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K. OMIDVAR

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TABLES OF BRANCHING RATIOS FOR ELECTRIC DIPOLE TRANSITIONS BETWEEN ARBITRARY LEVELS OF HYDROGEN-LIKE ATOMS

K. Omidvar

Theoretical Studies Group
NASA/Goddard Space Flight Center
Greenbelt, Maryland 20771

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TABLES OF BRANCHING RATIOS FOR ELECTRIC DIPOLE TRANSITIONS BETWEEN ARBITRARY LEVELS OF HYDROGEN-LIKE ATOMS

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ABSTRACT

The branching ratios in hydrogen-like atoms due to the electric-dipole transitions are tabulated for the initial principal and azimuthal quantum numbers $n' \ell'$, and final principal and azimuthal quantum numbers $n \ell$, where $n < n' \leqslant 15$ and $0 \leqslant \ell' \leqslant n' - 1$ with the restriction $\ell' \leqslant 6$, and $1 \leqslant n \leqslant 5$ and $0 \leqslant \ell \leqslant n-1$. Average values with respect to ℓ' are given. The branching ratios not tabulated, including the initial states $n' \to \infty$, ℓ' corresponding to the threshold of the continuum, could be obtained by extrapolation.

I. FORMULATION

Excited states of atoms and ions are formed frequently in laboratory and astrophysical plasmas. The important processes leading to the atomic excited states are the radiative and dielectronic recombinations in plasmas, electron capture by ions through the charge exchange collisions with atoms and ions, and collisional excitation of atoms and ions with other charged particles. The dominant transitions for cascading of the excited states are the electric dipole transitions. It is convenient to have tables of values for the branching ratios for these transitions between the important initial and final levels. This tabulation is being done here for the case of the hydrogen-like atoms.

The values for the branching ratios for the low-lying levels can easily be obtained using the known values of the atomic electric dipole moments.[1] For the higher levels, although the values of the dipole moments are available, calculation of the branching ratios becomes tedious. This justifies the preparation of the present tables.

Let $n'\ell'$ and $n\ell$ represent the principal and angular momentum quantum numbers of the initial and final states. In the tables that follow the branching ratios for all the final states $n\ell$ corresponding to the first 5 principal quantum numbers, and all the initial states $n'\ell'$ extending to $n' = 15$, except for the cases where $\ell' \geq 7$, are tabulated. Average values with respect to ℓ' are given. The branching ratios that are not tabulated, including those corresponding to the threshold of the continuum as the initial states, could be obtained by extrapolation.

The electric dipole transition probability for initial and final states i and f is given by the Einstein's A coefficient:[1]

$$A(f, i) = \frac{1}{3}\alpha^3 (R_\infty/\hbar a_0^2)(\Delta E/R_\infty)^3 \times |\langle i|\underset{\sim}{r}|f\rangle|^2, \quad (1)$$

where $\langle i|\underset{\sim}{r}|f\rangle$ is the electric-dipole moment, $\Delta E/R_\infty$ is the energy difference between the levels in rydberg, and α and a_0 are the fine structure constant and the Bohr radius. For the case of the hydrogen-like atoms (1) reduces to

$$A(n\ell, n'\ell') = \frac{1}{3}\alpha^3 (R_\infty/\hbar a_0^2)\, \mu\, Z^4 \left(\frac{1}{n^2} - \frac{1}{n'^2}\right)^3 \times \sum_{m m'} |\langle n\ell m|\underset{\sim}{r}|n'\ell' m'\rangle|^2 \quad (2)$$

where μ is the reduced mass, and Z is the nuclear charge of the atom, m and m' are the magnetic quantum numbers of the final and initial states, and $|\langle n\ell\, m|\underset{\sim}{r}|n'\ell'\, m'\rangle|^2$ is the square of the electric dipole moment for the atomic hydrogen.

The branching ratio due to the direct transition between $n'\ell'$ and $n\ell$, which we call $\beta(n\ell, n'\ell')$, is given by

$$\beta(n\ell, n'\ell') = \frac{A(n\ell, n'\ell')}{\sum_{n''=1}^{n'-1} \sum_{\ell''=0}^{n''-1} A(n''\ell'', n'\ell')}. \tag{3}$$

However, many transitions from $n'\ell'$ to $n\ell$ take place by the transient electron passing through the intermediate states $n''\ell''$ before reaching $n\ell$. Let us designate the branching ratio due to all possible transitions by $\beta_T(n\ell, n'\ell')$. Then if the values of $\beta_T(n\ell, n''\ell'')$ are known for all values of $n''\ell''$ up to $n'' < n' - 1$, the value of $\beta_T(n\ell, n'\ell')$ is obviously given by the following formula:

$$\beta_T(n\ell, n'\ell') = \beta(n\ell, n'\ell')$$

$$+ \sum_{\nu=1}^{n'-n-1} \sum_{\lambda=0}^{n+\nu-1} \beta(n+\nu\lambda, n'\ell')\, \beta_T(n\ell, n+\nu\lambda) \tag{4}$$

Of usefulness are also the branching ratios obtained by averaging $\beta_T(n\ell, n'\ell')$ with respect to ℓ' and summing with respect to ℓ:

$$\beta_T(n\ell, n') = \frac{1}{n'^2} \sum_{\ell'=0}^{n'-1} (2\ell' + 1)\, \beta_T(n\ell, n'\ell'), \tag{5}$$

$$\beta_T(n, n') = \sum_{\ell=0}^{n-1} (2\ell + 1)\, \beta_T(n\ell, n') \tag{6}$$

In practice, for a given $n\ell$ Equation (4) is used to calculate $\beta_T(n\ell, n'\ell')$ for $n' = n + 1$. Then (4) is used successively to obtain the total branching ratio for any desired $n'\ell'$.

In evaluating $\beta_T(n\ell, n'\ell')$ the values of the dipole moments between $n'\ell'$ and $n\ell$ are needed. A general formula in terms of a hypergeometric function for the dipole moment is given by Bethe and Salpeter.[1] Alternatively, a closed expression can be found for the dipole moment as follows.

For convenience let m and m′ represent from now on the absolute values of the magnetic quantum numbers. This could be done since the electric dipole moment is invariant with respect to the change of sign of m and m′. Using now the closed expression for $\langle n\ell\, m | \exp(i\underset{\sim}{q}\cdot\underset{\sim}{r}) | n'\ell'\, m'\rangle$, (Ref. 2), we find with straightforward algebra

$$\langle n\ell m | \hat{q}\cdot\underset{\sim}{r} | n'\ell' m'\rangle = \delta(m, m') \times \left[\frac{\partial}{\partial(iq)} \langle n\ell m | e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} | n'\ell' m\rangle\right]_{q\to 0}$$

$$= \delta(m, m') \sum_{n_1=0}^{n-m-1} \sum_{n_1'=0}^{n'-m-1} \langle nn_1 m | n\ell m\rangle \langle n'n_1' m | n'\ell' m\rangle \langle nn_1 m | \hat{q}\cdot\underset{\sim}{r} | n'n_1' m\rangle \qquad (7)$$

where $\underset{\sim}{q}$ is an arbitrary vector, $n'n_1'm$ and nn_1m are the parabolic coordinate quantum numbers of the initial and final states, $\langle nn_1 m | n\ell m\rangle$ are elements of the transformation matrix between the spherical and parabolic coordinates eigenstates. This matrix is given in terms of the Wigner's 3j symbol by

$$\langle nn_1 m | n\ell m\rangle = (-)^m (2\ell+1)^{1/2} \begin{pmatrix} \frac{1}{2}(n-1) & \frac{1}{2}(n-1) & \ell \\ \frac{1}{2}(m-n_1+n_2) & \frac{1}{2}(m+n_1-n_2) & -m \end{pmatrix} \qquad (8)$$

Finally,

$$\langle nn_1 m | \hat{q} \cdot \underset{\sim}{r} | n'n'_1 m \rangle = \frac{1}{4} \left[\frac{4nn'}{(n+n')^2} \right]^{m+2}$$

$$\times \left[\frac{(n_1+m)!\,(n_2+m)!\,(n'_1+m)!\,(n'_2+m)!}{n_1!\,n_2!\,n'_1!\,n'_2!} \right]^{1/2}$$

$$\times \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \sum_{\nu'_1=0}^{n'_1} \sum_{\nu'_2=0}^{n'_2} \left(\frac{-2n'}{n+n'} \right)^{\nu'_1+\nu'_2} \left(\frac{-2n}{n+n'} \right)^{\nu'_1+\nu'_2} \binom{n_1}{\nu_1} \binom{n_2}{\nu_2} \binom{n'_1}{\nu'_1} \binom{n'_2}{\nu'_2}$$

$$\times \frac{(m+\nu_1+\nu'_1)!\,(m+\nu_2+\nu'_2)!}{(m+\nu_1)!\,(m+\nu'_1)!\,(m+\nu_2)!\,(m+\nu'_2)!} \left[\lambda_1(\lambda_1+1) - \lambda_2(\lambda_2+1) \right],$$

$$n_2 = n - m - 1 - n_1, \quad n'_2 = n' - m - 1 - n'_1,$$

$$\lambda_1 = m + 1 + \nu_1 + \nu'_1, \quad \lambda_2 = m + 1 + \nu_2 + \nu'_2 \tag{9}$$

Equations (7) to (9) can be combined with

$$|\langle n\ell m | \underset{\sim}{r} | n'\ell' m' \rangle|^2 = 3 |\langle n\ell m | \hat{q} \cdot \underset{\sim}{r} | n'\ell' m' \rangle|^2 \tag{10}$$

to obtain the values of the square of the dipole moments to be used in (2) for calculation of $A(n\ell, n'\ell')$. Then, Eqs. (3) – (6) can be used to find the values of $\beta(n\ell, n'\ell')$, $\beta_T(n\ell, n'\ell')$, $\beta_T(n\ell, n')$ and $\beta_T(n, n')$.

It should be noted that while $A(n\ell, n'\ell')$ increases linearly with the reduced mass and increases as the 4th power of Z with respect to the nuclear charge Z, the branching ratios are independent of these parameters.

The branching ratios are given in the tables that follow.

Due to the space limitations in the tables, the branching ratios for $\ell' > 7$ are not listed in the tables. Since the state $n'\ell'$ where $\ell' = n' - 1$ can decay only into $n'' = n' - 1$ and $\ell'' = n'' - 1$, and then into $n''' = n'' - 1$ and $\ell''' = n''' - 1$, etc., the following relationship holds

$$\beta_T(n\ell, n'\ell' = n' - 1) = \delta(\ell, n - 1) \tag{11}$$

A crude estimate for the branching ratios corresponding to the final states ℓ' where $6 < \ell' < n' - 1$ could be obtained by interpolating values for the branching ratios between $\ell' = 6$ given in the tables and $\ell' = n' - 1$ given by (11).

Similarly, the branching ratios for $n' > 15$ can crudely be estimated by extrapolating the branching ratios for $n' \leq 15$. In this way an estimate of the branching ratios from the threshold of the continuum to any final state given in the tables can be estimated.

All the excited states of the hydrogen-like atoms decay by the electric dipole transitions except the 2s state. Then the sum of the branching ratios of the 1s and 2s states should be equal to unity, as is verified in the tables.

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REFERENCES

1. H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms (Academic Press Inc., New York, 1957).

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3. J. W. B. Hughes, Proc. Phys. Soc. (London) 91, 810 (1967).

II. RESULTS

In Tables I through XV that follow the branching ratios $\beta_T(n\ell, n'\ell')$ and $\beta_T(n\ell, n')$ are tabulated for $1 \leq n \leq 5$, $0 \leq \ell \leq n - 1$, $n < n' \leq 15$, and $0 \leq \ell' \leq 6$.

Table 1
Final 1s State

n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
	← $\rho_T(1s, n'\ell')$ →							$\rho_T(1s, n')$
2	0.00	1.00						7.50-1*
3	1.00	8.82-1	1.00					9.61-1
4	9.51-1	8.81-1	9.70-1	1.000				9.65-1
5	9.35-1	8.81-1	9.60-1	9.89-1	1.000			9.72-1
6	9.28-1	8.82-1	9.55-1	9.84-1	9.95-1	1.00		9.77-1
7	9.24-1	8.82-1	9.52-1	9.81-1	9.92-1	9.98-1	1.0000	9.82-1
8	9.21-1	8.82-1	9.51-1	9.79-1	9.91-1	9.96-1	9.99-1	9.85-1
9	9.20-1	8.82-1	9.49-1	9.77-1	9.89-1	9.95-1	9.98-1	9.87-1
10	9.19-1	8.82-1	9.49-1	9.76-1	9.88-1	9.94-1	9.97-1	9.89-1
11	9.18-1	8.82-1	9.48-1	9.76-1	9.88-1	9.93-1	9.96-1	9.91-1
12	9.17-1	8.82-1	9.48-1	9.75-1	9.87-1	9.93-1	9.96-1	9.92-1
13	9.16-1	8.82-1	9.47-1	9.75-1	9.87-1	9.93-1	9.96-1	9.93-1
14	9.16-1	8.82-1	9.47-1	9.75-1	9.87-1	9.92-1	9.96-1	9.94-1
15	9.16-1	8.82-1	9.47-1	9.74-1	9.86-1	9.92-1	9.95-1	9.94-1

*The fourth digit indicates the power of 10 by which the entry should be raised.

Table II
Final 2s State

$\leftarrow \sigma_T(2s, n'\ell') \rightarrow \sigma_T(2s, n')$

n' \ ℓ'	0	1	2	3	4	5	6	⟨ℓ'⟩
3	0.00	1.18-1	0.00					3.94-2
4	4.92-2	1.19-1	3.01-2	0.00				3.48-2
5	6.48-2	1.19-1	4.03-2	1.09-2	0.00			2.79-2
6	7.20-2	1.18-1	4.51-2	1.63-2	4.87-3	0.00		2.25-2
7	7.60-2	1.18-1	4.78-2	1.94-2	7.70-3	2.49-3	0.00	1.84-2
8	7.85-2	1.18-1	4.95-2	2.13-2	9.49-3	4.08-3	1.40-3	1.53-2
9	8.03-2	1.18-1	5.06-2	2.26-2	1.07-2	5.16-3	2.36-3	1.29-2
10	8.15-2	1.18-1	5.14-2	2.35-2	1.16-2	5.94-3	3.04-3	1.10-2
11	8.24-2	1.18-1	5.20-2	2.42-2	1.22-2	6.50-3	3.55-3	9.46-3
12	8.31-2	1.18-1	5.24-2	2.47-2	1.27-2	6.94-3	3.93-3	8.24-3
13	8.37-2	1.18-1	5.28-2	2.51-2	1.31-2	7.27-3	4.23-3	7.23-3
14	8.41-2	1.18-1	5.30-2	2.54-2	1.34-2	7.54-3	4.46-3	6.40-3
15	8.45-2	1.18-1	5.32-2	2.57-2	1.36-2	7.75-3	4.66-3	5.70-3

Table III
Final 2p State

$\longleftarrow \beta_T(2p, n'\ell') \longrightarrow$ $\beta_T(2p, n')$

n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
3	1.00	0.00	1.00					6.67-1
4	5.84-1	4.20-2	7.46-1	1.00				7.15-1
5	4.64-1	5.61-2	6.65-1	9.08-1	1.00			7.72-1
6	4.09-1	6.29-2	6.27-1	8.63-1	9.59-1	1.00		8.17-1
7	3.80-1	6.68-2	6.06-1	8.38-1	9.35-1	9.79-1	1.00	8.50-1
8	3.61-1	6.93-2	5.93-1	8.22-1	9.20-1	9.66-1	9.88-1	8.76-1
9	3.49-1	7.10-2	5.85-1	8.12-1	9.10-1	9.56-1	9.80-1	8.95-1
10	3.40-1	7.23-2	5.79-1	8.04-1	9.03-1	9.50-1	9.74-1	9.11-1
11	3.33-1	7.33-2	5.74-1	7.99-1	8.98-1	9.45-1	9.70-1	9.23-1
12	3.28-1	7.40-2	5.71-1	7.94-1	8.94-1	9.42-1	9.67-1	9.33-1
13	3.24-1	7.46-2	5.68-1	7.91-1	8.91-1	9.39-1	9.64-1	9.41-1
14	3.21-1	7.51-2	5.66-1	7.89-1	8.88-1	9.37-1	9.62-1	9.48-1
15	3.18-1	7.55-2	5.65-1	7.87-1	8.86-1	9.35-1	9.61-1	9.53-1

Table IV
Final 3s State

n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
	← $\beta_T(3s, n'\ell')$						→	$\beta_T(3s, n')$
4	0.00	3.77-2	0.00	0.00				7.07-3
5	8.57-3	3.90-2	3.90-3	0.00	0.00			5.80-3
6	1.28-2	3.91-2	5.96-3	6.78-4	0.00	0.00		4.57-3
7	1.52-2	3.91-2	7.15-3	1.22-3	1.62-4	0.00	0.00	3.64-3
8	1.67-2	3.90-2	7.90-3	1.62-3	3.21-4	4.83-5	0.00	2.94-3
9	1.78-3	3.90-2	8.41-3	1.90-3	4.52-4	1.02-4	1.69-5	2.41-3
10	1.85-2	3.89-2	8.77-3	2.12-3	5.56-4	1.49-4	3.71-5	2.01-3
11	1.90-2	3.89-2	9.04-3	2.28-3	6.38-4	1.90-4	5.63-5	1.70-3
12	1.94-2	3.89-2	9.24-3	2.40-3	7.04-4	2.23-4	7.34-5	1.45-3
13	1.97-2	3.88-2	9.39-3	2.50-3	7.57-4	2.51-4	8.81-5	1.26-3
14	2.00-2	3.88-2	9.52-3	2.58-3	8.00-4	2.74-4	1.01-4	1.10-3
15	2.02-2	3.88-2	9.62-3	2.65-3	8.35-4	2.94-4	1.11-4	9.63-4

Table V
Final 3p State

	$\longleftarrow \beta_T(3p, n'\ell') \longrightarrow$							$\beta_T(3p, n')$
n' \ ℓ'	0	1	2	3	4	5	6	$\langle\ell'\rangle$
4	4.16-1	0.00	2.54-1	0.00				1.05-1
5	3.19-1	8.43-3	2.36-1	9.23-2	0.00			8.69-2
6	2.73-1	1.26-2	2.25-1	1.20-1	4.12-2	0.00		7.34-2
7	2.47-1	1.50-2	2.17-1	1.31-1	6.07-2	2.10-2	0.00	6.28-2
8	2.32-1	1.65-2	2.13-1	1.37-1	7.17-2	3.32-2	1.19-2	5.42-2
9	2.21-1	1.75-2	2.10-1	1.41-1	7.86-2	4.09-2	1.95-2	4.72-2
10	2.14-1	1.83-2	2.07-1	1.43-1	8.31-2	4.62-2	2.47-2	4.15-2
11	2.08-1	1.88-2	2.06-1	1.45-1	8.64-2	4.99-2	2.85-2	3.66-2
12	2.04-1	1.92-2	2.04-1	1.46-1	8.87-2	5.27-2	3.13-2	3.26-2
13	2.01-1	1.95-2	2.03-1	1.47-1	9.05-2	5.48-2	3.34-2	2.91-2
14	1.98-1	1.98-2	2.02-1	1.47-1	9.19-2	5.65-2	3.51-2	2.62-2
15	1.96-1	2.00-2	2.02-1	1.48-1	9.30-2	5.78-2	3.64-2	2.37-2

Table VI
Final 3d State

	$\beta_T(3d, n'\ell')$							$\beta_T(3d, n')$
n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
4	0.00	4.28-3	0.00	1.00				4.38-1
5	9.72-4	3.56-3	3.96-3	6.37-1	1.00			5.40-1
6	1.33-3	3.22-3	6.15-3	5.18-1	8.38-1	1.000		6.17-1
7	1.50-3	3.04-3	7.57-3	4.62-1	7.58-1	9.17-1	1.00	6.78-1
8	1.59-3	2.94-3	8.59-3	4.30-1	7.12-1	8.69-1	9.53-1	7.25-1
9	1.65-3	2.88-3	9.35-3	4.11-1	6.83-1	8.37-1	9.23-1	7.64-1
10	1.68-3	2.84-3	9.96-3	3.97-1	6.63-1	8.16-1	9.02-1	7.95-1
11	1.71-3	2.81-3	1.04-2	3.88-1	6.49-1	8.00-1	8.87-1	8.20-1
12	1.73-3	2.79-3	1.08-2	3.81-1	6.38-1	7.89-1	8.76-1	8.41-1
13	1.75-3	2.77-3	1.12-2	3.75-1	6.30-1	7.80-1	8.67-1	8.59-1
14	[illegible]76-3	2.76-3	1.15-2	3.71-1	6.24-1	7.73-1	8.60-1	8.74-1
15	1.77-3	2.75-3	1.17-2	3.68-1	6.19-1	7.68-1	8.55-1	8.87-1

Table VII
Final 4s State

n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
	← $\beta_T(4s, n'\ell')$						→	$\beta_T(4s, n')$
5	0.00	1.75-2	0.00	0.00	0.00			2.10-3
6	2.52-3	1.82-2	9.39-4	0.00	0.00	0.00		1.72-3
7	4.02-3	1.83-2	1.54-3	9.19-5	0.00	0.00	0.00	1.37-3
8	4.97-3	1.82-2	1.92-3	1.85-4	1.33-5	0.00	0.00	1.11-3
9	5.61-3	1.82-2	2.19-3	2.63-4	3.02-5	2.53-6	0.00	9.05-4
10	6.06-3	1.82-2	2.38-3	3.25-4	4.65-5	6.22-6	5.90-7	7.52-4
11	6.39-3	1.81-2	2.51-3	3.75-4	6.10-5	1.02-5	1.54-6	6.34-4
12	6.65-3	1.81-2	2.62-3	4.14-4	7.35-5	1.39-5	2.62-6	5.41-4
13	6.85-3	1.81-2	2.70-3	4.46-4	8.40-5	1.73-5	3.71-6	4.66-4
14	7.01-3	1.81-2	2.77-3	4.73-4	9.30-5	2.04-5	4.76-6	4.06-4
15	7.14-3	1.81-2	2.82-3	4.94-4	1.01-4	2.30-5	5.71-6	3.56-4

Table VIII
Final 4p State

$\leftarrow \beta_T(4p, n'\ell') \rightarrow \beta_T(4p, n')$

n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
5	2.27-1	0.00	1.04-1	0.00	0.00			2.98-2
6	1.92-1	2.66-3	1.03-1	1.80-2	0.00	0.00		2.33-2
7	1.70-1	4.23-3	1.00-1	2.70-2	4.30-3	0.00	0.00	1.86-2
8	1.57-1	5.22-3	9.84-2	3.20-2	7.72-3	1.28-3	0.00	1.52-2
9	1.49-1	5.88-3	9.70-2	3.52-2	1.02-2	2.54-3	4.48-4	1.26-2
10	1.43-1	6.35-3	9.60-2	3.72-2	1.20-2	3.59-3	9.49-4	1.06-2
11	1.38-1	6.69-3	9.52-2	3.87-2	1.33-2	4.43-3	1.40-3	9.09-3
12	1.35-1	6.96-3	9.46-2	3.97-2	1.44-2	5.10-3	1.79-3	7.84-3
13	1.33-1	7.16-3	9.41-2	4.05-2	1.52-2	5.64-3	2.12-3	6.82-3
14	1.30-1	7.33-3	9.37-2	4.11-2	1.58-2	6.08-3	2.39-3	5.99-3
15	1.29-1	7.46-3	9.34-2	4.16-2	1.63-2	6.44-3	2.62-3	5.29-3

Table IX
Final 4d State

	$\beta_T(4d, n'\ell')$							$\beta_T(4d, n')$
n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
5	0.00	4.48−3	0.00	3.63−1	0.00			1.02−1
6	6.43−4	3.84−3	1.93−3	3.09−1	1.62−1	0.00		1.01−1
7	9.49−4	3.49−3	3.13−3	2.78−1	2.00−1	8.27−2	0.00	9.56−2
8	1.11−3	3.28−3	3.94−3	2.60−1	2.13−1	1.19−1	4.66−2	8.88−2
9	1.21−3	3.15−3	4.53−3	2.48−1	2.18−1	1.38−1	7.26−2	8.17−2
10	1.28−3	3.06−3	4.97−3	2.40−1	2.21−1	1.50−1	8.87−2	7.50−2
11	1.32−3	3.00−3	5.32−3	2.34−1	2.22−1	1.57−1	9.94−2	6.87−2
12	1.35−3	2.96−3	5.60−3	2.29−1	2.22−1	1.62−1	1.07−1	6.29−2
13	1.38−3	2.92−3	5.84−3	2.26−1	2.23−1	1.65−1	1.12−1	5.77−2
14	1.40−3	2.90−3	6.03−3	2.23−1	2.23−1	1.68−1	1.17−1	5.30−2
15	1.41−3	1.41−3	2.88−3	2.21−1	2.23−1	1.70−1	1.20−1	4.88−2

Table X
Final 4f State

n' \ ℓ'	0	1	2	3	4	5	6	$\langle\ell'\rangle$
	$\beta_T(4f, n'\ell')$							$\beta_T(4f, n')$
5	0.00	0.00	3.52-3	0.00	1.00			3.61-1
6	0.00	1.38-5	2.56-3	3.34-3	5.54-1	1.00		4.45-1
7	1.36-6	2.04-5	2.12-3	4.90-3	4.13-1	7.72-1	1.00	5.15-1
8	2.72-6	2.43-5	1.90-3	5.83-3	3.48-1	6.61-1	8.72-1	5.75-1
9	3.84-6	2.67-5	1.77-3	6.47-3	3.12-1	5.98-1	7.96-1	6.25-1
10	4.75-6	2.85-5	1.69-3	6.95-3	2.89-1	5.58-1	7.48-1	6.67-1
11	5.48-6	2.98-5	1.64-3	7.33-3	2.74-1	5.30-1	7.14-1	7.03-1
12	6.07-6	3.08-5	1.60-3	7.64-3	2.63-1	5.11-1	6.91-1	7.34-1
13	6.56-6	3.16-5	1.57-3	7.90-3	2.55-1	4.96-1	6.73-1	7.61-1
14	6.97-6	3.23-5	1.55-3	8.12-3	2.49-1	4.85-1	6.59-1	7.84-1
15	7.31-6	3.29-5	1.54-3	8.31-3	2.44-1	4.76-1	6.48-1	8.03-1

Table XI
Final 5s State

n' \ ℓ'	0	1	2	3	4	5	6	⟨ℓ'⟩
	← $\beta_T(5s, n'\ell')$						→	$\beta_T(5s, n')$
6	0.00	9.92-3	0.00	0.00	0.00	0.00		8.26-4
7	9.82-4	1.03-2	3.17-4	0.00	0.00	0.00	0.00	6.81-4
8	1.63-3	1.03-2	5.41-4	1.95-5	0.00	0.00	0.00	5.52-4
9	2.07-3	1.03-2	6.96-4	4.22-5	1.85-6	0.00	0.00	4.52-4
10	2.38-3	1.02-2	8.08-4	6.30-5	4.62-6	2.40-7	0.00	3.76-4
11	2.61-3	1.02-2	8.90-4	8.08-5	7.60-6	6.59-7	3.96-8	3.16-4
12	2.78-3	1.02-2	9.53-4	9.57-5	1.05-5	1.16-6	1.16-7	2.70-4
13	2.91-3	1.01-2	1.00-3	1.08-4	1.31-5	1.69-6	2.16-7	2.32-4
14	3.02-3	1.01-2	1.04-3	1.19-4	1.54-5	2.21-6	3.28-7	2.02-4
15	3.11-3	1.01-2	1.07-3	1.27-4	1.75-5	2.69-6	4.42-7	1.77-4

Table XII
Final 5p State

← $\beta_T(5p, n'\ell')$ → $\beta_T(5p, n')$

n' \ ℓ'	0	1	2	3	4	5	6	$\langle\ell\rangle$
6	1.44-1	0.00	5.35-2	0.00	0.00	0.00		1.14-2
7	1.27-1	1.08-3	5.44-2	5.24-3	0.00	0.00	0.00	8.96-3
8	1.15-1	1.79-3	5.38-2	8.52-3	7.58-4	0.00	0.00	7.12-3
9	1.07-1	2.26-3	5.32-2	1.06-2	1.53-3	1.44-4	0.00	5.80-3
10	1.02-1	2.59-3	5.26-2	1.20-2	2.17-3	3.29-4	3.37-5	4.81-3
11	9.84-2	2.83-3	5.22-2	1.30-2	2.69-3	5.10-4	8.35-5	4.05-3
12	9.55-2	3.02-3	5.19-2	1.38-2	3.10-3	6.72-4	1.37-4	3.46-3
13	9.34-2	3.16-3	5.16-2	1.43-2	3.43-3	8.12-4	1.89-4	2.98-3
14	9.16-2	3.27-3	5.14-2	1.47-2	3.70-3	9.32-4	2.37-4	2.60-3
15	9.02-2	3.36-3	5.12-2	1.51-2	3.92-3	1.03-3	2.80-4	2.28-3

Table XIII
Final 5d State

n' \ ℓ'	$\rho_T(5d, n'\ell')$ 0	1	2	3	4	5	6	$\rho_T(5d, n')$ ⟨ℓ'⟩
6	0.00	3.92-3	0.00	1.74-1	0.00	0.00		3.41-2
7	3.88-4	3.42-3	9.47-4	1.63-1	4.15-2	0.00	0.00	3.13-2
8	5.99-4	3.12-3	1.58-3	1.54-1	6.06-2	1.24-2	0.00	2.78.2
9	7.24-4	2.93-3	2.03-3	1.48-1	7.05-2	2.19-2	4.33-3	2.15-2
10	8.03-4	2.80-3	2.35-3	1.43-1	7.63-2	2.86-2	8.55-3	2.16-2
11	8.58-4	2.72-3	2.61-3	1.39-1	7.99-2	3.35-2	1.20-2	1.91-2
12	8.97-4	2.65-3	2.80-3	1.37-1	8.24-2	3.70-2	1.48-2	1.69-2
13	9.26-4	2.61-3	2.97-3	1.35-1	8.41-2	3.97-2	1.70-2	1.51-2
14	9.49-4	2.57-3	3.10-3	1.33-1	8.54-2	4.17-2	1.87-2	1.35-2
15	9.67-4	2.54-3	3.21-3	1.32-1	8.63-2	4.33-2	2.02-2	1.21-2

Table XIV
Final 5f State

$\longleftarrow \beta_T(5f,\ n'\ell') \longrightarrow$ $\beta_T(5d,\ n')$

n' \ ℓ'	0	1	2	3	4	5	6	$\langle \ell' \rangle$
6	0.00	0.00	4.66-3	0.00	4.46-1	0.00		1.12-1
7	0.00	1.53-5	3.59-3	2.33-3	3.48-1	2.28-1	0.00	1.16-1
8	1.11-6	2.35-5	3.04-3	3.62-3	2.96-1	2.68-1	1.28-1	1.14-1
9	2.32-6	2.83-5	2.73-3	4.45-3	2.66-1	2.75-1	1.79-1	1.11-1
10	3.40-6	3.15-5	2.54-3	5.02-3	2.47-1	2.74-1	2.03-1	1.05-1
11	4.30-6	3.37-5	2.41-3	5.45-3	2.33-1	2.72-1	2.16-1	9.99-2
12	5.05-6	3.54-5	2.32-3	5.79-3	2.24-1	2.69-1	2.23-1	9.41-2
13	5.67-6	3.67-5	2.25-3	6.07-3	2.17-1	2.66-1	2.28-1	8.84-2
14	6.19-6	3.78-5	2.20-3	6.29-3	2.12-1	2.64-1	2.31-1	8.30-2
15	6.64-6	3.86-5	2.17-3	6.49-3	2.08-1	2.62-1	2.33-1	7.78-2

Table XV
Final 5g State

n' \ ℓ'	0	1	2	3	4	5	6	$\langle\ell'\rangle$
	$\leftarrow$ $\rho_T(5g, n'\ell')$						$\rightarrow$	$\rho_T(5g, n')$
6	0.00	0.00	0.00	2.73-3	0.00	1.00		3.06-1
7	0.00	0.00	1.29-5	1.76-3	2.78-3	4.89-1	1.00	3.76-1
8	0.00	3.56-8	1.81-5	1.35-3	3.84-3	3.36-1	7.12-1	4.37-1
9	1.98-9	7.39-8	2.07-5	1.14-3	4.37-3	2.67-1	5.76-1	4.92-1
10	5.08-9	1.08-7	2.22-5	1.03-3	4.72-3	2.29-1	5.00-1	5.41-1
11	8.58-9	1.37-7	2.32-5	9.56-4	4.96-3	2.06-1	4.52-1	5.83-1
12	1.21-8	1.61-7	2.40-5	9.08-4	5.15-3	1.90-1	4.20-1	6.21-1
13	1.54-8	1.82-7	2.45-5	8.75-4	5.31-3	1.80-1	3.97-1	6.54-1
14	1.85-8	1.99-7	2.50-5	8.52-4	5.45-3	1.71-1	3.80-1	6.83-1
15	2.16-8	2.14-7	2.54-5	8.35-4	5.56-3	1.65-1	3.67-1	7.09-1