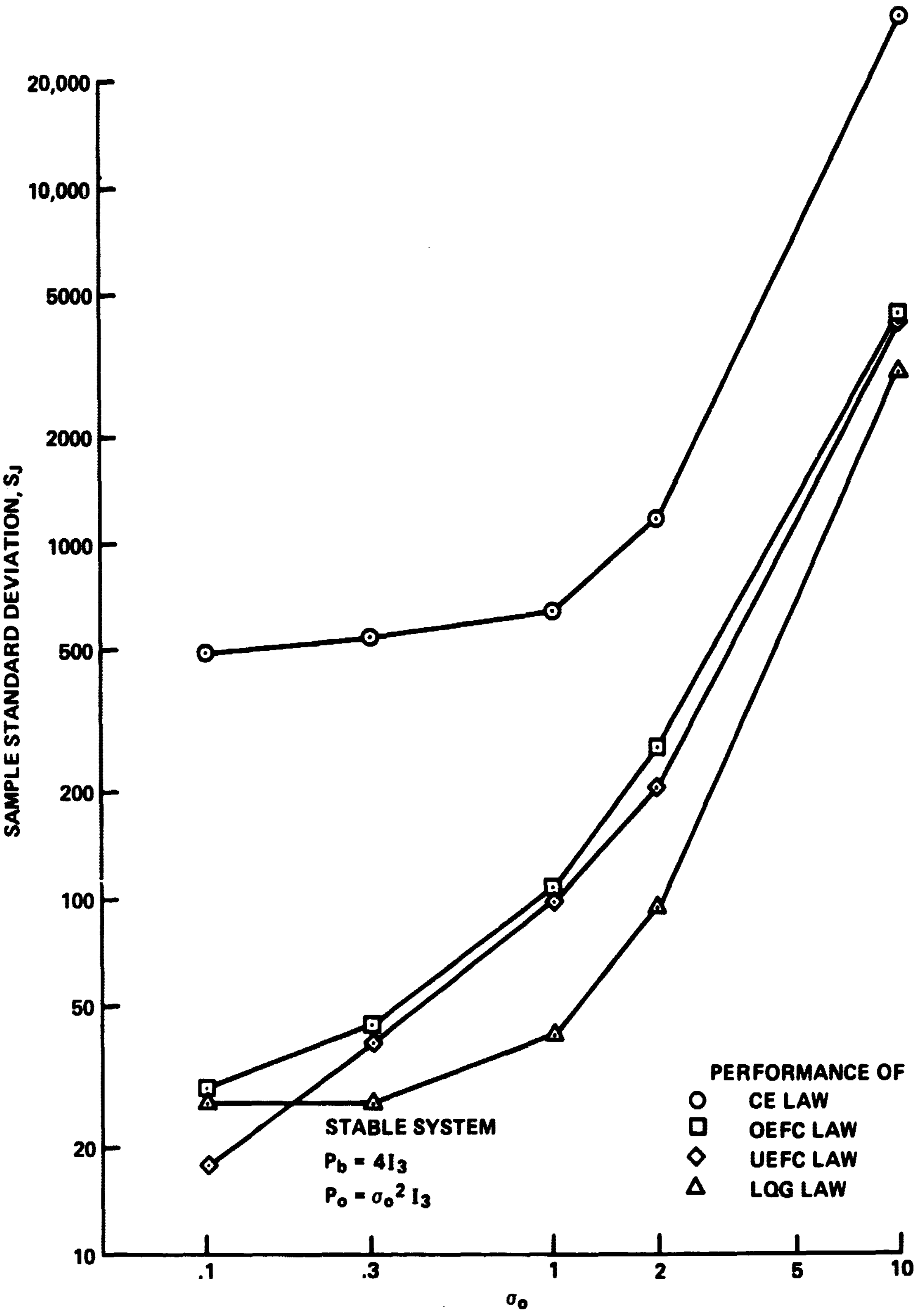

Figure 4. - Dependence of Monte Carlo performance on P_0 (sample standard deviation S_J).

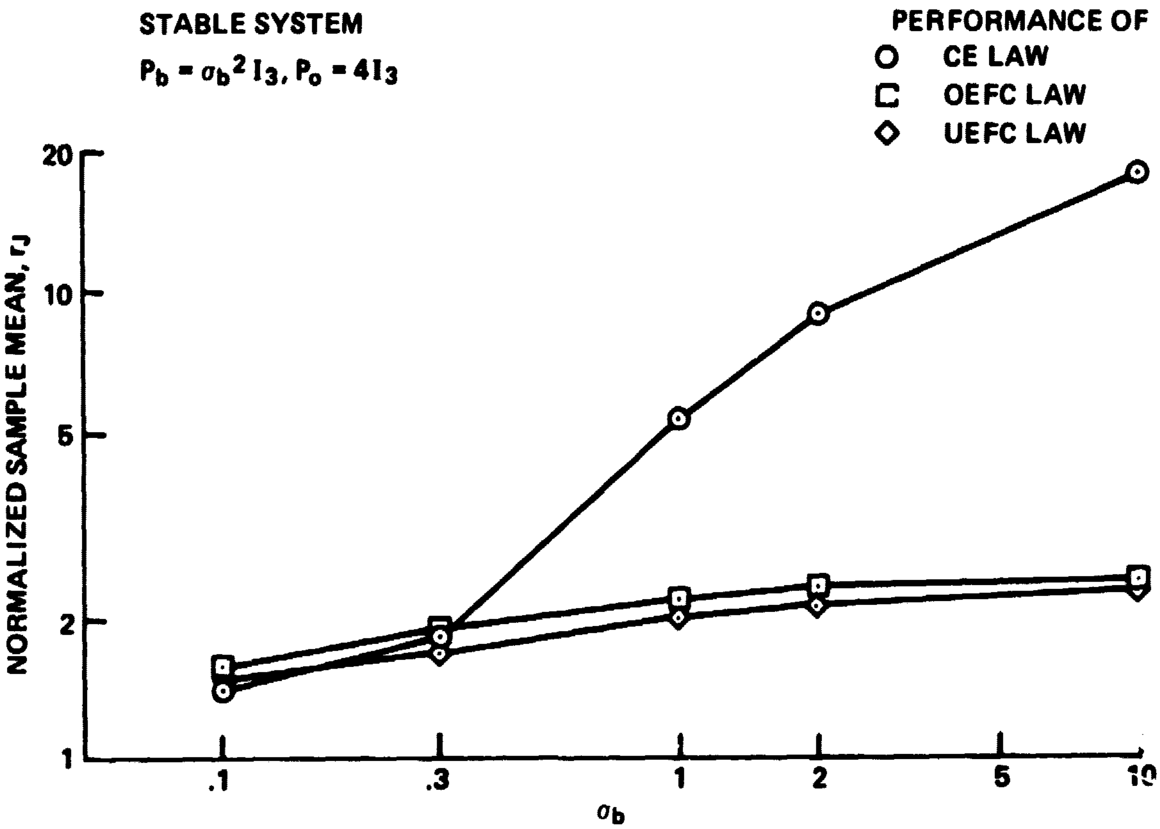

Figure 5. - Dependence of normalized sample mean r_J on P_b.

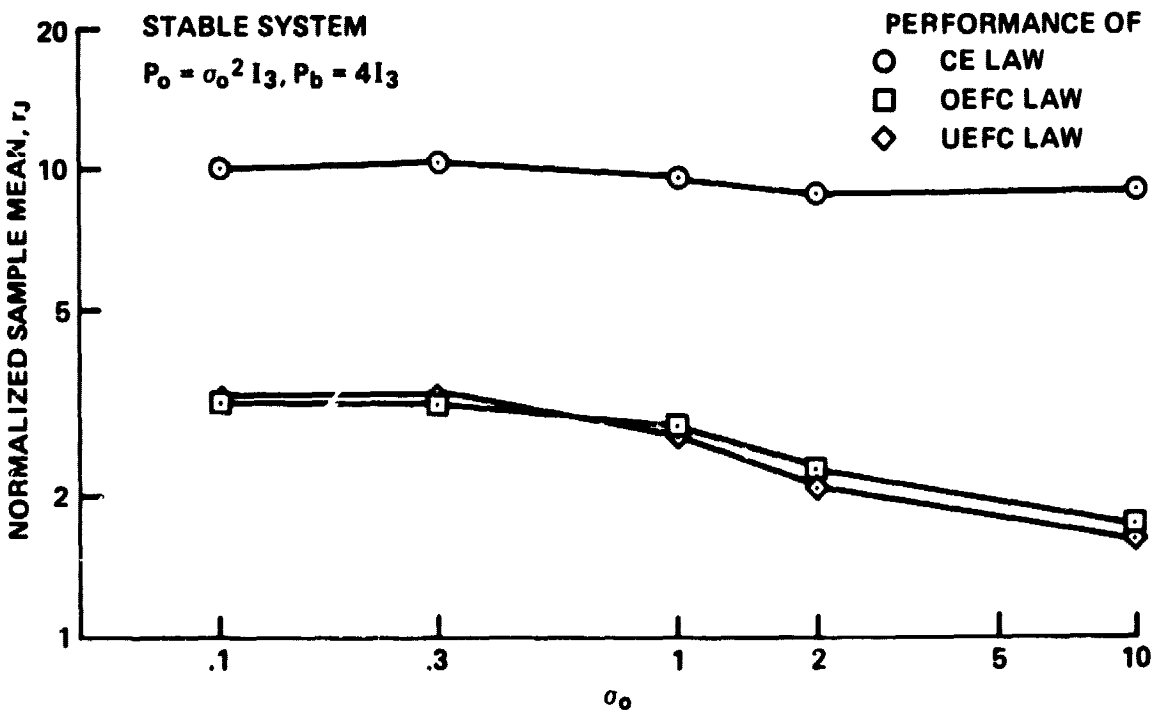

Figure 6. - Dependence of normalized sample mean r_J on P_0.

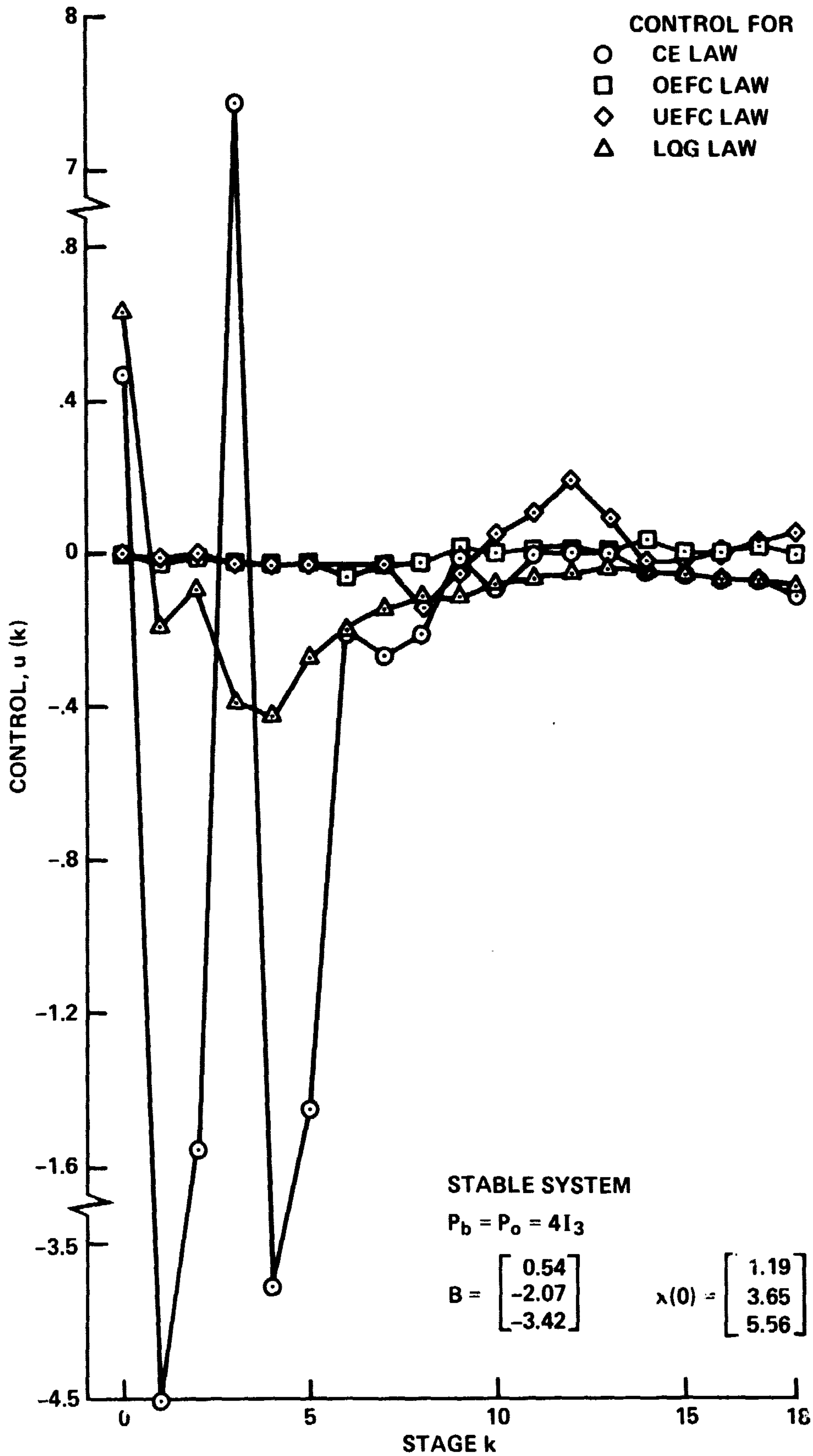

Figure 7. - Time history of control u(k) for the 4 control laws.

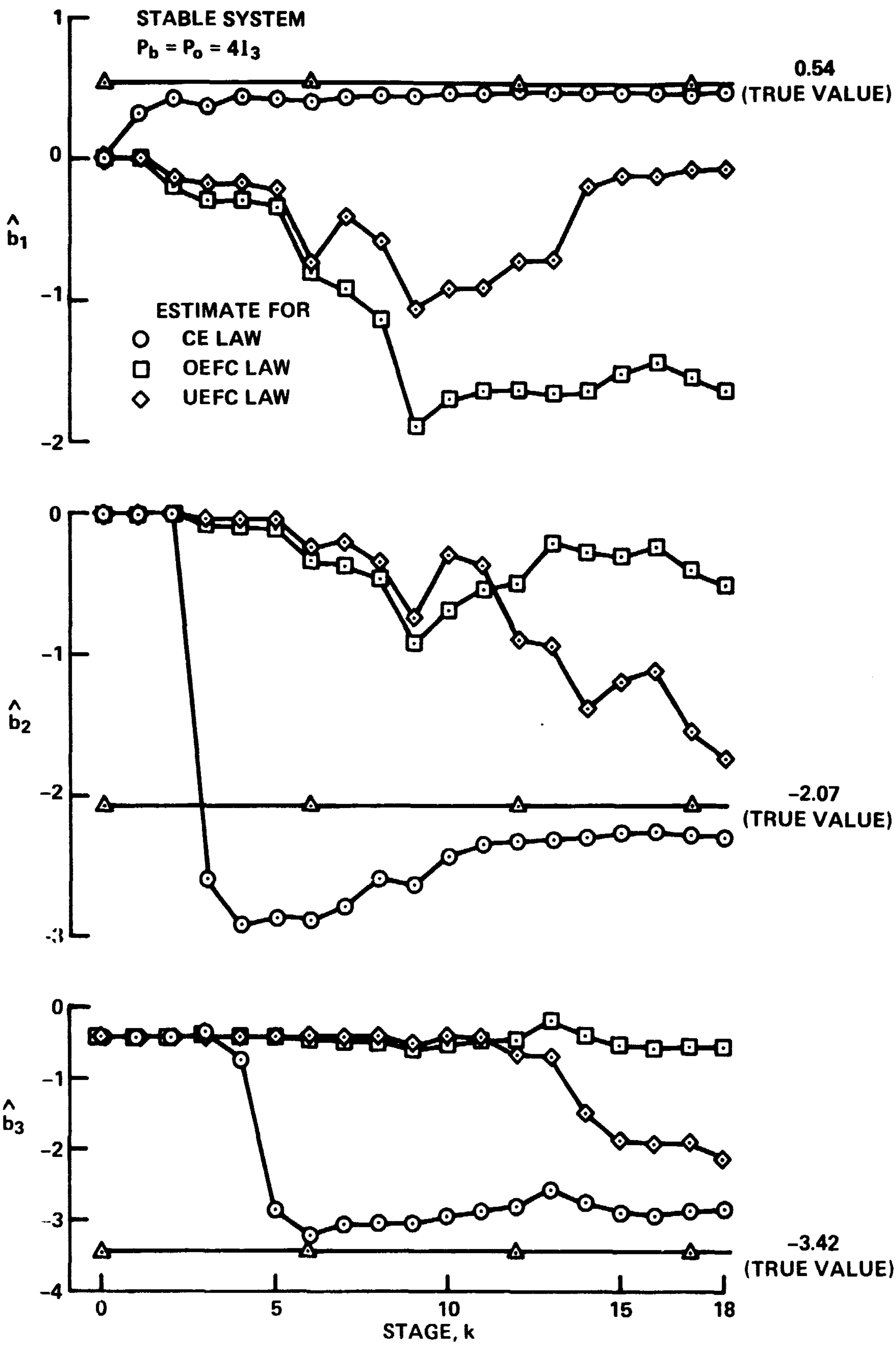

Figure 8. - Time history of estimate $\hat{b}(k|k)$ for the 4 control laws.

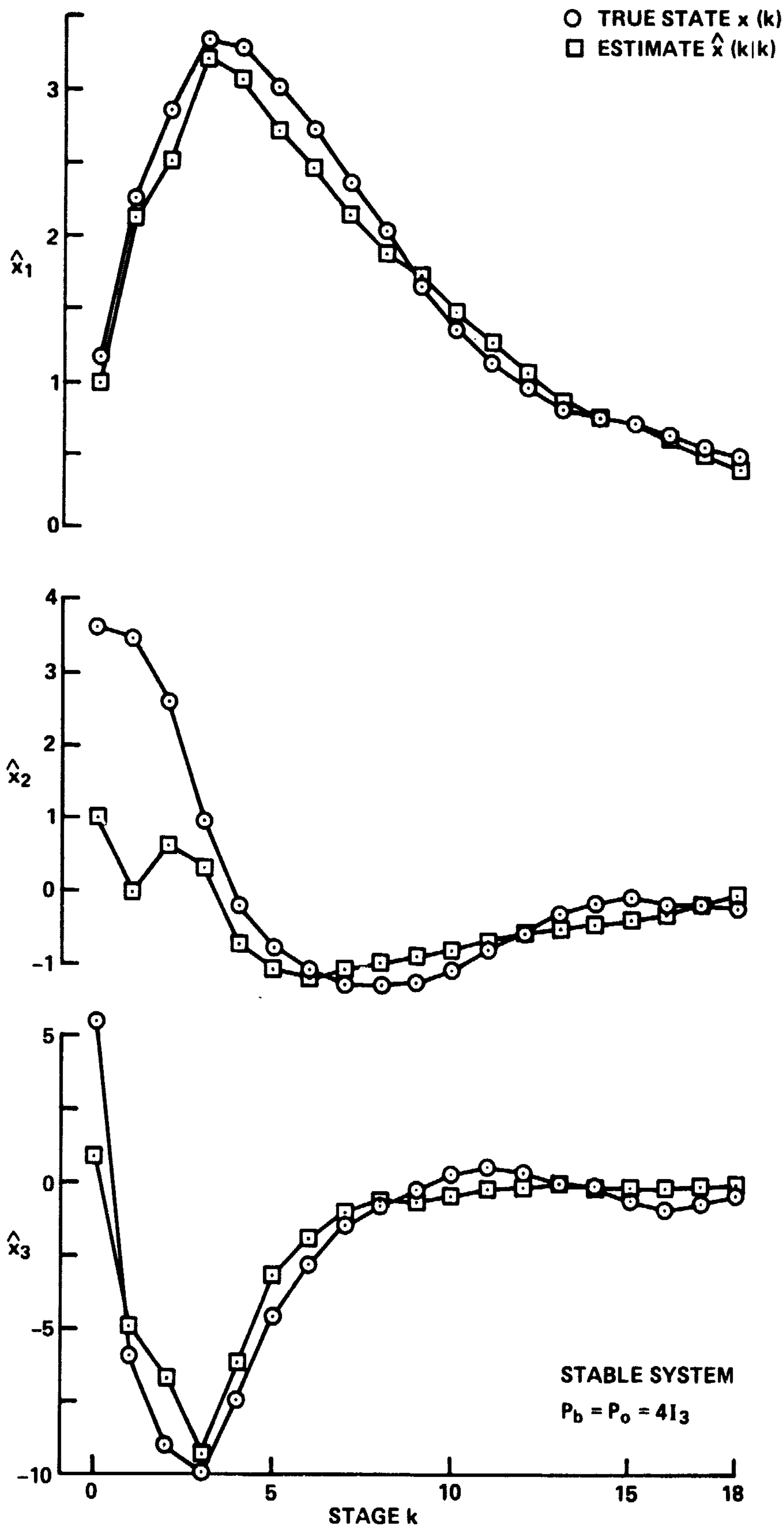

Figure 9. - Time history of true state $x(k)$ and estimate $\hat{x}(k|k)$ for LQG law.

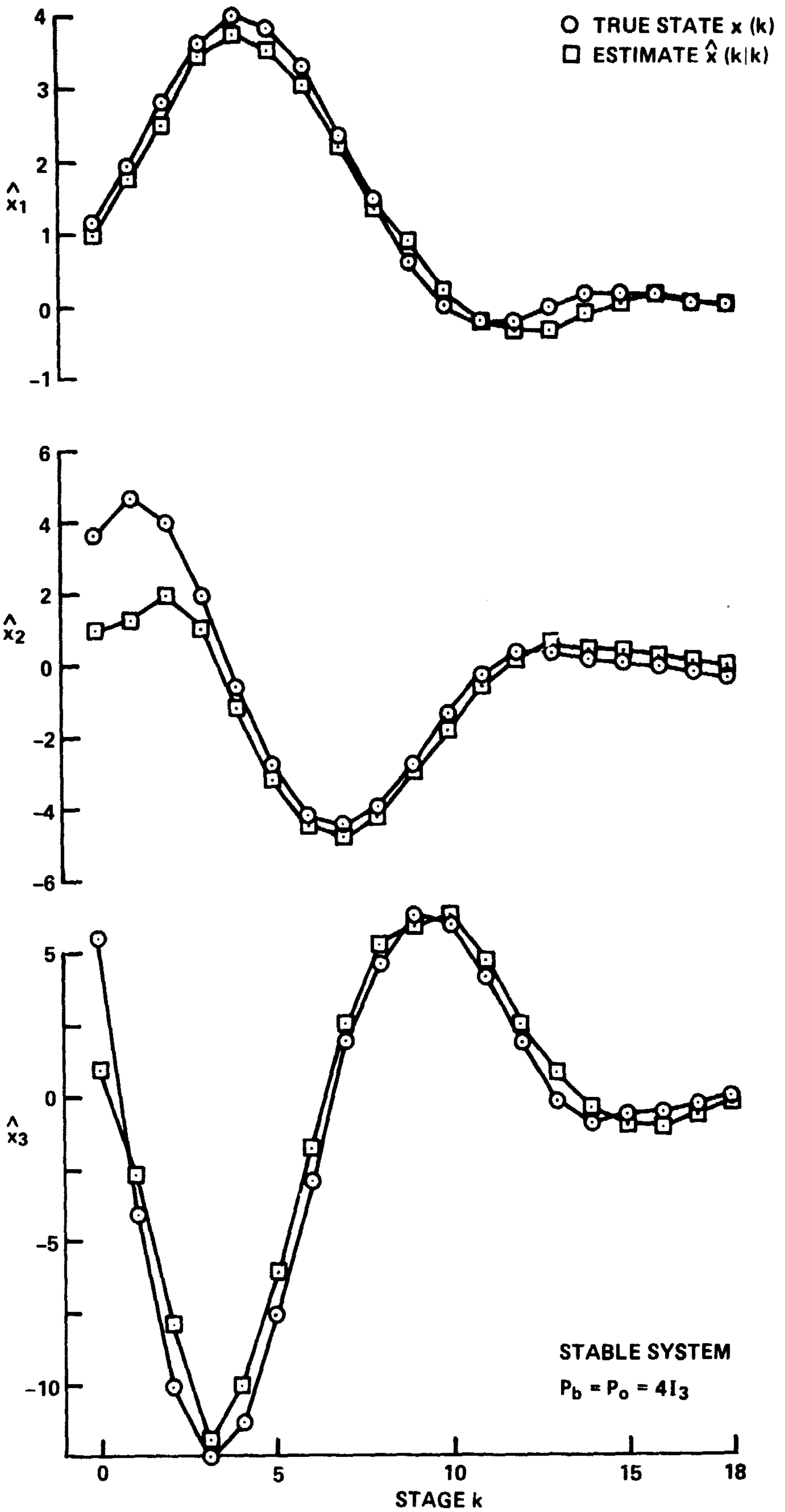

Figure 10. - Time history of true state $x(k)$ and estimate $\hat{x}(k|k)$ for UEFC law.

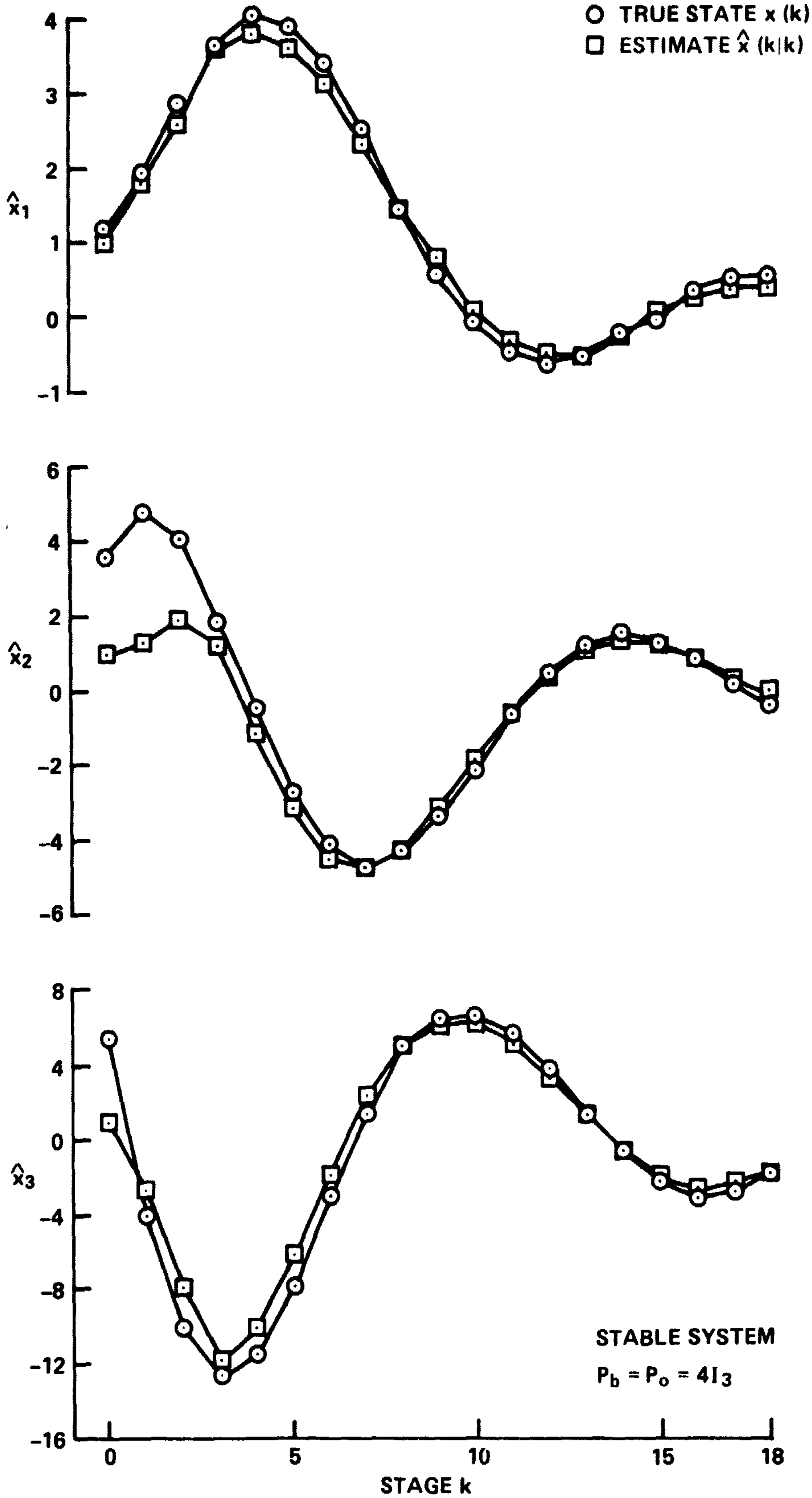

Figure 11. - Time history of true state $x(k)$ and estimate $\hat{x}(k|k)$ for OEFC law.

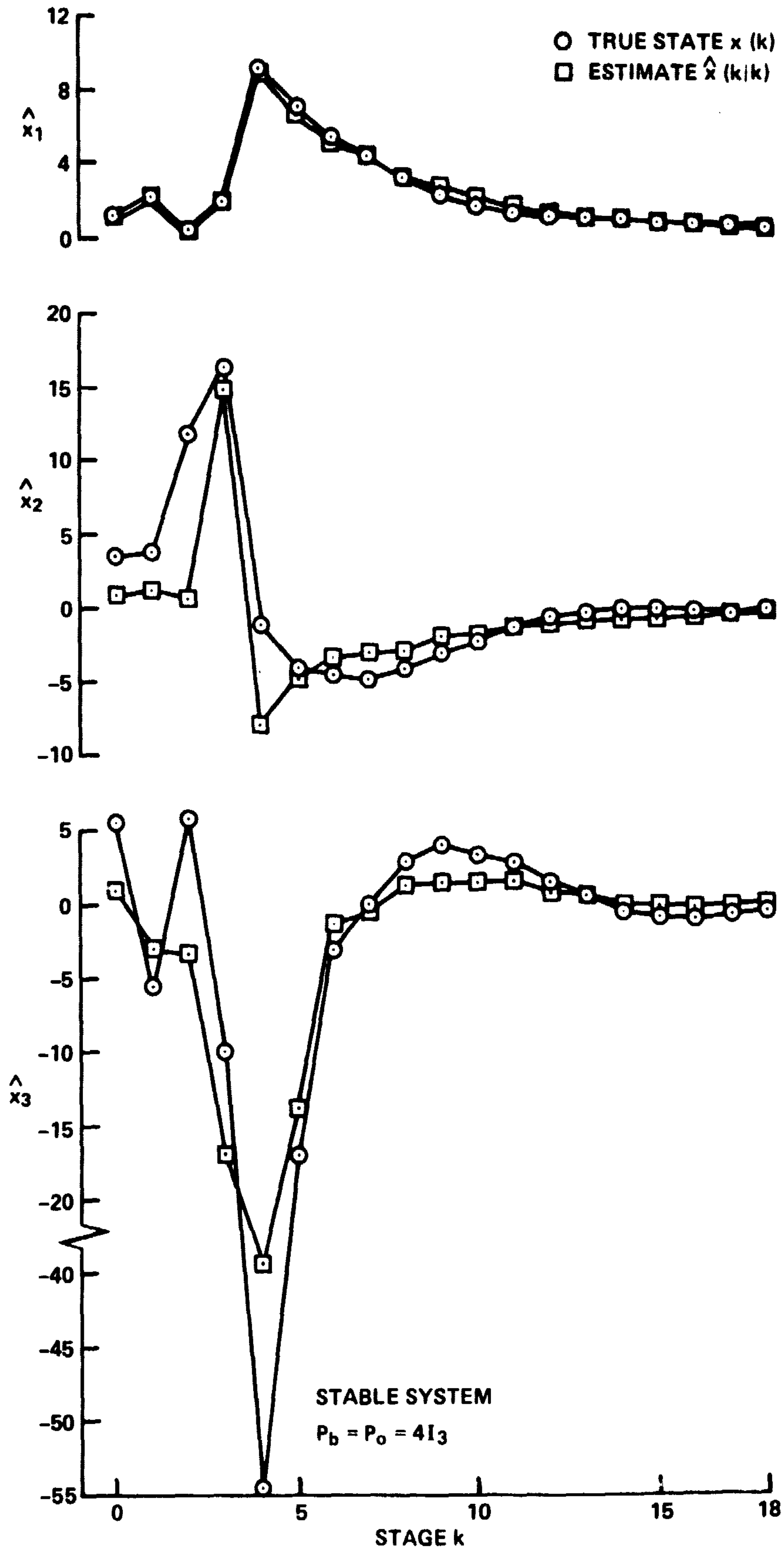

Figure 12. - Time history of true state $x(k)$ and estimate $\hat{x}(k|k)$ for CE law.

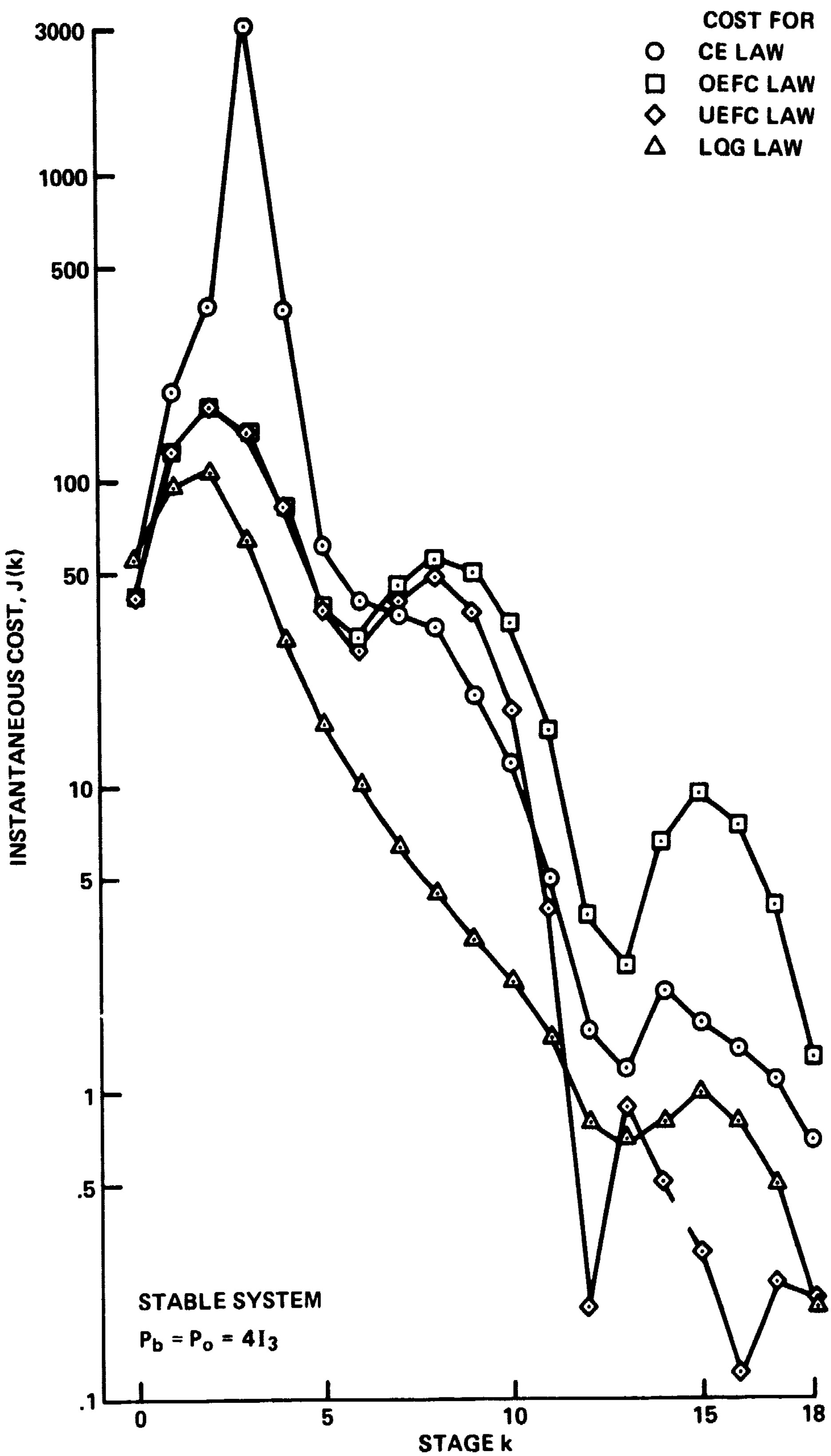

Figure 13. - Time history of instantaneous cost J(k) for the 4 control laws.

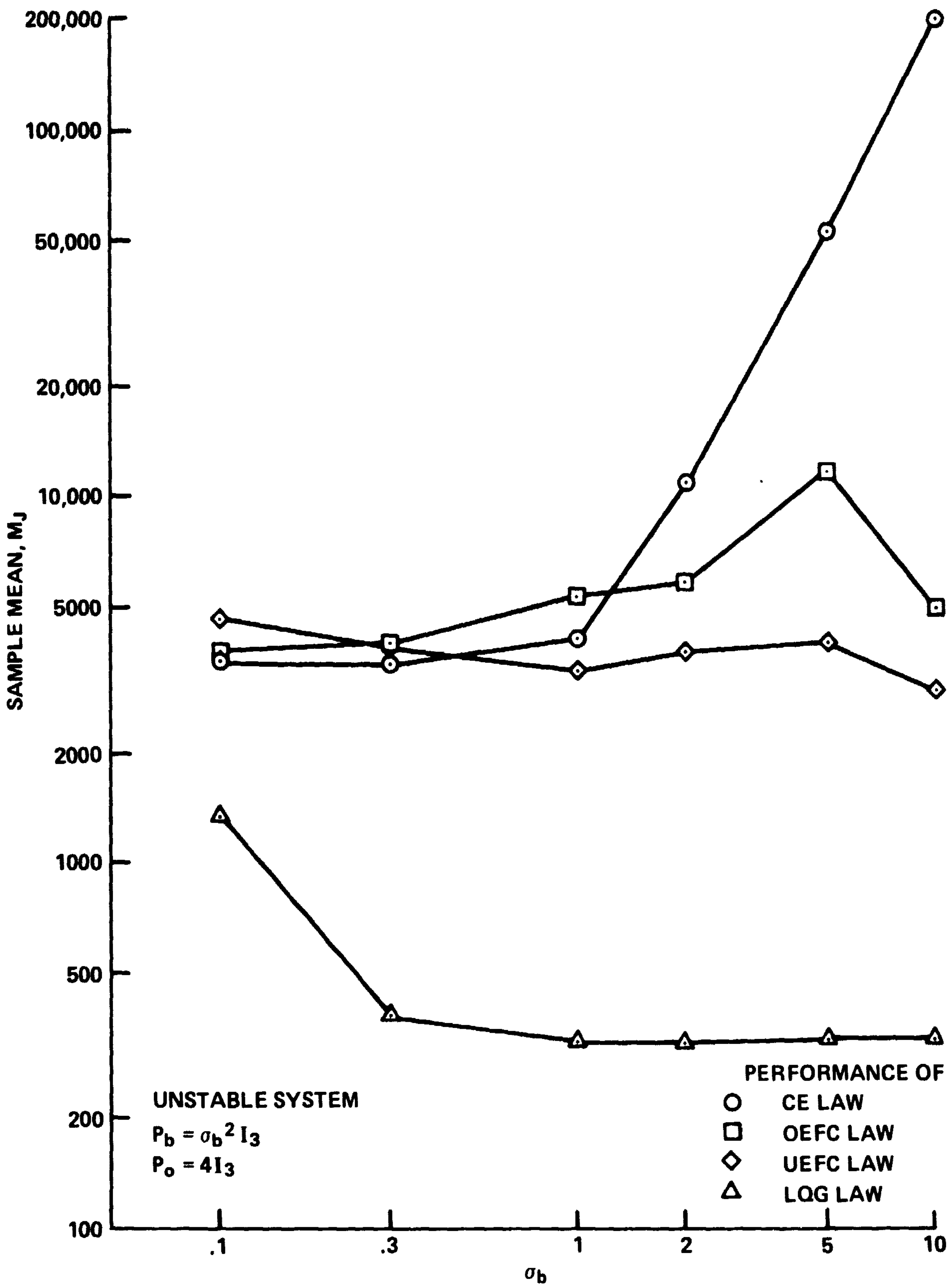

Figure 14. - Dependence of Monte Carlo performance on P_b (sample mean M_J).

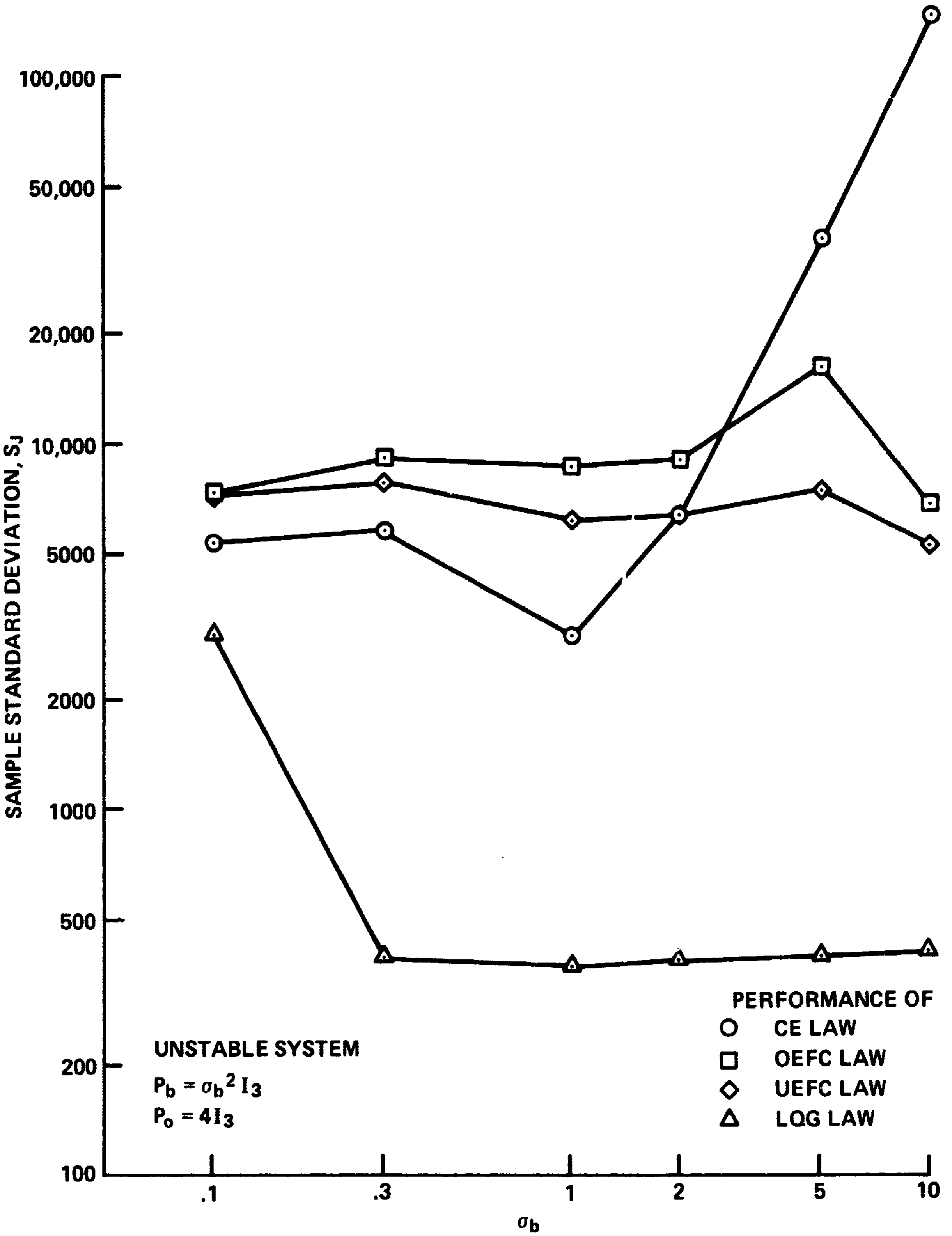

Figure 15. - Dependence of Monte Carlo performance on P_b (sample standard deviation S_J).

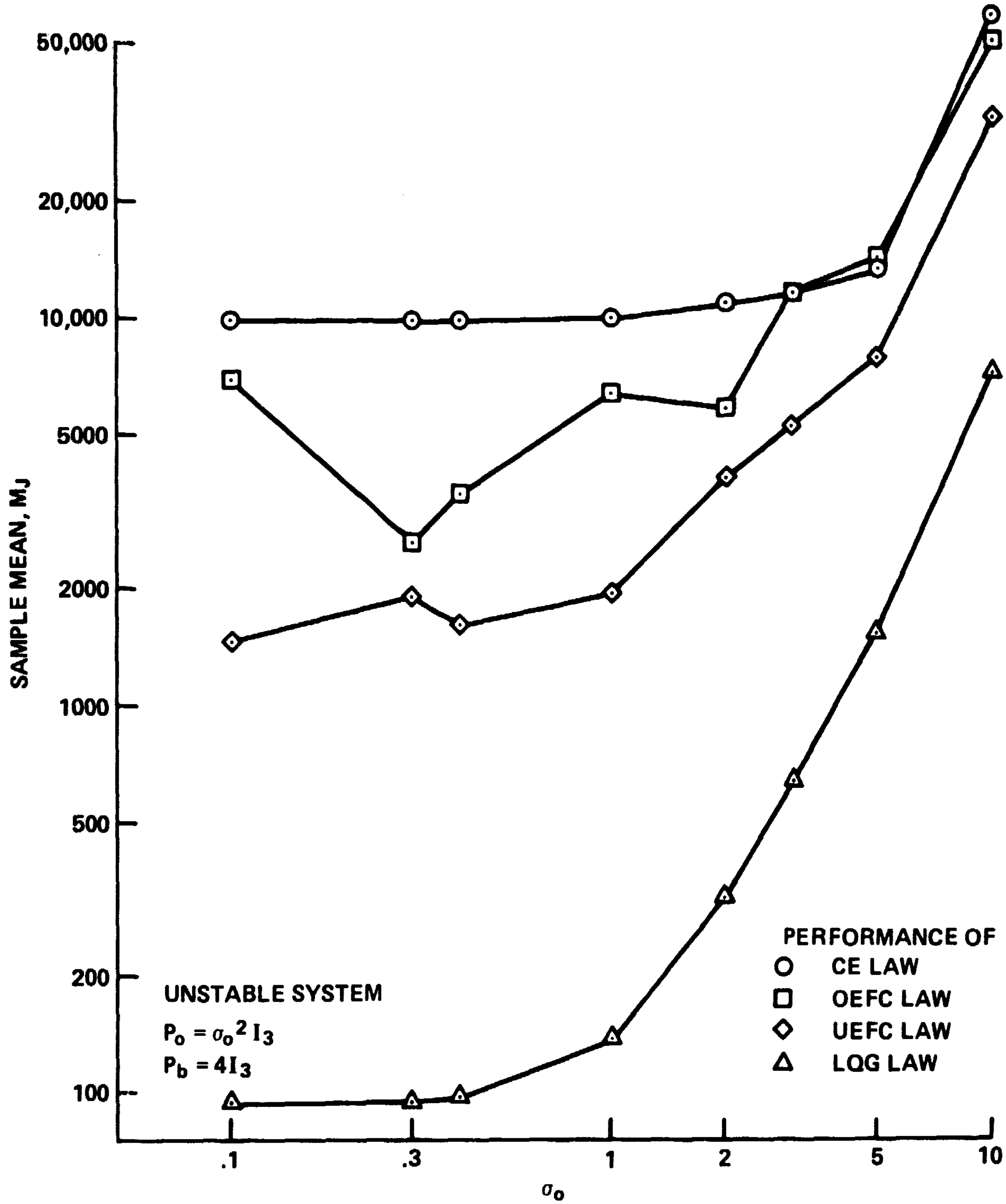

Figure 16.- Dependence of Monte Carlo performance on P_0 (sample mean M_J).

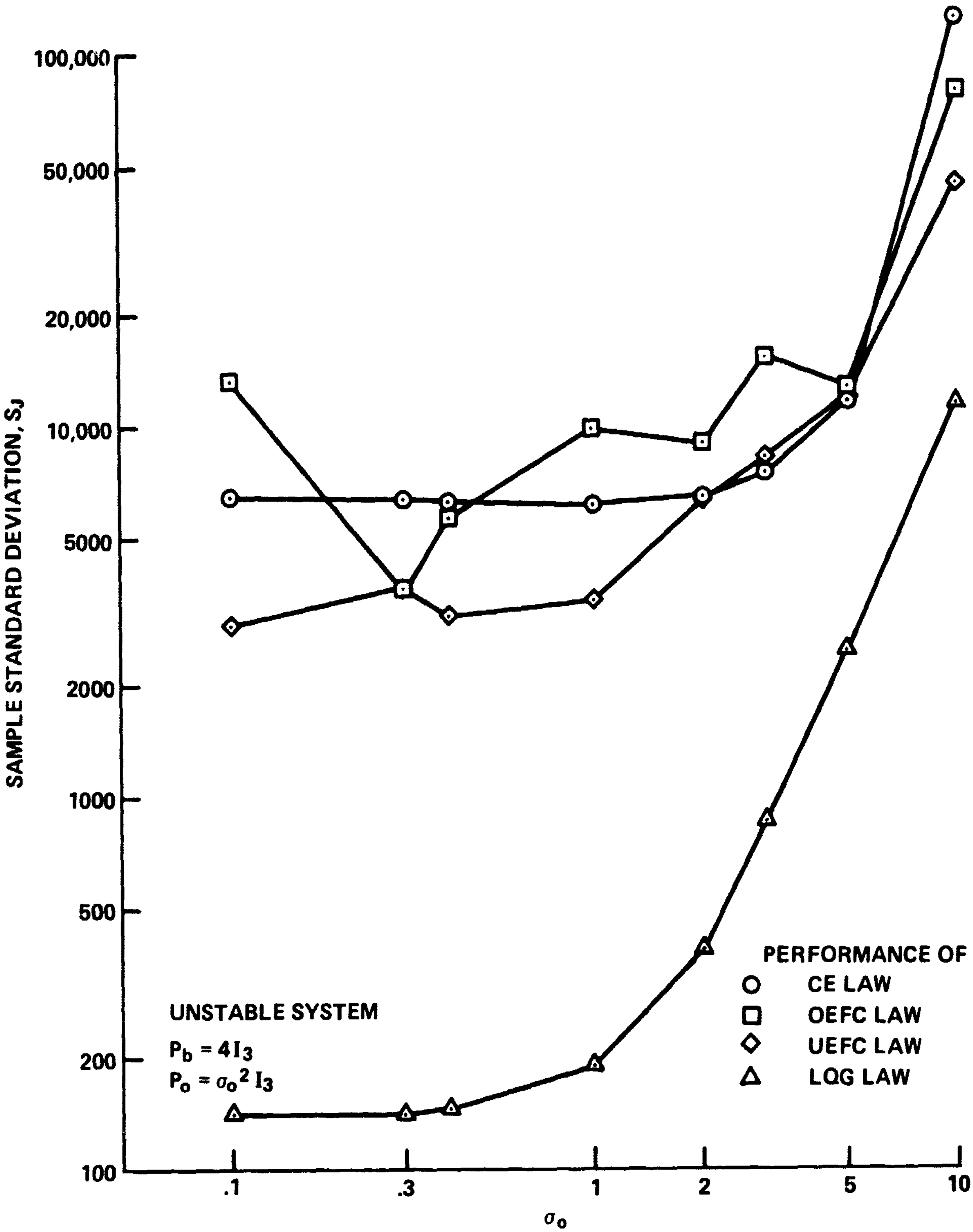

Figure 17. - Dependence of Monte Carlo performance on P_0 (sample standard deviation S_J).

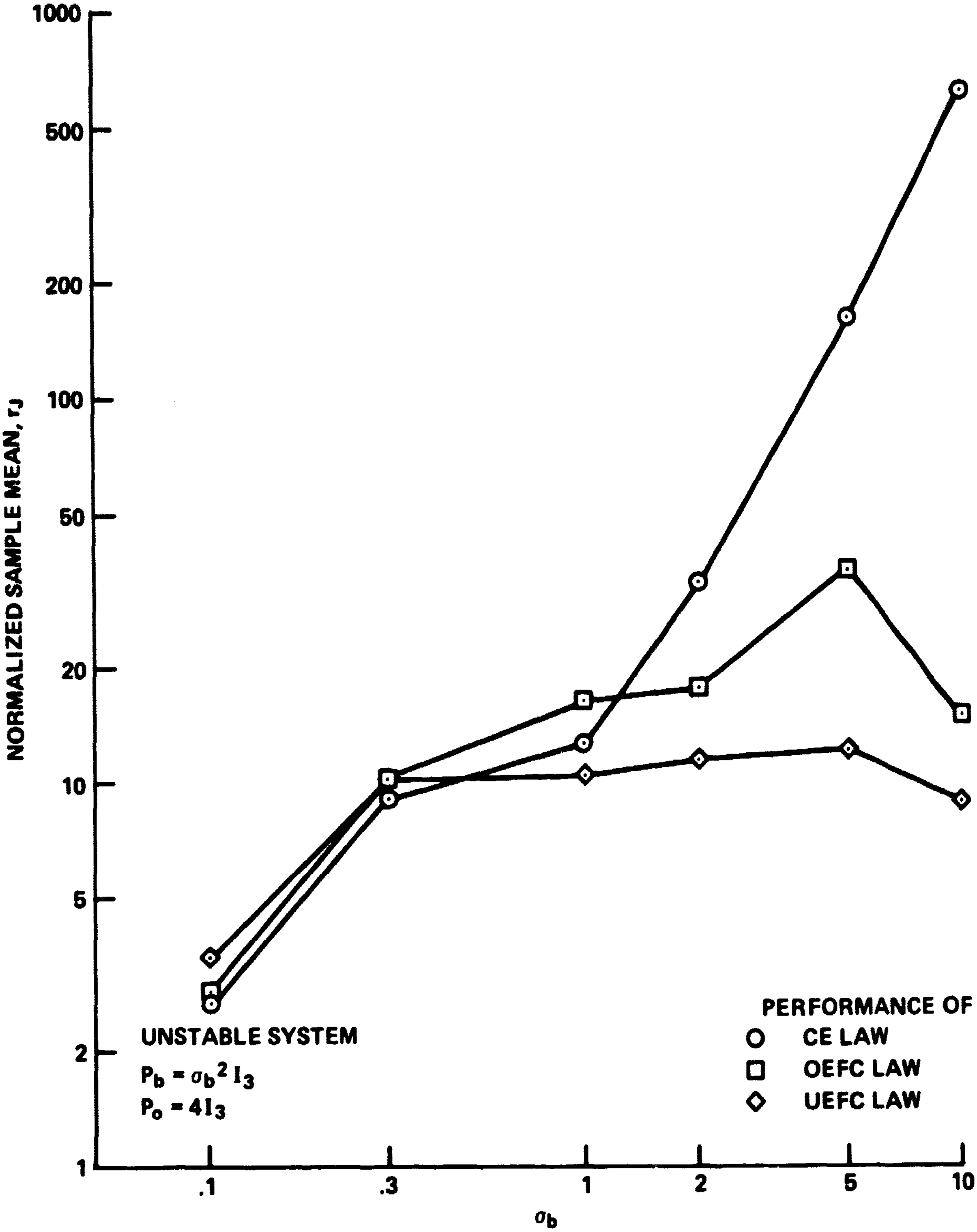

Figure 18. - Dependence of normalized sample mean r_J on P_b.

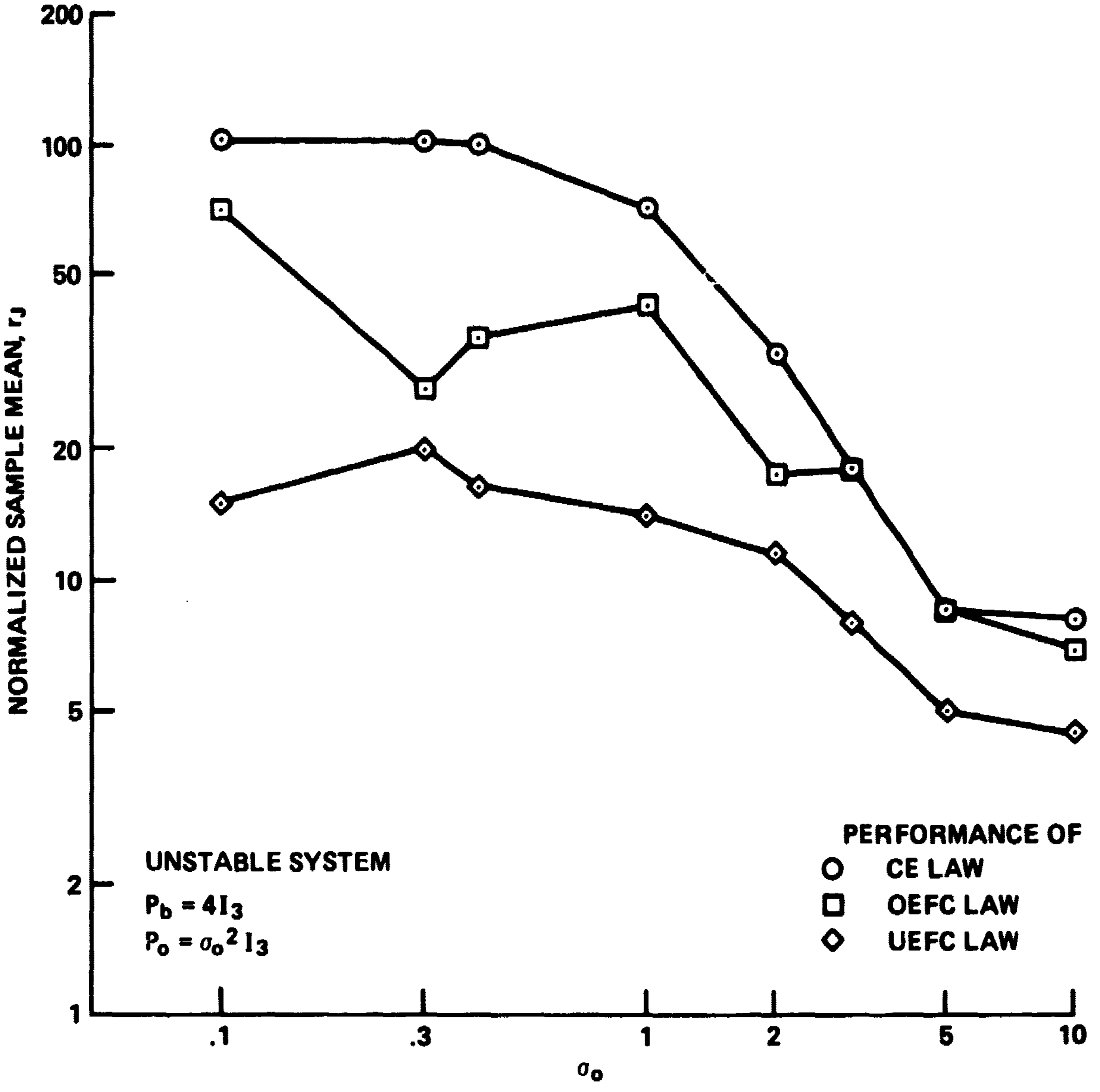

Figure 19. - Dependence of normalized sample mean r_J on P_0.

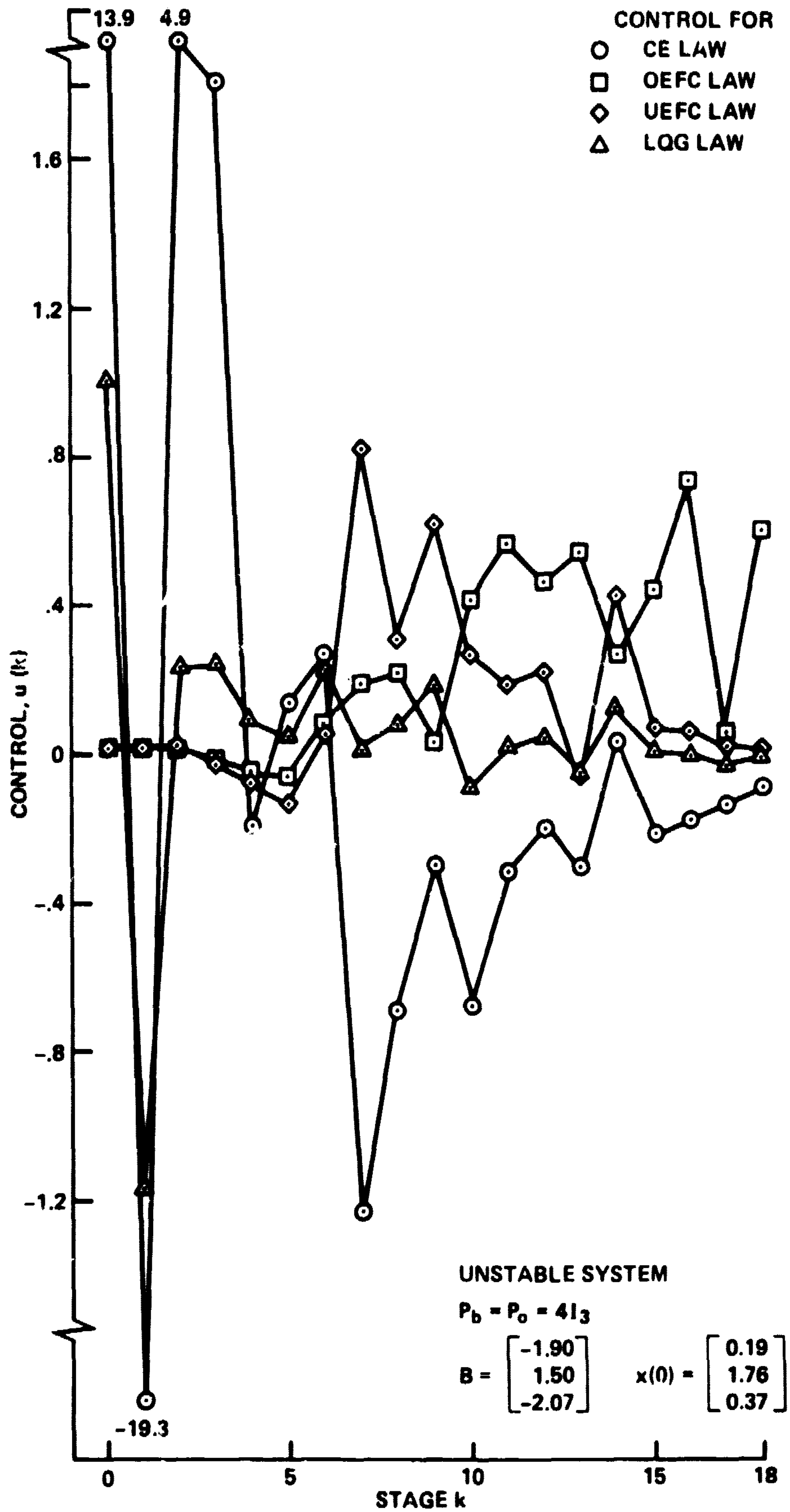

Figure 20. - Time history of control u(k) for the 4 control laws.

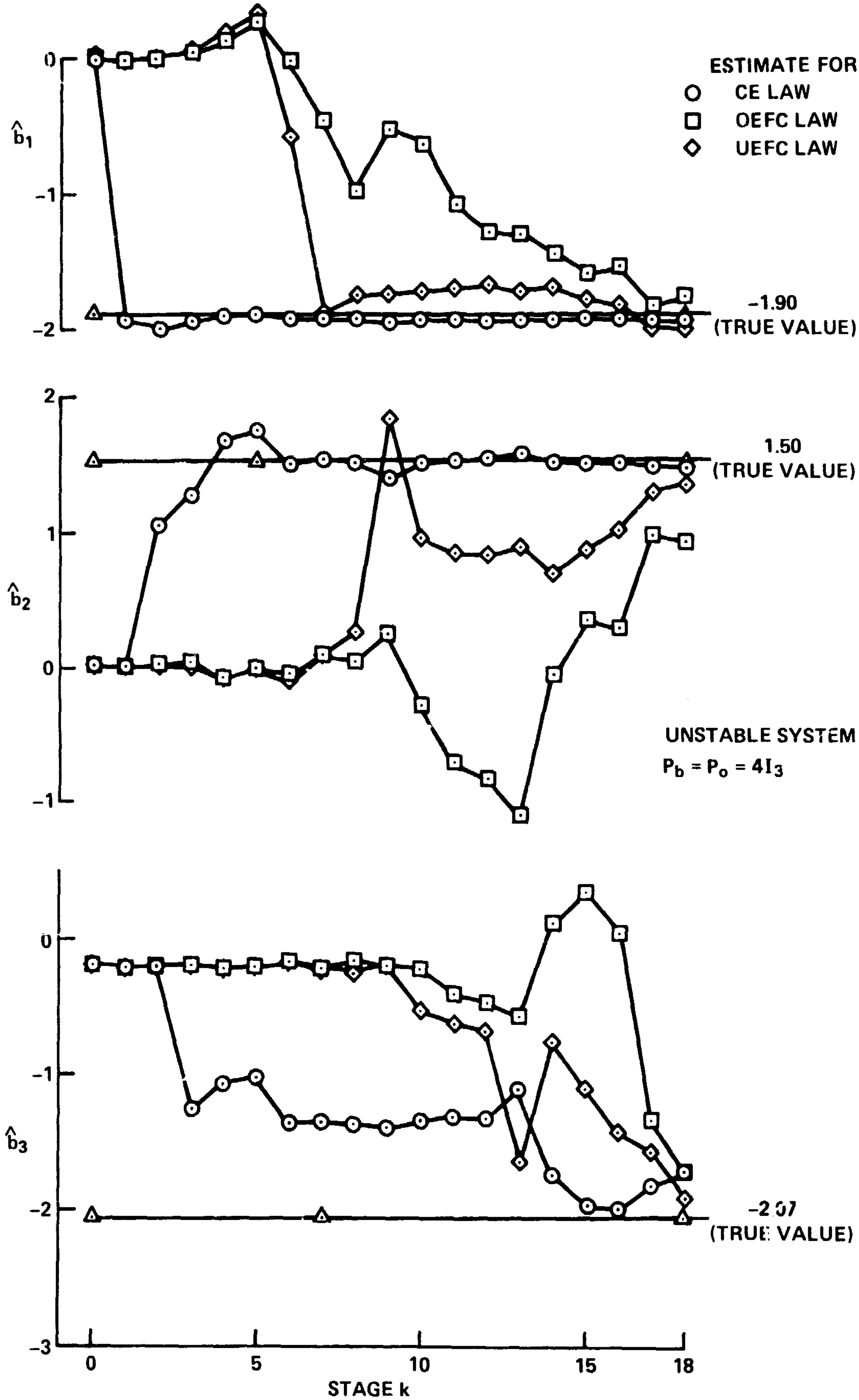

Figure 21. - Time history of estimate $\hat{b}(k|k)$ for the 4 control laws.

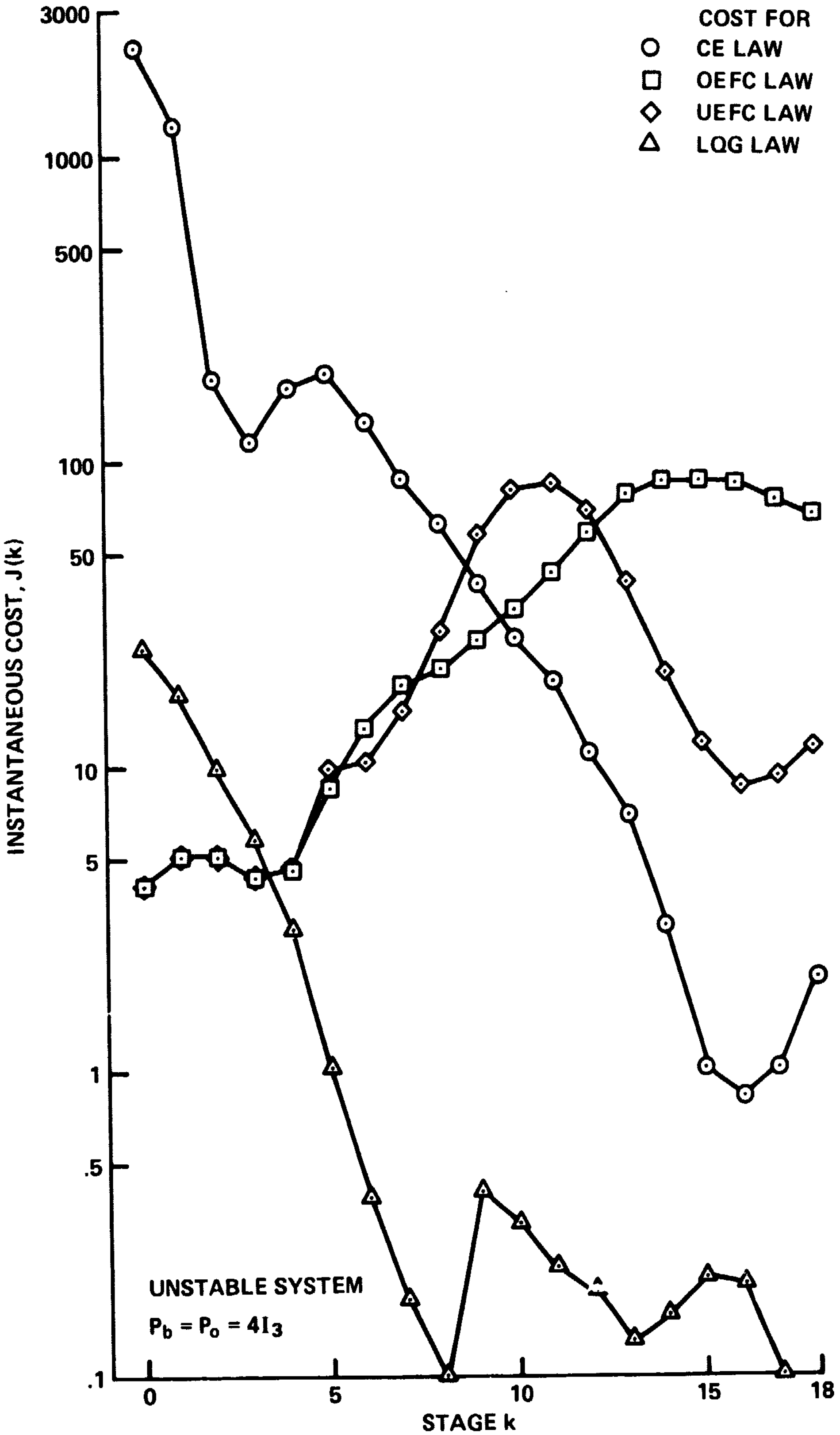

Figure 22. - Time history of instantaneous cost J(k) for the 4 control laws.