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16. Abstract

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THE GENERAL SOLUTION TO THE CLASSICAL PROBLEM OF FINITE EULER BERNOULLI BEAM

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SUMMARY

An analytical solution is obtained for the problem of free and forced vibrations of a finite Euler Bernoulli beam with arbitrary (partially fixed) boundary conditions. The effects of linear viscous damping, Winkler foundation, constant axial tension, a concentrated mass, and an arbitrary forcing function are included in the analysis. No restriction is placed on the values of the parameters involved, and the solution presented here contains all cited previous solutions as special cases.

1. INTRODUCTION

Since the classical theory of beam was evolved by Euler and Bernoulli, a large literature has accumulated on the subject, the major part of which deals with either infinite beams or finite beams with standard boundary conditions. The general solution for the dynamic response of the infinite Euler-Bernoulli beam with arbitrary initial conditions, subjected to an arbitrary load including the effects of damping, an elastic foundation and constant axial load, was obtained comparatively recently by Stadler and Shreeves (1). The analogous

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problem for a finite beam does not seem to have been solved in all generality. The first study allowing an elastically restrained edge condition, which is of particular interest in the aerospace field, appears to be that of Carmichael (2). Among subsequent investigations, those of Maltbaek (3), Hess (4), Amba-Rao (5), Sharp and Cobble (6) and Amba-Rao and Hussaini (7) are perhaps worth mentioning.

Maltbaek (3) solved the problem of a uniform beam with a rigid mass attached at its center and with elastic supports. Hess (4) gave a more complete solution. Amba-Rao (5) considered the free vibrations of an elastically restrained beam carrying an arbitrary number of discrete masses. He used the transform method, treating the added masses as impulses (represented by Dirac delta functions) in the mass density function. Sharp and Cobble (6) solved the problem of the arbitrarily loaded damped beam elastically restrained against rotation. Amba-Rao and Hussaini (7) presented a closed form solution to the problem of free vibrations of a partially fixed, linearly damped, beam on a Winkler foundation, carrying arbitrary masses. The present note extends the latter solution to include the case of forced vibrations under arbitrary initial conditions, and unifies all earlier investigations into a general theory.

2. STATEMENT OF THE PROBLEM

In addition to the usual assumptions of the Euler-Bernoulli theory, the beam is assumed to have constant axial tensile load, with viscous damping proportional to velocity, and to be resting on a Winkler foundation whose resistance is proportional to the transverse displacement. The relevant differential equation in operator form is

$$\mathcal{L}y(x,t) = q(x,t) \tag{2.1}$$

where

$$\mathcal{L} = EJ\frac{\partial^4}{\partial x^4} - P\frac{\partial^2}{\partial x^2} + f + d\frac{\partial}{\partial t} + \rho\frac{\partial^2}{\partial t^2}$$

Here $y(x,t)$ is the lateral deflection, EJ the flexural rigidity of the beam, ρ the mass density of the beam and concentrated masses, x the space coordinate measured along the length of the beam, P the tensile axial force on the beam, f the Winkler foundation constant, d the damping constant, t the time, and q the forcing function. The initial and boundary conditions are, respectively:

$$y = u(x)\ , \quad \frac{\partial y}{\partial t} = v(x) \qquad \text{at} \quad t = 0 \tag{2.2}$$

$$\left.\begin{aligned} \frac{1}{T_1}\frac{\partial^3 y}{\partial x^3} &= y \\ \frac{\partial^2 y}{\partial x^2} &= -R_1\frac{\partial y}{\partial x} \end{aligned}\right\} \quad \text{at} \quad x = 0 \quad \text{for all} \quad t \tag{2.3}$$

$$\left.\begin{aligned} \frac{1}{T_2}\frac{\partial^3 y}{\partial x^3} &= y \\ \frac{\partial^2 y}{\partial x^2} &= R_2\frac{\partial y}{\partial x} \end{aligned}\right\} \quad \text{at} \quad x = L \quad \text{for all} \quad t \tag{2.4}$$

where L is the length of the beam and T_1, T_2, R_1, and R_2 are spring constants (partial fixity factors). These boundary conditions are unmixed. The method of solution given here is applicable even if the conditions are mixed.

3. SOLUTION

The general theory of linear operators suggests two fundamental methods of solving equation (2.1). One method is to find the inverse of the operator $\mathcal{L}$, that is, in effect to obtain the Green's function. The operator $\mathcal{L}$ is not self-adjoint with respect to either x or t. However, if the Green's function G is evaluated satisfying the homogeneous initial and boundary conditions, then the problem can be solved in principle by an extended definition of $\mathcal{L}$ as given in equation (3.1) below:

$$
\begin{aligned}
\mathcal{L}y &= q + [\rho u(x)\delta'(t) + \rho v(x)\delta(t) + d\delta(t)] \\
&\quad + \text{terms due to inhomogeneous boundary conditions} \\
&= Q \text{ (say)}
\end{aligned}
\tag{3.1}
$$

where $\delta(t)$ is the delta function. The solution takes the form

$$y(x,t) = \int_0^L \int_0^\infty G(x,t;\, \xi,\tau)Q(\xi,\tau)d\xi\, d\tau \tag{3.2}$$

In the present paper, the solution is obtained by the second method using spectral representation of $\mathcal{L}$ in terms of x-eigenfunctions. First, the solution is sought as a sum of two functions, one of which is the solution of a homogeneous problem with inhomogeneous boundary conditions, while the other is the solution of an inhomogeneous equation with homogeneous boundary conditions. Thus, let

$$y(x,t) = y_1(x,t) + y_2(x,t) \tag{3.3}$$

where

$$\mathcal{L}y_1 = 0 \tag{3.4}$$

with the boundary conditions

$$\left.\begin{aligned} y_1'''(0,t) &= -T_1 y_1(0,t) \\ y_1'''(\pi,t) &= T_2 y_1(\pi,t) \\ y_1''(0,t) &= -R_1 y_1'(0,t) \\ y_1''(\pi,t) &= R_2 y_1'(\pi,t) \end{aligned}\right\} \quad (3.5)$$

and

$$\mathcal{L} y_2 = q \quad (3.6)$$

with the boundary conditions

$$\left.\begin{aligned} y_2(0,t) &= y_2(\pi,t) = 0 \\ y_2''(0,t) &= y_2''(\pi,t) = 0 \end{aligned}\right\} \quad (3.7)$$

and the initial conditions

$$\left.\begin{aligned} y_2(x,0) &= u(x) - y_1(x,0) = U(x) \\ \frac{\partial y_2}{\partial t}(x,0) &= v(x) - \frac{\partial y_1}{\partial t}(x,0) = V(x) \end{aligned}\right\} \quad (3.8)$$

where, by a proper choice of the origin and unit of length, the end points of the beam are at $x = 0$ and $x = \pi$, and primes denote differentiation with respect to x.

a) EVALUATION OF y_1

The solution y_1 of equation (3.4) is given in reference 7. A slightly different approach is presented here which yields results identical to those of reference 7. The approach is expected to be useful particularly if the series representing the inverse of the finite Fourier transform is not summable in closed form.

Write

$$y_1 = \psi(x)\exp(-\alpha t)\cos(\omega t) \tag{3.9}$$

Substituting this in equation (3.4), we find that $\psi(x)$ satisfies the ordinary differential equation

$$EJ\frac{d^4\psi}{dx^4} - P\frac{d^2\psi}{\partial x^2} + f\psi - \rho(\alpha^2 + \omega^2)\psi = 0 \tag{3.10}$$

where $2\alpha\rho = d$, and with concentrated masses being represented by delta function $\rho = m + M_j\delta(x - x_j)$. Thus, the concentrated mass is supposed to be situated at the point $x = x_j$; and d is assumed proportional to ρ.

The boundary conditions are

$$\left.\begin{array}{ll} \psi^{(3)}(0) = -T_1\psi(0)\ , & \psi^{(3)}(\pi) = T_2\psi(\pi) \\ \psi^{(2)}(0) = -R_1\psi^{(1)}(0) \quad \text{and} & \psi^{(2)}(\pi) = R_2\psi^{(1)}(\pi) \end{array}\right\} \tag{3.11}$$

where $\psi^{(i)}$ denotes ith derivative of ψ with respect to x.

Define

$$\&^i\bar{\psi} = \int_0^\pi \frac{\partial^i\psi}{\partial x^i}\cos nx\,dx$$

Integrating by parts, we establish the recurrence relation

$$\begin{aligned}\&^i\bar{\psi} = {} & \frac{1}{n^2}\left[(-1)^n\psi^{(i+1)}(\pi) - \psi^{(i+1)}(0)\right] \\ & - \frac{1}{n^4}\left[(-1)^n\psi^{(i+3)}(\pi) - \psi^{(i+3)}(0)\right] + \frac{1}{n^4}\&^{i+4}\bar{\psi}\end{aligned} \tag{3.12}$$

$$\begin{aligned}\bar{\psi}(n) = \&^0\bar{\psi} = {} & \frac{1}{n^2}\left[(-1)^n\psi^{(1)}(\pi) - \psi^{(3)}(0)\right] \\ & - \frac{1}{n^4}\left[(-1)^n\psi^{(3)}(\pi) - \psi^{(3)}(0)\right] + \frac{1}{n^4}\&^4\bar{\psi}\end{aligned} \tag{3.13}$$

The expression for $\&^4\bar{\psi}$ is obtained from the finite cosine transform of equation (3.4):

$$
\& ^4\bar{\psi}\left[1 + \frac{P}{EJ}\frac{1}{n^2}\right] = \bar{\psi}(n)\left[\frac{m}{EJ}(\alpha^2 + \omega^2) - \frac{f}{EJ}\right]
$$

$$
+ \frac{P}{EJ}\frac{1}{n^2}\left[(-1)^n\psi^{(3)}(\Lambda) - \psi^{(3)}(0)\right] + S_j \tag{3.14}
$$

where

$$
S_j = \frac{\alpha^2 + \omega^2}{EJ} M_j\psi(x_j)\cos nx_j
$$

Finally,

$$
\bar{\psi}(n) = \frac{1}{(n^2 + \ell^2)^2 - k^4}\left[(n^2 + 2\ell^2)\left\{(-1)^n\psi^{(1)}(\Lambda) - \psi^{(1)}(0)\right\}\right.
$$

$$
\left. - \left\{(-1)^n\psi^{(3)}(\Lambda) - \psi^{(3)}(0)\right\} + S_j\right] \tag{3.15}
$$

where

$$
\ell^2 = \frac{P}{EJ}, \quad -k^4 = -\left(\frac{P}{2EJ}\right)^2 + \frac{f}{EJ} - m\frac{(\alpha^2 + \omega^2)}{EJ}
$$

Then

$$
\psi(x) = \bar{\psi}(0) + \frac{2}{\Lambda}\sum_{m=1}^{\infty}\bar{\psi}(n)\cos nx
$$

or

$$
\psi(x) = \frac{1}{\Lambda}\frac{1}{\ell^4 - k^4}\left[\frac{\alpha^2 + \omega^2}{EJ}M_j\psi(x_j) - \left\{\psi^{(3)}(\Lambda) - \psi^{(3)}(0)\right\}\right.
$$

$$
\left. + 2\ell^2\left\{\psi^{(1)}(\Lambda) - \psi^{(1)}(0)\right\}\right] + \frac{\alpha^2 + \omega^2}{\Lambda EJ}M_j\psi(x_j)\left[I_3(x + x_j)\right.
$$

$$
\left. + I_3(|x - x_j|)\right] - \frac{2}{\Lambda}\left[\psi^{(1)}(0)\left\{I_1(x) + (\ell^2 + k^2)I_3(x)\right\}\right.
$$

$$
\left. + \psi^{(1)}(\lambda)\left\{I_2(x) + (\ell^2 + k^2)I_4(x)\right\} + \left\{\psi^{(3)}(0)I_3(x) - \psi^{(3)}(\Lambda)I_4(x)\right\}\right] \tag{3.16}
$$

where

$$I_1(x) = -\frac{1}{2(\ell^2 + k^2)} + \frac{\pi}{2\sqrt{\ell^2 + k^2}} \frac{\cosh(\pi - x)\sqrt{\ell^2 + k^2}}{\sinh \pi\sqrt{\ell^2 + k^2}}$$

$$I_3(x) = \frac{1}{2(\ell^4 - k^4)} - \frac{\pi}{4k^2}\left[\frac{\cosh(\pi - x)\sqrt{\ell^2 + k^2}}{\sqrt{\ell^2 + k^2}\,\sinh \pi\sqrt{\ell^2 + k^2}}\right.$$

$$\left.- \frac{\cosh(\pi - x)\sqrt{\ell^2 - k^2}}{\sqrt{\ell^2 - k^2}\,\sinh \pi\sqrt{\ell^2 - k^2}}\right]$$

$$I_2(x) = I_1(\pi - x) \text{ and } I_4(x) = I_3(\pi - x)$$

b) SOLUTION OF $\mathcal{L}y_2 = q(n,t)$

The operator $\mathcal{L}_2$ is a sum of two commutative operators L_x and L_t defined as

$$L_x = EJ\frac{\partial^4}{\partial x^4} - P\frac{\partial^2}{\partial x^2} + f \tag{3.17}$$

$$L_t = \rho\frac{\partial^2}{\partial t^2} + d\frac{\partial}{\partial t} \tag{3.18}$$

The inverse of $\mathcal{L}_2$ may be obtained by considering L_x as a constant. The results will be a function of the operator L_x and should be interpreted by using the spectral representation of L_x. Consider L_x: Its domain D_x is the set of all functions in S (which is a linear vector space of real-valued square integrable functions over $(0,\pi)$) and they have piecewise continuous fourth derivatives for $0 \le x < x_j$ and for $x_j < x \le \pi$, which satisfy the conditions

$$\psi(0) = \psi(\pi) = \psi''(0) = \psi''(\pi) = 0 \tag{3.19}$$

and are such that $L_x\psi$ belongs to S.

If ψ_i, ψ_j are in D_x, it can be easily shown following standard procedure

$$\left\langle \psi_j, L_x \psi_i \right\rangle = \left\langle L_x \psi_j, \psi_i \right\rangle \tag{3.20}$$

where the inner product is defined as

$$\left\langle \psi_i, \psi_j \right\rangle = \int_0^{\pi} \psi_i \psi_j \, dn \tag{3.21}$$

In other words, L_x is self-adjoint and its eigenfunctions are ψ_i. These ψ_i can be obtained from the analysis of section (a) by letting T_1, $T_2 \to \infty$ and R_1, $R_2 \to 0$. The ϕ_i defined by the relation

$$\phi_i = \frac{\psi_i}{\left[\left\langle \psi_2{}^2 \right\rangle\right]^{1/2}} \tag{3.22}$$

form the orthonormal basis for S. Hence q, U, V can be represented as

$$\left.\begin{aligned} q/\rho &= \sum_1^{\infty} q_i(t)\phi_i(x) \\ U &= \sum_1^{\infty} U_i \phi_i(x) \\ V &= \sum_1^{\infty} V_i \phi_i(x) \end{aligned}\right\} \tag{3.23}$$

where

$$\left.\begin{aligned} q_i &= \int_0^{\pi} \frac{q(\xi,t)}{\rho} \phi_i(\xi) d\xi \\ U_i &= \int_0^{\pi} U(\xi)\phi_i(\xi) d\xi \\ V_i &= \int_0^{\pi} V(\xi)\phi_i(\xi) d\xi \end{aligned}\right\} \tag{3.24}$$

The equation (3.6) is written as

$$\left(\rho \frac{\partial^2}{\partial t^2} + d \frac{\partial}{\partial t} + L_x\right) y_2 = q(x,t) \ , \quad y_2(x,0) = U(x) \ , \quad \frac{\partial y_2}{\partial t}(x,0) = V(x) \tag{3.25}$$

Treating L_x as a constant, we find the solution of (3.25) (by the method of Laplace transforms or by variation of parameters) as

$$y_2 = \int_0^t \exp[-\alpha(t - \tau)] \frac{\sinh \beta(t - \tau)}{\beta} \frac{q(\tau,x)d\tau}{\rho}$$

$$+ \exp(-\alpha t) \frac{\sinh \beta t + \cosh \beta t}{\beta} U(x)$$

$$+ \exp(-\alpha t) \frac{\sinh \beta t}{\beta} V(x) \qquad (3.26)$$

where $\beta = \sqrt{\alpha^2 - L_x/\rho}$ and $2\,\alpha\rho = d$. It is to be noted that the function of the operator is written before the function on which it acts and it is interpreted accordingly. For instance,

$$f(L_x)U(x) = \sum_1^\infty f(\lambda_n)U_n\phi_n(x)$$

where λ_n are the eigenvalues of L_x and $\phi_n(x)$ the associated normalized eigenfunctions. Thus,

$$y_2 = \sum_{n=1}^\infty \Big[F_n(t)*q_n(t) + F_n(t)(\alpha U_n + V_n)$$

$$\exp(-\alpha t)\cosh \beta_n t\ U_n\Big]\phi_n(x) \qquad (3.27)$$

where $F_n(t)*q_n(t) = \int_0^t F_n(t - \tau)q_n(\tau)d\tau$

$$F_n(t) = \exp(-\alpha t)\frac{\sinh \beta_n t}{\beta_n}, \text{ and } \beta_n = \sqrt{\alpha^2 - \lambda_n/\rho}$$

DISCUSSION

Substitution of (3.16) in (3.11) yields four equations for five unknowns $\psi^{(1)}(0)$, $\psi^{(1)}(\bar{\Lambda})$, $\psi^{(3)}(o)$, $\psi^{(3)}(\bar{\Lambda})$ and $\psi(x_j)$. Evaluation of ψ at $x = x_j$ from (3.16) provides the necessary fifth equation. The determinant of this system set equal to zero is the equation for natural frequencies. In the above procedure, if equation (3.19) is used instead of (3.11), we finally obtain the equation for the eigenvalues λ_n of the operator L_x. From equation (?.10) it is obvious that λ_n have the form

$$\lambda_n = \rho(\alpha^2 + \omega_n{}^2)$$

and then

$$\beta_n = \sqrt{-\omega_n{}^2}$$

where ω_n are the natural frequencies of the beam simply supported at both ends. The corresponding eigenfunctions are obtained from equation (3.16). In summary, a general solution has been presented for the free and forced vibrations of a finite Euler-Bernoulli beam.

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