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LOUIS JOSEPH IPPOLITO, JR.

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ABSTRACT

An investigation of the multiple scattering effects on wave propagation through a volume of discrete scatterers was accomplished. The mean field and intensity for a distribution of scatterers was developed using a discrete random media formulation, and second order series expansions for the mean field and total intensity derived for one-dimensional and three-dimensional configurations. The volume distribution results were shown to proceed directly from the one-dimensional results.

The analyses demonstrated that either discrete or continuous techniques may be employed for the mean field and intensity expansions, as long as care is taken to insure non-overlapping scatterers in the formulation.

The multiple scattering intensity expansion was compared to the classical 'single scattering' intensity and the classical result was found to represent only the first three terms in the total intensity expansion. The Foldy approximation to the mean field was applied to develop the coherent intensity, and was found to exactly represent all coherent terms of the total intensity.

An incoherent intensity term, secular in L, in path length, was found which was not accounted for in the Foldy approximation result or in the 'single scattering' formulation.

The study has demonstrated the feasibility of using discrete random media techniques for the determination of multiple scattering effects in propagation through a volume of discrete scatterers, and has provided some insight to the more general problem of multiple scattering in a rain volume.

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WITH IMPLICATIONS TO PROPAGATION THROUGH RAIN

CHAPTER 1

INTRODUCTION

The influence of rain on the propagation of electromagnetic radiation has long been recognized as a potential problem for many applications in radio science and engineering. Investigators, beginning in the early 1940's, noted that radar echoes at centimeter wavelengths were modified during rain conditions. Radio operators observed an increase in the level of noise and interference when rain was present in the vicinity of the receiver.

A comprehensive study by Ryde (1946) provided a theoretical basis for the measurement and prediction of meteorological effects on radio waves, and much subsequent work was based on his early efforts.

The rain medium was recognized as a configuration of isolated dielectric spheres which absorb and scatter incident energy in accordance with classical field theory. The solution of the rain problem was based on studying the characteristics of the single rain drop and applying a simple summation of intensity to determine the bulk effects of the rain volume on the propagating wave.

Gunn & East (1954) applied the theory developed by Gustav Mie (1908) for the diffraction of an electromagnetic wave by a dielectric sphere and described the scattering and absorption of rain by equivalent cross-sections for the spherical raindrops. They found semi-empirical relationships between rainfall rate and attenuation which could be described by a simple power law dependence.

Medhurst (1965) provided an extensive systematic analysis of rain attenuation for a wide range of frequencies and rainfall rates, and numerically calculated attenuation coefficients from the Ryde formulation. He determined the Mie coefficients for spherical drops based on the measured drop size distributions of Laws & Parsons (1943). Medhurst compared his calculations to twelve sets of measured data and found that the measured attenuation tended to exceed the maximum possible theoretical attenuation in most instances.

The advent of extensive microwave communications links in the late 1960's produced a large body of measured rain attenuation effects and studies extending from 2 to well above 100 GHz. Improved computer technology provided more refined calculations for microwave attenuation calculations from the classical Mie theory, Setzer (1970), Chu & Hogg (1968). Satellite communications links provided further information on rain attenuation at frequencies above 10 GHz. Two recent survey articles, by Crane (1971) and Hogg & Chu (1975), provide an extensive summary of the theory and measurements available on propagation factors influencing wave propagation through rain.

The classical development of the theory of wave propagation through rain has progressed as basically a 'deterministic' formulation, even though the phenomena involved are random in nature. The intensity of the wave propagating through the rain volume is assumed to decrease exponentially as it progresses through the precipitation. The quantities which describe the rain structure, such as drop size distribution, precipitation rate, refractive index, temperature, etc., are usually assumed as averaged quantities with deterministic characteristics.

Since the decrease in intensity, or attenuation, of the wave propagating through the rain is determined as a simple summation of the power absorbed and scattered by each isolated rain drop, multiple scattering effects between drops are not accounted for directly. Because of this situation, the classical attenuation development has often been termed the 'single scattering' approximation in the literature, for example Van de Hulst (1957), Medhurst (1965), Ishimaru & Lin (1972), even though no such explicit approximation appears in the theory.

Some investigators have recognized the lack of multiple scattering considerations in the classical formulation, but disagree on its relative importance. Medhurst (p. 563) states that the neglect of multiple scattering effects along the path could be one source of error in the disagreement between his calculated values and experimental measurements. Hogg & Chu (p. 1311) state that multiple scattering effects are implicitly accounted for in the classical formulation since the incident wave at each layer in the rain volume takes into account attenuation by preceeding layers, therefore the bulk of the multiple scattering contributions are included. Crane (p. 179) states that the classical attenuation calculations strictly apply only to a nonscattering medium, and that neither adequate measurements nor an adequate theoretical treatment of multiple scattering effects have been made.

Setzer (1970) applied the equation of radiative transfer given by Chandrasekhar (1960) and computed the first order Born scattering contribution for a line-of-site propagation link with a very narrow angular field of view. He concludes that, for a 150 mm/hr rainfall rate with a Laws & Parsons drop distribution, the multiple scattering term is completely negligible at 30 GHz, and of minor significance even at optical wavelengths.

Crane (1970), using the same radiative transfer technique, but apparently without the assumption of a narrow field of view, presents curves (his Fig. 9) which show that multiple scattering in a Laws & Parsons rain will be important at frequencies above 20 GHz for even moderate rain rates. The curves show, for example, that at the 150 mm/hr rain rate and for 30 GHz, multiple scattering effects become significant at a path length of about 200 meters.

The subject of multiple scattering has been treated extensively in the study of random media problems. In contrast with the 'deterministic' techniques employed by classical rain attenuation methods, the study of random media requires a 'statistical' approach and 'averaged' results are of interest. The study of random media is approached by both continuous and discrete formulations. Frisch (1968) provides a comprehensive survey of wave propagation in random media for both the continuous and discrete scatterer cases. Major contributions in the continuous random media area include those of Keller (1962, 1963), Rosenbaum (1969), Kupiec, et al. (1969), among many others.

A systematic procedure for the study of random media consisting of discrete scatterers was presented by Foldy (1945) and extended by Lax (1952), Bazer (1959), Lin & Ishimaru (1971, 1974), and Twersky (1962-1967), among many others.

The application of random media techniques to the rain scatter problem has not been very extensive, however. Lin & Ishimaru (1971) consider multiple scattering effects on millimeter wave propagation in an isotropically scattering discrete medium. In a later paper Ishimaru & Lin (1972), the case of a narrow beam line-of-sight propagation path is assumed. Both studies present integral equations for the mean field and coherent intensity based on the Foldy approximation formulation for forward scattering only, and solve equations after a number of simplifying assumptions which appear to reduce the problem to the classical 'single scattering' formulation. The relative contributions of multiple scattering to the rain scatter attenuation are not discussed.

The objective of this dissertation is an investigation of multiple scattering in propagation through a distribution of discrete point scatterers by a rigorous application of random media techniques. Chapter 2 reviews the continuous and discrete random media formulations important to the study. Chapter 3 establishes a treatment of propagation in one-dimensional media, with emphasis on the techniques applicable to the volume scatter problem. Chapter 4 extends the development to a volume distribution of discrete point scatterers, and develops expansion solutions for the mean field, coherent intensity, and incoherent intensity. Chapter 5 presents the general conclusions of the study of discrete scattering in a random medium and discusses some implications of the results to the volume rain scattering problem.

CHAPTER 2

WAVE PROPAGATION AND SCATTERING IN RANDOM MEDIA

The subject of wave propagation in random media covers a vast range of topics and techniques, all sharing a common factor of statistical solutions to problems that are non-deterministic in nature. As an electromagnetic wave propagates through a random medium, such as a turbulent atmosphere, fog, rain, ocean surface, or rough terrain, the randomness of the medium becomes the source of secondary waves, referred to as scattered waves or 'scattering'.

The scattering medium may be characterized in one of three ways; (a) a random continuum, where the randomness can be considered a continuous function of space and time, (b) discrete scattering, where the medium consists of a random distribution of discrete scatterers, and a knowledge of the characteristics of a single scatterer is required, and (c) rough surface scattering, where a random surface function, rather than a volume function, is used.

The study of rain scattering effects on electromagnetic propagation involves elements of both the random continuum and discrete scattering. This chapter reviews the basic formulation necessary for the developments which follow in Chapters 3 and 4. Most of the information in this chapter has been obtained from the published literature, and adapted, where necessary, for the particular notations and configurations of this study. The major references for the material summarized in this chapter are Frisch (1968), Foldy (1945), Twersky (1964), and Keller (1964).

2-1. CONTINUOUS RANDOM MEDIA TECHNIQUES

A medium which is considered continuously random in position and time can be described statistically by a dielectric constant, ϵ_r, which is a random function of position and time,

$$\epsilon_r = \epsilon_r(r, t) = n^2(r, t) \tag{2-1}$$

where n^2 is the index of refraction, also a random function. We make the assumption that the time variations of the medium are slow compared to the time of wave propagation through the medium, which is a reasonable assumption for the rain scatter channel. Under this condition

$$\epsilon_r = \epsilon_r(r) = n^2(r) \tag{2-2}$$

which implies that a propagating wave will maintain its frequency.

Consider Maxwell's equations for the medium, in complex vector notation

$$\nabla \times E(r) = i\omega u_o H(r) \tag{2-3a}$$

$$\nabla \times H(r) = -i\omega\epsilon_o \epsilon_r(r) E(r) \tag{2-3b}$$

where a time dependence of $e^{-i\omega t}$ is assumed. The electric field vector, for example, is available from

$$\epsilon(r, t) = \mathrm{Re}\left\{E(r)\, e^{-i\omega t}\right\} \tag{2-4}$$

where $\mathrm{Re}\{\ \}$ denotes 'real part of', and E(r) is a vector with complex components.

The vector wave equation for the medium is developed by combining Equations (2-3a, b),

$$\nabla \times \nabla \times E(r) - \omega^2 u_o \epsilon_o \epsilon_r(r) E(r) = 0 \tag{2-5}$$

with

$$\nabla \cdot [\epsilon_r(r) E(r)] = 0 \tag{2-6}$$

Equation (2-5) reduces to

$$\nabla^2 E(r) + k_o^2 \epsilon_r(r) E(r) - \nabla\left[\frac{\nabla \epsilon_r(r)}{\epsilon_r(r)} \cdot E(r)\right] = 0 \tag{2-7}$$

where

$$k_o^2 = \omega^2 u_o \epsilon_o$$

The last term in Equation (2-7) can be neglected if the variations in ϵ_r over distances comparable to the wavelength of the incident radiation are very small. In neglecting this term, polarization effects are also neglected since three uncoupled scalar wave equations result.

The resulting vector wave equation is therefore

$$\nabla^2 E(r) + k_o^2 \epsilon_r(r) E(r) = 0 \tag{2-8}$$

or in terms of the refractive index

$$\nabla^2 E(r) + k_o^2 n^2(r) E(r) = 0 \tag{2-9}$$

Each component of the vector wave equation above, $\Psi(r)$, satisfies the scalar wave equation

$$\nabla^2 \Psi(r) + k_0^2 n^2(r) \Psi(r) = 0$$

or

$$[\nabla^2 + k_0^2 n^2(r)] \Psi(r) = 0 \quad (2\text{-}10)$$

Consider a random medium imbedded is free space with a relative dielectric constant close to 1. The refractive index then has the form

$$n^2(r) = 1 + \epsilon u(r) \quad (2\text{-}11)$$

where the random function $u(r)$ is defined by a mean $\langle u(r) \rangle$ and correlation $\langle u(r)u(r') \rangle$, and ϵ is a small quantity.

The wave equation then has the form

$$\left\{\nabla^2 + k_0^2 [1 + \epsilon u(r)]\right\} \Psi(r) = 0$$

or

$$[\nabla^2 + k_0^2] \Psi(r) = -\epsilon k_0^2 u(r) \Psi(r) \quad (2\text{-}12)$$

This equation can be converted to an integral equation by use of the Green's function method. The solution is assumed as the sum of an incident wave which obeys Equation (2-10) and a scattered wave. The associated boundary conditions require that the scattered wave be composed of outgoing waves at infinity, and that $\Psi(r)$ is continuous and vanishes at infinity. With these boundary conditions, the wave equation converts to

$$\Psi(r) = \Psi_0(r) - \epsilon k_0^2 \int G(r, r') u(r') \Psi(r') dr' \quad (2\text{-}13)$$

where $G(r, r')$ is the free space Green's function

$$G(r, r') = \frac{-e^{ik |r - r'|}}{4\pi |r - r'|} \quad (2\text{-}14)$$

which is a solution to the wave equation

$$(\nabla^2 + k^2) G(r, r') = \delta(r - r') \quad (2\text{-}15)$$

The incident wave $\Psi_o(r)$ is a solution to the homogeneous wave equation

$$(\nabla^2 + k^2)\Psi_o(r) = 0 \tag{2-16}$$

Most of the field analyses in this study will begin with the integral form of the wave equation discussed above.

Born Approximation

An approximate solution for the integral form of the wave equation for scattering in random media is available by assuming the solution is of the form

$$\Psi(r) = \sum_{n=0}^{\infty} \epsilon^n \Psi_n(r) \tag{2-17}$$

The first iteration solution, for n = 1, from Equation (2-13), is

$$\Psi(r) = \Psi_o(r) - \epsilon k_o^2 \int G(r, r') u(r') \Psi_o(r') dr' + O(\epsilon^2) \tag{2-18}$$

where $O(\epsilon^2)$ indicates additional terms of order ϵ^2.

The second iteration assumes Equation (2-18) for a solution to Equation (2-15), i.e.

$$\begin{aligned} \Psi(r) = \Psi_o(r) &- \epsilon k_o^2 \int G(r, r') u(r')[\Psi_o(r') \\ &- \epsilon k_o^2 \int G(r', r'') u(r'') \Psi_o(r'') dr''] dr' + O(\epsilon^3) \end{aligned} \tag{2-19}$$

or

$$\begin{aligned} \Psi(r) = \Psi_o(r) &- \epsilon k_o^2 \int G(r, r') u(r') \Psi_o(r') dr' \\ &+ \epsilon^2 k_o^2 \int G(r, r') G(r', r'') u(r') u(r'') \Psi_o(r'') dr' dr'' + O(\epsilon^3) \end{aligned} \tag{2-20}$$

Additional terms of the Born series can be applied to obtain higher order expansions. For this study expansions to $O(\epsilon^2)$ will be utilized.

The expectation, or average, value of $\Psi(r)$ is available from Equation (2-20)

$$\begin{aligned} \langle\Psi(r)\rangle = \Psi_o(r) &- \epsilon k_o^2 \int G(r, r') \langle u(r')\rangle \Psi_o(r') dr' \\ &+ \epsilon^2 k_o^2 \int G(r, r') G(r', r'') \langle u(r') u(r'')\rangle \Psi_o(r'') dr' dr'' + O(\epsilon^3) \end{aligned} \tag{2-20a}$$

where $\langle u(r')\rangle$ and $\langle u(r')\, u(r'')\rangle$ are the first and second order statistics which define the function $u(r)$.

2-2. DISCRETE RANDOM MEDIA TECHNIQUES

Consider a distribution of identical isotropic point scatterers in the presence of an incident wave $\Psi_o(r)$. The scatterers are located at r_j. The scattering properties of the individual scatterer are characterized by a scattering function, g_j, which is a deterministic function determined from the physical characteristics of the individual scatterer. The distribution of the scatterers is given by the distribution function $f(r)$. We assume that g_j, $f(r)$, and $\Psi_o(r)$ are known. We require the average of the wave function, $\langle\Psi(r)\rangle$ and other related quantities.

For a particular configuration of scatterers, the wave function at the point r can be represented as (Foldy, 1945, p. 110)

$$\Psi(r) = \Psi_o(r) + \sum_j g_j\, \Psi(r_j)\, G(r, r_j) \qquad (2\text{-}21)$$

This equation represents the field as the sum of the incident wave and spherical waves diverging from each of the scatterers. The external field acting on the jth scatterer is then

$$\Psi(r_j) = \Psi_o(r_j) + \sum_{\substack{j' \\ j' \neq j}} g_{j'}\, \Psi(r_{j'})\, G(r, r_{j'}) \qquad (2\text{-}22)$$

Equations (2-21) and (2-22) represent the fundamental equations of multiple scattering.

The first order approximation to the solution of the above equations assumes that the external field on the j' th scatterer, $\Psi(r_{j'})$, is the incident field at that scatterer, i.e.

$$\Psi(r_{j'}) \cong \Psi_o(r_{j'}) \qquad (2\text{-}23)$$

The field at r is again represented as the sum of the incident wave and spherical waves diverging from each of the isotropic scatterers, i.e.

$$\Psi(r) = \Psi_o(r) + \sum_j g_j\, G(r, r_j)\, \Psi(r_j) \qquad (2\text{-}25)$$

where $\Psi(r_j)$ is the external field acting on the j th scatterer, and $G(r, r_j)$ is the Green's function representing the spherical wave of the j th scatterer.

But now, in the first order approximation, $\Psi(r_j)$ is the sum of the incident wave at r_j plus the sum of the fields produced by the other scatterers excited by the incident wave, i.e.

$$\Psi(r_j) = \Psi_0(r_j) + \sum_{\substack{j' \\ j' \neq j}} g_{j'} G(r_j, r_{j'}) \Psi_0(r_{j'}) \tag{2-26}$$

where $G(r_j, r_{j'})$ is the field produced at r_j from the scatterer at $r_{j'}$ excited by the incident wave $\Psi_0(r_{j'})$.

Substituting Equation (2-26) into Equation (2-25)

$$\Psi(r) = \Psi_0(r) + \sum_j g_j G(r, r_j) \left[\Psi_0(r_j) + \sum_{\substack{j' \\ j \neq j'}} g_{j'} G(r_j, r_{j'}) \Psi_0(r_{j'}) \right]$$

or

$$\Psi(r) = \Psi_0(r) + \sum_j g_j G(r, r_j) \Psi_0(r_j) + \sum_j \sum_{j'}{}' g_j g_{j'} G(r, r_j) G(r_j, r_{j'}) \Psi_0(r_{j'}) \tag{2-27}$$

where the prime in the summation indicates $j \neq j'$. Equation (2-27) represents the fundamental equation for the evaluation of multiple scattering by discrete isotropic scatterers, and will be the starting point for the discrete formulation expansion solutions in this study.

Foldy Approximation

The direct method of solution for the discrete multiple scattering field, represented by Equation (2-27), consists of solving the set of simultaneous linear equations for the $\Psi(r_j)$, given by Equation (2-26), and substituting these into Equation (2-27) to find $\Psi(r)$ as a function of position and the parameters of the scatterers. This is not a feasible method. The problem is even more severe when $\langle\Psi(r)\rangle$ is desired. In that case, integrations must be carried out over all the simultaneous equations to obtain $\langle\Psi(r)\rangle$.

This problem was studied by Foldy (1945), and resulted in the development of a procedure for determining the mean field integral equation by applying an appropriate approximation for the external field acting on each scatterer. This important development has been used extensively to study multiple scattering effects, and serves as the basis for most present treatments of discrete scattering problems, Twersky (1964), Ishimaru & Lin (1972), Lax (1951), Bazer (1959), among others.

The detailed development of the Foldy approximation will be applied in this study for the one-dimensional case, Section 3-3, and for the volume scattering case, Section 4-2. The Foldy approximation expansion for $\langle\Psi(r)\rangle$ and $|\langle\Psi(r)\rangle|^2$ will be compared with directly evaluated expansions for the slab and volume scatter cases, and the limits of applicability to the rain scatter channel evaluated.

CHAPTER 3

PROPAGATION IN ONE-DIMENSIONAL MEDIA

As a prelude to the direct evaluation of multiple scattering in a volume of scatterers, we begin by first establishing a rigorous treatment of the problem of the multiple scattering of waves by distributions of point scatterers. In the next chapter, similar techniques will be applied and extended to the three-dimensional volume scatter problem.

A one-dimensional slab of Poisson distributed point scatterers will be considered, and expansions of the mean field and intensity will be developed, utilizing continuous and discrete techniques. The particular areas of interest to be investigated include, (a) the relative advantages of discrete versus continuous techniques for point scatterer problems, (b) the applicability and range of validity of the Foldy approximation in the mean and intensity expansions, (c) the determination of a 'weighting factor' for the point scatterer which represents a dielectric finite scatterer in the limit as its width approaches zero, and, finally, (d) the implications of the one-dimensional results and their applicability to the solution of the rain scatter problem.

Many of the techniques and methods of evaluation established in this chapter will be found to be directly applicable, with some extensions, to the three-dimensional volume scattering problem, hence the results here will provide a useful guide in the method of analysis for this study.

3-1. POISSON IMPULSE REPRESENTATION FOR DISCRETE SCATTERERS

The point scatterer will be utilized in this study to represent a physical scatterer, thus it is important, at least in the mathematical sense, for its properties to reasonably represent the characteristic properties of the scatterer. It will be useful in the statistical evaluation of the moments of a configuration of point scatterers to consider the scatterers as a uniform distribution of Poisson impulses, with an appropriate 'weighting factor' which represents the physical drop characteristics, such as radius and dielectric constant. In this way the point scatterer formalism is maintained for ease of mathematical evaluation, yet the physical properties of the scatterer are maintained and available for practical interpretation of the results obtained. We begin by considering a single point scatterer and developing the 'weighting factor' which represents a dielectric scatterer within specified bounds of size and dielectric constant.

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(a) Single Delta-Function Scatterer

Consider a single impulse scatterer located at x_i in free space, as in Figure 3-1(a), on the x axis, with an incident wave propagating from the negative x direction of

$$\Psi_i(x) = e^{ikx} \tag{3-1}$$

where an $e^{-i\omega t}$ time dependence is assumed. The field $\Psi(x)$ obeys the reduced one-dimensional wave equation

$$\frac{d^2}{dx^2} \Psi(x) + k^2 n^2(x) \Psi(x) = 0 \tag{3-2}$$

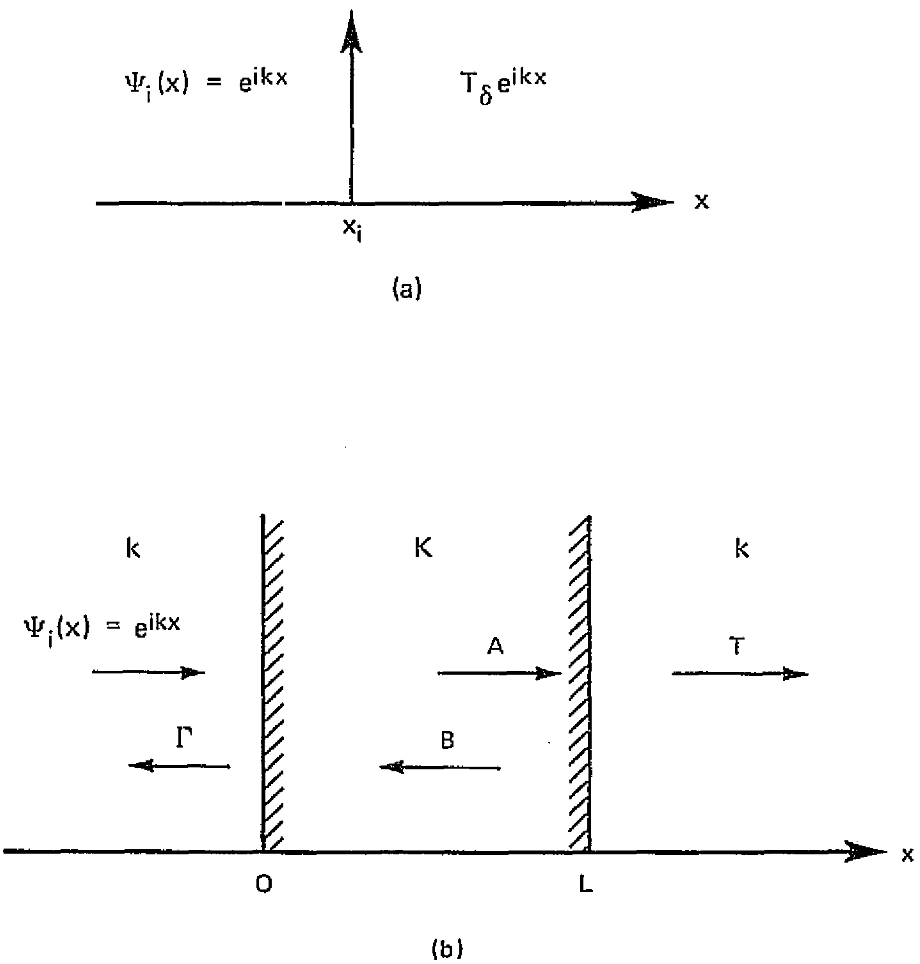

Figure 3-1. One-dimensional Scatterer Configurations

where n(x) is the index of refraction and k is the free space wave number. The scatterer at x_i results in

$$n^2(x) = 1 + \sigma\,\delta(x_i) \tag{3-3}$$

where $\delta(x_i)$ is the Dirac delta function and σ is a complex 'weighting factor', to be determined later. The wave equation is therefore

$$\frac{d^2}{dx^2}\,\Psi(x) + [k^2 + k^2\sigma\,\delta(x_i)]\,\Psi(x) = 0$$

or

$$\left[\frac{d^2}{dx^2} + k^2\right]\Psi(x) = -k^2\sigma\,\delta(x_i) \tag{3-4}$$

The above equation converts into the integral equation

$$\Psi(x) = \Psi_i(x) - k^2\sigma \int G_o(x, x')\,\delta(x' - x_i)\,\Psi(x')\,dx' \tag{3-5}$$

where $G_o(x, x')$ is the free space Green's function, i.e.

$$G_o(x, x') = \frac{e^{ik|x - x'|}}{2ik} \tag{3-6}$$

The delta function evaluates the integral at $x' = x_i$

$$\Psi(x) = \Psi_i(x) - k^2\sigma\,\frac{e^{ik|x - x_i|}}{2ik}\,\Psi(x_i) \tag{3-7}$$

At $x = x_i$

$$\Psi(x_i) = e^{ikx_i} - \frac{k\sigma}{2i}\,\Psi(x_i)$$

or

$$\Psi(x_i) = \left[1 + \frac{k\sigma}{2i}\right]^{-1} e^{ikx_i} \tag{3-8}$$

Substitute $\Psi(x_i)$ back into Equation (3-7)

$$\Psi(x) = \Psi_i(x) - \frac{k\sigma}{2i} \frac{e^{ik|x - x_i|}}{\left[1 + \frac{k\sigma}{2i}\right]} e^{ikx_i} \qquad (3\text{-}9)$$

For $x > x_i$

$$\Psi(x) = \Psi_i(x) - \frac{k\sigma}{2i\left[1 + \frac{k\sigma}{2i}\right]} e^{ikx}$$

or

$$\Psi(x) = \left[\frac{1}{1 - i\frac{k\sigma}{2}}\right] e^{ikx} \qquad (3\text{-}10)$$

The term in brackets is seen to be the transmission coefficient T_δ, of the delta function scatterer, i.e.

$$T_\delta = \frac{1}{1 - i\frac{k\sigma}{2}} \qquad (3\text{-}11)$$

(b) Evaluation of Weighting Factor from Transmission Coefficient

In this section the transmission coefficient for a slab of random media will be compared with the delta function transmission coefficient and a weighting factor, σ, developed.

The transmission coefficient, T, for a dielectric slab of width L was developed in Appendix A, Equation (A-12)

$$T = \frac{4kKe^{iKL}}{(k + K)^2 - (k - K)^2 e^{i2KL}} \qquad (3\text{-}12)$$

where K is the propagation constant in the slab and k is the free space value (see Fig. 3-1(b) for the slab configuration). Let ϵ_2 be the complex relative dielectric constant of the slab, then

$$K = k\sqrt{\epsilon_2} \qquad (3\text{-}13)$$

We impose the following conditions on ϵ_2 and L:

$$\begin{aligned} &(1)\ \epsilon_2 \text{ is } \gg 1, \text{ but} \\ &(2)\ L \text{ is small, such that} \\ &(3)\ k\sqrt{\epsilon_2}\, L \ll 1, \text{ i.e. } KL \ll 1. \end{aligned} \tag{3-14}$$

Expanding the denominator terms in T

$$(k + K)^2 = k^2(1 + \sqrt{\epsilon_2})^2 = k^2\, \epsilon_2 \left(1 + \frac{1}{\sqrt{\epsilon_2}}\right)^2$$

$$(k - K)^2 = k^2(1 - \sqrt{\epsilon_2})^2 = k^2\, \epsilon_2 \left(1 - \frac{1}{\sqrt{\epsilon_2}}\right)^2$$

From condition (1) above

$$\left(1 + \frac{1}{\sqrt{\epsilon_2}}\right)^2 \cong 1 + \frac{2}{\sqrt{\epsilon_2}}$$

and

$$\left(1 - \frac{1}{\sqrt{\epsilon_2}}\right)^2 \cong 1 - \frac{2}{\sqrt{\epsilon_2}}$$

From condition (3)

$$e^{iKL} \cong 1 + ik\sqrt{\epsilon_2}\, L$$

and

$$e^{i2KL} \simeq 1 + i2k\sqrt{\epsilon_2}\, L$$

Therefore, under the above conditions, T will be

$$T \cong \frac{4k^2\sqrt{\epsilon_2}\,(1 + ik\sqrt{\epsilon_2}\, L)}{k^2\, \epsilon_2 \left[\left(1 + \frac{2}{\sqrt{\epsilon_2}}\right) - \left(1 - \frac{2}{\sqrt{\epsilon_2}}\right)(1 + i2k\sqrt{\epsilon_2}\, L)\right]}$$

or

$$T = \frac{4(1 + ik\sqrt{\epsilon_2}\, L)}{4 + (1 - \sqrt{\epsilon_2})\, i2k\sqrt{\epsilon_2}\, L} \tag{3-15}$$

With conditions (1) and (3) employed above, T reduces to

$$T = \frac{4}{4 - i2k\epsilon_2 L}$$

and finally

$$T = \frac{1}{1 - i\,\frac{k\epsilon_2 L}{2}} \tag{3-16}$$

Comparing this result with T_δ, Equation (3-11)

$$\frac{1}{1 - i\,\frac{k\sigma}{2}} \Leftrightarrow \frac{1}{1 - i\,\frac{k\epsilon_2 L}{2}} \tag{3-17}$$

we see that

$$\sigma = \epsilon_2 L \tag{3-18}$$

Thus, with the weighting factor $\epsilon_2 L$, a delta function scatterer can represent a physical dielectric scatterer of dielectric constant ϵ_2 and width L, under the limiting condition

$$k\sqrt{\epsilon_2}\, L \ll 1 \tag{3-19}$$

With k replaced by $2\pi/\lambda$, this condition requires that

$$\frac{2\pi L}{\lambda} \ll \frac{1}{\sqrt{\epsilon_2}} \ll 1 \tag{3-20}$$

which is seen to be the Rayleigh scattering criteria, i.e. the size of the scatterer must be much smaller than the incident wavelength.

(c) Mean Field - Continuous Formulation

In this section we consider the propagation of a plane wave incident on a slab of random medium, where the medium is described by a continuous function of position and time. This description, where the medium can be considered a "random continuum," differs from the discrete scatterer description, which will be developed in the next section.

Figure 3-2 shows the configuration under consideration; a slab of width L, with complex refractive index n(x), and a plane wave incident from the negative x direction. The refractive index of the slab can, in general, be a random function of position and time, n(x, t), but we assume that the time fluctuations of the medium are slow compared to the time of wave propagation, therefore the refractive index can be considered a function of position only.

The scalar field $\Psi(x)$ satisfies the reduced one-dimensional wave equation

$$\frac{d^2}{dx^2}\Psi(x) + k^2 n^2(x)\Psi(x) = 0 \qquad (3\text{-}21)$$

We assume $n^2(x)$ in the slab is

$$n_s^2(x) = 1 + \epsilon u(x) \quad 0 \leqslant x \leqslant L \qquad (3\text{-}22)$$

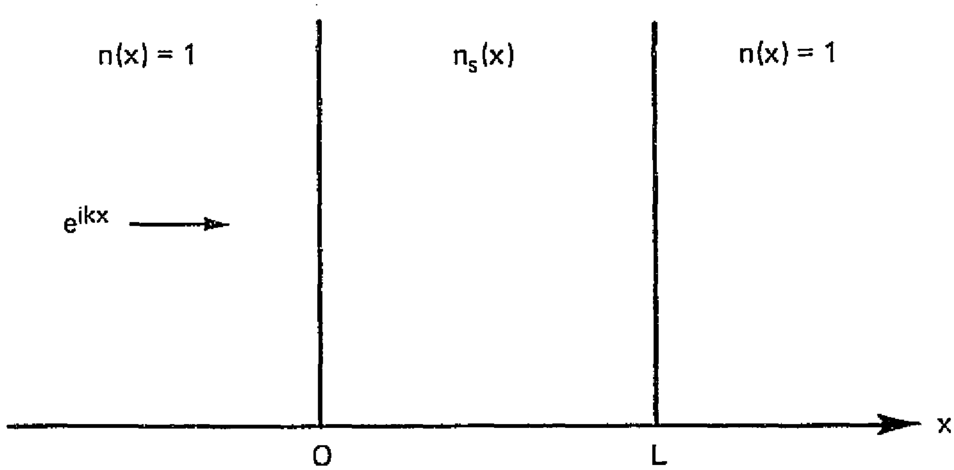

Figure 3-2. Plane Wave Incident on a One-dimensional Slab of Random Medium

where u(x) is a random function of x, defined by a mean $\langle u(x)\rangle$, and correlation $\langle u(x)\, u(x')\rangle$. Let the function u(x) be a Poisson impulse process of the form

$$u(x) = \sum_{j=1}^{N} \sigma\, \delta(x - x_j) \tag{3-22a}$$

Then the mean and correlation are (Papoulis, 1965)

$$\langle u(x)\rangle = \sigma\lambda \tag{3-23}$$

$$\langle u(x)\, u(x')\rangle = \sigma^2\lambda^2 + \sigma^2\lambda\, \delta(x - x') \tag{3-24}$$

where $\lambda = N/L$, the uniform number density.

The wave equation is then

$$\left[\frac{d^2}{dx^2} + k^2\right] \Psi(x) = -k^2\, \epsilon\, u(x)\, \Psi(x) \tag{3-25}$$

The above wave equation and its associated boundary conditions convert into the integral equation

$$\Psi(x) = \Psi_o(x) - k^2\, \epsilon \int_o^L \frac{e^{ik|x - x'|}}{2ik}\, u(x')\, \Psi(x')\, dx' \tag{3-26}$$

where $\Psi_o(x)$ is the incident wave, a solution to the homogeneous wave equation

$$\left[\frac{d^2}{dx^2} + k^2\right] \Psi(x) = 0 \tag{3-27}$$

To obtain the Born expansion solution, let $\Psi(x)$ be of the form

$$\Psi(x) = \sum_{n=0}^{\infty} \epsilon^n\, \Psi_n(x) \tag{3-28}$$

The first term of the expansion gives

$$\Psi(x) = e^{ikx} - k^2\, \epsilon \int_o^L \frac{e^{ik|x - x'|}}{2ik}\, u(x')\, e^{ikx'}\, dx' + O(\epsilon^2) \tag{3-29}$$

Substitute $\Psi(x)$ from Equation (3-29) into the integral of Equation (3-26)

$$\Psi(x) = e^{ikx} - k^2 \epsilon \int_o^L \frac{e^{ik|x-x'|}}{2ik} u(x') \left[e^{ikx'} - k^2 \epsilon \int_o^L \frac{e^{ik|x'-x''|}}{2ik} u(x'') e^{ikx''} dx'' \right] dx' + O(\epsilon^3)$$

or

$$\Psi(x) = e^{ikx} - k^2 \epsilon \int_o^L \frac{e^{ik|x-x'|}}{2ik} u(x') e^{ikx'} dx' + k^4 \epsilon^2 \int_o^L \frac{e^{ik|x-x'|}}{2ik} u(x') \int_o^L \frac{e^{ik|x'-x''|}}{2ik} u(x'') e^{ikx''} dx'' dx' + O(\epsilon^3) \tag{3-30}$$

Combining the exponent terms

$$\Psi(x) = e^{ikx} - k^2 \epsilon \int_o^L \frac{e^{ik(|x-x'|+x')}}{2ik} u(x') dx' + k^4 \epsilon^2 \int_o^L \int_o^L \frac{e^{ik(|x-x'|+|x'-x''|+x'')}}{-4k^2} u(x') u(x'') dx' dx'' + O(\epsilon^3) \tag{3-31}$$

Equation (3-31) gives $\Psi(x)$ to $O(\epsilon^2)$.

Additional terms of the Born series can be applied to obtain higher order expansions, i.e Equation (3-31) is substituted back into the integral of Equation (3-26), etc., to obtain $\Psi(x)$ to $O(\epsilon^3)$. For this study $\Psi(x)$ to $O(\epsilon^2)$ will be utilized. Taking the average of Equation (3-31), we have

$$\langle \Psi(x) \rangle = e^{ikx} - k^2 \epsilon \int_o^L \frac{e^{ik(|x-x'|+x')}}{2ik} \langle u(x') \rangle dx' + k^4 \epsilon^2 \int_o^L \int_o^L \frac{e^{ik(|x-x'|+|x'-x''|+x'')}}{-4k^2} \langle u(x') u(x'') \rangle dx' dx'' \tag{3-32}$$

Substitute in the mean and correlation of u(x) from Equations (3-23) and (3-24)

$$\langle\Psi(x)\rangle = e^{ikx} - \frac{k\epsilon}{2i}\,\sigma\lambda \int_0^L e^{ik(|x-x'|+x')}\,dx'$$

$$-\frac{k^2\epsilon^2}{4}\,\sigma^2\left[\lambda^2 \int_0^L\int_0^L e^{ik(|x-x'|+|x'-x''|+x'')}\,dx'\,dx''\right. \quad (3\text{-}33)$$

$$\left. + \lambda \int_0^L\int_0^L e^{ik(|x-x'|+|x'-x''|+x'')}\,\delta(x'-x'')\,dx'\,dx''\right]$$

The delta function in the third integral evaluates the integral at $x' = x''$, i.e. the third integral term is

$$-\frac{k^2\epsilon^2}{4}\,\sigma^2\lambda \int_0^L e^{ik(|x-x'|+x')}\,dx' \quad (3\text{-}34)$$

The resulting mean field is therefore

$$\langle\Psi(x)\rangle = e^{ikx} - \frac{k\epsilon}{2i}\,\sigma\lambda \int_0^L e^{ik(|x-x'|+x')}\,dx'$$

$$-\frac{k^2\epsilon^2}{4}\,\sigma^2\left[\lambda \int_0^L e^{ik(|x-x'|+x')}\,dx'\right. \quad (3\text{-}35)$$

$$\left. + \lambda^2 \int_0^L\int_0^L e^{ik(|x-x'|+|x'-x''|+x'')}\,dx'\,dx''\right]$$

The above equation expresses the mean field inside and outside the slab, to $O(\epsilon^2)$, and was derived using the classical Born expansion technique for an assumed continuous random medium.

(d) Mean Field - Discrete Formulation

We consider here the same one-dimensional slab problem treated in Section (c), under the assumption that the random medium consists of a uniform distribution of N discrete scatterers whose position x_j is defined by the Poisson impulse process

$$\sigma \sum_{j=1}^{N} \delta(x - x_j) \quad 0 \leqslant x \leqslant L \tag{3-36}$$

where σ is a complex weighting function defined by the characteristics of the scatterer.

We proceed to find the solution for the field $\Psi(x)$ for any value of x. The field satisfies the reduced homogeneous wave equation outside the slab

$$\left[\frac{d^2}{dx^2} + k^2\right] \Psi_0(x) = 0 \quad \begin{matrix} x < 0 \\ x > L \end{matrix} \tag{3-37}$$

and

$$\Psi_0(x) = e^{ikx} \tag{3-38}$$

In the slab, the wave equation for the field is

$$\frac{d^2}{dx^2} \Psi(x) + k^2 \left[1 + \sigma \sum_{j=1}^{N} \delta(x - x_j)\right] \Psi(x) = 0 \tag{3-39}$$

or

$$\left[\frac{d^2}{dx^2} + k^2\right] \Psi(x) = -k^2 \sigma \sum_{j=1}^{N} \delta(x - x_j)\, \Psi(x) \tag{3-40}$$

The above wave equation and its associated boundary conditions convert to the integral equation

$$\Psi(x) = \Psi_0(x) - k^2 \sigma \sum_{j=1}^{N} \int_0^L \frac{e^{ik|x - x'|}}{2ik}\, \delta(x' - x_j)\, \Psi(x')\, dx' \tag{3-41}$$

The delta function evaluates the integral at $x' = x_j$

$$\Psi(x) = \Psi_0(x) + i\,\frac{k\sigma}{2} \sum_{j=1}^{N} e^{ik|x - x_j|}\, \Psi(x_j) \tag{3-42}$$

This equation represents the field at x as the sum of the incident field in the absence of the scatterers, and waves of the form $e^{ik|x - x_j|}$ with an external exciting field $\Psi(x_j)$ eminating from each scatterer.

The external field at the j th scatterer is the sum of the incident wave at x_j plus the sum of the fields produced by the other N-1 scatterers, i.e.

$$\Psi(x_j) = e^{ikx_j} + i\frac{k\sigma}{2}\sum_{\substack{l=1 \\ l \neq j}}^{N} e^{ik|x_j - x_l|}\, e^{ikx_l} \tag{3-43}$$

where $e^{ik|x_j - x_l|}$ is the field produced at x_j from the scatterer at x_l excited by the incident wave e^{ikx_l}.

Substituting $\Psi(x_j)$ from Equation (3-43) into Equation (3-42),

$$\Psi(x) = e^{ikx} + i\frac{k\sigma}{2}\sum_{j=1}^{N} e^{ik|x - x_j|}\left[e^{ikx_j} + i\frac{k\sigma}{2}\sum_{\substack{l=1 \\ l \neq j}}^{N} e^{ik|x_j - x_l|}\,\Psi(x_l)\right]$$

or

$$\Psi(x) = e^{ikx} + i\frac{k\sigma}{2}\sum_{j=1}^{N} e^{ik(|x - x_j| + x_j)} - \frac{k^2\sigma^2}{4}\sum_{j=1}^{N}\sum_{l=1}^{N}{}' e^{ik(|x - x_j| + |x_j - x_l| + x_l)} \tag{3-44}$$

where the ' denotes $l \neq j$.

It is of interest at this point to compare this result with a Born expansion solution of $\Psi(x)$, Equation (3-42). Assume

$$\Psi(x) = \sum_{n=0}^{\infty} \sigma^n\, \Psi_n(x) \tag{3-45}$$

The first term of the expansion, from Equation (3-42), gives

$$\Psi(x) = e^{ikx} + i\frac{k\sigma}{2}\sum_{j=1}^{N} e^{ik|x - x_j|}\, e^{ikx_j} + O(\sigma^2) \tag{3-46}$$

Substituting $\Psi(x)$ from Equation (3-46) into the summation term in Equation (3-42)

$$\Psi(x) = e^{ikx} + i\frac{k\sigma}{2}\sum_{j=1}^{N} e^{ik|x-x_j|}\left[e^{ikx_j} + i\frac{k\sigma}{2}\sum_{l=1}^{N} e^{ik|x_j - x_l|} e^{ikx_l}\right] + O(\sigma^3) \tag{3-47}$$

or

$$\Psi(x) = e^{ikx} + i\frac{k\sigma}{2}\sum_{j=1}^{N} e^{ik(|x-x_j| + x_j)} - \frac{k^2\sigma^2}{4}\sum_{j=1}^{N}\sum_{l=1}^{N} e^{ik(|x-x_j| + |x_j - x_l| + x_l)} + O(\sigma^3) \tag{3-48}$$

Comparing Equation (3-48) with Equation (3-44), the results are seen to be identical, except for the prime in the summation of Equation (3-44), i.e. $l \neq j$. Thus the Born expansion represents the discrete scatterer field if $l \neq j$, i.e. if the scatterers are not allowed to overlap. The significance of this point will be discussed in more detail later in this section.

Proceeding to the average of Equation (3-44), we have

$$\langle\Psi(x)\rangle = e^{ikx} + i\frac{k\sigma}{2}\left\langle\sum_{j=1}^{N} e^{ik(|x-x_j| + x_j)}\right\rangle - \frac{k^2\sigma^2}{4}\left\langle\sum_{j=1}^{N}\sum_{l=1}^{N}{}' e^{ik(|x-x_j| + |x_j - x_l| + x_l)}\right\rangle \tag{3-49}$$

The evaluation of the mean terms above is accomplished in Appendix B, Section (a). From Equation (B-9)

$$\left\langle\sum_{j=1}^{N} e^{ik(|x-x_j| + x_j)}\right\rangle = \lambda\int_0^L e^{ik(|x-x'| + x')}\,dx' \tag{3-50}$$

and from Equation (B-18),

$$\left\langle \sum_{j=1}^{N} \sum_{l=1}^{N}{}' e^{ik(|x - x_j| + |x_j - x_l| + x_l)} \right\rangle = \lambda^2 \int_0^L \int_0^L e^{ik(|x - x'| + |x' - x''| + x'')} dx' dx'' \tag{3-60}$$

where

$$\lambda = \frac{N}{L}$$

The resulting mean field is then

$$\langle \Psi(x) \rangle = e^{ikx} + i \frac{k\sigma}{2} \lambda \int_0^L e^{ik(|x - x'| + x')} dx' - \frac{k^2 \sigma^2}{4} \lambda^2 \int_0^L \int_0^L e^{ik(|x - x'| + |x' - x''| + x'')} dx' dx'' \tag{3-61}$$

Equation (3-61) represents the mean field for a set of uniformly distributed non-overlapping point scatterers whose position is described by the Poisson impulse process of Equation (3-36).

Comparing the discrete formulation result above with the mean derived from the continuous random medium formulation, Equation (3-35) with $\epsilon = 1$, the results are seen to be identical except for one term which appears in the continuous result,

$$- \frac{k^2 \sigma^2}{4} \lambda \int_0^L e^{ik(|x - x'| + x')} dx'$$

This term is produced for the condition where the N scatterers are allowed to overlap, i.e. when $x' = x''$, which is allowed in the continuous formulation. The term is produced from the delta function term in the correlation for $u(x)$, Equation (3-24), and results in the third integral of Equation (3-33), which is evaluated by the delta function at $x' = x''$, Equation (3-34).

Note also that the Born expansion for the discrete formulation, Equation (3-48), allows overlapping scatterers and would also produce the term above.

The comparison of the continuous and discrete formulations has brought out a very important factor relating to the condition of overlapping scatterers. For application to the physical rain scatter channel, the overlapping scatterer is not acceptable, and the inclusion of the overlapping scatterer term in the mean field expansion would lead to incorrect results.

Therefore, if the continuous formulation is used, the correlation function for u(x) must be modified to

$$\langle u(x)\, u(x') \rangle = \sigma^2 \lambda^2 \qquad (3\text{-}62)$$

which can be shown, (Papoulis, 1965, p. 285), to represent the Poisson process with non-overlapping intervals. The continuous formulation mean field would then be identical to Equation (3-61), the discrete result.

One final point, regarding the discrete formulation utilizing the Born expansion: since overlapping scatterers are not allowed, the Born result Equation (3-48) must be modified with a prime in the double summation to produce the desired field.

In summary, as long as care is taken in the formulation of the solution to insure non-overlapping scatterers, either discrete or continuous techniques can be used for the development of the mean field and its associated parameters.

3-2. SERIES EXPANSION SOLUTIONS FOR THE ONE-DIMENSIONAL SLAB

The mean field was developed in Section 3-1(d), Equation (3-61), for a slab of width L consisting of N uniformly distributed non-overlapping point scatterers whose position is described by the Poisson impulse process of Equation (3-36). In this section we proceed to evaluate expansions to $O(\sigma^2)$ for the mean field $\langle \Psi(x) \rangle$, coherent intensity $\langle \Psi(x) \rangle \; \langle \Psi^*(x) \rangle$, and total intensity $\langle \Psi(x)\, \Psi^*(x) \rangle$, for the region $x > L$.

(a) <u>Mean Field</u>

Consider the integral form of the mean field, Equation (3-61). We desire a solution for $x > L$, therefore with $x > x_j,\ x_l$

$$(|x - x_j| + x_j) = x - x_j + x_j = x$$

$$(|x - x_j| + |x_j - x_l| + x_l = x + |x_j - x_l| + x_l - x_j$$

Then

$$\langle\Psi(x)\rangle = e^{ikx} + i\,\frac{k\sigma}{2}\,\lambda\, e^{ikx}\int_0^L dx'$$

$$- \frac{k^2\sigma^2}{4}\,\lambda^2\, e^{ikx}\int_0^L\int_0^L e^{ik(|x'-x''|+x''-x')}\,dx'\,dx'' \tag{3-63}$$

Note that

$$\int_0^L dx' = L$$

The double integral is evaluated as I_1 in Appendix C, Equation (C-5)

$$\langle\Psi(x)\rangle = e^{ikx} + i\,\frac{k\sigma}{2}\,\lambda\, L e^{ikx} - \frac{k^2\sigma^2}{4}\,\lambda^2\, e^{ikx}\left[\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{ik2L}}{4k^2} - \frac{L}{2ik}\right]$$

or

$$\langle\Psi(x)\rangle = e^{ikx}\left[1 + i\,\frac{k\sigma}{2}\,\lambda L - \frac{k^2\sigma^2}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{ik2L}}{4k^2} - \frac{L}{2ik}\right)\right] \tag{3-64}$$

(b) Coherent Intensity

The complex conjugate of the mean, from Equation (3-64) above, is

$$\langle\Psi^*(x)\rangle = e^{-ikx}\left[1 - i\,\frac{k\sigma^*}{2}\,\lambda L - \frac{k^2(\sigma^2)^*}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{-ik2L}}{4k^2} + \frac{L}{2ik}\right)\right] \tag{3-65}$$

The coherent intensity is found as

$$\langle\Psi(x)\rangle\langle\Psi^*(x)\rangle = |\langle\Psi(x)\rangle|^2 = 1 + i\frac{k\sigma}{2}\,\lambda L - \frac{k^2\sigma^2}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{ik2L}}{4k^2} - \frac{L}{2ik}\right)$$

$$- i\,\frac{k\sigma^*}{2}\,\lambda L - \frac{k^2(\sigma^2)^*}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{-ik2L}}{4k^2} + \frac{L}{2ik}\right)$$

$$+ \frac{k^2\sigma\sigma^*}{4}\,\lambda^2 L^2 + O(\sigma^3)$$

or

$$|\langle\Psi(x)\rangle|^2 = 1 + i\frac{k}{2}\lambda L(\sigma - \sigma^*) + \frac{k^2\sigma\sigma^*}{4}\lambda^2 L^2$$

$$- \frac{k^2\sigma^2}{4}\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{ik2L}}{4k^2} - \frac{L}{2ik}\right) \tag{3-66}$$

$$- \frac{k^2(\sigma^2)^*}{4}\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{-ik2L}}{4k^2} + \frac{L}{2ik}\right)$$

(c) Total Intensity

The total intensity, $\langle\Psi(x)\ \Psi^*(x)\rangle$, is developed from the discrete formulation for $\Psi(x)$, Equation (3-44). Taking the complex conjugate

$$\Psi^*(x) = e^{-ikx} - i\frac{k\sigma^*}{2}\sum_{j'=1}^{N} e^{-ik(|x - x_{j'}| + x_{j'})}$$

$$- \frac{k^2(\sigma^2)^*}{4}\sum_{j'=1}^{N}{\sum_{l'=1}^{N}}' e^{-ik(|x - x_{j'}| + |x_{j'} - x_{l'}| + x_{l'})} \tag{3-67}$$

Then

$$\Psi(x)\,\Psi^*(x) = 1 + i\frac{k\sigma}{2}e^{-ikx}\sum_{j=1}^{N} e^{ik(|x - x_j| + x_j)}$$

$$- \frac{k^2\sigma^2}{4}e^{-ikx}\sum_{j=1}^{N}{\sum_{l=1}^{N}}' e^{ik(|x - x_j| + |x_j - x_l| + x_l)}$$

$$- i\frac{k\sigma^*}{2}e^{ikx}\sum_{j'=1}^{N} e^{-ik(|x - x_{j'}| + x_{j'})} \tag{3-68}$$

$$+ \frac{k^2\sigma\sigma^*}{4}\sum_{j=1}^{N} e^{ik(|x - x_j| + x_j)}\sum_{j'=1}^{N} e^{-ik(|x - x_{j'}| + x_{j'})}$$

$$- \frac{k^2(\sigma^2)^*}{4}e^{ikx}\sum_{j'=1}^{N}{\sum_{l'=1}^{N}}' e^{-ik(|x - x_{j'}| + |x_{j'} - x_{l'}| + x_{l'})} + O(\sigma^3)$$

The expectation value of Equation (3-68) is

$$
\begin{aligned}
\langle \Psi(x)\,\Psi^*(x)\rangle = 1 &+ i\,\frac{k\sigma}{2}\,e^{-ikx}\left\langle \sum_{j=1}^{N} e^{ik(|x-x_j|+x_j)}\right\rangle \\
&- i\,\frac{k\sigma^*}{2}\,e^{ikx}\left\langle \sum_{j=1}^{N} e^{-ik(|x-x_j|+x_j)}\right\rangle \\
&- \frac{k^2\,\sigma^2}{4}\,e^{-ikx}\left\langle \sum_{j=1}^{N}\sum_{l=1}^{N}{}' e^{ik(|x-x_j|+|x_j-x_l|+x_l)}\right\rangle \\
&- \frac{k^2(\sigma^2)^*}{4}\,e^{ikx}\left\langle \sum_{j=1}^{N}\sum_{l=1}^{N}{}' e^{-ik(|x-x_j|+|x_j-x_l|+x_l)}\right\rangle \\
&+ \frac{k^2\,\sigma\,\sigma^*}{4}\left\langle \sum_{j=1}^{N}\sum_{j'=1}^{N} e^{ik(|x-x_j|+x_j-|x-x_{j'}|-x_{j'})}\right\rangle
\end{aligned}
\qquad (3\text{-}69)
$$

The mean functions above are evaluated in Appendix B, Equation (B-9) for the first two terms, Equation (B-18) for the next two terms, and Equation (B-17) for the last term. Applying these results

$$
\begin{aligned}
\langle \Psi(x)\,\Psi^*(x)\rangle = \langle|\Psi(x)|\rangle^2 = 1 &+ i\frac{k\sigma}{2}\lambda e^{-ikx}\int_0^L e^{ik(|x-x'|+x')}\,dx' \\
&- i\frac{k\sigma^*}{2}\lambda e^{ikx}\int_0^L e^{-ik(|x-x'|+x')}\,dx' \\
&- \frac{k^2\,\sigma^2}{4}\lambda^2 e^{-ikx}\int_0^L\!\!\int_0^L e^{ik(|x-x'|+|x'-x''|+x'')}\,dx'\,dx'' \\
&- \frac{k^2(\sigma^2)^*}{4}\lambda^2 e^{ikx}\int_0^L\!\!\int_0^L e^{-ik(|x-x'|+|x'-x''|+x'')}\,dx'\,dx'' \\
&+ \frac{k^2\,\sigma\,\sigma^*}{4}\left\{\lambda^2\int_0^L\!\!\int_0^L e^{ik(|x-x'|+x'-|x-x''|-x'')}\,dx'\,dx''\right. \\
&\left.+ \lambda\int_0^L e^{ik(|x-x'|+x'-|x-x'|-x')}\,dx'\right\}
\end{aligned}
\qquad (3\text{-}70)
$$

where $\lambda = N/L$. Solving the integrals in Equation (3-70), and recalling that

$$x > x', \; x > x''$$

we get

$$\langle|\Psi(x)|^2\rangle = 1 + i\,\frac{k\sigma}{2}\,\lambda L - i\,\frac{k\sigma^*}{2}\,\lambda L$$
$$- \frac{k^2\sigma^2}{4}\,\lambda^2 \int_0^L\int_0^L e^{ik(|x'-x''|+x''-x')}\,dx'\,dx''$$
$$- \frac{k^2(\sigma^2)^*}{4}\,\lambda^2 \int_0^L\int_0^L e^{-ik(|x'-x''|+x''-x')}\,dx'\,dx'' \tag{3-71}$$
$$+ \frac{k^2\sigma\sigma^*}{4}\,\lambda^2 L^2 + \frac{k^2\sigma\sigma^*}{4}\,\lambda L$$

The first double integral is evaluated in Appendix C, Equation (C-5)

$$I_1 = \left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{i2kL}}{4k^2} - \frac{L}{2ik}\right) \tag{3-72}$$

The second integral is the complex conjugate of I_1 above, evaluated in Appendix C, Equation (C-7)

$$I_2 = \left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{-i2kL}}{4k^2} + \frac{L}{2ik}\right) \tag{3-73}$$

Therefore

$$\langle|\Psi(x)|^2\rangle = 1 + i\,\frac{k}{2}\,\lambda L(\sigma - \sigma^*) - \frac{k^2\sigma^2}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{i2kL}}{4k^2} - \frac{L}{2ik}\right)$$
$$- \frac{k^2(\sigma^2)^*}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{-i2kL}}{4k^2} + \frac{L}{2ik}\right) + \frac{k^2\sigma\sigma^*}{4}\,\lambda^2 L^2 + \frac{k^2\sigma\sigma^*}{4}\,\lambda L \tag{3-74}$$

or

$$\langle|\Psi(x)|^2\rangle = 1 + i\,\frac{k}{2}\,\lambda\,L(\sigma - \sigma^*) + \frac{k^2\,\sigma\,\sigma^*}{4}\,\lambda\,L$$
$$+ \frac{k^2}{4}\,\lambda^2\left[\left(\sigma\,\sigma^* - \frac{\sigma^2 + (\sigma^2)^*}{2}\right)L^2 - \frac{\sigma^2 + (\sigma^2)^*}{4k^2}\right. \tag{3-75}$$
$$\left. + \frac{\sigma^2}{4k^2}\,e^{i2kL} + \frac{(\sigma^2)^*}{4k^2}\,e^{-i2kL} + \frac{\sigma^2 - (\sigma^2)^*}{2ik}\,L\right]$$

With

$$\sigma = \sigma_r + i\,\sigma_i$$
$$\sigma - \sigma^* = i\,2\sigma_i$$
$$\sigma^2 + (\sigma^2)^* = 2(\sigma_r^2 - \sigma_i^2)$$
$$\sigma\,\sigma^* = \sigma_r^2 + \sigma_i^2$$
$$\sigma^2 - (\sigma^2)^* = i\,4\sigma_r\,\sigma_i$$

Thus Equation (3-74) is further reduced to

$$\langle|\Psi(x)|^2\rangle = 1 - k\,\lambda\,L\,\sigma_i + \frac{k^2}{4}\,\lambda(\sigma_r^2 + \sigma_i^2)\,L + \frac{k^2}{4}\,\lambda^2\left[2\sigma_i^2\,L^2 - \frac{(\sigma_r^2 - \sigma_i^2)}{2k^2} + \frac{2\sigma_r\,\sigma_i}{k}\,L\right.$$
$$\left. + \frac{(\sigma_r^2 - \sigma_i^2)}{2k^2}\cos 2\,k\,L + \frac{\sigma_r\,\sigma_i}{4k^2}\sin 2\,k\,L\right] \tag{3-76}$$

The above result is the total intensity in the region $x > L$ for the slab of width L. For $L = 0$, $\langle|\Psi(x)|^2\rangle = 1$, as expected. We further note that for $\sigma = i\,\sigma_i$ (pure complex)

$$\langle|\Psi(x)|^2\rangle = 1 - k\,\lambda\,L\,\sigma_i + \frac{k^2}{4}\,\lambda\,\sigma_i^2\,L + \frac{k^2}{4}\,\lambda^2\left[2\sigma_i^2\,L^2 + \frac{\sigma_i^2}{2k^2}\,(1 - \cos 2\,k\,L)\right] \tag{3-77}$$

And for $\sigma = \sigma_r$ (real)

$$\langle|\Psi(x)|^2\rangle = 1 + \frac{k^2}{4}\,\lambda\,\sigma_r^2\,L + \frac{k^2}{4}\,\lambda^2\left[-\frac{\sigma_r^2}{2k^2}\,(1 - \cos 2\,k\,L)\right] \tag{3-78}$$

If the total intensity expansion, Equation (3-74), is compared with the coherent intensity expansion, Equation (3-66), all terms are the same except for

$$+ \frac{k^2 \sigma \sigma^*}{4} \lambda L$$

which is present in the total intensity expression. This term represents the first term in the incoherent intensity expansion, i.e.

$$\langle |\Psi(x)|^2 \rangle - |\langle \Psi(x) \rangle|^2 = \frac{k^2 \sigma \sigma^*}{4} \lambda L + O(\sigma^3) \qquad (3\text{-}79)$$

and is seen to be of $O(\sigma^2 L)$.

3-3. APPLICATION OF THE FOLDY APPROXIMATION TO THE ONE-DIMENSIONAL SLAB

The problem of multiple scattering of particles by a random distribution, treated by Foldy (1945), resulted in the development of a procedure for determining the mean field integral equation by applying an appropriate approximation for the external field acting on each scatterer. This important technique, later utilized by Lax (1951), Twersky (1964, 1967), Ishimaru & Lin (1972), and others, is reviewed in Section 2-2.

The Foldy approximation is applied here to the one-dimensional slab problem, the mean field $\langle \Psi(x) \rangle$ developed, and the resulting expansions compared to the expansions previously obtained.

Consider the one-dimensional wave equation for the slab of width L, consisting of a uniform distribution of N scatterers, given by Equation (3-40)

$$\left[\frac{d^2}{dx^2} + k^2 \right] \Psi(x) = -k^2 \sigma \sum_{j=1}^{N} \delta(x - x_j) \Psi(x) \qquad (3\text{-}80)$$

The above wave equation plus the associated boundary conditions produce the integral equation

$$\Psi(x) = e^{ikx} - k^2 \sigma \sum_{j=1}^{N} \int_0^L \frac{e^{ik|x - x'|}}{2ik} \delta(x' - x_j) \Psi(x') \, dx' \qquad (3\text{-}81)$$

Take the average of Equation (3-81)

$$\langle \Psi(x) \rangle = e^{ikx} - k^2 \sigma \sum_{j=1}^{N} \int_0^L \frac{e^{ik|x-x'|}}{2ik} \langle \Psi(x') \, \delta(x' - x_j) \rangle \, dx' \qquad (3\text{-}82)$$

The mean of $\langle \Psi(x') \, \delta(x' - x_j) \rangle$ is

$$\langle \Psi(x') \, \delta(x' - x_j) \rangle = \int \Psi(x') \, \delta(x' - x_j) \, f(x_1, x_2, \ldots x_N) \, dx_1 \, dx_2 \ldots dx_N \quad (3\text{-}83)$$

where

$$f(x_1, x_2, \ldots x_N) = f(x_j)$$

is the joint probability density function of the x_j.

With the x_j statistically independent

$$\begin{aligned} f(x_1, x_2, \ldots x_N) &= f(x_1) \, f(x_2) \ldots f(x_N) \\ &= \prod_{k=1}^{N} f(x_k) \end{aligned} \qquad (3\text{-}84)$$

Equation (3-83) is therefore

$$\langle \Psi(x') \, \delta(x' - x_j) \rangle = \int \Psi(x') \, \delta(x' - x_j) \prod_{k=1}^{N} f(x_k) \, dx_k \qquad (3\text{-}85)$$

The product term can be expressed as

$$\prod_{k=1}^{N} f(x_k) \, dx_k = f(x_j) \, dx_j \prod_{\substack{k=1 \\ k \neq j}}^{N} f(x_k) \, dx_k \qquad (3\text{-}86)$$

Therefore, from Equation (3-85)

$$\langle \Psi(x') \, \delta(x' - x_j) \rangle = \int \Psi(x') \prod_{\substack{k=1 \\ k \neq j}}^{N} f(x_k) \, dx_k \, \delta(x' - x_j) \, f(x_j) \, dx_j \qquad (3\text{-}87)$$

Note that

$$\langle \Psi(x') \mid x' \rangle = \int\int \ldots \int \prod_{\substack{k=1 \\ k \neq j}}^{N} f(x_k)\, dx_k \tag{3-88}$$

$$= \text{the field at } x' \text{ averaged over all possible configurations of the other N-1 scatterers.}$$

Then, from Equation (3-82)

$$\langle \Psi(x) \rangle = e^{ikx} - k^2 \sigma \sum_{j=1}^{N} \int_o^L \frac{e^{ik|x-x'|}}{2ik} \left[\int \langle \Psi(x') \mid x' \rangle\, \delta(x' - x_j)\, f(x_j)\, dx_j \right] dx' \tag{3-89}$$

The integral over x_j is evaluated at the delta function, i.e.

$$\int \langle \Psi(x') \mid x' \rangle\, \delta(x' - x_j)\, f(x_j)\, dx_j = \langle \Psi(x') \mid x' \rangle\, f(x') \tag{3-90}$$

Thus, from Equation (3-89)

$$\langle \Psi(x) \rangle = e^{ikx} - k^2 \sigma \sum_{j=1}^{N} \int_o^L \frac{e^{ik|x-x'|}}{2ik} \langle \Psi(x') \mid x' \rangle\, f(x')\, dx' \tag{3-91}$$

There are no j dependent terms in the integral, therefore the summation evaluates the integral N times,

$$\langle \Psi(x) \rangle = e^{ikx} - k^2 \sigma N \int_o^L \frac{e^{ik|x-x'|}}{2ik} \langle \Psi(x') \mid x' \rangle\, f(x')\, dx' \tag{3-92}$$

The rigorous solution of this equation requires solving the set of simultaneous linear algebraic equations for each scatterer and then performing the integrations to obtain $\langle \Psi(x') \mid x' \rangle$, (see Eq. 3-88). This procedure is not feasible.

The approximation introduced by Foldy consists of replacing $\langle \Psi(x') \mid x' \rangle$, the field at the j th scatterer averaged over all possible configurations of the other N-1 scatterers, with $\langle \Psi(x') \rangle$, the average field at the j th scatterer position, when that scatterer is not present. As Foldy notes, these averages differ only by a term of the order 1/N, thus if N is large, the approximation is reasonable.

Therefore, with

$$\langle \Psi(x') \mid x' \rangle \simeq \langle \Psi(x') \rangle \tag{3-93}$$

the mean field, from Equation (3-92), will be

$$\langle \Psi(x) \rangle = e^{ikx} - k^2 \sigma N \int_0^L \frac{e^{ik|x - x'|}}{2ik} \langle \Psi(x') \rangle f(x') \, dx' \tag{3-94}$$

In addition, if the x_j are uniformly distributed,

$$f(x') = \frac{1}{L} \tag{3-95}$$

Therefore

$$\langle \Psi_F(x) \rangle = e^{ikx} - k^2 \sigma \lambda \int_0^L \frac{e^{ik|x - x'|}}{2ik} \langle \Psi(x') \rangle \, dx' \tag{3-96}$$

where $\lambda = N/L$.

The above integral equation represents the mean field for the slab under the Foldy approximation to the external field acting on each scatterer.

(a) Series Expansion For the Mean Field and Coherent Intensity

The Foldy Approximation to the mean field $\langle \Psi_F(x) \rangle$, Equation (3-96), with the addition of the boundary conditions of the slab, represents the solution to a wave equation of the form

$$\left[\frac{d^2}{dx^2} + k^2 \right] \langle \Psi_F(x) \rangle = -k^2 \sigma \lambda \langle \Psi_F(x) \rangle$$

or

$$\left[\frac{d^2}{dx^2} + k^2 (1 + \sigma \lambda) \right] \langle \Psi_F(x) \rangle = 0 \tag{3-97}$$

A solution to Equation (3-97) is

$$\langle \Psi_F(x) \rangle = e^{ik(1 + \sigma \lambda)^{1/2} x}$$

or

$$\langle \Psi_F(x) \rangle = e^{iKx} \tag{3-98}$$

where

$$K = k(1 + \sigma \lambda)^{1/2} \tag{3-99}$$

Thus, under the Foldy approximation for the mean field, the slab of discrete scatterers can be represented as a continuous medium with an 'effective' propagation constant K, given above. The mean field for $x > L$ is then available from the transmission coefficient T, see Figure 3-3,

$$\langle \Psi_F(x) \rangle = T e^{ik(x - L)} \qquad x > L \tag{3-100}$$

The mean field and coherent intensity $\langle \Psi_F(x) \rangle \ \langle \Psi_F^*(x) \rangle$ can now be developed from the expansion solution for T to $O(\sigma^2)$.

The transmission coefficient for a slab of width L and propagation constant K was derived in Appendix A, Equation (A-12),

$$T = \frac{4kKe^{iKL}}{(k+K)^2 - (k-K)^2 e^{i2KL}} \tag{3-101}$$

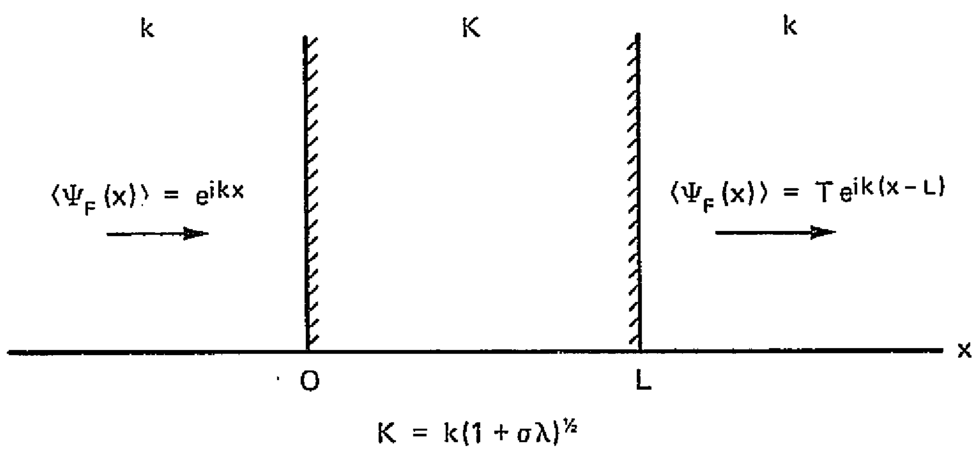

Figure 3-3. Foldy Approximation Representation for the Mean Field

Expanding K to $O(\sigma^2)$, (recall that σ may be complex),

$$K = k(1 + \sigma\lambda)^{1/2} = k\left[1 + \frac{\sigma\lambda}{2} - \frac{(\sigma\lambda)^2}{8} + O(\sigma^3)\right] \qquad (3\text{-}102)$$

Then

$$e^{iKL} = e^{ikL}\, e^{ik\frac{\sigma\lambda}{2}L}\, e^{-ik\frac{(\sigma\lambda)^2}{8}}$$

$$e^{iKL} = e^{ikL}\left[1 + ik\frac{\sigma\lambda}{2}L - k^2\frac{(\sigma\lambda)^2}{8}L^2\right] \times \left[1 - ik\frac{(\sigma\lambda)^2}{8} + \,..\,O(\sigma^4)\right]$$

$$= e^{ikL}\left[1 + ik\frac{\sigma\lambda}{2}L - k^2\frac{(\sigma\lambda)^2}{8}L^2 - ik\frac{(\sigma\lambda)^2}{8}\right] \qquad (3\text{-}103)$$

and

$$e^{i2KL} = e^{i2kL}\left[1 + ik\sigma\lambda L - k^2\frac{(\sigma\lambda)^2}{2}L^2\right]\left[1 - ik\frac{(\sigma\lambda)^2}{4}L\ldots O(\sigma^4)\right]$$

$$= e^{i2kL}\left[1 + ik\sigma\lambda L - k^2\frac{(\sigma\lambda)^2}{2}L^2 - ik\frac{(\sigma\lambda)^2}{4}L\right] \qquad (3\text{-}104)$$

Applying these expansions, we find

$$(k + K)^2 = k^2\left(4 + 2\sigma\lambda - \frac{(\sigma\lambda)^2}{4}\right)$$

$$(k - K)^2\, e^{i2KL} = k^2\left[\frac{(\sigma\lambda)^2}{4}\right]e^{i2kL}$$

$$(k + K)^2 - (k - K)^2\, e^{i2KL} = k^2\left[4 + 2\sigma\lambda - \frac{(\sigma\lambda)^2}{4} - \frac{(\sigma\lambda)^2}{4}e^{i2kL}\right]$$

$$4kK\, e^{iKL} = 4k^2\, e^{ikL}\left[1 + ik\frac{\sigma\lambda}{2}L + \frac{\sigma\lambda}{2} + ik\frac{(\sigma\lambda)^2}{8}L - k^2\frac{(\sigma\lambda)^2}{8}L^2 - \frac{(\sigma\lambda)^2}{8}\right]$$

The transmission coefficient is then

$$T = \frac{4k^2 e^{ikL}\left[1 + \left(ik\frac{\lambda}{2}L + \frac{\lambda}{2}\right)\sigma + \left(ik\frac{\lambda^2}{8}L - k^2\frac{\lambda^2}{8}L^2 - \frac{\lambda^2}{8}\right)\sigma^2\right]}{4k^2\left[1 + \left(\frac{\lambda}{2}\right)\sigma + \left(-\frac{\lambda^2}{16} - \frac{\lambda^2}{16}e^{i2kL}\right)\sigma^2\right]} \quad (3\text{-}105)$$

This is of the form

$$T = \frac{e^{ikL}[1 + A\sigma + B\sigma^2]}{[1 + a\sigma + b\sigma^2]}$$

where the A, B, a, and b are the coefficients in Equation (3-105).

Expanding the denominator

$$[1 + a\sigma + b\sigma^2]^{-1} = 1 - a\sigma - b\sigma^2 + a^2\sigma^2 + O(\sigma^3)$$

$$= 1 - a\sigma + (a^2 - b)\sigma^2$$

Therefore

$$T = e^{ikL}[1 + A\sigma + B\sigma^2]\ [1 - a\sigma + (a^2 - b)\sigma^2]$$

$$= e^{ikL}[1 + (A - a)\sigma + (a^2 - b - aA + B)\sigma^2]$$

$$= e^{ikL}\left[1 + ik\frac{\lambda\sigma}{2}L - \frac{(\lambda\sigma)^2}{16} - ik\frac{(\lambda\sigma)^2}{8}L\right.$$

$$\left. - k^2\frac{(\sigma\lambda)^2}{8}L^2 + \frac{(\sigma\lambda)^2}{16}e^{i2kL}\right]$$

Rearranging terms

$$T = e^{ikL}\left[1 + i\frac{k\sigma}{2}\lambda L - \frac{k^2\sigma^2}{4}\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{i2kL}}{4k^2} - \frac{L}{2ik}\right)\right] \quad (3\text{-}106)$$

The resulting mean field is therefore

$$\langle\Psi_F(x)\rangle = T e^{ik(x - L)}$$

or

$$\langle\Psi_F(x)\rangle = e^{ikx}\left[1 + i\,\frac{k\sigma}{2}\,\lambda L - \frac{k^2\,\sigma^2}{4}\,\lambda^2\left(\frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{i2kL}}{4k^2} - \frac{L}{2ik}\right)\right] \quad (3\text{-}107)$$

A comparison of the above result with the discrete solution expansion for the mean field, Equation (3-64), shows that the Foldy approximation represents the mean field exactly to $O(\sigma^2)$, i.e.

$$\langle\Psi_F(x)\rangle = \langle\Psi(x)\rangle \text{ to } O(\sigma^2) \quad (3\text{-}108)$$

The coherent intensity $\langle\Psi_F(x)\rangle\,\langle\Psi_F^*(x)\rangle$ is found from Equation (3-107) and its complex conjugate, and the result will be identical to Equation (3-66), the discrete solution expansion.

The Foldy approximation therefore represents the coherent intensity exactly to $O(\sigma^2)$, i.e.

$$|\langle\Psi_F(x)\rangle|^2 = |\langle\Psi(x)\rangle|^2 \text{ to } O(\sigma^2) \quad (3\text{-}109)$$

The total intensity expansion $\langle\Psi_F(x)\ \Psi_F^*(x)\rangle$, however, will not agree with the discrete solution expansion for $\langle\Psi(x)\ \Psi^*(x)\rangle$ to $O(\sigma^2)$. The Foldy approximation does not account for the incoherent intensity term

$$+\frac{k^2\,\sigma\,\sigma^*}{4}\,\lambda\,L$$

which is present in the discrete solution expansion, Equation (3-76).

The implications of the Foldy approximation results developed here are discussed in the following section.

3-4. SUMMARY OF ONE-DIMENSIONAL MEDIA RESULTS

The one-dimensional study of a slab of discrete scatterers presented in this chapter has produced a number of useful techniques and results which are applicable to the volume scatter rain channel. They are summarized briefly below:

(a) the Poisson distributed point scatterer, with an appropriate 'weighting function', can represent a distribution of discrete finite scatterers – the weighting function provides the physical characteristics of the scatterer for practical interpretation of results obtained.

(b) either discrete or continuous techniques may be employed for the development of the mean field and intensity expansions, as long as care is taken in the formulation to insure non-overlapping scatterers.

(c) series expansion of the mean field to $O(\sigma^2)$ produces secular terms in L for both $O(\sigma)$ and $O(\sigma^2)$ - the slab width limitation for series expansion requires that

$$L < \frac{2}{k|\sigma|\lambda}$$

(d) the Foldy approximation expansion for the mean field agrees exactly, to $O(\sigma^2)$, with the discrete Born series expansion. The Foldy approximation can be used to represent the coherent intensity, however it is not applicable to the total intensity, where an incoherent intensity term of $O(\sigma^2)$ was found which is not accounted for in the Foldy expansion. The Foldy approximation has the same slab width limitation as given above.

(e) an incoherent intensity term of $O(\sigma^2)$ was found, which is secular in L, and which may be significant in the volume scatter case for certain wavelength/scatterer size ranges.

The applicability of the one-dimensional results to the volume rain scatter case will be observed as the volume configuration solution is developed in the next chapter.

CHAPTER 4

PROPAGATION THROUGH A VOLUME DISTRIBUTION OF DISCRETE SCATTERERS

A study of the multiple scattering of waves propagating through a volume of discrete scatterers is developed in this chapter. The scope of the study will be directed at determining a suitable representation applicable to the rain scatter channel and evaluating the statistical parameters of the field by discrete random media techniques, some of which were employed in the previous chapter for the one-dimensional case.

The scattering volume is described by a uniform distribution of point scatterers whose position is described by a Poisson impulse process. The scattering characteristics of a single scatterer are represented, as in the one-dimensional case, by an appropriate 'weighting factor'. Integral equations are developed for the mean field and intensity; the results are solved by Fourier transform techniques; and expansions for the far field are derived. The Foldy approximation to the mean field is applied and limitations on its range of validity determined.

4-1. POINT SOURCE INCIDENT ON A VOLUME OF DISCRETE SCATTERERS

We begin the development of the volume scattering study by defining the model used to describe the scattering medium. An isotropic spherical wave is assumed incident on a volume V of point scatterers whose position is determined by a Poisson impulse process. The isotropic incident wave was chosen because of its generality and ease of extension to other types of incident radiation such as the plane wave and beam wave.

Consider the incident wave source point at r_o, in an orthogonal coordinate system whose origin is located within the scattering volume V, as described in Figure 4-1. The medium consists of N isotropic point scatterers randomly distributed in the volume V, with r_n the position vector of each scatterer (a random variable), defined by the Poisson impulse process

$$u(r) = \sum_{n=1}^{N} \sigma\, \delta(r - r_n) \qquad (4\text{-}1)$$

where σ is the complex weighting factor determined from the physical characteristics of a single scatterer.

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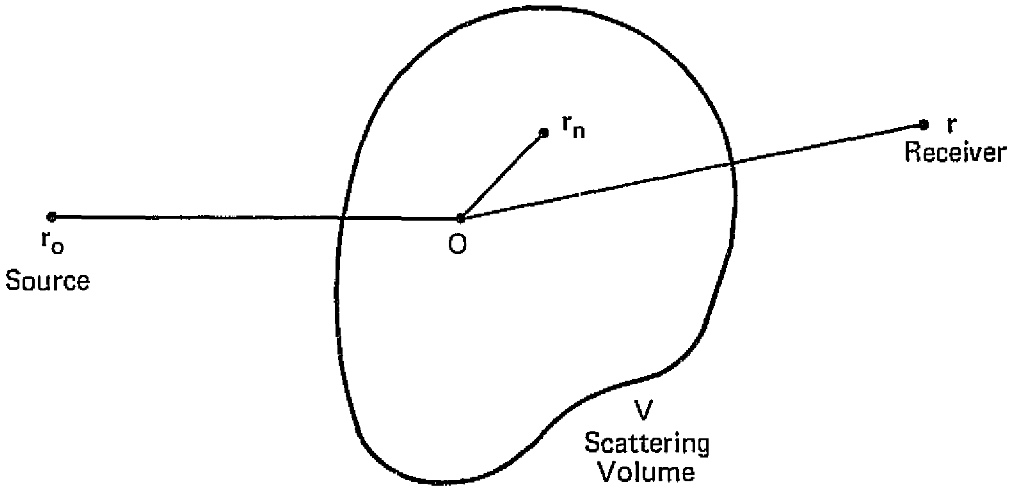


Figure 4-1. Coordinate Notation for Volume Distribution of Discrete Scatterers

The resultant field at the receiver point, $\Psi(r)$, is desired. The field $\Psi(r)$ obeys the reduced wave equation

$$[\nabla^2 + k^2 n^2(r)] \Psi(r) = 0 \tag{4-2}$$

where $n(r)$ is the refractive index and k is the free space propagation constant, $k = 2\pi/\lambda$. An e^{-iwt} time dependence is assumed.

For the configuration defined above, the wave equation will be

$$\left\{\nabla^2 + k^2 \left[1 + \sigma \sum_{n=1}^{N} \delta(r - r_n)\right]\right\} \Psi(r, r_o) = 0 \tag{4-3}$$

The above wave equation and its associated boundary conditions convert to the integral equation

$$\Psi(r, r_o) = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} \int_V G(r, r') \, \delta(r' - r_n) \, \Psi(r', r_o) \, dr' \tag{4-4}$$

where $G(r, r')$ is the free space Green's function

$$G(r, r') = \frac{e^{ik|r - r'|}}{4\pi|r - r'|} \tag{4-5}$$

and $G_o(r, r_o)$ is the incident spherical wave at r_o, i.e.

$$G_o(r, r_o) = \frac{e^{ik|r - r_o|}}{4\pi|r - r_o|} \tag{4-6}$$

The delta function evaluates the integral at $r' = r_n$

$$\Psi(r, r_o) = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} G(r, r_n)\, \Psi(r_n, r_o) \tag{4-7}$$

This equation represents the field at r as the sum of the incident wave in the absence of scatterers, and spherical waves, $G(r, r_n)$, with an external field $\Psi(r_n, r_o)$ eminating from each of the N scatterers.

The external field at the nth scatterer is the sum of the incident wave at r_n plus the sum of the fields produced by all the other N-1 scatterers, i.e.

$$\Psi(r_n, r_o) = G_o(r_n, r_o) - k^2 \sigma \sum_{\substack{p=1 \\ p \neq n}}^{N} G(r_p, r_o)\, G(r_n, r_p) \tag{4-8}$$

where $G(r_n, r_p)$ is the spherical wave produced at r_n from the scatterer at r_p excited by the incident wave $G(r_p, r_o)$.

Substituting $\Psi(r_n, r_o)$ from Equation (4-8) into Equation (4-7)

$$\Psi(r, r_o) = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} G(r, r_n) \left[G_o(r_n, r_o) - k^2 \sigma \sum_{\substack{p=1 \\ p \neq n}}^{N} G(r_p, r_o)\, G(r_n, r_p) \right]$$

or

$$\Psi(r, r_o) = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} G(r, r_n)\, G_o(r_n, r_o) + k^4 \sigma^2 \sum_{n=1}^{N} {\sum_{p=1}^{N}}' G(r, r_n)\, G(r_p, r_o)\, G(r_n, r_p) \tag{4-9}$$

where the prime in the p summation indicates "except for p = n," i.e. $p \neq n$.

This result represents the fundamental multiple scattering equation for the field at r produced by a configuration of N Poisson distributed scatterers at r_n in the volume V excited by a point source at r_o.

It can be shown that the resulting field, Equation (4-9), is also obtained by a Born expansion solution to $\Psi(r, r_o)$ where

$$\Psi(r, r_o) = \sum_{j=0}^{\infty} \sigma^j \Psi_j(r, r_o) \tag{4-10}$$

and the expansion is carried out to $O(\sigma^2)$. This result was demonstrated for the one-dimensional case in Section 3-3(d); the procedure is similar for the three-dimensional case and need not be repeated here.

Proceeding to the average of Equation (4-9)

$$\begin{aligned}\langle\Psi(r, r_o)\rangle = G_o(r, r_c) - k^2 \sigma \left\langle \sum_{n=1}^{N} G(r, r_n)\, G_o(r_n, r_o) \right\rangle \\ + k^4 \sigma^2 \left\langle \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G(r, r_n)\, G(r_p, r_o)\, G(r_n, r_p) \right\rangle\end{aligned} \tag{4-11}$$

The averaged summation terms above were developed in Appendix B. From Equation (B-25)

$$\left\langle \sum_{n=1}^{N} G(r, r_n)\, G_o(r_n, r_o) \right\rangle = \rho \int_V G(r, r_n)\, G_o(r_n, r_o)\, dr_n \tag{4-12}$$

and from Equation (B-33)

$$\left\langle \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G(r, r_n)\, G(r_p, r_o)\, G(r_n, r_p) \right\rangle = \rho^2 \int_V \int G(r, r_n)\, G(r_p, r_o)\, G(r_n, r_p)\, dr_n\, dr_p \tag{4-13}$$

where

$$\rho = \frac{N}{V} \tag{4-14}$$

The mean field from Equation (4–11) is therefore

$$\langle\Psi(\mathbf{r},\mathbf{r}_o)\rangle = G_o(\mathbf{r},\mathbf{r}_o) - k^2\sigma\rho\int_V G(\mathbf{r},\mathbf{r}')\,G(\mathbf{r}',\mathbf{r}_o)\,d\mathbf{r}'$$

$$+ k^4\sigma^2\rho^2\int_V\int G(\mathbf{r},\mathbf{r}')\,G(\mathbf{r}',\mathbf{r}'')\,G(\mathbf{r}'',\mathbf{r}_o)\,d\mathbf{r}'\,d\mathbf{r}'' \qquad (4\text{–}15)$$

(a) <u>Mean Field Expansion by Application of Transverse Fourier Transforms</u>

The solution of the mean field integrals in Equation (4–15) are now considered for the case of a slab volume of width L in the direction of propagation z, and extending to ±∞ in the x and y dimensions. The volume integrals can then be transformed into one-dimensional integrals in z by application of the two-dimensional Fourier transform in the transverse x – y plane.

With

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$$

and

$$\mathbf{k} = k_x\mathbf{a}_x + k_y\mathbf{a}_y + k_z\mathbf{a}_z$$

let

$$\mathbf{r}_t = x\mathbf{a}_x + y\mathbf{a}_y \qquad (4\text{–}16a)$$

and

$$\mathbf{k}_t = k_x\mathbf{a}_x + k_y\mathbf{a}_y \qquad (4\text{–}16b)$$

The transverse Fourier transform pair for a function f(**r**) is then

$$\tilde{f}(z,\mathbf{k}_t) = \iint f(\mathbf{r})\,e^{-i\mathbf{k}_t\cdot\mathbf{r}}\,d\mathbf{r}_t \qquad (4\text{–}17a)$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2}\iint \tilde{f}(z,\mathbf{k}_t)\,e^{i\mathbf{k}_t\cdot\mathbf{r}}\,d\mathbf{k}_t \qquad (4\text{–}17b)$$

where;

$$dk_t = dk_x dk_y, dr_t = dxdy$$

and

$$k_t \cdot r = k_x x + k_y y = k_t \cdot r_t$$

For the Green's function

$$G(r, r') = \frac{-e^{ik|r - r'|}}{4\pi|r - r'|}$$

the transform pair is of the form

$$\tilde{G}(z - z', k_t) = \iint G(r - r')\, e^{-ik_t \cdot (r - r')}\, dr_t \qquad (4\text{-}18a)$$

$$G(r - r') = \frac{1}{(2\pi)^2} \iint \tilde{G}(z - z', k_t)\, e^{ik_t \cdot (r - r')}\, dk_t \qquad (4\text{-}18b)$$

The Green's function $G(r, r')$ is a solution to the wave equation

$$(\nabla^2 + k^2)\, G(r, r') = \delta(r - r') \qquad (4\text{-}19)$$

If Equation (4-19) is multiplied by $e^{ik_t \cdot r}$ and integrated over dr_t to $\pm\infty$, the inverse transform of $G(r, r')$ is found to be (Tyras, 1969, p. 113)

$$\tilde{G}(z - z', k_t) = \frac{e^{ik_z|z - z'|}}{2ik_z} \qquad (4\text{-}20)$$

where

$$k_z = \sqrt{k^2 - |k_t|^2} \qquad (4\text{-}21)$$

Utilizing the formulations of Equations (4-18) and (4-20), the Green's functions in $\langle\Psi(r, r_0)\rangle$, Equation (4-15), can be represented by their inverse Fourier transforms, i.e.

$$\langle\Psi(\mathbf{r},\mathbf{r}_o)\rangle = \frac{1}{(2\pi)^2}\iint \frac{e^{ik_z|z-z_o|}}{2ik_z}\, e^{i\mathbf{k}_t\cdot(\mathbf{r}-\mathbf{r}_o)}\, d\mathbf{k}_t$$

$$- k^2\,\sigma\,\rho\int\left[\frac{1}{(2\pi)^2}\iint \frac{e^{ik_z|z-z'|}}{2ik_z}\, e^{i\mathbf{k}_t\cdot(\mathbf{r}-\mathbf{r}')}\, d\mathbf{k}_t\right.$$

$$\left.\frac{1}{(2\pi)^2}\iint \frac{e^{ik_z'|z'-z_o|}}{2ik_z'}\, e^{i\mathbf{k}_t'\cdot(\mathbf{r}'-\mathbf{r}_o)}\, d\mathbf{k}_t'\right] d\mathbf{r}'$$

$$+ k^4\,\sigma^2\,\rho^2\iint\left[\frac{1}{(2\pi)^2}\iint \frac{e^{ik_z|z-z'|}}{2ik_z}\, e^{i\mathbf{k}_t\cdot(\mathbf{r}-\mathbf{r}')}\, d\mathbf{k}_t\right. \tag{4-22}$$

$$\frac{1}{(2\pi)^2}\iint \frac{e^{ik_z'|z'-z''|}}{2ik_z'}\, e^{i\mathbf{k}_t'\cdot(\mathbf{r}'-\mathbf{r}'')}\, d\mathbf{k}_t'$$

$$\left.\frac{1}{(2\pi)^2}\iint \frac{e^{ik_z''|z''-z_o|}}{2ik_z''}\, e^{i\mathbf{k}_t''\cdot(\mathbf{r}''-\mathbf{r}_o)}\, d\mathbf{k}_t''\right] d\mathbf{r}'\, d\mathbf{r}''$$

The integral term over $d\mathbf{r}'$ in Equation (4-22) can be rearranged as

$$I_1 = -k^2\,\sigma\,\rho\,\frac{1}{(2\pi)^4}\int\left\{\left[\iint \frac{e^{ik_z|z-z'|}}{2ik_z}\, d\mathbf{k}_t \iint \frac{e^{ik_z'|z'-z_o|}}{2ik_z'}\, d\mathbf{k}_t'\right]\right. \tag{4-23}$$

$$\left. e^{i[\mathbf{k}_t\cdot(\mathbf{r}-\mathbf{r}')+\mathbf{k}_t'\cdot(\mathbf{r}'-\mathbf{r}_o)]}\right\} d\mathbf{r}'$$

With

$$d\mathbf{r}' = d\mathbf{r}_t'\, dz'$$

the integral in Equation (4-23) over $d\mathbf{r}'$ is

$$\int e^{i[\mathbf{k}_t\cdot\mathbf{r}-\mathbf{k}_t'\cdot\mathbf{r}_o]}\, e^{-i\mathbf{r}'\cdot(\mathbf{k}_t-\mathbf{k}_t')}\, d\mathbf{r}' =$$

$$e^{i[\mathbf{k}_t\cdot\mathbf{r}-\mathbf{k}_t'\cdot\mathbf{r}_o]}\int\left[\iint e^{-i\mathbf{r}'\cdot(\mathbf{k}_t-\mathbf{k}_t')}\, d\mathbf{r}_t'\right] dz' \tag{4-24}$$

From the delta function definition

$$\delta(x - x') = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} e^{-ik_x(x - x')} \, dk_x \tag{4-25}$$

we find that

$$\delta(k_t - k_t') = \frac{1}{(2\pi)^2} \iint e^{-ir \cdot (k_t - k_t')} \, dr \tag{4-26}$$

Therefore the bracketed integral term in Equation (4-24) is seen to be

$$\iint e^{-ir' \cdot (k_t - k_t')} \, dr_t' = (2\pi)^2 \, \delta(k_t - k_t') \tag{4-27}$$

Therefore I_1 is

$$I_1 = -k^2 \sigma \rho \frac{1}{(2\pi)^2} \int \frac{e^{ik_z|z - z'|}}{2ik_z} \frac{e^{ik_z'|z' - z_o|}}{2ik_z'} e^{i[k_t \cdot r - k_t' \cdot r_o]} \, \delta(k_t - k_t') \, dk_t \, dk_t' \, dz' \tag{4-28}$$

The delta function evaluates one of the k_t integrals at $k_t = k_t'$, i.e.

$$I_1 = -k^2 \sigma \rho \frac{1}{(2\pi)^2} \int \frac{e^{ik_z|z - z'|}}{2ik_z} \frac{e^{ik_z|z' - z_o|}}{2ik_z} e^{i[k_t \cdot r - k_t \cdot r_o]} \, dk_t \, dz'$$

or

$$I_1 = -k^2 \sigma \rho \frac{1}{(2\pi)^2} \int \frac{e^{ik_z(|z - z'| + |z' - z_o|)}}{-4k_z^2} e^{ik_t \cdot (r - r_o)} \, dk_t \, dz' \tag{4-29}$$

The integral term over dr' dr'' in Equation (4–22) can be rearranged as

$$I_2 = k^4 \sigma^2 \rho^2 \frac{1}{(2\pi)^6} \iint \frac{e^{ik_z|z - z'|}}{2ik_z} dk_t \iint \frac{e^{ik_z'|z' - z''|}}{2ik_z'} dk_t'$$

$$\iint \frac{e^{ik_z''|z'' - z_o|}}{2ik_z''} dk_t'' e^{i[k_t \cdot r - k_t'' \cdot r_o]} \tag{4–30}$$

$$\int \left[\iint e^{-ir' \cdot (k_t - k_t')} dr_t'\right] dz' \int \left[\iint e^{-ir'' \cdot (k_t' - k_t'')} dr_t''\right] dz''$$

The bracketed terms above over dr_t' and dr_t'' are seen, from Equation (4–26), to be delta functions in $(k_t - k_t')$ and $(k_t' - k_t'')$. Therefore

$$I_2 = k^4 \sigma^2 \rho^2 \frac{1}{(2\pi)^2} \iint \cdots \int \frac{e^{ik_z|z - z'|}}{2ik_z} \frac{e^{ik_z'|z' - z''|}}{2ik_z'} \frac{e^{ik_z''|z'' - z_o|}}{2ik_z''} \tag{4–31}$$

$$e^{i[k_t \cdot r - k_t'' \cdot r_o]} \delta(k_t - k_t') \delta(k_t' - k_t'') \quad dk_t\, dk_t''\, dk_t''\, dz'\, dz''$$

The delta functions evaluate two of the dk_t integral pairs, at $k_t' = k_t$ and at $k_t'' = k_t'$. First at $k_t'' = k_t'$

$$I_2 = k^4 \sigma^2 \rho^2 \frac{1}{(2\pi)^2} \iint \cdots \int \frac{e^{ik_z|z - z'|}}{2ik_z} \frac{e^{ik_z'|z' - z''|}}{2ik_z'} \frac{e^{ik_z'|z'' - z_o|}}{2ik_z'} \tag{4–32}$$

$$e^{i[k_t \cdot r - k_t' \cdot r_o]} \delta(k_t - k_t')\, dk_t\, dk_t'\, dz'\, dz''$$

then at $k_t' = k_t$

$$I_2 = k^4 \sigma^2 \rho^2 \frac{1}{(2\pi)^2} \iint \frac{e^{ik_z(|z - z'| + |z' - z''| + |z'' - z_o|)}}{-8ik_z^3} \tag{4–33}$$

$$e^{ik_t \cdot (r - r_o)} dk_t\, dz'\, dz''$$

The results of Equations (4-29) and (4-33) can now be applied to Equation (4-22) to get the mean field

$$\langle\Psi(r, r_o)\rangle = \frac{1}{(2\pi)^2} \iint \frac{e^{ik_z|z - z_o|}}{2ik_z} e^{ik_t \cdot (r - r_o)} dk_t$$

$$- k^2 \sigma \rho \frac{1}{(2\pi)^2} \iint \frac{e^{ik_z(z - z'| + |z' - z_o|)}}{-4k_z^2} e^{ik_t \cdot (r - r_o)} dk_t\, dz' \qquad (4\text{-}34)$$

$$+ k^4 \sigma^2 \rho^2 \frac{1}{(2\pi)^2} \iint \frac{e^{ik_z(|z - z'| + |z' - z''| + |z'' - z_o|)}}{-8ik_z^3} e^{ik_t \cdot (r - r_o)} dk_t\, dz'\, dz''$$

where

$$k_z = \sqrt{k^2 - |k_t|^2} \qquad (4\text{-}35)$$

C⁻· .ider the second term integral over dk_t dz' in Equation (4-34)

$$I_A = \int_{-\infty}^{+\infty} \int dk_t\, e^{ik_t \cdot (r - r_o)} \int_o^L \frac{e^{ik_z(|z - z'| + |z' - z_o|)}}{-4k_z^2} dz' \qquad (4\text{-}36)$$

We select the origin, in rectangular coordinates, at the leading edge of the slab of width L, as in Figure 4-2. Then $z_o < z' < z$, and the integral over z' above will be

$$\int_o^L \frac{e^{ik_z(z - z_o)}}{-4k_z^2} dz' = - \frac{L}{4} \frac{e^{ik_z(z - z_o)}}{k_z^2} \qquad (4\text{-}37)$$

Therefore

$$I_A = - \frac{L}{4} \iint \frac{e^{ik_z(z - z_o)}}{k_z^2} e^{ik_t \cdot (r - r_o)} dk_t$$

$$= \iint \frac{e^{ik_z(z - z_o)}}{2ik_z} \left[\frac{L}{2ik_z}\right] e^{ik_t \cdot (r - r_o)} dk_t \qquad (4\text{-}38)$$

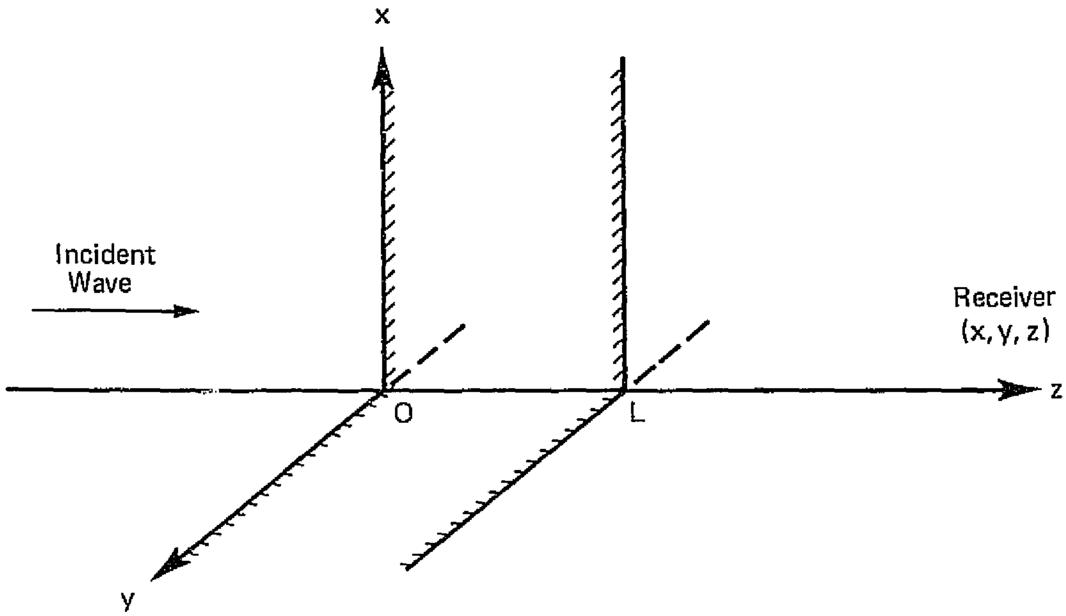


Figure 4-2. Volume Slab in Rectangular Coordinates

Consider the third term integral over $dk_t\ dz'\ dz''$ in Equation (4-34)

$$I_B = \int \frac{e^{ik_z(|z-z'|+|z'-z''|+|z''-z_o|)}}{-8ik_z^3}\ e^{ik_t \cdot (r-r_o)}\ dk_t\ dz'\ dz'' \qquad (4\text{-}39)$$

With $z_o < z',\ z'' < z$,

$$(|z-z'| + |z'-z''| + |z''-z_o|) = (z-z_o) + (|z'-z''| - z' + z'')$$

and

$$I_B = \int_{-\infty}^{+\infty} \int dk_t\ e^{ik_t \cdot (r-r_o)}\ \frac{e^{ik_z(z-z_o)}}{-8ik_z^3} \int_o^L \int_o^L dz'\ dz''\ e^{ik_z(|z'-z''|-z'+z'')} \qquad (4\text{-}40)$$

The double integral over $dz'\ dz''$ in Equation (4-40) is evaluated in Appendix C, Equation (C-2). The results are given by Equation (C-5). Then

$$I_B = \int \frac{e^{ik_z(z-z_o)}}{2ik_z}\left[\frac{L^2}{-8k_z^2} + \frac{L}{i8k_z^3} - \frac{1}{16k_z^4} + \frac{e^{ik_z 2L}}{16k_z^4}\right] e^{ik_t \cdot (r-r_o)}\ dk_t \qquad (4\text{-}41)$$

Finally, the resulting integrals I_A and I_B, from Equations (4-38) and (4-41) are substituted back into Equation (4-34) to give

$$\langle\Psi(r, r_o)\rangle = \frac{1}{(2\pi)^2} \iint \left\{ \frac{e^{ik_z(z - z_o)}}{2ik_z} \left[1 - k^2 \sigma \rho \frac{L}{2ik_z} + k^4 \sigma^2 \rho^2 \left(\frac{L^2}{-8k_z^2} + \frac{L}{i8k_z^3} - \frac{1}{16k_z^4} + \frac{e^{ik_z 2L}}{16k_z^4} \right) \right] \right\} e^{ik_t \cdot (r - r_o)}\, dk_t \tag{4-42}$$

From Equation (4-18b) the term in brackets $\{\ \}$ above is seen to be the inverse Fourier transform of $\langle\Psi(r, r_o)\rangle$, i.e. $\langle\widetilde{\Psi}(z - z_o, k_t)\rangle$. Let

$$\epsilon = \frac{\sigma\rho}{2}\, ; \quad \epsilon^2 = \frac{\sigma^2 \rho^2}{4} \tag{4-43}$$

Then

$$\langle\widetilde{\Psi}(z - z_o, k_t)\rangle = \frac{e^{ik_z(z - z_o)}}{2ik_z} \left[1 - \frac{k^2}{ik_z} L\epsilon - \frac{k^4}{2k_z^2} L^2 \epsilon^2 + \frac{k^4}{2ik_z^3} L\epsilon^2 - \frac{1}{4}\left(\frac{k}{k_z}\right)^4 \epsilon^2 + \frac{1}{4}\left(\frac{k}{k_z}\right)^4 \epsilon^2 e^{ik_z 2L} \right] \tag{4-44}$$

Equations (4-42) and (4-44) represent the expansion solutions for the Fourier transform pair of the mean field of a semi-infinite slab of width L, to $O(\epsilon^2)$. We now proceed to develop an asymptotic expression for $\langle\Psi(r, r_o)\rangle$ for large r, i.e. with the receiver in the far field of the incident and scattered waves.

(b) Far Field Solution for the Mean Field

Consider the integral expression for $\langle\Psi(r, r_o)\rangle$ developed to $O(\sigma^2)$ in Equation (4-42). We re-express the equation as

$$\langle\Psi(r, r_o)\rangle = \frac{1}{(2\pi)^2} \iint B(k_z)\, e^{ik_t \cdot (r - r_o)}\, dk_t \tag{4-45}$$

where

$$B(k_z) = \frac{e^{ik_z(z - z_o)}}{2ik_z} \left[1 - k^2 \sigma \rho \frac{L}{2ik_z} + \frac{k^4 \sigma^2}{k_z^2} \rho^2 \left(-\frac{L^2}{8} + \frac{L}{i8k_z} - \frac{1}{16k_z^2} + \frac{e^{ik_z 2L}}{16k_z^2} \right) \right] \tag{4-46}$$

and

$$k_z = \sqrt{k^2 - |k_t|^2}\ ,\ dk_t = dk_x\, dk_y$$

The field quantities are now represented by angular displacement parameters, as defined by Figure 4-3. From Figure 4-3a

$$(x - x_o) = R \cos\theta \tag{4-47a}$$

$$(y - y_o) = R \sin\theta \tag{4-47b}$$

and from Figure 4-3b

$$k_x = k_t \cos\phi \tag{4-48a}$$

$$k_y = k_t \sin\phi \tag{4-48b}$$

$$dk_x\, dk_y = k_t\, dk_t\, d\phi \tag{4-48c}$$

Note also that

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy = \int_{o}^{\infty} k_t\, dk_t \int_{-\pi}^{+\pi} d\phi \tag{4-49}$$

Therefore

$$\begin{aligned} k_t \cdot (r - r_o) &= k_x(x - x_o) + k_y(y - y_o) \\ &= k_t \cos\phi\, R \cos\theta + k_t \sin\phi\, R \sin\theta \\ &= R\, k_t [\cos\phi \cos\theta + \sin\phi \sin\theta] \\ &= R\, k_t \cos(\phi - \theta) \end{aligned} \tag{4-50}$$

The integral over dk_t in Equation (4-45), called I, is then

$$I = \int_{o}^{\infty} k_t\, dk_t \int_{-\pi}^{+\pi} d\phi\, B(k_t)\, e^{i R k_t \cos(\phi - \theta)} \tag{4-51}$$

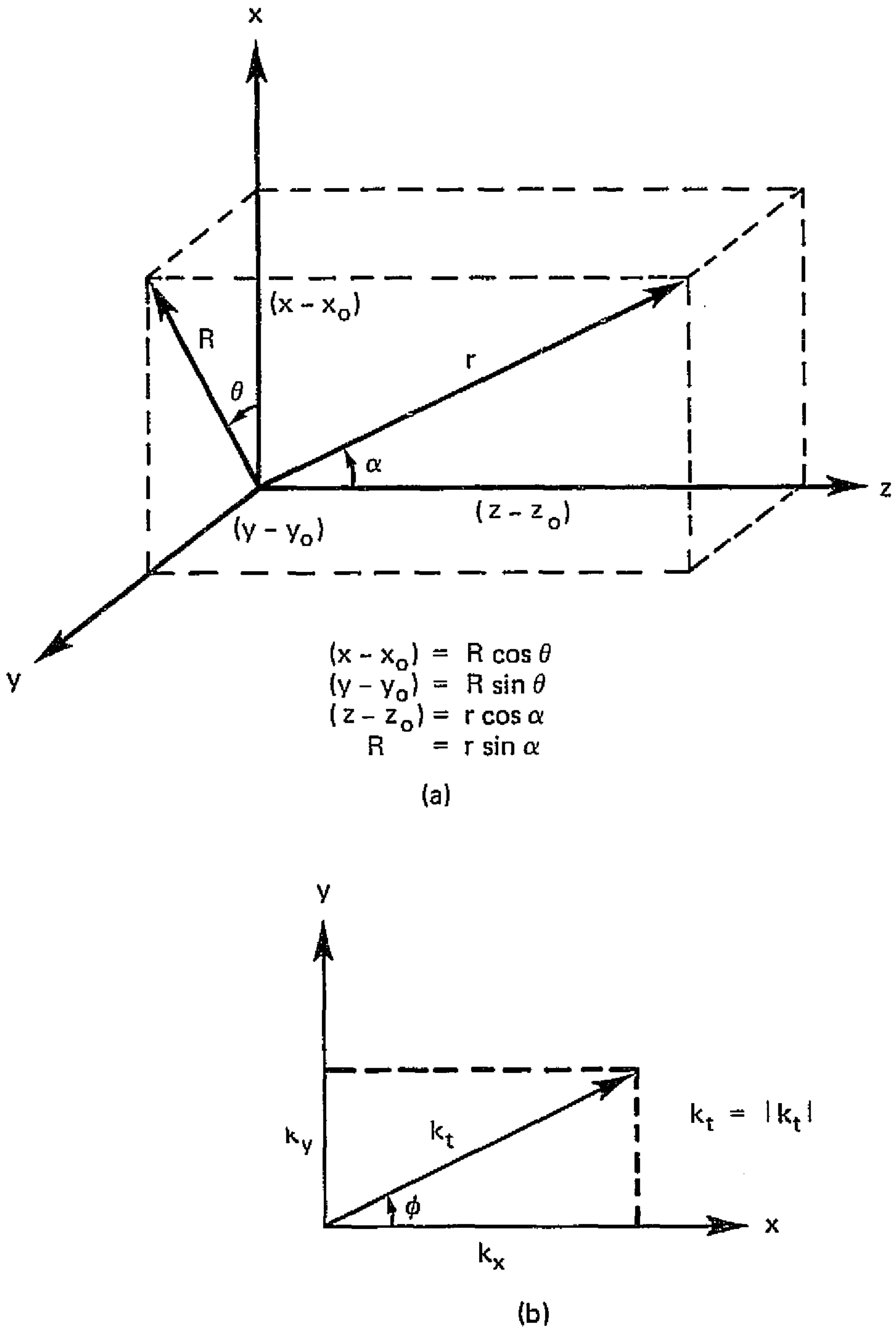


Figure 4-3. Angular Displacement Representation for Field Quantities

With $k_z = \sqrt{k^2 - k_t^2}$, $B(k_z)$ is re-expressed as

$$B(k_t) = \frac{e^{i\sqrt{k^2 - k_t^2}\,(z - z_o)}}{2i\sqrt{k^2 - k_t^2}} \left[1 + i\,\frac{k^2 \sigma \rho L}{2\sqrt{k^2 - k_t^2}} + \frac{k^4 \sigma^2 \rho^2}{(k^2 - k_t^2)} \left(-\frac{L^2}{8} - i\,\frac{L}{8\sqrt{k^2 - k_t^2}} - \frac{1 - e^{i2L\sqrt{k^2 - k_t^2}}}{16(k^2 - k_t^2)} \right) \right] \qquad (4\text{–}52)$$

Equation (4–51) is rearranged as

$$I = \int_0^\infty k_t\, B(k_t) \left[\int_{-\pi}^{+\pi} e^{i R\, k_t \cos(\phi - \theta)}\, d\phi \right] dk_t \qquad (4\text{–}53)$$

The term in brackets is of the form of the zero-th order Bessel function (Abramowitz, 1964),

$$J_o(z) = \frac{1}{\pi} \int_0^\pi e^{i z \cos \beta}\, d\beta$$

or

$$J_o(z) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i z \cos \beta}\, d\beta \qquad (4\text{–}54)$$

Let

$$\cos(\phi - \theta) = \cos\beta\,,\ \therefore \beta = \phi - \theta\,,\ d\beta = d\phi \qquad (4\text{–}55)$$

The integral limits are

$$\phi = +\pi \rightarrow \beta = \pi - \theta$$

$$\phi = -\pi \rightarrow \beta = -\pi - \theta$$

Since the integrand is an even, periodic function

$$\int_{-\pi}^{+\pi} d\phi \rightarrow \int_{-\pi - \theta}^{\pi - \theta} d\beta \rightarrow \int_{-\pi}^{+\pi} d\beta$$

Therefore the term in brackets of I, Equation (4-53), is

$$\left[\int_{-\pi}^{\pi} e^{i R k_t \cos(\phi - \theta)} d\phi\right] = \int_{-\pi}^{+\pi} e^{i R k_t \cos\beta} d\beta \tag{4-56}$$

$$= (2\pi) J_o(k_t R)$$

and

$$= 2\pi \int_o^\infty k_t B(k_t) J_o(k_t R) dk_t \tag{4-57}$$

Substituting I above back into Equation (4-45)

$$\langle\Psi(r, r_o)\rangle = \frac{1}{2\pi} \int_o^\infty k_t B(k_t) J_o(k_t R) dk_t \tag{4-58}$$

This integral can be converted to one extending from $-\infty$ to $+\infty$ by expressing the Bessel function in terms of Hankel functions (Tyras, 1969)

$$J_n(z) = \frac{1}{2} [H_n^{(1)}(z) + H_n^{(2)}(z)] \tag{4-59}$$

$$H_n^{(2)}(z e^{-i\pi}) = -e^{in\pi} H_n^{(1)}(z) \tag{4-60}$$

Therefore

$$J_o(k_t R) = \frac{1}{2} [H_o^{(1)}(k_t R) + H_o^{(2)}(k_t R)] \tag{4-61}$$

and

$$H_o^{(2)}(k_t R e^{-i\pi}) = -H_o^{(1)}(k_t R) \tag{4-62}$$

Since $k_t R$ is real, the $H_o^{(2)}(k_t R e^{-i\pi})$ term is equivalent to the $H_o^{(1)}(k_t R)$ term along the negative real axis, 0 to $-\infty$. Therefore, from Equation (4-58)

$$\langle\Psi(r, r_o)\rangle = \frac{i}{4\pi} \int_{-\infty}^{+\infty} k_t B(k_t) H_o^{(1)}(k_t R) dk_t \tag{4-63}$$

The integral of Equation (4-63) is evaluated asymptotically in Appendix D for large r by; (a) expressing the Hankel function in its asymptotic approximation, (b) employing appropriate coordinate transformations, (c) evaluation of saddle points, and (d) determination of a steepest descent (constant phase) path of integration.

Application of the results of Appendix D, Equation (D-36), to Equation (4-63) gives the far field expansion for the mean field,

$$\langle \Psi(r) \rangle \simeq \frac{e^{ikr}}{4i\pi r} \left[1 + i \frac{k \sigma \rho L}{2 \cos \alpha} - \frac{k^2 \sigma^2 \rho^2 L^2}{8 \cos^2 \alpha} - i \frac{k \sigma^2 \rho^2 L}{8 \cos^3 \alpha} - \frac{\sigma^2 \rho^2}{16 \cos^4 \alpha} + \frac{\sigma^2 \rho^2}{16 \cos^4 \alpha} e^{i2kL \cos \alpha} \right] \tag{4-64}$$

where α is the angle between the +z axis and the receiver point r, as seen in Figure 4-3.

The far field expansion contains one first order term, and secular terms in L and L^2. A comparison of the far field expansion above with the one-dimensional result, Equation (3-64), shows that the terms are similar, and with $\alpha = 0$, the forward scatter direction, the terms are identical in form. Note also that for $L = 0$,

$$\langle \Psi(r) \rangle = \frac{e^{ikr}}{4i\pi r} \tag{4-65}$$

which is the far field incident wave approximation.

(c) Total Intensity Expansion

The total intensity

$$\langle |\Psi(r, r_o)|^2 \rangle = \langle \Psi(r, r_o) \Psi^*(r, r_o) \rangle \tag{4-66}$$

is now evaluated, to $O(\sigma^2)$, for the integral field representation developed in Section 4-1. From Equation (4-9)

$$\Psi(r, r_o) = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} G(r, r_n) G_o(r_n, r_o) + k^4 \sigma^2 \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G(r, r_n) G(r_p, r_o) G(r_n, r_p)$$

the complex conjugate is found

$$\Psi^*(r, r_o) = G_o^*(r, r_o) - k^2 \sigma^* \sum_{n=1}^{N} G^*(r, r_n) G_o^*(r_n, r_o)$$

$$+ k^4 (\sigma^2)^* \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G^*(r, r_n) G^*(r_p, r_o) G^*(r_n, r_p) \tag{4-67}$$

The product is

$$\Psi(r, r_o) \Psi^*(r, r_o) = G_o G_o^* - k^2 \sigma G_o^* \sum_{n=1}^{N} G(r, r_n) G_o(r_n, r_o)$$

$$+ k^4 \sigma^2 G_o^* \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G(r, r_n) G(r_p, r_o) G(r_n, r_p)$$

$$- k^2 \sigma^* G_o \sum_{n=1}^{N} G^*(r, r_n) G_o^*(r_n, r_o) \tag{4-68}$$

$$+ k^4 (\sigma^2)^* G_o \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G^*(r, r_n) G^*(r_p, r_o) G^*(r_n, r_p)$$

$$+ k^4 \sigma \sigma^* \sum_{n=1}^{N} G(r, r_n) G_o(r_n, r_o) \sum_{n'=1}^{N} G^*(r, r_{n'}) G_o^*(r_n, r_o)$$

Taking the expectation value of Equation (4-68)

$$\langle \Psi(r, r_o) \Psi^*(r, r_o) \rangle = \langle |\Psi(r, r_o)|^2 \rangle = \frac{1}{(4\pi)^2 |r - r_o|^2}$$

$$- k^2 \sigma G_o^* \left\langle \sum_{n=1}^{N} G(r, r_n) G_o(r_n, r_o) \right\rangle \tag{4-69}$$

$$+ k^4 \sigma^2 G_o^* \left\langle \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G(r, r_n) G(r_p, r_o) G(r_n, r_p) \right\rangle$$

$$- k^2 \sigma^* G_o \left\langle \sum_{n=1}^{N} G^*(r, r_n)\, G_o^*(r_n, r_o) \right\rangle$$

$$+ k^4 (\sigma^2)^* G_o \left\langle \sum_{n=1}^{N} \sum_{p=1}^{N}{}' G^*(r, r_n)\, G^*(r_p, r_o)\, G^*(r_n, r_p) \right\rangle \tag{4-69 (Cont.)}$$

$$+ k^4 \sigma \sigma^* \left\langle \sum_{n=1}^{N} G(r, r_n)\, G_o(r_n, r_o) \sum_{p=1}^{N} G^*(r, r_p)\, G_o^*(r_p, r_o) \right\rangle$$

The expectation values of the summation terms were evaluated in Appendix B. The first and third (single) summations are found from Equation (B-25), the second and fourth (double) summations from Equation (B-33), and the last (unprimed) summation from Equation (B-31). The resulting total intensity is then

$$\begin{aligned}
\langle |\Psi(r, r_o)|^2 \rangle = {} & \frac{1}{(4\pi)^2 |r - r_o|^2} - k^2 \sigma G_o^* \rho \int_V G(r, r')\, G_o(r', r_o)\, dr' \\
& - k^2 \sigma^* G_o \rho \int_V G^*(r, r')\, G_o^*(r', r_o)\, dr' \\
& + k^4 \sigma^2 G_o^* \rho^2 \int_V \int G(r, r')\, G(r'', r_o)\, G(r', r'')\, dr'\, dr'' \\
& + k^4 (\sigma^2)^* G_o \rho^2 \int_V \int G^*(r, r')\, G^*(r'', r_o)\, G^*(r', r'')\, dr'\, dr'' \\
& + k^4 \sigma \sigma^* \Big[\rho^2 \int_V \int G(r, r')\, G_o(r', r_o)\, G^*(r, r'')\, G_o^*(r'', r_o)\, dr'\, dr'' \\
& + \rho \int_V G(r, r')\, G_o(r', r_o)\, G^*(r, r')\, G_o^*(r', r_o)\, dr' \Big]
\end{aligned} \tag{4-70}$$

The solution of Equation (4-70) is accomplished by application of the transverse Fourier transform to each term, as was done for $\langle\Psi(r, r_o)\rangle$ in Section 4-1(a). With each Green's function replaced by its transform from Equation (4-18b)

$$G(r, r') = \frac{1}{(2\pi)^2} \iint \widetilde{G}(z - z', k_t)\, e^{ik_t \cdot (r - r')}\, dk_t$$

and evaluated, a solution of the form of Equation (4-42) is obtained. A comparison of Equation (4-70) with Equation (4-15) shows that the first two single integral terms for the intensity are identical in form to the single integral term for the mean, and the first two double integral terms are likewise the same as the double integral mean term. Thus the results of the mean field integral evaluations can be directly applied to solve the first five terms of Equation (4-70). Since the far field solution is desired for this study, the explicit development of the intensity expansion for the far field will be considered.

Before solving for the far field total intensity, however, the coherent intensity will be developed and compared with the results given above, and the last integral in Equation (4-70) will be seen to represent an incoherent intensity term.

(d) Coherent Intensity

The coherent intensity

$$|\langle\Psi(r, r_o)\rangle|^2 = \langle\Psi(r, r_o)\rangle\langle\Psi^*(r, r_o)\rangle \tag{4-71}$$

is found from Equation (4-15)

$$\langle\Psi(r, r_o)\rangle = G_o(r, r_o) - k^2\,\sigma\,\rho \int_V G(r, r')\, G(r', r_o)\, dr'$$

$$+ k^4\,\sigma^2\,\rho^2 \int_V \int G(r, r')\, G(r', r'')\, G(r'', r_o)\, dr'\, dr''$$

Taking the complex conjugate

$$\langle\Psi^*(r, r_o)\rangle = G_o^*(r, r_o) - k^2\,\sigma^*\,\rho \int_V G^*(r, r')\, G^*(r, r_o)\, dr'$$

$$+ k^4(\sigma^2)^*\,\rho^2 \int_V \int G^*(r, r')\, G^*(r', r'')\, G^*(r'', r_o)\, dr'\, dr'' \tag{4-72}$$

The coherent intensity is the product

$$|\langle\Psi(r, r_o)\rangle|^2 = \langle\Psi(r, r_o)\rangle\langle\Psi^*(r, r_o)\rangle = G_o G_o^* - k^2 \sigma \rho G_o^* \int_V G(r, r') G(r', r_o) dr'$$

$$-k^2 \sigma^* \rho G_o \int_V G^*(r, r') G^*(r', r_o) dr'$$

$$+ k^4 \sigma^2 \rho^2 G_o^* \int_V \int G(r, r') G(r', r'') G(r'', r_o) dr' dr'' \qquad (4\text{-}73)$$

$$+ k^4 (\sigma^2)^* \rho^2 G_o \int_V \int G^*(r, r') G^*(r', r'') G^*(r'', r_o) dr' dr''$$

$$+ k^4 \sigma \sigma^* \rho^2 \int_V \int G(r, r') G(r', r_o) G^*(r, r'') G^*(r'', r_o) dr' dr''$$

Comparison of this result with Equation (4-70) for the total intensity shows that all terms are identical except for the last term in the total intensity, which is not present in the coherent expansion. Therefore, to $O(\sigma^2)$

$$\langle|\Psi(r, r_o)|^2\rangle = |\langle\Psi(r, r_o)\rangle|^2 + k^4 \sigma \sigma^* \rho \int_V G(r, r') G_o(r', r_o) G^*(r, r') G^*(r', r_o) dr' \qquad (4\text{-}74)$$

The last term of Equation (4-74) is of interest, since it represents an incoherent intensity term, I_{inc}, in the total intensity expansion. I_{inc} will be evaluated in the next section by asymptotic expansion for large r.

(e) Incoherent Intensity

The incoherent intensity, from Equation (4-74), is

$$I_{inc} = k^4 \sigma \sigma^* \rho \int_V G(r, r') G(r', r_o) G^*(r, r'') G^*(r'', r_o) dr' \qquad (4\text{-}75)$$

with $r' = r''$.

We proceed to solve I_{inc} by application of the transverse Fourier transform to Equation (4-75), as was accomplished in Section 4-1(a) for the mean field. Replacing each Green's function in Equation (4-75) by its inverse Fourier transform, Equation (4-18b)

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^8} \iint [\tilde{G}(z - z', k_t) e^{ik_t \cdot (r - r')} dk_t$$

$$\tilde{G}(z' - z_o, k_t') e^{ik_t' \cdot (r' - r_o)} dk_t' \tag{4-76}$$

$$\tilde{G}(z - z'', k_t'') e^{ik_t'' \cdot (r - r'')} dk_t''$$

$$\tilde{G}(z'' - z_o, k_t''') e^{ik_t''' \cdot (r'' - r_o)} dk_t'''] \, dr'$$

The inverse functions are replaced by their one-dimensional equivalent solutions, from Equation (4-20)

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^8} \int \frac{e^{ik_z|z - z'|}}{2ik_z} \frac{e^{ik_z'|z' - z_o|}}{2ik_z'} \frac{e^{-ik_z''|z - z''|}}{-2ik_z''} \frac{e^{-ik_z'''|z'' - z_o|}}{-2ik_z'''} \tag{4-77}$$

$$e^{i[k_t \cdot (r - r') + k_t' \cdot (r' - r_o) + k_t''(r - r'') + k_t''' \cdot (r'' - r_o)]} \quad dr' \, dk_t \, dk_t' \, dk_t'' \, dk_t'''$$

The k_t exponential can be expressed as

$$e^{-ir' \cdot (k_t - k_t')} e^{-ir'' \cdot (k_t'' - k_t''')} \quad e^{i[(k_t + k_t'') \cdot r - (k_t' + k_t''') \cdot r_o]} \tag{4-78}$$

Since $r' = r''$

$$\int e^{-ir' \cdot [(k_t + k_t'') - (k_t' + k_t''')]} dr_t = (2\pi)^2 \, \delta[(k_t + k_t'') - (k_t' + k_t''')] \tag{4-79}$$

Therefore

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^4} \int \frac{e^{i[k_z|z - z'| + k_z'|z' - z_o| - k_z''|z - z'| - k_z'''|z' - z_o|]}}{16 \, k_z \, k_z' \, k_z'' \, k_z'''} \tag{4-80}$$

$$e^{i[(k_t + k_t'') \cdot r - (k_t' + k_t''') \cdot r_o]} \, \delta(k_t - k_t') \, \delta(k_t'' - k_t''') \quad dk_t \, dk_t' \, dk_t'' \, dk_t''' \, dz'$$

The delta functions evaluate the k_t integrals at $k_t = k_t'$ and $k_t'' = k_t'''$

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^4} \int \frac{e^{i[k_z(z - z') + k_z|z' - z_o| - k_z''|z - z'| - k_z''|z' - z_o|]}}{16 k_z^2 k_z''^2} \quad (4\text{–}81)$$

$$e^{i[(k_t + k_t'') \cdot (r - r_o)]} \, dk_t \, dk_t'' \, dz'$$

With $z_o < z' < z$

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^4} \int \frac{e^{ik_z(z - z_o)} e^{-ik_z''(z - z_o)}}{16 k_z^2 k_z''^2} e^{i[(k_t + k_t'') \cdot (r - r_o)]} \, dk_t \, dk_t'' \, dz' \quad (4\text{–}82)$$

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^4} \int \frac{e^{ik_z(z - z_o)} e^{-ik_z''(z - z_o)}}{16 k_z^2 k_z''^2} e^{i[(k_t + k_t'') \cdot (r - r_o)]} \, dk_t \, dk_t'' \int_o^L dz' \quad (4\text{–}83)$$

Evaluating the integral over dz'

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{(2\pi)^4} L \int \frac{e^{ik_z(z - z_o)} e^{-ik_z''(z - z_o)}}{16 k_z^2 k_z''^2} e^{i[(k_t + k_t'') \cdot (r - r_o)]} \, dk_t \, dk_t'' \quad (4\text{–}84)$$

Rearranging terms

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho}{16(2\pi)^4} L \int \frac{e^{ik_z(z - z_o)}}{k_z^2} e^{ik_t \cdot (r - r_o)} \, dk_t \int \frac{e^{-ik_z'(z - z_o)}}{k_z'^2} e^{ik_t' \cdot (r - r_o)} \, dk_t' \quad (4\text{–}85)$$

The two integrals in Equation (4-85) are seen to be transverse Fourier transforms of conjugate functions, i.e. (see Eq. 4-17b)

$$\int \frac{e^{ik_z(z - z_o)}}{k_z^2} e^{ik_t \cdot (r - r_o)} \, dk_t = (2\pi)^2 f(r - r_o) \quad (4\text{–}86a)$$

and

$$\int \frac{e^{-ik_z'(z - z_o)}}{k_z'^2} e^{ik_t' \cdot (r - r_o)} \, dk_t' = (2\pi)^2 f^*(r - r_o) \quad (4\text{–}86b)$$

Therefore, from Equation (4-85)

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho L}{16} f(r - r_o) f^*(r - r_o) \tag{4-87}$$

Since

$$f(r - r_o) f^*(r - r_o) = |f(r - r_o)|^2$$

Then

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho L}{16} |f(r - r_o)|^2 \tag{4-88}$$

The function $f(r - r_o)$ is now evaluated in the far field by the method of stationary phase. With

$$f(r - r_o) = \frac{1}{(2\pi)^2} \iint \frac{e^{ik_z(z - z_o)}}{k_z^2} e^{ik_t \cdot (r - r_o)} dk_t \tag{4-89}$$

The above integral is seen to be identical to the integral in Equation (4-45), with

$$B(k_z) = \frac{e^{ik_z(z - z_o)}}{k_z^2} \tag{4-90}$$

which was evaluated in the far field in Section 4-1(b), and Appendix D.

The method is applied here to $f(r - r_o)$, with only the major steps listed. From Equation (D-4)

$$f(r - r_o) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8 \pi^3 R}} \int_{-\infty}^{+\infty} \sqrt{k_t} B(k_t) e^{ik_t R} dk_t \tag{4-91}$$

where k_t and R are as defined in Figure 4-3. With

$$k_t = k \sin W$$

$$k_z = k \cos W$$

$$\sqrt{k_t} = \sqrt{k \sin W}$$

$$e^{ik_t R} = e^{ikR \sin W} = e^{ikr \sin W \sin \alpha}$$

$$B(k_t) = \frac{e^{ikr \sin W \sin \alpha}}{k^2 \cos^2 W}$$

$$dk_t = k \cos W \, dW$$

Therefore

$$\begin{aligned} f(r - r_o) &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8\pi^3 R}} \int \frac{\sqrt{k}\sqrt{\sin W}\, e^{ikr \cos (W - \alpha)}}{k^2 \cos^2 W} k \cos W \, dW \\ &= \sqrt{\frac{1}{8\pi^3 k R}}\, e^{-i\frac{\pi}{4}} \int \frac{\sqrt{\sin W}\, e^{ikr \cos (W - \alpha)}}{\cos W} dW \end{aligned} \quad (4\text{-}92)$$

The SDP determined in Appendix D is applicable for the above integral also, and the asymptotic representation given by Equation (D-27) can be applied. With

$$\hat{q}(W_s) = k, \; W_s = \alpha, \; |\hat{q}''(z)| = k$$

$$f(W_s) = \frac{\sqrt{\sin \alpha}}{\cos \alpha}$$

Therefore, from Equation (D-27), for large r

$$\begin{aligned} f(r - r_o) &\simeq \sqrt{\frac{1}{8\pi^3 k R}}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2\pi}{rk}} \frac{\sqrt{\sin \alpha}}{\cos \alpha} e^{irk + i\frac{\pi}{4}} \\ &= \frac{e^{ikr}}{2\pi k r \cos \alpha} \end{aligned} \quad (4\text{-}93)$$

and

$$|f(r - r_o)|^2 = [4\pi^2 k^2 r^2 \cos^2 \alpha]^{-1} \quad (4\text{-}94)$$

Then, from Equation (4-88)

$$I_{inc} = \frac{k^4 \sigma \sigma^* \rho L}{64 \pi^2 k^2 r^2 \cos^2 \alpha}$$

with

$$\sigma \sigma^* = (\sigma_r^2 + \sigma_i^2)$$

$$I_{inc} = \frac{k^2 (\sigma_r^2 + \sigma_i^2) \rho L}{64 \pi^2 r^2 \cos^2 \alpha} \qquad (4\text{-}95)$$

(f) Far Field Solution for Total Intensity

The total intensity $\langle |\Psi(r, r_o)|^2 \rangle$ was found, to $O(\sigma^2)$, in Section (d) as, Equation (4-74)

$$\langle |\Psi(r, r_o)|^2 \rangle = \langle \Psi(r, r_o) \rangle \langle \Psi^*(r, r_o) \rangle + I_{inc}$$

where I_{inc} is the incoherent intensity term given by Equation (4-75) and by Equation (4-95) for large r.

The far field approximation for the total intensity can now be developed from the far field expansion to the mean field, Equation (4-64). The complex conjugate of Equation (4-64) is

$$\langle \Psi^*(r) \rangle \simeq \frac{e^{-ikr}}{-4i \pi r} \left[1 - i \frac{k \sigma^* \rho L}{2 \cos \alpha} - \frac{k^2 (\sigma^2)^* \rho^2 L^2}{8 \cos^2 \alpha} + i \frac{k (\sigma^2)^* \rho^2 L}{8 \cos^3 \alpha} - \frac{(\sigma^2)^* \rho^2}{16 \cos^4 \alpha} + \frac{(\sigma^2)^* \rho^2}{16 \cos^4 \alpha} e^{-i2kL \cos \alpha} \right] \qquad (4\text{-}96)$$

The coherent intensity, to $O(\sigma^2)$ is then

$$\langle \Psi(r) \rangle \langle \Psi^*(r) \rangle = \frac{1}{16 \pi^2 r^2} \left[1 + i \frac{k \sigma \rho L}{2 \cos \alpha} - \frac{k^2 \sigma^2 \rho^2 L^2}{8 \cos^2 \alpha} - i \frac{k \sigma^2 \rho^2 L}{8 \cos^3 \alpha} - \frac{\sigma^2 \rho^2}{16 \cos^4 \alpha} \right.$$

$$+ \frac{\sigma^2 \rho^2}{16 \cos^4 \alpha} e^{i2kL \cos \alpha} - i \frac{k \sigma^* \rho L}{2 \cos \alpha}$$

$$+ \frac{k^2 \sigma \sigma^* \rho^2 L^2}{4 \cos^2 \alpha} - \frac{k^2 (\sigma^2)^* \rho^2 L^2}{8 \cos^2 \alpha} + i \frac{k (\sigma^2)^* \rho^2 L}{8 \cos^3 \alpha}$$

$$\left. - \frac{(\sigma^2)^* \rho^2}{16 \cos^4 \alpha} + \frac{(\sigma^2)^* \rho^2}{16 \cos^4 \alpha} e^{-i2kL \cos \alpha} \right]$$

Or

$$\langle\Psi(r)\rangle\langle\Psi^*(r)\rangle = \frac{1}{16\pi^2 r^2}\Bigg[1 + i\,\frac{k(\sigma-\sigma^*)\rho L}{2\cos\alpha} + \frac{k^2\rho^2 L^2}{4\cos^2\alpha}\left(\sigma\sigma^* - \frac{\sigma^2+(\sigma^2)^*}{2}\right) - i\,\frac{k(\sigma^2-(\sigma^2)^*)\rho^2 L}{8\cos^3\alpha} - \frac{(\sigma^2+(\sigma^2)^*)\rho^2}{16\cos^4\alpha} + \frac{\rho^2}{16\cos^4\alpha}\left(\sigma^2 e^{i2kL\cos\alpha} + (\sigma^2)^* e^{-i2kL\cos\alpha}\right)\Bigg] \tag{4-97}$$

Rearranging terms

$$\langle\Psi(r)\rangle\langle\Psi^*(r)\rangle = \frac{1}{16\pi^2 r^2}\Bigg[1 - \frac{k\sigma_i\rho L}{\cos\alpha} + \frac{k^2\sigma_i^2\rho^2 L^2}{2\cos^2\alpha} + \frac{k\sigma_r\sigma_i\rho^2 L}{2\cos^2\alpha} + \frac{\rho^2\sigma_r\sigma_i}{16\cos^4\alpha}\sin(2kL\cos\alpha) - \frac{(\sigma_r^2-\sigma_i^2)\rho^2}{8\cos^4\alpha} + \frac{\rho^2(\sigma_r^2-\sigma_i^2)}{8\cos^4\alpha}\cos(2kL\cos\alpha)\Bigg] \tag{4-98}$$

The addition of the far field result for I_{inc}, from Equation (4-95), results in the total intensity far field expansion

$$\langle|\Psi(r)|^2\rangle = \frac{1}{16\pi^2 r^2}\Bigg[1 - \frac{k\sigma_i\rho L}{\cos\alpha} + \frac{k^2(\sigma_r^2+\sigma_i^2)\rho L}{4\cos^2\alpha} + \frac{k^2\sigma_i^2\rho^2 L^2}{2\cos^2\alpha} + \frac{k\sigma_r\sigma_i\rho^2 L}{2\cos^2\alpha} + \frac{\sigma_r\sigma_i\rho^2}{16\cos^4\alpha}\sin(2kL\cos\alpha) - \frac{(\sigma_r^2-\sigma_i^2)\rho^2}{8\cos^4\alpha} + \frac{(\sigma_r^2-\sigma_i^2)\rho^2}{8\cos^4\alpha}\cos(2kL\cos\alpha)\Bigg] \tag{4-99}$$

For $\alpha = 0$, i.e. the forward scattering direction, Equation (4-99) gives

$$\frac{\langle|\Psi(r)|^2\rangle}{|\Psi_0(r)|^2} = \Bigg\{1 - k\rho\sigma_i L + k^2\rho^2\sigma_i^2\frac{L^2}{2} + \left[\frac{k^2\rho(\sigma_r^2+\sigma_i^2)L}{4}\right] + \frac{k\rho^2\sigma_r\sigma_i L}{2} + \frac{\sigma_r\sigma_i\rho^2}{16}\sin(2kL) - \frac{(\sigma_r^2-\sigma_i^2)\rho^2}{8}(1-\cos 2kL)\Bigg\} \tag{4-100}$$

where the term in brackets [ ] is the incoherent intensity term.

The far field total intensity expansion contains one first order term, and secular terms in L and L^2. A comparison of Equation (4-100) and Equation (3-76), the one-dimensional result, shows that all terms of the expansion are identical in form.

Also, for L = 0,

$$\langle |\Psi(r)^2\rangle = \frac{1}{16\,\pi^2\, r^2}$$

which is seen from Equation (4-65) to be the far field intensity of the incident wave.

The applicability of the Foldy approximation to the volume scattering case will now be investigated.

4-2. FOLDY APPROXIMATION FOR A VOLUME OF DISCRETE SCATTERERS

The Foldy approximation to the external field acting on a volume of discrete scatterers is applied here to the volume slab configuration described in the previous section. The mean field $\langle \Psi_F(r, r_o)\rangle$ is developed, and the resulting expansions compared to the mean and intensity expansions derived in the previous section. The development is similar to the one-dimensional slab case presented in Section 3-3.

Consider the wave equation for the spherical wave incident on a volume of discrete scatterers, as described in Figure 4-1, and by Equation (4-3)

$$\left\{\nabla^2 + k^2\left[1 + \sigma \sum_{n=1}^{N} \delta(r - r_n)\right]\right\}\Psi(r, r_o) = 0 \qquad (4\text{-}101)$$

The wave equation and its associated boundary conditions convert to the integral equation

$$\Psi(r, r_o) = G_o(r, r_o) - k^2\,\sigma \sum_{n=1}^{N} \int_V G(r, r')\,\delta(r' - r_n)\,\Psi(r', r_o)\,dr' \qquad (4\text{-}102)$$

where

$$G_o(r, r_o) = \frac{-e^{ik|r - r_o|}}{4\pi|r - r_o|}$$

and

$$G(r, r') = \frac{-e^{ik|r - r'|}}{4\pi|r - r'|}$$

The average of Equation (4-102) is

$$\langle\Psi(r, r_o)\rangle = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} \int_V G(r, r') \langle\delta(r' - r_n) \Psi(r', r_o)\rangle \, dr' \quad (4\text{-}103)$$

The mean of $\delta(r' - r_n) \Psi(r', r_o)$ is

$$\langle\delta(r' - r_n) \Psi(r', r_o)\rangle = \int \delta(r' - r_n) \Psi(r', r_o) \quad f(r_1, r_2, \ldots r_N) \, dr_1, dr_2, \ldots dr_N \quad (4\text{-}104)$$

where

$$f(r_1, r_2, \ldots r_N) = f(r_n)$$

is the joint probability function of the r_n.

With the r_n statistically independent

$$f(r_1, r_2, \ldots r_N) = f(r_1) f(r_2) \ldots f(r_N) = \prod_{p=1}^{N} f(r_p) \quad (4\text{-}105)$$

Therefore Equation (4-104) is

$$\langle \delta(r' - r_n)\,\Psi(r', r_o)\rangle = \int \delta(r' - r_n)\,\Psi(r', r_o) \prod_{p=1}^{N} f(r_p)\,dr_p \qquad (4\text{-}106)$$

The product term can be expressed as

$$\prod_{p=1}^{N} f(r_p)\,dr_p = f(r_n)\,dr_n \prod_{\substack{p=1 \\ p \neq n}}^{N} f(r_p)\,dr_p \qquad (4\text{-}107)$$

Therefore Equation (4-106) is

$$\langle \Psi(r', r_o)\,\delta(r' - r_n)\rangle = \iint \cdots \int \Psi(r', r_o) \prod_{\substack{p=1 \\ p \neq n}}^{N} f(r_p)\,dr_p\,\delta(r' - r_n)\,f(r_n)\,dr_n \qquad (4\text{-}108)$$

Note that

$$\iint \cdots \int \Psi(r', r_o) \prod_{\substack{p=1 \\ p \neq n}}^{N} f(r_p)\,dr_p = \langle \Psi(r', r_o) | r' \rangle \qquad (4\text{-}109)$$

which is the field at r' averaged over all possible configurations of the other N-1 scatterers. Therefore, from Equation (4-108)

$$\langle \Psi(r', r_o)\,\delta(r' - r_n)\rangle = \int \langle \Psi(r', r_o) | r' \rangle\,\delta(r' - r_n)\,f(r_n)\,dr_n \qquad (4\text{-}110)$$

The integral is evaluated by the delta function at $r' = r_n$, i.e.

$$\langle \Psi(r', r_o)\,\delta(r' - r_n)\rangle = \langle \Psi(r', r_o) | r' \rangle\,f(r') \qquad (4\text{-}111)$$

Therefore from Equation (4-103)

$$\langle \Psi(r, r_o)\rangle = G_o(r, r_o) - k^2 \sigma \sum_{n=1}^{N} \int_V G(r, r')\,\langle \Psi(r', r_o) | r' \rangle\,f(r')\,dr' \qquad (4\text{-}112)$$

There are no n dependent terms in the integral, so the integral is evaluated N times by the summation

$$\langle \Psi(r, r_o) \rangle = G_o(r, r_o) - k^2 \sigma N \int_V G(r, r') \langle \Psi(r', r_o) | r' \rangle f(r') \, dr' \qquad (4\text{-}113)$$

The approximation introduced by Foldy consists of replacing $\langle \Psi(r', r_o) | r' \rangle$, the field at the n th scatterer averaged over all possible configurations of the other N-1 scatterers, with $\langle \Psi(r', r_o) \rangle$, the average field at the n th scatterer position, when that scatterer is not present. These averages differ only by a term of the order 1/N, thus for large N, the approximation is reasonable.

Therefore with

$$\langle \Psi(r', r_o) | r' \rangle \simeq \langle \Psi(r', r_o) \rangle \qquad (4\text{-}114)$$

the mean field, from Equation (4-113) will be

$$\langle \Psi_F(r, r_o) \rangle = G_o(r, r_o) - k^2 \sigma N \int_V G(r, r') \langle \Psi(r', r_o) \rangle f(r') \, dr'$$

Also, since the scatterers are uniformly distributed in V,

$$f(r') = \frac{1}{V} \qquad (4\text{-}115)$$

Therefore

$$\langle \Psi_F(r, r_o) \rangle = G_o(r, r_o) - k^2 \sigma \rho \int_V G(r, r') \langle \Psi(r', r_o) \rangle \, dr' \qquad (4\text{-}116)$$

where

$$\rho = \frac{N}{V}$$

The above integral equation represents the mean field for the volume V of discrete scatterers under the Foldy approximation to the external field acting on each scatterer.

(a) Mean Field Expansion

The Foldy approximation integral equation for the mean field $\langle \Psi_F(r, r_0) \rangle$, Equation (4-116), with the addition of the boundary conditions of the scattering volume, is seen to represent a wave equation for $\langle \Psi_F(r, r_0) \rangle$ of the form

$$[\nabla^2 + k^2] \langle \Psi_F(r, r_0) \rangle = -k^2 \sigma \rho \langle \Psi_F(r, r_0) \rangle \quad (4\text{-}117)$$

or

$$[\nabla^2 + k^2(1 + \sigma \rho)] \langle \Psi_F(r, r_0) \rangle = 0 \quad (4\text{-}118)$$

The above equation defines an 'effective' propagation constant, K', for which the mean wave propagates in the scattering medium. The mean wave, with an incident spherical wave, is thus represented in the slab as

$$\langle \Psi_F(r, r_0) \rangle = \frac{-e^{iK'|r - r_0|}}{4\pi |r - r_0|} \quad (4\text{-}119)$$

where

$$K' = k(1 + \sigma \rho)^{1/2} \quad (4\text{-}120)$$

Expanding K' to $O(\sigma^2)$

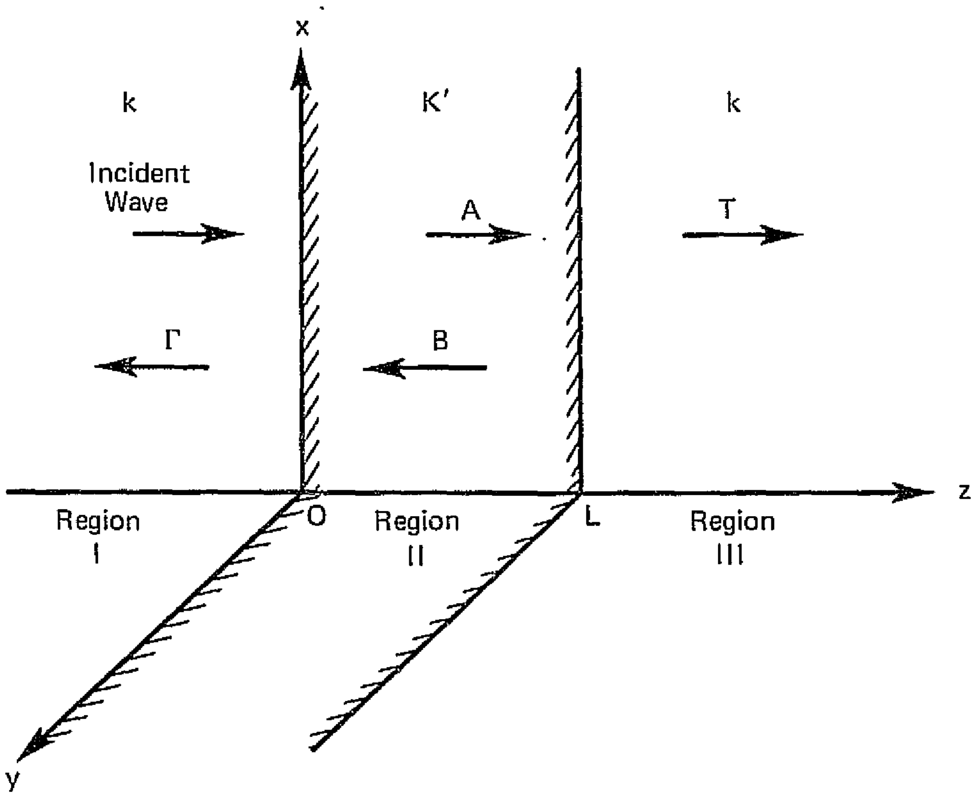
$$K' = k\left[1 + \frac{\sigma\rho}{2} - \frac{(\sigma\rho)^2}{8}\right]$$

or

$$K' = k + k\frac{\sigma\rho}{2} - k\frac{(\sigma\rho)^2}{8} \quad (4\text{-}121)$$

The results thus far are seen to parallel directly with the one-dimensional case, with K as given in Equation (3-102), and λ replaced by ρ.

Under the Foldy approximation for the mean field, the volume slab of discrete scatterers can be replaced by a 'continuous' medium with an effective propagation constant K', as displayed in Figure 4-4. The field expansion for $\langle \Psi_F(r, r_0) \rangle$ in Region III, $z > L$, is desired.



$$K' = k(1 + \sigma\rho)^{1/2}$$

Figure 4-4. Foldy Approximation Representation for Mean Field of a Volume Slab of Discrete Scatterers.

The solution for $\langle \Psi_F(r, r_0) \rangle$ is reduced to a one-dimensional evaluation by considering the transverse Fourier transform in each region, i.e.

$$\langle \widetilde{\Psi}_I(z, k_t) \rangle = \frac{e^{ik_z|z - z_0|}}{2ik_z} + \Gamma e^{-ik_z z}, \quad z < 0 \tag{4-122a}$$

$$\langle \widetilde{\Psi}_{II}(z, k_t) \rangle = A e^{iK'_z z} + B e^{-iK'_z z}, \quad 0 \leqslant z \leqslant L \tag{4-122b}$$

$$\langle \widetilde{\Psi}_{III}(z, k_t) \rangle = T e^{ik_z z}, \quad z > L \tag{4-122c}$$

where

$$k_z = \sqrt{k^2 - k_t^2} \tag{4-123a}$$

$$K'_z = \sqrt{K'^2 - k_t^2} \tag{4-123b}$$

The boundary conditions at z = 0 and z = L are

$$\langle\widetilde{\Psi}_I\rangle = \langle\widetilde{\Psi}_{II}\rangle \quad , z = 0 \tag{4-124a}$$

$$\langle\widetilde{\Psi}_{II}\rangle = \langle\widetilde{\Psi}_{III}\rangle \quad , z = L \tag{4-124b}$$

$$\frac{\partial}{\partial z}\langle\widetilde{\Psi}_I\rangle = \frac{\partial}{\partial z}\langle\widetilde{\Psi}_{II}\rangle \, , \; z = 0 \tag{4-124c}$$

$$\frac{\partial}{\partial z}\langle\widetilde{\Psi}_{II}\rangle = \frac{\partial}{\partial z}\langle\widetilde{\Psi}_{III}\rangle , \; z = L \tag{4-124d}$$

The resulting equations from the boundary conditions are

$$A + B - \Gamma = \frac{e^{-ik_z z_o}}{2ik_z} \tag{4-125a}$$

$$A\, e^{iK'_z L} + B\, e^{-iK'_z L} - T\, e^{ik_z L} = 0 \tag{4-125b}$$

$$A\, K'_z - B\, K'_z + \Gamma\, k_z = \frac{e^{-ik_z z_o}}{2i} \tag{4-125c}$$

$$A\, K'_z\, e^{iK'_z L} - B\, K'_z\, e^{-iK'_z L} - T\, k_z\, e^{ik_z L} = 0 \tag{4-125d}$$

For ease of notation, as before, Equation (4-43), let

$$\epsilon = \frac{\sigma\rho}{2} \;\therefore\; \epsilon^2 = \frac{(\sigma\rho)^2}{4} \tag{4-126}$$

Then

$$K' = k + k\,\epsilon - k\,\frac{\epsilon^2}{2} \tag{4-127}$$

and

$$\begin{aligned} K'_z &= \sqrt{K'^2 - k_t^2} \quad = \sqrt{k^2 + 2\,\epsilon\, k^2 - k_t^2} \\ &= \sqrt{k_z^2 + 2\,\epsilon\, k^2} = k_z \sqrt{1 + 2\,\epsilon\left(\frac{k}{k_z}\right)^2} \end{aligned} \tag{4-128}$$

Expanding the square root term to $O(\epsilon^2)$

$$K_z' = k_z\left[1 + \left(\frac{k}{k_z}\right)^2 \epsilon - \frac{1}{2}\left(\frac{k}{k_z}\right)^4 \epsilon^2\right] \tag{4-129a}$$

$$= g\, k_z \tag{4-129b}$$

where g is the term in brackets of Equation (4-129a).

The expansion for the exponentials in K_z' is

$$e^{iK_z'L} = e^{ik_z\left[1 + \left(\frac{k}{k_z}\right)^2 \epsilon - \frac{1}{2}\left(\frac{k}{k_z}\right)^4 \epsilon^2\right] L}$$

$$= e^{ik_zL}\left[1 + i\,\frac{k^2}{k_z}\,\epsilon\, L - \frac{1}{2}\,\frac{k^4}{k_z^3}\,\epsilon^2 (k_z\, L^2 + i\, L)\right] \tag{4-130a}$$

$$= p\, e^{ik_zL} \tag{4-130b}$$

and

$$e^{-iK_z'L} = p^*\, e^{-ik_zL} \tag{4-131}$$

where p is the term in brackets, Equation (4-130a), and p* denotes its complex conjugate.

The expansions of K_z', $e^{iK_z'L}$, and $e^{-iK_z'L}$ are applied to the four simultaneous equations, Equations (4-125) to give

$$A + B - \Gamma = \frac{e^{-ik_z z_o}}{2ik_z} \tag{4-132a}$$

$$pA + p^*\, e^{-i2k_zL}\, B - T = 0 \tag{4-132b}$$

$$gA - gB + \Gamma = \frac{e^{-ik_z z_o}}{2ik_z} \tag{4-132c}$$

$$gpA - gp^*\, e^{-i2k_zL}\, B - T = 0 \tag{4-132d}$$

The solution for the transmission coefficient T was developed by a Cramer's reduction of the determinent for the four simultaneous equations, Equations (4-132).

The result, in series form to $O(\epsilon^2)$, was found, and the resulting field $\langle\widetilde{\Psi}_{III}(z, k_t)\rangle$ from Equation (4-122c) is available. The inverse transform of the field, in Region III, from Equation (4-122c) and the T expansion is

$$\langle\widetilde{\Psi}_{III}(z, k_t)\rangle = T e^{ik_z z}$$

$$= \frac{e^{ik_z(z - z_o)}}{2ik_z}\left[1 + i\frac{k^2}{k_z} L\epsilon - i\frac{1}{2}\frac{k^4}{k_z^3} L\epsilon^2 - \frac{1}{2}\frac{k^4}{k_z^2} L^2\epsilon^2 - \frac{1}{4}\left(\frac{k}{k_z}\right)^4 \epsilon^2 + \frac{1}{4}\left(\frac{k}{k_z}\right)^4 e^{i2k_z L}\epsilon^2\right] \quad (4\text{-}133)$$

A comparison of the above result with the discrete solution expansion for the mean field inverse Fourier transform, Equation (4-44), shows the two expansions to be exactly equivalent. Thus the Foldy approximation expansion to the mean field for the volume scatter case has been shown, as in the one-dimensional case, to represent the mean field exactly, i.e.

$$\langle\Psi_F(r, r_o)\rangle = \langle\Psi(r, r_o)\rangle \text{ to } O(\sigma^2) \quad (4\text{-}134)$$

The far field expansion for the mean developed in Section 4-1 (b), Equation (4-64), is therefore also applicable to the Foldy expansion result.

(b) Intensity

The results of the previous section have shown that the mean field $\langle\Psi(r, r_o)\rangle$ can be represented, to $O(\sigma^2)$ by $\langle\Psi_F(r, r_o)\rangle$, thus the coherent intensity $\langle\Psi(r, r_o)\rangle \langle\Psi^*(r, r_o)\rangle$ will likewise be represented by the Foldy approximation, to $O(\sigma^2)$.

While the Foldy approximation represents the coherent intensity, it does not, however, account for the incoherent intensity term, found in the far field expansion, Equation (4-95), as

$$I_{inc} = \frac{k^2 (\sigma_r^2 + \sigma_i^2)\rho L}{64\pi^2 r^2 \cos^2\alpha}$$

which is present in the total intensity far field expansion, Equation (4-99).

The Foldy approximation to the mean field is therefore found to be a useful technique in the representation of the mean field and coherent intensity by considering the volume slab of discrete scatterers as a continuous random medium with an 'effective' propagation constant. However, the Foldy approximation does not represent the total intensity, where an incoherent intensity term of $O(\sigma^2)$ was found which was not accounted for in the Foldy expansion.

CHAPTER 5

IMPLICATIONS OF DISCRETE SCATTERING RESULTS TO MULTIPLE SCATTERING IN RAIN

The two previous chapters have investigated the intensity of a wave propagating through a distribution of discrete scatterers by application of random media techniques. In this chapter, the study concludes with a discussion of some implications of the results obtained to the rain scatter channel.

First, some general comments can be made on the mean field and intensity expansions. The analyses of Chapter 3 have demonstrated that either discrete or continuous techniques may be employed for the development of the mean field and intensity expansions, as long as care is taken in the formulations to insure non-overlapping scatterers. The resulting expansions for the mean field were shown to be equivalent to a first order Born series expansion solution, and the limit of convergence for the expansions is equivalent to the Born expansion limit.

The mean field and intensity expansions for a distribution of discrete scatterers, developed in Chapters 3 and 4, have shown that the volume distribution results proceed directly from the one-dimensional distribution findings. The similarity of the volume scattering results to the one-dimensional case was not unexpected, however the term by term agreement, even to the incoherent intensity term, indicates that a one-dimensional evaluation could provide useful quantative as well as qualitative predictions of multiple scattering effects in rain scattering.

Next, the applicability of the isotropic scatterer assumption to the rain scatter problem is discussed. The volume distribution results of Chapter 4 were obtained for isotropic point scatterers, and extension to the rain scatter case is dependent on the validity of this assumption for the case of a distribution of dielectric spheres. The induced or scattered field produced by a dielectric sphere in the presence of an incident electric field can be shown by a direct solution of spherical wave functions of the incident and induced waves to be a vector field with components which depend on the angle of observation to the sphere (Stratton, p. 566). Hence the spherical rain drop cannot, in general, be considered as an 'isotropic scatterer'.

If, however, the discussion of the rain scatterer is limited to the Rayleigh scattering region and to the forward direction only, the scattered field can be shown to be a scalar function of the incident field (Stratton, p. 435), and the results of Chapter 4 can be considered.

Thus, the question of how the multiple scattering results developed in this study relate to the classical 'single scattering' method can be considered by comparing expansions of both methods in the forward scattering direction, where angular dependence is not a factor.

A summary of the classical technique used to determine rain attenuation is presented in Appendix E. The second order series expansion for the classical 'single scattering' intensity of a wave propagating through rain is given in Equation (E-14). When this expansion is compared with the forward direction (α = O) multiple scattering expansion given by Equation (4-100), the 'single scattering' formulation is seen to be similar in form to the first three terms of the multiple scattering expansion. All other terms, including the incoherent intensity term, are not included.

It is also of interest to note that the Foldy approximation expansion was found to represent all terms of the coherent intensity (Section 4-2b), thus the Foldy approximation appears to be more representative of the total intensity than is the classical 'single scattering' formulation.

It must be emphasized again that these observations are made for the forward scattering direction only, and the complete vector scatter solution may produce additional factors which are not evident in the scalar results discussed here.

In conclusion, the study has demonstrated the feasibility of using discrete random media techniques for the determination of multiple scattering effects in propagation through a volume of discrete scatterers, and has provided some insight into the more general problem of multiple scattering in a volume of rain.

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APPENDIX A

TRANSMISSION COEFFICIENT FOR A ONE-DIMENSIONAL RANDOM MEDIUM SLAB

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TRANSMISSION COEFFICIENT FOR A ONE-DIMENSIONAL RANDOM MEDIUM SLAB

Consider a slab of width, L, with a propagation constant K, with an incident wave $\Psi_O(x)$ as shown in Figure A-1. In regions I and III the propagation constant is k, and transmission coefficients Γ, A, B, and T, as shown in the figure, are assumed.

The field in region I is

$$\Psi_I(x) = e^{ikx} + \Gamma e^{-ikx} \tag{A-1}$$

In region II

$$\Psi_{II}(x) = Ae^{ikx} + Be^{-ikx} \tag{A-2}$$

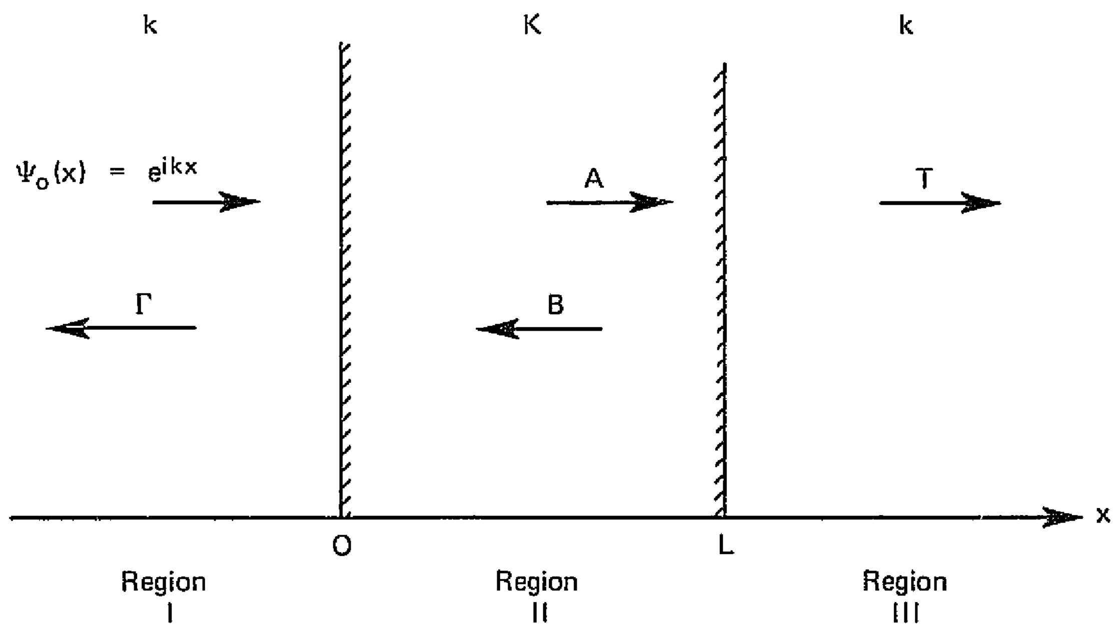

Figure A-1. One-Dimensional Slab Configuration

In region III

$$\Psi_{III}(x) = Te^{ik(x-L)} \tag{A-3}$$

The boundary conditions at x = O and x = L require that

$$\Psi_I(O) = \Psi_{II}(O)$$

$$\Psi_{II}(L) = \Psi_{III}(L)$$

$$\frac{\partial}{\partial x}\Psi_I(O) = \frac{\partial}{\partial x}\Psi_{II}(O)$$

$$\frac{\partial}{\partial x}\Psi_{II}(L) = \frac{\partial}{\partial x}\Psi_{III}(L)$$

From the boundary condition equations

$$1 + \Gamma = A + B$$

$$Ae^{iKL} + Be^{-iKL} = T$$

$$ik - ik\Gamma = iKA - iKB$$

$$iKAe^{iKL} - iKBe^{-iKL} = ikT$$

Rearranging terms

$$A + B - \Gamma = 1 \tag{A-4}$$

$$Ae^{iKL} + Be^{-iKL} = T \tag{A-5}$$

$$KA - KB + k\Gamma = k \tag{A-6}$$

$$KAe^{iKL} - KBe^{-iKL} = kT \tag{A-7}$$

Multiply Equation (A-4) by k, and add to Equation (A-6)

$$(k + K)A + (k - K)B = 2k$$

Therefore

$$A = \frac{2k}{k + K} - \frac{k - K}{k + K} B \qquad \text{(A-8)}$$

Multiply Equation (A-5) by k, and subtract Equation (A-7)

$$(k - K) Ae^{iKL} + (k + K) Be^{-iKL} = O$$

Therefore

$$A = - \frac{k + K}{k - K} e^{-i2KL} B \qquad \text{(A-9)}$$

Equate Equations (A-8) and (A-9)

$$B = \frac{2k(k - K)}{(k - K)^2 - (k + K)^2 e^{-i2KL}} \qquad \text{(A-10)}$$

Substitute B into Equation (A-9) to find A

$$A = \frac{-2(K + k) ke^{-i2KL}}{(k - K)^2 - (k + K)^2 e^{i2KL}} \qquad \text{(A-11)}$$

Finally, substitute Equations (A-10) and (A-11) into Equation (A-5) to find T, the desired transmission coefficient

$$T = \frac{-4kKe^{iKL}}{(k - K)^2 - (k + K)^2 e^{-i2KL}}$$

To express T as a positive function, multiply by $-e^{i2KL} / -e^{i2KL}$, i.e.

$$T = \frac{4kKe^{iKL}}{(k + K)^2 - (k - K)^2 e^{i2KL}} \qquad \text{(A-12)}$$

APPENDIX B

EVALUATION OF MEAN FUNCTIONS FOR ONE-DIMENSIONAL AND VOLUME DISTRIBUTIONS OF DISCRETE SCATTERERS

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EVALUATION OF MEAN FUNCTIONS FOR ONE-DIMENSIONAL AND VOLUME DISTRIBUTIONS OF DISCRETE SCATTERERS

In the development of the mean fields and intensities produced by a distribution of point scatterers, the evaluation of means for a series of random functions are required. This Appendix develops the means for these functions, for the one-dimensional case, Section (a), and the volume distribution, Section (b).

(a) <u>One-Dimensional Distribution of Scatterers</u>

Consider $h(x_i)$, a function of the random variable set x_i. The mean value of $h(x_i)$ is:

$$\langle h(x_i) \rangle = \int h(x_i)\, f(x_1, x_2, \ldots x_N)\, dx_1, dx_2, \ldots dx_N \tag{B-1}$$

where $f(x_1, x_2, \ldots x_N)$ is the joint probability density function of the x_i. If the x_i are statistically independent and uniformly distributed along O to L,

$$\begin{aligned} f(x_1, x_2, \ldots x_N) &= f(x_1)\, f(x_2) \ldots . f(x_N) \\ &= \frac{1}{L}\frac{1}{L} \ldots . \frac{1}{L} = \left(\frac{1}{L}\right)^N \end{aligned} \tag{B-2}$$

Consider the function

$$\sum_{n=1}^{N} h(x_n) \tag{B-3}$$

From above

$$\begin{aligned} \left\langle \sum_{n=1}^{N} h(x_n) \right\rangle &= \int \left[\sum_{n=1}^{N} h(x_n) \right] f(x_1, x_2, x_3, \ldots x_N)\, dx_1\, dx_2 \ldots dx_N \\ &= \left(\frac{1}{L}\right)^N \int [h(x_1) + h(x_2) + \ldots + h(x_N)]\, dx_1\, dx_2 \ldots dx_N \end{aligned} \tag{B-4}$$

Each term of the series is of the form

$$\int h(x_i)\, dx_i \iint \cdots \int \prod_{\substack{j=1 \\ j \neq i}}^{N} dx_j \qquad \text{(B-5)}$$

There are N of these terms, therefore

$$\left\langle \sum_{n=1}^{N} h(x_n) \right\rangle = \left(\frac{1}{L}\right)^N N \int h(x_i)\, dx_i \iint \cdots \int \prod_{\substack{j=1 \\ j \neq i}}^{N} dx_j \qquad \text{(B-6)}$$

Since

$$\int_0^L \int_0^L \cdots \int_0^L \prod_{\substack{j=1 \\ j \neq i}}^{N} dx_j = L^{N-1} \qquad \text{(B-7)}$$

Therefore

$$\begin{aligned} \left\langle \sum_{n=1}^{N} h(x_n) \right\rangle &= \frac{N}{L^N} L^{N-1} \int_0^L h(x_i)\, dx_i \\ &= \frac{N}{L} \int_0^L h(x_i)\, dx_i \end{aligned} \qquad \text{(B-8)}$$

Let $\lambda = N/L$, the uniform number density then

$$\left\langle \sum_{n=1}^{N} h(x_n) \right\rangle = \lambda \int_0^L h(x_i)\, dx_i \qquad \text{(B-9)}$$

Consider now a function $h(x_1, x_2)$ of two random variables of the set x_i,

$$\begin{aligned} \langle h(x_1,x_2) \rangle &= \int h(x_1,x_2)\, f(x_1,x_2, \ldots x_N)\, dx_1\, dx_2 \ldots dx_N \\ &= \left(\frac{1}{L}\right)^N \int h(x_1,x_2)\, dx_1\, dx_2 \ldots dx_N \end{aligned} \qquad \text{(B-10)}$$

Similarly

$$\langle h(x_1,x_2) + h(x_3,x_4) \rangle = \left(\frac{1}{L}\right)^N \int [h(x_1,x_2) + h(x_3,x_4)]\, dx_1\, dx_2 \ldots dx_N \quad \text{(B-11)}$$

and

$$\left\langle \sum_{j=1}^{N} \sum_{l=1}^{N} h(x_j, x_l) \right\rangle = \left(\frac{1}{L}\right)^N \int \left[\sum_{j=1}^{N} \sum_{l=1}^{N} h(x_j, x_l) \right] dx_1\, dx_2 \ldots dx_N$$

$$= \left(\frac{1}{L}\right)^N \left[N \int h(x_j, x_j)\, dx_j \int\int \cdots \int \prod_{\substack{m=1 \\ m \neq j}}^{N} dx_m \right. \quad \text{(B-12)}$$

$$\left. + (N^2 - N) \int h(x_j, x_l)\, dx_j\, dx_l \int\int \cdots \int \prod_{\substack{m=1 \\ m \neq j,l}}^{N} dx_m \right]$$

i.e., the first term is for N terms where $j = l$, second term is for N^2-N terms where $j \neq l$.

Since

$$\int_0^L \int_0^L \cdots \int_0^L \prod_{\substack{m=1 \\ m \neq j}}^{N} dx_m = L^{N-1} \quad \text{(B-13)}$$

and

$$\int_0^L \int_0^L \cdots \int_0^L \prod_{\substack{m=1 \\ m \neq j,l}}^{N} dx_m = L^{N-2} \quad \text{(B-14)}$$

Therefore

$$\left\langle \sum_{j=1}^{N} \sum_{l=1}^{N} h(x_j, x_l) \right\rangle = \left(\frac{1}{L}\right)^N \Bigg[N(L^{N-1}) \int h(x_j, x_j)\, dx_j$$

$$+ (N^2 - N)(L^{N-2}) \iint h(x_j, x_l)\, dx_j\, dx_l \Bigg]$$

(B-15)

$$= \frac{N}{L} \int h(x_j, x_j)\, dx_j$$

$$+ \frac{N^2 - N}{L^2} \iint h(x_j, x_l)\, dx_j\, dx_l$$

For large N

$$\frac{N^2 - N}{L^2} \to \frac{N^2}{L^2} = \lambda^2 \qquad \text{(B-16)}$$

Therefore

$$\left\langle \sum_{j=1}^{N} \sum_{l=1}^{N} h(x_j, x_l) \right\rangle = \lambda^2 \int_0^L \int_0^L h(x_j, x_l)\, dx_j\, dx_l$$

(B-17)

$$+ \lambda \int_0^L h(x_j, x_j)\, dx_j$$

The first integral results from the N^2-N terms of the series where $j \neq l$, while the second integral results from the N terms where $j = l$, i.e., where the scatterers are allowed to overlap.

For the case where scatterers are not allowed to overlap, only the first integral is present, i.e.,

$$\left\langle \sum_{j=1}^{N} \sum_{l=1}^{N}{}' h(x_j, x_l) \right\rangle = \lambda^2 \int_0^L \int_0^L h(x_j, x_l)\, dx_j\, dx_l \qquad \text{(B-18)}$$

where the prime $'$ denotes $j \neq l$.

(b) Volume Distribution of Scatterers

In this section, the same summed averages as above are derived for the volume distribution case. The development is nearly identical, and only major results are presented.

Consider, $h(R_n)$, a function of the random variable set R_n. With the R_n statistically independent and uniformly distributed in the volume V,

$$f(r_1, r_2, \ldots r_N) = f(r_1)\, f(r_2) \ldots f(r_N) \qquad \text{(B-19)}$$

and

$$f(r_i) = \frac{1}{V} \qquad i = 1, 2, \ldots N \qquad \text{(B-19a)}$$

Then the mean value of $h(R_1)$ is evaluated as

$$\begin{aligned} \langle h(R_1) \rangle &= \int_V h(r_1)\, f(r_1)\, f(r_2) \ldots f(r_N)\; dr_1\, dr_2 \ldots dr_N \\ &= \left(\frac{1}{V}\right)^N \int h(r_1)\, dr_1\, dr_2 \ldots dr_N \end{aligned} \qquad \text{(B-20)}$$

and

$$\langle h(R_1) + h(R_2) \rangle = \left(\frac{1}{V}\right)^N \int [h(r_1) + h(r_2)]\, dr_1\, dr_2 \ldots dr_N$$

so

$$\left\langle \sum_{n=1}^{N} h(R_n) \right\rangle = \left(\frac{1}{V}\right)^N \int [h(r_1) + h(r_2) + \ldots + h(r_N)]\, dr_1\, dr_2 \ldots dr_N$$

$$= \left(\frac{1}{V}\right)^N \left[\int [h(r_1)\, dr_1]\, dr_2\, dr_3 \ldots dr_N \right.$$

$$\left. + \int [h(r_2)\, dr_2]\, dr_1\, dr_3 \ldots dr_N + \ldots\ldots \right]$$

Each term of the series is of the form

$$\int h(r_i)\, dr_i \int\int \cdots \int \prod_{\substack{j=1 \\ j \neq i}}^{N} dr_j$$

There are N of these terms, therefore

$$\left\langle \sum_{n=1}^{N} h(R_n) \right\rangle = \left(\frac{1}{V}\right)^N N \int h(r_i)\, dr_i \int\int \cdots \int \prod_{\substack{j=1 \\ j \neq i}}^{N} dr_j \qquad \text{(B-21)}$$

Since

$$\int\int \cdots \int \prod_{\substack{j=1 \\ j \neq i}}^{N} dr_j = V^{N-1} \qquad \text{(B-22)}$$

Therefore

$$\left\langle \sum_{n=1}^{N} h(R_n) \right\rangle = \frac{N}{V} \int_V h(r_n)\, dr_n \qquad \text{(B-23)}$$

C-2

Let the number density be ρ i.e.,

$$\rho = \frac{N}{V} \tag{B-24}$$

then

$$\left\langle \sum_{n=1}^{N} h(R_n) \right\rangle = \rho \int_V h(r_n)\, dr_n \tag{B-25}$$

Consider now a function $h(R_1, R_2)$ of two random variables of the set R_n,

$$\begin{aligned} \langle h(R_1, R_2) \rangle &= \int h(r_1, r_2)\, p(r_1)\, p(r_2) \ldots p(r_N)\, dr_1\, dr_2 \ldots dr_N \\ &= \left(\frac{1}{V}\right)^N \int h(r_1, r_2)\, dr_1\, dr_2 \ldots dr_N \end{aligned} \tag{B-26}$$

Similarly

$$\langle h(R_1, R_2) + h(R_3, R_4) \rangle = \left(\frac{1}{V}\right)^N \int [h(R_1, R_2) + h(R_3, R_4)]\, dr_1\, dr_2 \ldots dr_N \tag{B-27}$$

The sum is therefore

$$\begin{aligned} \left\langle \sum_{n=1}^{N} \sum_{p=1}^{N} h(R_n, R_p) \right\rangle &= \left(\frac{1}{V}\right)^N \int \left[\sum_{n=1}^{N} \sum_{p=1}^{N} h(r_n, r_p) \right] dr_1\, dr_2 \ldots dr_N \\ &= \left(\frac{1}{V}\right)^N \left[(N^2 - N) \int h(r_n, r_p)\, dr_n\, dr_p \iint \cdots \int \prod_{\substack{j=1 \\ j \neq n, p}}^{N} dr_j \right. \\ &\quad \left. + N \int h(r_n, r_n)\, dr_n \iint \cdots \int \prod_{\substack{j=1 \\ j \neq n}}^{N} dr_j \right] \end{aligned} \tag{B-28}$$

The first integral consists of the $(N^2 - N)$ terms where $n \neq p$, the second integral the N terms where $n = p$.

Since

$$\iint \cdots \int \prod_{\substack{j=1 \\ j \neq n,p}}^{N} dr_j = V^{N-2} \tag{B-29}$$

and

$$\iint \cdots \int \prod_{\substack{j=1 \\ j \neq n}}^{N} dr_j = V^{N-1} \tag{B-30}$$

Therefore

$$\left\langle \sum_{n=1}^{N} \sum_{p=1}^{N} h(R_n, R_p) \right\rangle = \rho^2 \iint_V h(r_n, r_p)\, dr_n\, dr_p + \rho \int_V h(r_n, r_n)\, dr_n \tag{B-31}$$

where, for large N

$$\frac{N^2 - N}{V^2} \rightarrow \frac{N^2}{V^2} = \rho^2 \tag{B-32}$$

For the case where the scatterers are not allowed to overlap, i.e. $j \neq 1$, only the first integral is present, i.e.,

$$\left\langle \sum_{n=1}^{N} \sum_{p=1}^{N}{}' h(R_n, R_p) \right\rangle = \rho^2 \iint_V h(r_n, r_p)\, dr_n\, dr_p \tag{B-33}$$

where the $'$ indicates $n \neq p$.

APPENDIX C

SOLUTION OF DOUBLE INTEGRALS REQUIRED FOR FIELD SOLUTIONS

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SOLUTION OF DOUBLE INTEGRALS REQUIRED FOR FIELD SOLUTIONS

Two integrals which occur in the analysis of the mean field and coherent intensity of a slab of width L will be evaluated here.

Consider the first integral,

$$I_1 = \int_o^L \int_o^L e^{ik(|x - x'| + |x' - x''| + x'' - x)} \, dx' \, dx'' \qquad \text{(C-1)}$$

for $x > L$, $x > x'$, x'', see Figure C-1a

$$I_1 = \int_o^L \int_o^L e^{ik(|x' - x''| + x'' - x')} \, dx' \, dx'' \qquad \text{(C-2)}$$

The integral is evaluated in two regions, I and II, as shown in Figure C-1b. In Region I, $x' > x''$ and $(|x' - x''| + x'' - x') = 0$ and the integral limits are

$$\int_o^L dx' \int_o^{x'} dx'' \qquad \text{(C-3)}$$

In Region II, $x'' > x'$ and $(|x' - x''| + x'' - x') = 2x'' - 2x'$ and the integration limits are

$$\int_o^L dx' \int_{x'}^L dx'' \qquad \text{(C-4)}$$

Therefore:

$$I_1 = \int_o^L dx' \left[x' + e^{-ik2x'} \int_{x'}^L e^{ik2x''} \, dx'' \right]$$

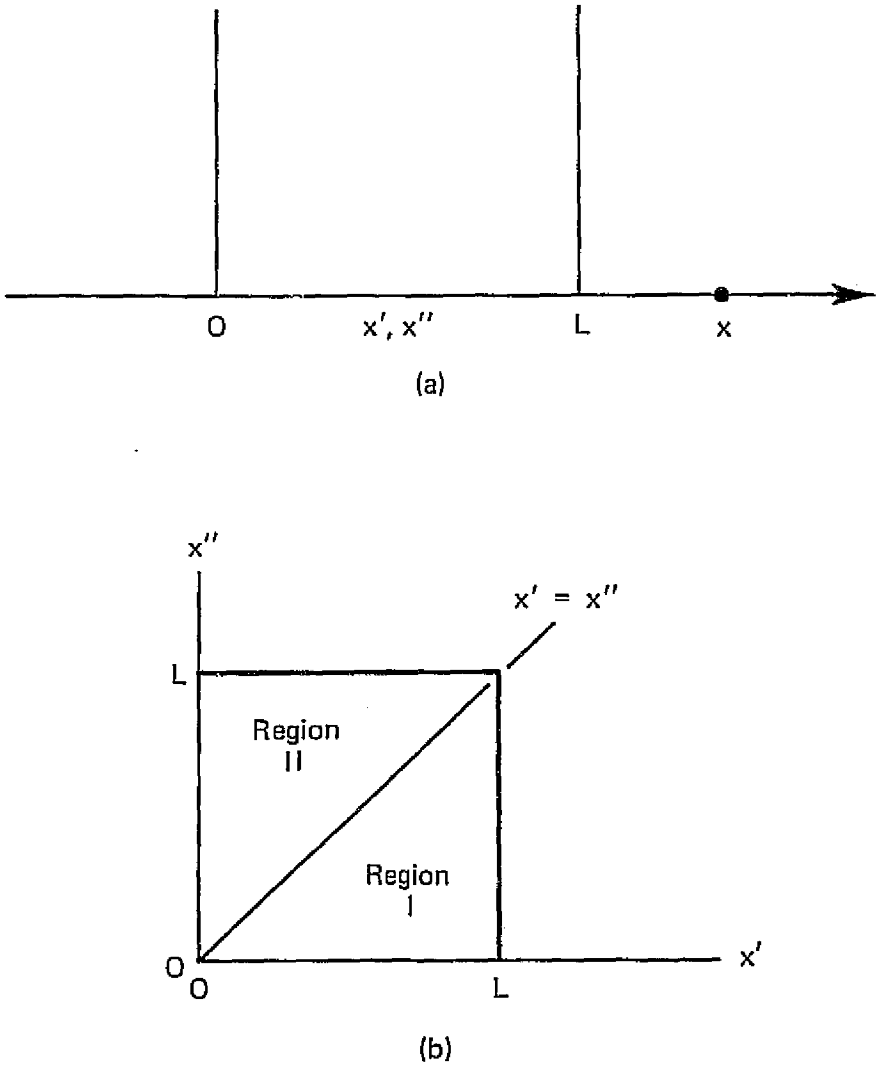

Figure C-1. Regions of Evaluation for Double Integrals
One-dimensional Slab

Continuing the evaluation

$$I_1 = \int_0^L dx' \left[x' + e^{-ik2x'} \frac{e^{ik2L} - e^{ik2x'}}{2ik} \right]$$

$$= \int_0^L x'\, dx' + \int_0^L \frac{e^{ik2L}\, e^{-ik2L} - 1}{2ik}\, dx'$$

$$= \frac{L^2}{2} + \frac{e^{ik2L}}{2ik} \int_0^L e^{-ik2x'}\, dx' - \frac{1}{2ik} \int_0^L dx'$$

$$= \frac{L^2}{2} + \frac{e^{ik2L}}{2ik} \left[\frac{e^{-ik2L} - 1}{-2ik} \right] - \frac{L}{2ik}$$

$$I_1 = \frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{ik2L}}{4k^2} - \frac{L}{2ik} \qquad \text{(C-5)}$$

The second integral to be evaluated

$$I_2 = \int_0^L \int_0^L e^{-ik(|x - x'| + |x' - x''| + x'' - x)}\, dx'\, dx'' \qquad \text{(C-6)}$$

is seen to be the complex conjugate of I_1. The evaluation therefore follows directly from Equation (C-5).

$$I_2 = \frac{L^2}{2} + \frac{1}{4k^2} - \frac{e^{-ik2L}}{4k^2} + \frac{L}{2ik} \qquad \text{(C-7)}$$

APPENDIX D

ASYMPTOTIC EVALUATION OF MEAN FIELD INTEGRAL

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ASYMPTOTIC EVALUATION OF MEAN FIELD INTEGRAL

Consider the integral given by Equation (4-63),

$$I_1 = \frac{1}{4\pi} \int_{-\infty}^{+\infty} k_t \, B(k_t) \, H_o^{(1)}(k_t \, R) \, dk_t \tag{D-1}$$

and the required radiation condition

$$\mathrm{Im} \sqrt{k^2 - k_t^2} > 0 \tag{D-2}$$

To evaluate for large r, we apply the asymptotic approximation for large arguments for $H_o^{(1)}(k_t R)$ (Abramowitz #9.2.3)

$$H_o^{(1)}(k_t \, R) \sim \sqrt{\frac{2}{\pi \, k_t \, R}} \, e^{i\left(k_t \, R - \frac{\pi}{4}\right)} \tag{D-3}$$

with

$$-\pi < \arg(k_t \, R) < 2\pi$$

therefore, Equation (D-1) is

$$I_1 = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8 \, \pi^3 \, R}} \int_{-\infty}^{+\infty} \sqrt{k_t} \, B(k_t) \, e^{ik_t \, R} \, dk_t \tag{D-4}$$

where

$$k_t = \sqrt{k^2 - k_z^2}$$

or

$$k_z = \sqrt{k^2 - k_t^2} \tag{D-5}$$

The square root function k_z is a multi-valued function of k_t, and its properties are investigated in a two plane "Reimann sheet" diagram for k_t, shown in Figure D-1(a). The top sheet, characterized by $\mathrm{Im}\, k_z > 0$, is the proper sheet, where the radiation condition is maintained; the bottom sheet, where $\mathrm{Im}\, k_z < 0$,

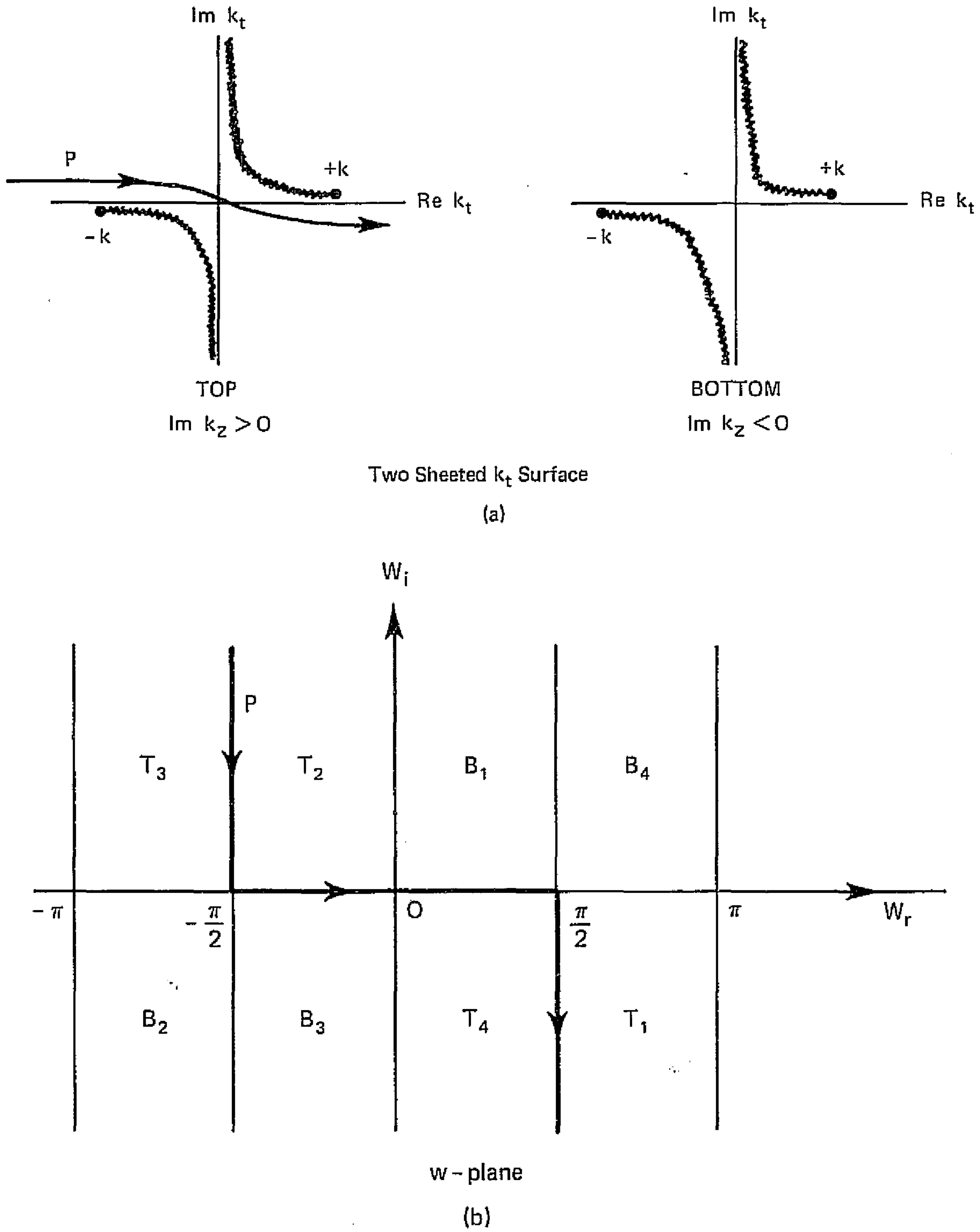


Figure D-1. Riemann Surface for k_t and Mapping of k_t Sheets into the W-Plane is an improper sheet. In order to cross from one sheet of the Rieman surface to another, the cut, called a branch cut, must be crossed.

In the complex k_t planes, the sheets will have branch cuts along Im $k_t = 0$, and branch points at $k_t = \pm k$ (k real).

The path of integration P is shown on the top sheet of Figure D-1(a) where an infinitesimal amount of loss has been introduced (Im $k > 0$), which results in a slight displacement of the branch points at ±k. The four quadrants of the two k_t sheets have been labeled T_1, . . . T_4, B_1, . . . B_4.

To eliminate the branch points, we introduce a new complex variable, $W = W_r + i\, W_i$, and define a W-plane by the transformation

$$k_t = k \sin W \tag{D-6}$$

$$= k(\sin W_r \cosh W_i + i \cos W_r \sinh W_i) \tag{D-8}$$

and also

$$k_z^2 = k^2 - k_t^2 = k^2 \cos^2 W \tag{D-9}$$

$$k_z = k \cos W \tag{D-10}$$

where the + root is chosen to allow $k_z = k$ when $k_t = 0$. The k_t plane is thus mapped into the W-plane as shown in Figure D-1(b). $k_t = k \sin W$ is single valued, and the path of integration P proceeds as shown, with each sector of the W-plane representing the various quadrants of the k_t-plane as labled.

We replace the displacement $(z - z_0)$ with its angular equivalent, (see Fig. 4-3),

$$(z - z_0) = r \cos \alpha \tag{D-11}$$

and

$$R = r \sin \alpha \tag{D-12}$$

Therefore, Equation (D-4) in the W-plane can be developed

$$\sqrt{k^2 - k_t^2} = k\sqrt{1 - \sin^2 W}$$

$$= k \cos W$$

$$\sqrt{k_t} = \sqrt{k \sin W}$$

$$e^{ik_t R} = e^{ikR \sin W} = e^{ikr \sin W \sin \alpha}$$

$$dk_t = k \cos W \, dW$$

And, from Equation (4-52)

$$B(k_t) = e^{ikr \cos W \cos \alpha} \left[\frac{I}{2ik \cos W} + \frac{\sigma \rho L}{4 \cos^2 W} - i \frac{k \sigma^2 \rho^2}{2 \cos^3 W} \left(- \frac{L^2}{8} - i \frac{L}{8 k \cos W} - \frac{1 - e^{i2Lk \cos W}}{16 k^2 \cos^2 W} \right) \right] \tag{D-13}$$

or

$$B(k_t) = T(\cos W) \, e^{ikr \cos W \cos \alpha} \tag{D-14}$$

where T(cos W) is the bracketed term [ ] in Equation (D-13). Therefore, Equation (D-4) is

$$I_1 = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8 \pi^3 R}} \int_{-\infty}^{+\infty} \sqrt{k} \sqrt{\sin W} \, T(\cos W) \quad e^{ikr \cos W \cos \alpha} \, e^{ikr \sin W \sin \alpha} \, k \cos W \, dW \tag{D-15}$$

Let

$$F(W) = \sqrt{\sin W} \, T(\cos W) \cos W \tag{D-16}$$

Then

$$I_1 = \sqrt{\frac{k^3}{8 \pi^3 R}} \, e^{-i\frac{\pi}{4}} \int_{\overline{P}} F(W) \, e^{ikr(\cos W \cos \alpha + \sin W \sin \alpha)} \, dW$$

$$I_1 = \sqrt{\frac{k^3}{8 \pi^3 R}} \, e^{-i\frac{\pi}{4}} \int_{\overline{P}} F(W) \, e^{ikr \cos(W - \alpha)} \, dW \tag{D-17}$$

The above integral can be evaluated for large r asymptotically by deforming the integration path $\overline{P}$ into a path of steepest descent, SDP, through the saddle points.

To determine the SDP, we need to examine the phase characteristics of the exponent in Equation (D-17), since the steepest descent path is a constant phase path in the neighborhood of the saddle points. Recall that, (see Fig. 4-3)

$$|r| = \sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2} \tag{D-18}$$

and

$$r = |r| e^{i\alpha}, \quad -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2} \tag{D-19}$$

The exponent of Equation (D-19) is then

$$\begin{aligned} ikr\cos(W - \alpha) &= ik\frac{|r|}{2}\left[e^{i(W - \alpha)} + e^{-i(W - \alpha)}\right] \\ &= ik\frac{|r|}{2}\left[e^{i(W_r + i W_i - \alpha)} + e^{-i(W_r + i W_i - \alpha)}\right] \\ &= ik\frac{|r|}{2}\left[e^{-W_i} e^{i(W_r - \alpha)} + e^{W_i} e^{-i(W_r - \alpha)}\right] \end{aligned} \tag{D-20}$$

and

$$\mathrm{Re}[ikr\cos(W - \alpha)] = k\frac{|r|}{2}\left[e^{W_i} - e^{-W_i}\right]\sin(W_r - \alpha) \tag{D-21}$$

$$\mathrm{Im}[ikr\cos(W - \alpha)] = k\frac{|r|}{2}\left[e^{W_i} - e^{-W_i}\right]\cos(W_r - \alpha) \tag{D-22}$$

As seen on the W-plane plot, the integration path begins at $W_i = +\infty$ and ends at $W_i = -\infty$. We require that the integrand of I_1, (Eq. (D-17)) decays at these end points, i.e.,

$$\mathrm{Re}[ikr\cos(W - \alpha)] \to 0 \text{ as } W_i \to \pm\infty$$

From Equation (D-21), this is found to require

$$W_r \to -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \text{ as } W_i \to +\infty \tag{D-23a}$$

$$W_r \rightarrow -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \text{ as } W_i \rightarrow -\infty \quad \text{(D-23b)}$$

Since $-\pi/2 < \alpha < \pi/2$, the above conditions are seen to be met and the integrand will decay at the end points of the path defined on the W-plane plot, since the path remains within the allowable region.

The steepest descent path, SDP, is a constant phase path, i.e.,

$$\text{Im}[ikr\cos(W - \alpha)] = \text{constant as } W_i \rightarrow \pm\infty \quad \text{(D-24)}$$

From Equation (D-22), this requires that

$$W_r \rightarrow \alpha - \frac{\pi}{2}, \text{ as } W_i \rightarrow +\infty \quad \text{(D-25a)}$$

$$W_r \rightarrow \alpha + \frac{\pi}{2}, \text{ as } W_i \rightarrow -\infty \quad \text{(D-25b)}$$

We see that the ends of the path lie within the allowable regions of the W-plane since $-\pi/2 < \alpha < \pi/2$ by its definition (Fig. 4-3). Thus the path $\overline{P}$ can be deformed into SDP, and the integral can be evaluated asymptotically. Figure D-2 shows the $\overline{P}_{SD}$ where a small amount of loss $\alpha \rightarrow \alpha + i\epsilon$ has been introduced to move the poles at $\pm W_p$ to allow $\overline{P}$ to be deformed into $\overline{P}_{SD}$ without introducing a singularity.

The method outlined by Felson and Marcuvitz 1973 (Chapter 4) will be used to develop the first order approximation for I_1, (Eq. (D-17)). Felson and Marcuvitz (Eq. (4.2a), (1b)) show that an SDP integral of the form

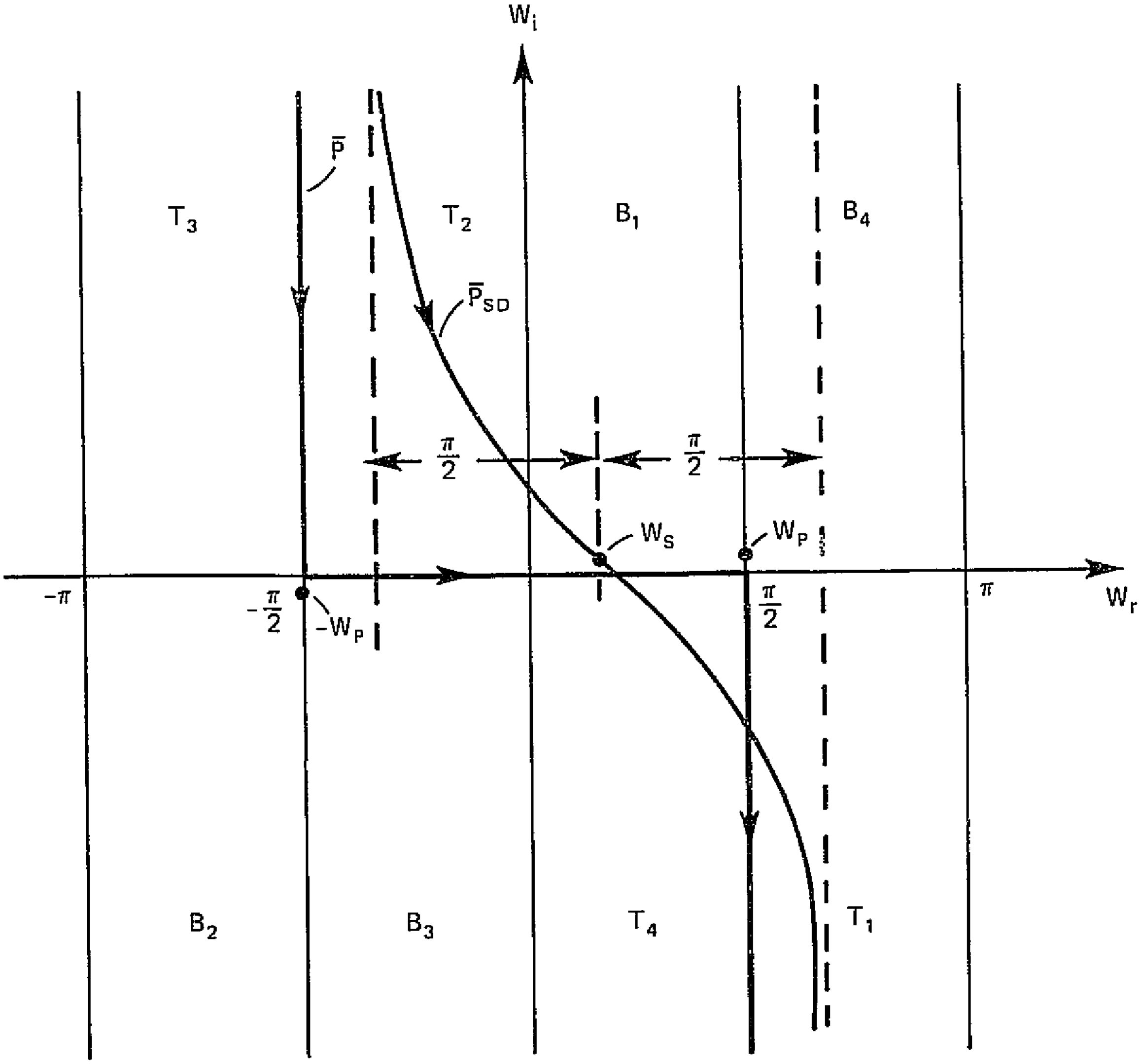
$$I(\Omega) = \int_{\overline{P}_{SD}} f(z)\, e^{i\Omega\hat{q}(z)}\, dz \quad \text{(D-26)}$$

can be asymptotically represented by

$$I(\Omega) \sim \sqrt{\frac{2\pi}{\Omega\,|\hat{q}''(z)|}}\, f(z_S)\, e^{i\Omega\hat{q}(z_S) \pm i\frac{\pi}{4}} \quad \text{(D-27)}$$

as $\Omega \rightarrow \infty$ where z_S is a first order saddle point, (real) and

$$q(z) = i\,\hat{q}(z), \quad \hat{q} \text{ real}$$



$$W_P = \pm\left(\frac{\pi}{2} + i\epsilon\right)$$

$$W_S = \alpha + i\epsilon$$

Figure D-2. Path of Steepest Descent, P_{SD}, for Integral I_1

For Equation (D-17), $\Omega = r$

$$q(W) = ik\cos(W - \alpha) \tag{D-28}$$

$$\hat{q}(W) = k\cos(W - \alpha) \tag{D-29}$$

The saddle point, W_S, is found at $q'(W) = 0$

$$\begin{aligned} q'(W) &= -ik\sin(W - \alpha) \\ W_S &= \alpha \end{aligned} \tag{D-30}$$

Therefore

$$\hat{q}(W_S) = k \tag{D-31}$$

$$\begin{aligned} \hat{q}'(W) &= -k\sin(W - \alpha) \\ \hat{q}'(W_S) &= 0 \end{aligned} \tag{D-32}$$

$$\begin{aligned} \hat{q}''(W) &= -k\cos(W - \alpha) \\ \hat{q}''(W_S) &= -k \end{aligned} \tag{D-33}$$

$$f(W) = F(W) = \sqrt{\sin W}\, T(\cos W)\cos W$$

$$\begin{aligned} f(W_S) = \sqrt{\sin\alpha}\Bigg[\frac{1}{2ik} + \frac{\sigma\rho L}{4\cos\alpha} + i\,\frac{k\sigma^2\rho^2 L^2}{16\cos^2\alpha} \\ + \frac{\sigma^2\rho^2 L}{16\cos^3\alpha} + i\,\frac{\sigma^2\rho^2}{32k\cos^4\alpha} - i\,\frac{\sigma^2\rho^2}{32k\cos^4\alpha}\, e^{i2kL\cos\alpha}\Bigg] \end{aligned} \tag{D-34}$$

The asymptotic representation for I_1, Equation (D-27) is then found from Equation (D-17) with the above substitutions:

$$I_1 \simeq \sqrt{\frac{k^3}{8\pi^3 R}}\, e^{-i\frac{\pi}{4}} \sqrt{\frac{2\pi}{rk}}\sqrt{\sin\alpha}\, T(\cos\alpha)\; e^{ikr + i\frac{\pi}{4}} \tag{D-35}$$

where the + sign for the $i\,\pi/4$ term is selected to provide zero argument at $r = 0$, and $T(\cos\alpha)$ is the bracketed term [ ] in Equation (D-34). Recalling that

$R = r \sin\alpha$, the above equation reduces to

$$I_1 = \frac{e^{ikr}}{4 i \pi r}\left[1 + i\,\frac{k\,\sigma\,\rho\,L}{2\cos\alpha} - \frac{k^2\,\sigma^2\,\rho^2\,L^2}{8\cos^2\alpha} + i\,\frac{k\,\sigma^2\,\rho^2\,L}{8\cos^3\alpha} - \frac{\sigma^2\,\rho^2}{16\cos^4\alpha} + \frac{\sigma^2\,\rho^2}{16\cos^4\alpha}\,e^{i2kL\cos\alpha}\right] \qquad \text{(D-36)}$$

APPENDIX E

SUMMARY OF CLASSICAL METHODS IN RAIN ATTENUATION AT MILLIMETER WAVELENGTHS

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SUMMARY OF CLASSICAL METHODS IN RAIN ATTENUATION AT MILLIMETER WAVELENGTHS

We begin the development of the rain attenuation discussion by briefly reviewing the classical techniques used to predict rain effects on electromagnetic propagation. Most previous studies of rain attenuation, including the often referenced works of Gunn and East (1954), Medhurst (1965), Setzer (1970), Hogg (1968, 1969), Oguchi (1960, 1964), and Hogg and Chu (1975), among many others, have used the so-called 'single scattering' approximation either directly or through the assumptions made in their analyses.

While the method does not rigorously postulate single scattering or specifically exclude multiple scattering effects, the assumed exponential decay for the wave intensity, the use of cross-sections based on forward scattering only, and the assumption of independent drop statistics with superposition of intensity, all lead to a model which does not account directly for multiple scattering.

The classical 'single scattering' formulation assumes an exponential decay for the intensity of the wave as it propagates through a rain volume. Let $\Psi(r)$ be the wave propagated through the rain, and $\Psi_0(r)$ the wave in the absence of rain. The intensity of the wave is then of the form

$$\frac{|\Psi(r)|^2}{|\Psi_0(r)|^2} = e^{-\rho Q_t L} \tag{E-1}$$

where ρ is the drop density, the number of drops per unit volume; Q_t is the attenuation cross section of a single spherical drop, defined below; and L is the path length.

The attenuation cross section of the single spherical raindrop is defined as

$$Q_t = \frac{\text{Total power extracted from the incident wave (watts)}}{\text{Incident power density (watts/m}^2\text{)}}$$

where the units of Q_t are m^2.

The cross section of a spherical drop can be expressed in terms of the Mie coefficients a_n and b_n (Stratton, p. 569)

$$Q_t = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n + 1) \, \mathrm{Re}[a_n + b_n] \qquad \text{(E-2)}$$

where the a_n and b_n are complex functions with arguments dependent on drop radius, wavelength, and m, the complex refractive index of the water drop. Q_t is temperature dependent through m, which varies by as much as a factor of 100% over the temperature range of 0 to 40°C (Ray, 1972).

For a distribution of drops with varying radii,

$$\rho Q_t = \int Q_t(a) \, n(a) \, da \qquad \text{(E-3)}$$

where n(a)da is the number of drops per unit volume with radii between a and a + da, and the integration is over the range of drop radii.

The drop size distribution, n(a), has been determined by direct observation to be approximately proportional to the rainfall rate, R, by an exponential relation of the form (Laws and Parsons, 1943; Marshall and Palmer, 1948)

$$n(a) = n(o) \, e^{-\alpha a R^{-.21}} \qquad \text{(E-4)}$$

where n(O) and α are constants determined from fits of measured distributions.

The resulting intensity for the case of a distribution of drop sizes, at the wavelength λ, for a rainfall rate R is then, from Equation (E-1)

$$\frac{|\Psi(r)|^2}{|\Psi_0(r)|^2} = e^{-L \int Q_t(a, \lambda) \, n(a, R) \, da} \qquad \text{(E-5)}$$

If the rainfall rate varies along the path L, the path variable must also be integrated, i.e.

$$\frac{|\Psi(r)|^2}{|\Psi_0(r)|^2} = e^{-\int_0^L dx \int da \, Q_t(a, \lambda) \, n[a, R(x)]} \qquad \text{(E-6)}$$

The intensity ratio above is usually expressed as a dB ratio, referred to as the 'attenuation' of the wave. From Equation (E-6)

$$A(dB) = 10 \log_{10} \left[e^{-\int_0^L dx \int da\, Q_t(a,\lambda)\, n[a, R(x)]} \right] \tag{E-7}$$

$$= 4.343 \int_0^L dx \int Q_t(a, \lambda)\, n[a, R(x)]\, da$$

The classical or 'single scattering' formulation for rain induced attenuation was shown to be of the form

$$\frac{|\Psi(r)|^2}{|\Psi_0(r)|^2} = e^{-\rho Q_t L}$$

where the total cross section, in terms of the Mie coefficients, is

$$Q_t = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n + 1)\, \mathrm{Re}[a_n + b_n]$$

Consider the region where $2\pi a << \lambda$, i.e., the Rayleigh scattering region, where the drop size is much smaller than the incident wavelength. The Mie coefficient expansions then converge very rapidly, and only the first two coefficients need be considered. As developed by Stratton (1941, p. 571),

$$a_1 \simeq \frac{i}{45} (m^2 - 1) \left(\frac{2\pi a}{\lambda}\right)^5 \tag{E-8}$$

$$b_1 = -\frac{2}{3} i \frac{m^2 - 1}{m^2 + 2} \left(\frac{2\pi a}{\lambda}\right)^3 + \frac{1}{15} i \frac{m^4 - 1}{m^2 + 2} \left(\frac{2\pi a}{\lambda}\right)^5 \tag{E-9}$$

When the ratio of radius to wavelength is so small that $(2\pi a/\lambda)^5$ can be neglected with respect to $(2\pi a/\lambda)^3$, only b_1 is significent. Then the cross section will be

$$Q_t = \frac{2\pi}{k^2} 3\, \mathrm{Re}(b_1) = \frac{2\pi}{k^2} 3 \left[\frac{2}{3} (ka)^3\, \mathrm{Im}\left(\frac{m^2 - 1}{m^2 + 2}\right)\right] \tag{E-10}$$

where

$$k = \frac{2\pi}{\lambda}$$

Therefore

$$Q_t = k4\pi a^3 \, \mathrm{Im} \left(\frac{m^2 - 1}{m^2 + 2} \right) \qquad \text{(E-11a)}$$

$$= k\beta_i \qquad \text{(E-11b)}$$

where

$$\beta_i = 4\pi a^3 \, \mathrm{Im} \left(\frac{m^2 - 1}{m^2 + 2} \right) \qquad \text{(E-12)}$$

The intensity ratio is then

$$\frac{|\Psi(r)|^2}{|\Psi_0(r)|^2} = e^{-k\rho\beta_i L} \qquad \text{(E-13)}$$

With $|k \rho \beta_i L| < 1$, which corresponds to attenuation values of 4.343 dB or less, the exponential can be expressed in series form, i.e.,

$$\frac{|\Psi(r)|^2}{|\Psi_0(r)|^2} = \left[1 - k\rho\beta_i L + \frac{k^2 \rho^2 \beta_i^2 L^2}{2} + O([\rho\beta_i L]^3) \right] \qquad \text{(E-14)}$$