UTL

PARTICLE ACCELERATION IN AXISYMMETRIC, MAGNETIZED NEUTRON STARS

by

K.B. Baker and P.A. Sturrock

National Aeronautics and Space Administration
Grant NGR 05-020-668

SUIPR Report No. 695

May 1977

INSTITUTE FOR PLASMA RESEARCH
STANFORD UNIVERSITY, STANFORD, CALIFORNIA

PARTICLE ACCELERATION IN AXISYMMETRIC, MAGNETIZED NEUTRON STARS

by

K.B. Baker* and P.A. Sturrock †

National Aeronautics and Space Administration

Grant NGR 05-020-668

SUIPR Report No. 695

May 1977

* Also Department of Physics

† Also Department of Applied Physics

PARTICLE ACCELERATION IN AXISYMMETRIC, MAGNETIZED NEUTRON STARS

K.B. Baker* and P.A. Sturrock†
Institute for Plasma Research
Stanford University
Stanford, California

ABSTRACT

The potential drop in the polar cap region of a rotating, magnetized neutron star can be found assuming that the magnetic field is dipolar, with the field aligned (or anti-aligned) with the rotation axis. The curvature of the field lines is of critical importance. Charge flow is assumed to be along magnetic field lines. The electric field has a maximum at radius $1.5R_*$ and the magnitude and functional form of the current can be determined.

---

* Also Department of Physics

† Also Department of Applied Physics

I. Introduction

The source of the accelerating potential in pulsars continues to be a significant problem in our understanding of these objects. It is clear that particles are accelerated but the source of the accelerating electric field is not yet understood. The most widely accepted picture of a pulsar is one in which particles are accelerated in the region of the magnetic polar cap, resulting (via some kind of plasma or MHD instability) in a pencil beam of radio emission (Radhakrishnan and Cooke, 1969). Specific models along these lines have been proposed by a number of investigators (Pacini and Rees (1970), Sturrock (1971), Roberts and Sturrock (1972), Ruderman and Sutherland (1975)).

In this paper we will investigate the form of the electric potential in the polar cap region of an aligned magnetic star. Clearly this cannot be a realistic model of a pulsar, but it can give us some insight into the physics of a real pulsar. We shall assume that the magnetic field is dipolar and the polar cap region is bounded by the first closed field line (i.e. the first line that does not connect to the surrounding nebula). The field lines are defined by the equation

$$\frac{\sin^2 \theta}{r} = \text{const.} \tag{1.1}$$

In particular, the polar cap angle is defined as

$$\frac{\sin^2 \theta_p}{R_*} = \frac{1}{R_{max}} \tag{1.2}$$

where R_* is the radius of the neutron star and R_{max} is a free parameter which is fixed by a more detailed model. In the work of Sturrock (1971) and Ruderman and Sutherland (1975), R_{max} is the radius of the light cylinder, whereas the work of Roberts and Sturrock (1972) it is the "force-balance" radius.

II. The Polar Cap Region

In the region of the polar cap, the magnetic field is much larger than any expected electric field, so that it is reasonable to assume that charged particles will be tied to the magnetic field lines. In this case the current density is approximately proportional to the magnetic field strength

$$j(r) \approx j_* \frac{R_*^3}{r^3} \qquad (2.1)$$

If the stream flows in one direction only, e.g. outward, the charge density is given by $\rho = j/v$. We will consider the situtation where $v \simeq c$ and thus $\rho \simeq j/c$. Clearly this is not true in the region immediately above the polar cap, but we will show that it is a good approximation when the distance above the cap is small compared with the overall dimensions of the system. We also note that in this situation the component of $\vec{j}$ rotating with the star is negligible until the particles approach the light cylinder. In the Roberts and Sturrock model (1972), the polar-cap region extends only to the force-balance radius and hence the co-rotating component of $\vec{j}$ is always negligible.

Poisson's equation can now be written as

$$\nabla^2 \Phi = - \frac{4\pi j_*}{c} \frac{R_*^3}{r^3} \qquad (2.2)$$

At this point we introduce orthonormal dipole coordinates defined by

$$\xi = \frac{(\cos\,\theta)^{\frac{1}{2}}}{r} \quad , \qquad \eta = \frac{\sin\,\theta}{r^{\frac{1}{2}}} \quad . \tag{2.3}$$

The curves of constant η are the magnetic field lines and curves of constant ξ are the lines of constant magnetic potential. The remaining coordinate is the azimuthal angle φ. Since the system is azimuthally symmetric, we can now write Poisson's equation as

$$\nabla^2\Phi = \frac{1+3\cos^2\,\theta}{4r^5(\cos\,\theta)^{\frac{1}{2}}}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}\;\frac{\partial\Phi}{\partial\xi}\right) + {}^{\xi}/_{\eta}\;\frac{\partial}{\partial\eta}\left(r^3\eta\;\frac{\partial\Phi}{\partial\eta}\right)\right] \tag{2.4}$$

where r and θ are the usual spherical co-ordinates and are implicitly defined by ξ, η.

We now make a major simplifying assumption, that the contribution of the divergence of the electric field due to the field component transverse to the magnetic field is small compared with that due to the field component parallel to the magnetic field. This is opposed to the assumption of Goldreich and Julian (1969), according to which the parallel component vanishes. We are implicitly assuming that the assumption of current outflow from the polar cap necessitates the development of a parallel electric field, and we are studying the effect of this component, which is associated with current outflow, independently of the transverse electric field component, which is associated with corotation. A complete solution of the electric field configuration of pulsar magnetospheres will require the simultaneous incorporation of both effects. However, since much work has been based on the Goldreich-Julian simplification, it seems not unreasonable to explore briefly the implications of the converse simplification.

We therefore now assume that the η derivatives can be neglected with respect to the ξ derivatives:

$$\nabla^2\Phi \simeq \frac{1+3\cos^2\theta}{4r^5(\cos\theta)^{\frac{1}{2}}} \frac{\partial}{\partial\xi}\left(\frac{1}{\xi}\frac{\partial\Phi}{\partial\xi}\right) = -\frac{4\pi j_*}{c} \frac{R_*^3}{r^3} \qquad (2.5)$$

From equation (2.5), it immediately follows that

$$\frac{1}{\xi}\frac{\partial\Phi}{\partial\xi}\Bigg|_{\xi_*}^{\xi} = -\frac{4\pi j_*}{c} R_*^3 \int_{\xi_*}^{\xi} \frac{4r^2(\cos\theta)^{\frac{1}{2}}}{1+3\cos^2\theta} d\xi \qquad (2.6)$$

The integration is along a field line η=const., and we can therefore express $d\xi$ in terms of dr. Since

$$d\xi = -\frac{(\cos\theta)^{\frac{1}{2}} dr}{r^2} - \frac{\sin\theta\, d\theta}{2r(\cos\theta)^{\frac{1}{2}}}$$

and

$$d\eta = -\frac{\sin\theta}{2r^{3/2}} dr + \frac{\cos\theta}{r^{\frac{1}{2}}} d\theta \equiv 0, \qquad (2.7)$$

we see that

$$d\xi = -\frac{1+3\cos^2\theta}{4r^2(\cos\theta)^{3/2}} dr \quad , \qquad (2.8)$$

so that equation (2.6) becomes

$$\frac{1}{\xi}\frac{d\Phi}{d\xi}\Bigg|_{\xi_*}^{\xi} = \frac{4\pi j_*}{c} R_*^3 \int_R^r \frac{dr}{\cos\theta} \qquad (2.9)$$

The equation of a field line is then used to write

$$\cos\theta = (1-\alpha r)^{\frac{1}{2}} \qquad (2.10)$$

where α is a constant for each field line, given by

$$\alpha = \sin^2 \theta_*/R_* \qquad (2.11)$$

The integration is now trivial:

$$\frac{d\Phi}{d\xi} = \left[\frac{8\pi j_*}{c} \quad \frac{R_*^3}{\alpha} \left[(1-\alpha R_*)^{\frac{1}{2}} - (1-\alpha r)^{\frac{1}{2}}\right] + \frac{1}{\xi_*} \frac{d\Phi}{d\xi} \Big|_{\xi_*} \right]\xi \qquad (2.12)$$

We may integrate equation (2.10) again, using the condition that we are integrating along a field line, and so obtain

$$\Phi = \frac{8\pi j_* R_*^3}{c\alpha} \left[\frac{1}{2r}\left(\frac{3}{2}\alpha - \frac{1}{r}\right) - \frac{1}{2R_*}\left(\frac{3}{2}\alpha - \frac{1}{R_*}\right) + (1-\alpha R_*)^{\frac{1}{2}} \frac{\xi^2 - \xi^2}{2}\right]$$

$$+ \frac{\xi^2 - \xi_*^2}{2} \frac{d\Phi}{d\xi}\Big|_{\xi_*} + \Phi(\xi_*) \qquad (2.13)$$

At this point we note that the dependence of Φ on η (or angle θ_*) is completely determined by specifying Φ and $d\Phi/d\xi$ at $\xi = \xi_*$ as functions of η. We also note that the value of ξ_* is itself a function of η (i.e. of the particular field line chosen). However, one may find a surface $\xi = \xi_*$ which is very close to the surface of the star in the polar-cap region. Since a neutron star is a nearly perfect conductor, the force on a charged particle rotating with the star must be zero. This means that, on the surface of the star,

$$\vec{E}_* = -(\vec{\Omega} \times \vec{r})/c \times \vec{B}, \qquad (2.14)$$

which in turn implies that on the stellar surface

$$\Phi_* = -\frac{\Omega B_* R_*^2}{2c}\left(\cos^2\theta_* - \cos^2\theta_p\right) . \qquad (2.15)$$

We have used our freedom to change Φ by a constant to set $\Phi_*=0$ on the closed field lines that bound the polar-cap regions.

If we now require that the charge flow be space-charge limited (this condition has been used in a number of models, e.g. Sturrock (1971), we require that the electric field parallel to the magnetic field lines be zero at the stellar surface. This implies

$$\left.\frac{d\Phi}{d\xi}\right|_{\xi_*} = 0 \tag{2.16}$$

Finally, if we can determine either Φ or $d\Phi/d\xi$ at an outer boundary $\xi = \xi_{max}$, we can determine j_*. Specifically, we note that $d\Phi/d\xi \neq 0$ for any finite distance above the stellar surface.

To see more clearly what these results mean, we can look at equation (2.12) in the limit $\theta_* << 1$:

$$\Phi \simeq \frac{8\pi j_*}{c} R_*^{\,3}\left[\frac{1}{2r} - \frac{1}{4R_*} - \frac{R_*}{4r^2}\right] \tag{2.17}$$

Thus we see that in this limit Φ appears to behave in the same way as the free-space electric field produced by a monopole and a dipole. We note also that the maximum electric field occurs at $r = {}^3/_2\, R_*$.

III. Two-Dimensional Approximation

In this section we shall attempt to improve the model by considering the two-dimensional case. The potential Φ can be expanded in a power series in η. The symmetry of the problem guarantees that the series has only even powers of η. Since $\eta_{max} = 1/R_{max}^{1/2}$, η may be treated as small and we shall consider the first two terms only in the expansion. The boundary condition on the field lines $\eta = \eta_{max}$ is $\Phi = 0$, which determines the expansion:

$$\Phi(\xi,\eta) = \Phi(\xi,0)\left(1 - \eta^2/\eta_{max}^2\right) \qquad (3.1)$$

On substituting this expression into equation 2.4, we obtain

$$\nabla^2\Phi = \frac{1+3\cos^2\theta}{4r^5(\cos\theta)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}\frac{\partial\Phi(\xi,0)}{\partial\xi}\right) - \frac{\Phi(\xi,0)}{r^3\eta_{max}^2}(1+3\cos 2\theta)$$

$$= \frac{-4\pi j_\star}{c}\frac{R_\star^3}{r^3} \qquad (3.2)$$

We may now choose the field line $\eta=0$ on which to solve equation (3.2). On this field line $\theta=0$ and $r=1/\xi$ so that

$$\nabla^2\Phi = \xi^5\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}\frac{\partial\Phi}{\partial\xi}\right) - \frac{4\xi^3\Phi}{\eta_{max}^2} = -\frac{4\pi j_\star}{c}R_\star^3\xi^3 \qquad (3.3)$$

A particular solution to the inhomogeneous equation is

$$\Phi_p(\xi,0) = \frac{\pi j_\star}{c}R_\star^3\eta_{max}^2 \qquad (3.4)$$

The homogeneous equation can be solved by making the substitution

$$u = \xi^{\frac{1}{2}} \quad \text{and} \quad \Phi(\xi,0) = u^2B(u,0) \qquad (3.5)$$

The function B(u) must then satisfy the equation

$$\frac{d^2B}{du^2} + \frac{1}{u}\frac{dB}{du} - \left[\frac{4}{u^2} + \frac{16}{\eta_{max}^2}\right]B = 0 \qquad (3.6)$$

This is the modified Bessel equation of order 2 and parameter $4/\eta_{max}$.

We now have

$$\Phi(\xi,\eta) = \left[c_1\xi I_2\left(\frac{4\sqrt{\xi}}{\eta_{max}}\right) + c_2\xi K_2\left(\frac{4\sqrt{\xi}}{\eta_{max}}\right) + \frac{\pi j_* R_*^3 \eta_{max}^2}{c}\right]\left(1- \frac{\eta^2}{\eta_{max}^2}\right). \quad (3.7)$$

We require that $\Phi \to 0$ as $\xi \to 0$ $(r \to \infty)$. The I_2 term automatically satisfies this condition. As $z \to 0$, $K_2(z) \sim \frac{2}{z^2}$ so that we have

$$\Phi(0,0) = \frac{c_2}{8}\eta_{max}^2 + \frac{\pi j_*}{c} R_*^3 \eta_{max}^2 \qquad (3.8)$$

which gives

$$c_2 = -\frac{8\pi j_*}{c} R_*^3 \quad . \qquad (3.9)$$

At $\xi = {}^1/_{R_*}$, we want

$$\Phi(\xi_*,0) = -\frac{\Omega B_* R_*^3}{2c}\eta_{max}^2 \quad . \qquad (3.10)$$

Since $4\sqrt{\xi_*}/\eta_{max} >> 1$, we can use the asymptotic expansions of I_2 and K_2 to determine the constant c_1.

$$I_2 \sim \frac{e^z}{(2\pi z)^{1/2}} \; ; \quad K_2 \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \quad \text{with } z = \frac{4\sqrt{\xi}}{\eta_{max}} \qquad (3.11)$$

Hence

$$c_1 = -\left[\frac{\Omega B_*}{2} R_* \eta_{max}^2 + \pi j_* R_* \eta_{max}^2 - 8\pi j_*\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\right] \frac{R_*^3}{c} (2\pi z)^{1/2} e^{-z} \quad . \qquad (3.12)$$

If we now require the flow to be space-charge limited, we can determine j_*. We write $d\Phi/d\xi$ in terms of z (as defined in equation 3.11):

$$\frac{d\Phi(\xi,0)}{d\xi} = \frac{8c_1}{z\eta_{max}^2}\frac{d}{dz}\left[\frac{z^2}{16}\eta_{max}^2 I_2(z)\right] + \frac{8c_2}{Z\eta_{max}^2}\frac{d}{dz}\left[\frac{z^2}{16}\eta_{max}^2 K_2(z)\right] \qquad (3.13)$$

Using

$$\frac{d}{dz}\left[z^2 I_2(z)\right] = z^2 I_1(z) \qquad (3.14)$$

we find

$$\left.\frac{d\Phi(\xi,0)}{d\xi}\right|_{\xi_*} = \frac{c_1}{2Z}Z^2 I_1(z) - \frac{c_2}{2z}z^2 K_1(z) = 0 \quad . \qquad (3.15)$$

Thus

$$c_1 = +\pi e^{-2z}c_2 = \frac{-8\pi^2 j_*}{c} R_*^3 e^{-2z} \qquad (3.16)$$

Combining (3.16) with (3.12) then gives us the following condition on j_*:

$$j_*\left[16\pi - (2\pi z)^{\frac{1}{2}}e^z R_*\eta_{max}^2\right] = \left(\frac{z}{2\pi}\right)^{\frac{1}{2}}\Omega B_* R_*\eta_{max}^2 e^z \qquad (3.17)$$

Since $z>>1$, we can approximate j_* by

$$j_* \simeq -\frac{\left(\frac{z}{2\pi}\right)^{\frac{1}{2}}\Omega B_* R_*\eta_{max}^2 e^z}{(2\pi z)^{\frac{1}{2}}e^z\eta_{max}^2 R_*} = -\frac{\Omega B_*}{2\pi} \qquad (3.18)$$

which is the value estimated by Sturrock (1971).

We would also like to know how j_* varies across the surface of the star. To determine this, we return to equation (2.4) and write it as an expansion in powers of η. From the lefinition of ξ and η we can write

$$\xi^4 r^4 + r \eta^2 = 1 \tag{3.19}$$

We then solve for r implicitly, giving us

$$r = \frac{1}{\xi}\left(1 + \frac{\eta^2}{r^3\xi^4}\right)^{-\frac{1}{4}} \tag{3.20}$$

This becomes (to ord-- η^2)

$$r = \frac{1}{\xi}\left(1 - \frac{\eta^2}{4\xi}\right) + 0\left({}^{\eta^4}/_{\xi^2}\right) \tag{3.21}$$

This expansion is valid as long as $\eta^2 < 4\xi$ (i.e. as long as $\theta \leq 85°$). We also must expand the potential to order η^2 giving us

$$\Phi(\xi,\eta) = \Phi(\xi,0)\left[1 - \frac{\eta^2}{\eta_{max}^2} + \alpha(\xi)\eta^4\right] + 0(\eta^6) \tag{3.22}$$

and we expand the current density to order η^2 giving

$$j_*(\eta) = j_*(0)\ (1 + d_1\ \eta^2\ ) + 0(\eta^4) \tag{3.23}$$

Equation (2.4) then becomes

$$\begin{aligned}\nabla^2\Phi = \left(1 + \frac{3}{4}\frac{\eta^2}{\xi}\right)\xi^5 \Bigg\{&\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}\frac{\partial\Phi(\xi,0)}{\partial\xi}\right)\left(1 - \frac{\eta^2}{\eta_{max}^2}\right) \\ &+ \frac{1}{\xi^2}\Phi(\xi,0)\left[-\frac{4}{\eta_{max}^2} + \left(\frac{6}{\xi\eta_{max}^2} + 16\alpha(\xi)\right)\eta^2\right]\Bigg\} \\ = -\frac{4\pi j_*(0)}{c}\ &R_*^3\ \xi^3\left(1 + \frac{3}{4}\frac{\eta^2}{\xi}\right)\ (1 + d_1\eta^2)\ .\end{aligned} \tag{3.24}$$

The zeroth order equation is equation (3.3) while the second-order equation is

$$\xi^5\left(\frac{3}{4}\frac{1}{\xi} - \frac{1}{\eta_{max}^2}\right)\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}\frac{\partial\Phi}{\partial\xi}\right) - \frac{3\xi^2}{\eta_{max}^2}\Phi + \xi^3\left(\frac{6}{\xi\eta_{max}^2} + 16\alpha\right)\Phi = - \frac{4\pi j_*}{c} R_*^3 \xi^3\left(\frac{3}{4}\frac{1}{\xi} + d_1\right) \quad . \tag{3.25}$$

Using the zeroth order equation to simplify equation (3.25), we obtain

$$- \frac{4\Phi(\xi,0)}{\eta_{max}^4} + \frac{4\pi j_*(0)}{c}\frac{R_*^3}{\eta_{max}^2} + \left(\frac{6}{\xi\eta_{max}^2} + 16\alpha(\xi)\right)\Phi(\xi,0) = - \frac{4\pi j_*(0)}{c} R_*^3 d_1 \quad . \tag{3.26}$$

Since the variation of j with respect to r has already been included to all orders ($j \sim 1/r^3$) we know that d_1 is independent of ξ. Equation (3.26) is satisfied for any form of Φ if

$$\alpha(\xi) = \frac{1}{4\eta_{max}^2}\left(\frac{1}{\eta_{max}^2} - \frac{3}{2\xi}\right) \tag{3.27}$$

and

$$d_1 = - \frac{1}{\eta_{max}^2} \quad . \tag{3.28}$$

Thus

$$j_*(\eta) = - \frac{\Omega B_*}{2\pi}\left(1 - \frac{\eta^2}{\eta_{max}^2}\right) \tag{3.29}$$

and we have also determined the corrections to Φ of order η^4.

The fact that j_* has the same η dependence (to second order) as Φ suggests that we may be able to find other solutions of Poisson's equation for which this is true. We therefore write equation (2.4) in the small angle approximation under this assumption. We write Φ and j_* as follows:

$$\begin{aligned} \Phi(\xi,\eta) &= \psi(\xi)H(\eta) \\ j_*(\eta) &= j_* H(\eta) \end{aligned} \qquad (3.30)$$

In the small angle approximation Poisson's equation now becomes

$$\begin{aligned} \xi H(\eta)\ \psi''(\xi) - H(\eta)\psi'(\xi) + \psi(\xi)H''\eta + \frac{1}{\eta}\ \psi(\xi)H^1(\eta) \\ = - \frac{4\pi j_*}{c}\ R_*^3\ H(\eta) \quad . \end{aligned} \qquad (3.31)$$

Dividing by ψH, we obtain two ordinary differential equations

$$H''(\eta) + \frac{1}{\eta}\ H^1(\eta) + \lambda^2 H(\eta) = 0 \qquad (3.32)$$

$$\xi\ \psi''(\xi) - \psi^1(\xi) - \lambda^2\psi(\xi) = - \frac{4\pi j_*}{c}\ R_*^3 \qquad (3.33)$$

The solution of equation (3.32) is

$$H(\eta) = J_0(\lambda\eta) \qquad (3.34)$$

while the solution of equation (3.33) is

$$\psi(\xi) = c_1 \xi I_2(\lambda\sqrt{\xi}) + c_2 \xi K_2(\lambda\sqrt{\xi}) + \frac{4\pi j_*}{\lambda^2 c}\ R_*^3 \qquad (3.35)$$

The boundary condition on the closed field lines implies that

$$J_0(\lambda\eta_{max}) = 0 \qquad (3.36)$$

To complete the solution, the constants c_1 and c_2 must be determined by the boundary conditions at the stellar surface and at infinity. This requires that ψ be a sum of the form:

$$\psi(\xi) = \sum_i \left\{ c_{1i}\xi I_2(\lambda_i\sqrt{\xi}) + c_{2i}\xi K_2(\lambda_i\sqrt{\xi}) + \frac{4\pi j_{*i}}{\lambda_i{}^2 \, L} \; R_*^3 \right\} \qquad (3.37)$$

where the λ_i are determined by the zeros of equation (3.36) and

$$j_*(\eta) = \sum_i j_{*i} J_0(\lambda_i \eta) \quad . \qquad (3.38)$$

IV. Conclusion

We have determined the analytic behavior of Φ by several different methods. The firm conclusion of these various approaches is that the current density is of the order $\frac{-\Omega B_*}{2\pi}$ at the surface. The one dimensional approximation indicates that the electric field has a maximum on the order of one star radius above the surface. We have shown that it is possible to find a two-dimensional solution which satisfies the boundary condition on the star, goes to zero at large distances and results in space-charge limited current flow.

We are currently preparing a numerical analysis of the two-dimensional solution in order to plot the surfaces of constant electric potential and determine the location of the maximum electric field for various values of η_{max}, B_*, and Ω.

Acknowledgements

This work was supported by the National Aeronautics and Space Administration under grant NGR 05-020-668.

References

Goldriech, P. and Julian, W.H. 1969, Ap.J. 157, 869.

Pacini, F. and Rees, M.J. 1970, Nature, 226, 622.

Radhakrishnan, V. and Cooke, D.J. 1969, Astrophys. Let., 3, 225.

Roberts, D.H. and Sturrock, P.A. 1972, Ap. J., 172, 435.

Ruderman, M.A. and Sutherland, P.G. 1975, Ap.J., 196, 51.

Sturrock, P.A. 1971, Ap. J., 164, 529.