X-692-77-92
PREPRINT

TmX-71338

# TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE III. THE APPROACH TO EQUILIBRIUM

(NASA-TM-X-71338) TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE. 3: THE APPROACH TO EQUILIBRIUM (NASA) 55 p HC A04/MF A01 CSCL 20H
N77-27864
Unclas
G3/70 37029

A. J. KLIMAS
G. SANDRI*
J. D. SCUDDER
D. R. HOWELL



MAY 1977

GSFC
GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

X-692-77-92
PREPRINT

# TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE III. THE APPROACH TO EQUILIBRIUM

A. J. Klimas

G. Sandri*

J. D. Scudder

D. R. Howell

Laboratory for Extraterrestrial Physics

May 1977

*A.R.A.P., Inc., 50 Washington Road, Princeton, N.J. 08540

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

# TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE III. THE APPROACH TO EQUILIBRIUM

A. J. Klimas

G. Sandri

J. D. Scudder

D. R. Howell

## ABSTRACT

The asymptotic behavior, for large time, of the quasi-linear diabatic solutions and their local approximations is considered. A time averaging procedure is introduced which yields the averages of these solutions over time intervals which contain only large time values. A discussion of the quasi-linear diabatic solutions which is limited to those solutions that are bounded from below as functions of time is given. It is shown that as the upper limit of the time averaging interval is allowed to approach infinity the time averaged quasi-linear diabatic solutions must approach isotropy ($\mu$-independence). The first derivative with respect to $\mu$ of these solutions is also considered. This discussion is limited to first derivatives which are bounded functions of time. It is shown that as the upper limit of the time averaging interval is allowed to approach infinity the time averaged first derivative must approach zero everywhere in $\mu$ except at $\mu = 0$ where it must approach a large value which is calculated. The impact of this large derivative on the quasi-linear expansion scheme is discussed. An H-theorem for the first local approximation to the quasi-linear

diabatic solutions is constructed. Without time averaging, the H-theorem is used to determine sufficient conditions for the first local approximate solutions to asymptote, with increasing time, to exactly the same final state which the time averaged quasi-linear diabatic solutions must approach as discussed above. An explanation, based on our numerical results, of how this peculiar final state develops is given. A formal discussion of the integro-differential equation, equation II.1 of paper II. in this series, which contains all of the fundamental approaches to statistical mechanics is given. The exact solution of this equation, as well as all of its local approximations is considered. For this discussion the local approximation concept is generalized to include all of the approximations which result when any of the Pade' approximants to the Laplace transformed kernal of equation II.1 are substituted for the exact kernel transform instead of only those Pade' approximants which lie near the diagonal of the Pade' table and correspond to the kernel convergents. This discussion is limited to those solutions and their local approximations which are bounded from below as functions of time, and to those local approximations which are generated by a Pade' approximant whose range is not reduced compared to the range of the exact kernel transform. Then, it is shown that the time average of the exact solution and also of all its local approximations must asymptote, with increasing time, to a unique final state which is calculated.

# TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE
## III. THE APPROACH TO EQUILIBRIUM

## I. INTRODUCTION

In the first paper of this series[1] (paper I.) we demonstrated the failure of the familiar Fokker-Planck description of the propagation of a charged test particle in a simple model of plasma turbulence. It had been shown previously[2] that in the axi-symmetric slab model of the magnetostatic turbulence the Fokker-Planck equation for the ensemble and gyro-phase averaged probability distribution function follows from the quasi-linear diabatic equation through the application of the adiabatic approximation. We showed in paper I. that the adiabatic approximation is the leading term in the standard linear time-scale[3] expansion in a small parameter. As is typical in the derivation of many kinetic theories this small parameter can be expressed as the ratio of two time scales which are relevant to the mechanical system under consideration and which are assumed to be well separated; i.e., one of the time scales is assumed to be much larger than the other so that an appropriate ratio can be formed which is small and which then plays the role of an expansion parameter. In paper I. we showed that for the test particle propagation problem the assumed separation of the two relevant time scales is generally satisfied if the plasma turbulence is weak, however, if a strong mean magnetic field is present there is a small but important region of the particle phase space in which the two time scales become comparable, or mixed. This mixing of the two time scales was shown to lead to a non-uniformity in the expansion which contains the Fokker-Planck result as the leading term. This failure occurs in the vicinity of 90° pitch angle and is due to the presence of a strong mean magnetic field. Thus, we showed that the failure of the Fokker-Planck description is due to the mixing of time scales which were incorrectly assumed well separated.

In the second paper in this series[4] (paper II.) we introduced our new local approximation method. In that paper we showed that this approximation method can be applied to the fundamental equations of statistical mechanics if they are cast in the integrodifferential form of equation II.1. We demonstrated the use of this new method through its application to a very simple exactly soluble model equation and to the quasi-linear diabatic equation whose solution, as discussed above, could not be well approximated through the use of the more standard approximation techniques. We showed that for these problems the local approximation method does not require a small parameter for expansion purposes. Instead, the equation under consideration was Laplace transformed and approximated near the origin of the Laplace variable using the smallness of the Laplace variable in that vicinity. The purpose of this particular approach is identical to that of the more standard approximation methods; i.e., we seek an accurate representation of the slow and/or long time evolution of the probability distribution function. However, in the local approximation method, since we do not require a small parameter, we avoid the non-uniformity problems associated with the standard asymptotic expansion procedures. In particular, this method was used to replace the unsuccessful Fokker-Planck approximation with the remarkably successful first local approximation.

Except for our considerations of the very simple exactly soluble model equation in paper I., our characterizations of success or failure of the approximation methods discussed above rest on the results of numerical integrations of the equations which generate the quasi-linear diabatic solutions, the quasi-linear adiabatic (Fokker-Planck) solutions, and the first local solutions. The lengths of these numerical integrations are necessarily limited, and therefore, our comparisons of the approximate solutions with the exact ones typically extend over two to three Larmor periods of elapsed time. While this amount of time is sufficient to reject the Fokker-Planck approximation we are still left with many unanswered questions concerning both the quasi-linear diabatic solutions and their first local approximations. Do the diabatic solutions approach a final steady state? If so, what is this state? Do the first local approximations to the diabatic solutions approach a final steady state also? If so, how does this final state compare with that of the diabatic solutions? These questions are answered in this paper.

We find that the accuracy of the first local approximations to the diabatic solutions, which is evident during the first few Larmor periods from the numerical results, almost certainly persists for <u>all</u> time beyond the origin where the initial conditions on these solutions are specified.

In section II. of this paper we discuss the behavior of the quasi-linear diabatic solutions for very large times. We limit our discussion to solutions which are bounded from below as functions of time and consider magnetostatic turbulence whose power spectrum contains finite total power and has no zeroes in its high frequency tail except at infinite frequency. With these limitations we are able to show that the time average of these solutions, as defined by equation III.1 below, must asymptote, with increasing time, to a final state which is isotropic (independent of $\mu$). Even so, we are also able to show that the first derivative with respect to $\mu$ of these solutions, similarly time averaged, must asymptote to a large non-zero value at $\mu = 0$. In the last part of this section we make the further assumption that the solutions under consideration have a first derivative with respect to $\mu$ which is a bounded function of time. Then we are able to show that this derivative, again time averaged, must asymptote to zero everywhere that $\mu \neq 0$. Various evidence from our numerical results is presented later in this paper to give an indication of how this peculiar final state actually develops.

We have noted earlier (in paper I.) that the structure of the quasi-linear diabatic equation does not allow us to exclude the possibility that its solutions contain wave-like behavior for large times. Thus, the solutions to the quasi-linear diabatic equation need not approach a final state even though their time averages must. On the other hand, we have never found evidence from our numerical solutions, for as far as they go in time, for any such wavelike behavior. If these solutions actually approach a final steady state, then the discussion in the previous paragraph applies to the actual solutions rather than their time averages since in this case the time average of the steady state is identical to the steady state.

Section III. of this paper is devoted to the large time behavior of the first local approximation to the quasi-linear diabatic solutions. With some further assumptions on the properties of the magnetostatic turbulence which are made in this section and are further discussed in Appendix B, we are able to construct an H-theorem for the first local approximate solutions. From the H-theorem we find that these solutions must asymptote, with increasing time, to an isotropic steady state with a first derivative with respect to $\mu$ which is zero almost everywhere. The H-theorem does not require the first derivative in this final steady state to be zero at $\mu = 0$, and it actually allows the first derivative to be non-zero at other undetermined $\mu$-values as well. We continue by limiting our consideration to those solutions which conserve probability and which remain integrable in $\mu$ for all $\tau$. We also assume that the magnetostatic turbulence is such that the zero'th and first moments of the Laplace transformed kernel in the quasi-linear diabatic equation are continuous functions of $\mu$ with no zeroes in the first moment as a function of $\mu$. Then, we are able to show that the first derivative with respect to $\mu$ of the first local approximate solutions must be a bounded function of $\tau$ pointwise in $\mu$. Thus, these solutions must be continuous in the closed $\mu$-interval and bounded functions of $\mu$ pointwise in $\tau$. This information allows us to prove that the first derivative with respect to $\mu$ of the first local approximate solutions must be a continuous function of $\mu$ for any finite time, and when the time goes to infinity this first derivative must remain continuous in $\mu$ except at $\mu = 0$ where it is discontinuous and takes on the large value which we found the time average of the first derivative of the quasi-linear diabatic solutions must approach. Because we found from the H-theorem that the first derivative must asymptote, with increasing time, to zero almost everywhere in $\mu$, and since here we find that for $\mu \neq 0$ the first derivative must remain continuous in $\mu$ even as the time goes to infinity, we can finally conclude that the first derivative must actually asymptote to zero everywhere in $\mu$ except at $\mu = 0$. Thus, we find that the first local approximation to the quasi-linear diabatic solutions must asymptote, with increasing time, to exactly the same state which the time average of the quasi-linear diabatic solutions must also approach.

Given the very good agreement between the quasi-linear diabatic solutions and their first local approximation which is evident from our numerical results for short times, and given the manner of agreement discussed in the preceding paragraph as the time asymptotes to infinity, we feel that it is reasonable to assume that the first local approximation remains an excellent approximation for all time greater than zero.

In section IV. of this paper we discuss a general consequence of the local approximation method which has broad implications to statistical mechanical studies of the approach to equilibrium. We restrict our attention to those solutions of the general integro-differential equation given in paper II. by equation II.1 and to those local approximations which remain bounded functions of time. We generalize our conception of a local approximation in this section to include any of the approximations which result when the Laplace transform of the kernel in equation II.1 is replaced by any one of its Pade' approximants.[4] Then we show that the exact solution of equation II.1 and all of the local approximations which are generated by Pade' approximants that do not have a reduced range compared to the exact kernel transform have time averages (as defined through equation III.1) which must asymptote with increasing time to a single unique final state. As a corollary to this result we show that the failure of the adiabatic approximation can be understood on the grounds that it is in fact generated by a Pade' approximant which does have a reduced range.

## II. THE QUASI-LINEAR DIABATIC SOLUTIONS

In this section we consider the final behavior of the quasi-linear diabatic solutions of equation I.20 (paper I.). We introduce the following type of time averaged probability distribution function,

$$\langle f \rangle_T \equiv \left(\frac{1}{\tau - T}\right) \int_T^{\tau} ds\, \bar{f}(\mu, s) \qquad \text{III.1}$$

and show that $\langle f \rangle_T$ must asymptote to a final isotropic state (i.e., independent of $\mu$) as $\tau \to \infty$. In this expression T is any finite time which, however, can be made as large as necessary so long as $\tau$ is larger. Thus, equation III.1 gives a time average of the final evolution of the quasi-linear diabatic solutions. We also consider the first derivative of $\bar{f}(\mu, \tau)$ with respect to $\mu$. We introduce another time average,

$$\langle f' \rangle_T \equiv \left(\frac{1}{\tau - T}\right) \int_T^{\tau} ds\, \frac{\partial \bar{f}(\mu, s)}{\partial \mu} \qquad \text{III.2}$$

and show that at $\mu = 0$, $\langle f' \rangle_T$ does not asymptote to zero as might be expected in view of the approach to isotropy of $\langle f \rangle_T$, but instead asymptotes to a large but finite value given by,

$$\langle f' \rangle_T \underset{\tau \uparrow}{\sim} \frac{1/2 - N(0, 0)}{\eta^2} \qquad \text{III.3}$$

We also show that when $\mu \neq 0$, $\langle f' \rangle_T \sim 0$ as $\tau \to \infty$.
These results hold for "normalizable" quasi-linear diabatic solutions in magnetostatic turbulence which has a "simple" power spectrum associated with it. Both of these terms are defined immediately below.

In our numerical evaluations of the solutions of equation I.20 from paper I. (the quasi-linear diabatic equation), we have never found evidence for negative values of $\bar{f}(\mu, \tau)$ for any $\mu$ and $\tau$. Nevertheless, equation I.20 may contain solutions which do become negative. We limit our discussion here to those solutions which are bounded from below so that a sufficiently large time independent isotropic component of the probability distribution function can be added on to the quasi-linear diabatic solution to make the sum non-negative, and thereby qualify it as a probability distribution function. Thus, having made the restriction to quasi-linear diabatic solutions which are bounded from below, it is no further restriction to assume that these solutions are non-negative. We further restrict our consideration to solutions which conserve probability. From equation I.24 of paper I., we see that we expect $N(\pm 1, \tau)$ = constant so long as the integral on the right side of that equation remains bounded. Due to the finite range of the correlation function at $\mu = \pm 1$ (see equation I.21), we expect the integral to remain bounded unless

$$\frac{\partial^2 N(\mu, \tau)}{\partial \mu^2} = \frac{\partial \bar{f}(\mu, \tau)}{\partial \mu}$$

becomes unbounded. We assume a bounded derivative at $\mu = \pm 1$. Then $N(\pm 1, \tau)$ is a cc nstant in time. Since equation I.24 is homogeneous we further set $N(1, \tau) = 1$ with no loss of generality. We call solutions which satisfy the above requirements, "normalizable". Notice that for normalizable solutions,

$$0 \leq N(\mu, \tau) \leq 1 \qquad \text{III.4}$$

because $\bar{f}(\mu, \tau) \geq 0$. In the last part of this section we will further limit our discussion to those solutions which have a bounded first derivative with respect to $\mu$ as a function of $\tau$ in order to deduce the behavior of this derivative when $\mu \neq 0$ and $\tau \to \infty$.

From equation I.38 of paper I. we find,

$$\tilde{K}(\mu, 0) = \mathfrak{M}_0(\mu) = \frac{1}{|\mu|} P\left(\frac{1}{|\mu|}\right) \qquad \text{III.5}$$

in which $P(\omega)$ is the power spectral density, or more simply, the power spectrum. As in paper I., we will assume a "simple" power spectrum which is non-negative and which does not contain any zeroes in the frequency range, $1 \leq \omega < \infty$. In addition we will assume that

$$P(\omega) = o\left(\frac{1}{\omega}\right) \qquad (\omega \to \infty) \qquad \text{III.6}$$

so that the total power in the random field is finite. We note here that in our dimensionless notation, $\omega = 1$ corresponds to that wavelength in the magnetostatic turbulence which is equal to the correlation length. In the next section of this paper, where the first local approximation to the quasi-linear diabatic solutions is discussed, another special requirement will be imposed on the turbulence; the simple power spectrum is sufficient here.

The remainder of this section will be subdivided into four parts. In the first, we will develop certain properties of the Laplace modes and their associated eigenvalues which are necessary for the remainder of this section. In the second part we will prove that $\langle f \rangle_T$ asymptotes to isotropy, and in the third we will prove equation III.3 nevertheless. In the fourth part of this section we will prove that $\langle f' \rangle_T \sim 0$ as $\tau \to 0$ when $\mu \neq 0$.

a). The Laplace Modes

We reintroduce the Laplace mode expansion of the probability distribution function and prove several useful properties of the modes and their associated eigenvalues. On integrating equation I.64 of paper I. in $\mu$ from -1 to $\mu$, and then rearranging the result slightly, we find,

$$\left(\frac{p}{\lambda_m(p)}\right) \frac{\partial \psi_m(\mu, p)}{\partial \mu} = -\left(\frac{p}{\tilde{\mathfrak{D}}(\mu, p)}\right) \int_{-1}^{\mu} d\mu' \psi_m(\mu', p) \qquad \text{III.7}$$

in which

$$\tilde{\mathfrak{D}}(\mu, p) = (1 - \mu^2)\, \tilde{\mathfrak{K}}(\mu, p) \qquad \text{III.8}$$

where $\tilde{\mathfrak{K}}(\mu, p)$ is defined through,

$$\tilde{\mathfrak{K}}(\mu, p) \equiv \int_0^{\infty} d\tau\, e^{-p\tau} R(\mu\tau) \cos \tau \qquad \text{III.9}$$

(To obtain equation III.7 we have divided by $\lambda_m(p)$. Since $\lambda_0(p) = 0$, we exclude $m = 0$ in the following. We also point out that the divergence of the right side of equation III.7 at $\mu = \pm 1$ is only apparent; the integral also goes to zero at these points so that the ratio remains finite.) The quantity, $R(\zeta)$, is the two-point correlation function associated with the magnetostatic turbulence (see equation I.19) and is a function of the spatial separation, $\zeta$.

Having adopted the simple power spectrum, we find that when $\mu \neq 0$,

$$\frac{p}{\tilde{\mathfrak{D}}(\mu, p)} \underset{p\downarrow}{\sim} 0 \qquad (\mu \neq 0) \qquad \text{III.10}$$

and therefore that,

$$\left(\frac{p}{\lambda_m(p)}\right) \frac{\partial \psi_m(\mu, p)}{\partial \mu} \underset{p\downarrow}{\sim} 0 \qquad (\mu \neq 0) \qquad \text{III.11}$$

On the other hand, when $\mu = 0$ we have,

$$\tilde{K}(0, p) = \frac{p}{1 + p^2} \qquad \text{III.12}$$

(we have normalized the correlation function so that R (0) = 1) and, in contrast to equation III.10

$$\frac{p}{\tilde{\mathfrak{D}}(0, p)} \underset{p\downarrow}{\sim} 1 \qquad (\mu = 0) \qquad \text{III.13}$$

Thus,

$$\left(\frac{p}{\lambda_m(p)}\right) \frac{\partial \psi_m(0, p)}{\partial \mu} \underset{p\downarrow}{\sim} - \int_{-1}^{0} d\mu' \psi_m(\mu', 0) \qquad \text{III.14}$$

In paper I. (see discussion surrounding equation I.69) we have shown that the $\psi_m(\mu, 0)$ can be constructed through even and odd combinations of the eigen-functions which form a complete orthogonal set on the half-domain, $-1 \leq \mu \leq 0$. In this half-domain, and starting at $m = 0$, each pair of $\psi_m(\mu, 0)$ (e.g., $m = 0, 1$; $m = 2, 3$; etc.) are identical to each other. Thus, from the orthogonality of these eigenfunctions on the half-domain we can conclude that the integral on the right side of equation III.14 is non-zero for $m = 0$ and 1 only. Therefore,

$$\frac{p}{\lambda_m(p)} \frac{\partial \psi_m(0, p)}{\partial \mu} \underset{p\downarrow}{\sim} \begin{cases} -1 & (m = 1) \\ 0 & (m > 1) \end{cases} \qquad \text{III.15}$$

We have further seen in paper I. that,

$$\frac{\partial \psi_1(0, p)}{\partial \mu} \underset{p\downarrow}{\sim} -\infty \qquad \text{III.16}$$

Thus, even though $\lambda_1(p) \sim 0$ $(p \to 0)$, we still have,

$$\frac{p}{\lambda_1(p)} \underset{p\downarrow}{\sim} 0 \qquad \text{III.17}$$

For m > 1 we found in paper I. that $\lambda_1(p) \sim \lambda_m^{(A)}$ $(p \to 0)$ with all $\lambda_m^{(A)} > 0$. Therefore, we can make the general statement,

$$\frac{p}{\lambda_m(p)} \underset{p\downarrow}{\sim} 0 \qquad (m > 0) \qquad \text{III.18}$$

Finally, from equation III.7 we find,

$$\left(\frac{p}{\lambda_m(p)}\right) \frac{\partial \psi_m(\mu, p)}{\partial \mu} \left(\frac{\tilde{\mathfrak{D}}(\mu, p)}{\lambda_m(p)}\right) = -\left(\frac{p}{\lambda_m(p)}\right) \int_{-1}^{\mu} d\mu' \psi_m(\mu', p) \qquad \text{III.19}$$

and therefore,

$$\left(\frac{p}{\lambda_m(p)}\right) \frac{\partial \psi_m(\mu, p)}{\partial \mu} \left(\frac{\tilde{\mathfrak{D}}(\mu, p)}{\lambda_m(p)}\right) \underset{p\downarrow}{\sim} 0 \qquad m > 0 \qquad \text{III.20}$$

With this information, we can now proceed to a discussion of the final states of the time averaged quantities introduced above. We note in passing, that we have only assumed the simple power spectrum to obtain the results presented so far in this section.

b). Isotropy

From equations I.65 through I.67 of paper I., we find that the Laplace mode expansion of $p\tilde{f}(\mu, p)$, where $\tilde{f}(\mu, p)$ is the Laplace transform of the probability distribution function, is given by,

$$p\tilde{f}(\mu, p) = f_0(0) + \sum_{m=1}^{\infty} \frac{\left(\frac{p}{\lambda_m(p)}\right) \psi_m(\mu, p) \int_{-1}^{1} d\mu' \psi_m(\mu', p)\, \overline{f}(\mu', 0)}{E_m^2(p)\left[\eta^2 + \left(\frac{p}{\lambda_m(p)}\right)\right]} \qquad \text{III.21}$$

where,

$$f_0(0) \equiv \frac{1}{2}\int_{-1}^{1} d\mu' \, \overline{f}(\mu', 0) = \frac{1}{2} \qquad \text{III.22}$$

the last equality holding for normalizable solutions.

By construction, and through the properties of the Laplace modes which are solutions of the Sturm-Liouville problem (equation I.64, paper I.), we are guaranteed that this expansion converges absolutely and uniformly in $\mu$ for any $p > 0$. Furthermore, by our construction of this same expansion for $\tilde{f}(\mu, p)$ at $p = 0$ in terms of the adiabatic Laplace modes we are also assured that this expansion converges absolutely and uniformly in $\mu$ for $p = 0$ as well. From our discussion in section a). concerning the properties of the eigenvalues as $p \to 0$ we conclude,

$$p\tilde{f}(\mu, p) \underset{p\downarrow}{\sim} f_0(0) \qquad \text{III.23}$$

Because the normalizable solutions are non-negative we can apply the Tauberian theorem, theorem 4.3, Chapter V., of Widder, which is a special case of a theorem provided by Karamata, as well as a special case of Wiener's general Tauberian theorem, and which allows us to conclude from equation III.23 that,

$$\frac{1}{\tau}\int_0^{\tau} ds \, \overline{f}(\mu, s) \underset{\tau\uparrow}{\sim} f_0(0) \qquad \text{III.24}$$

From equation III.1 we can obtain,

$$<f>_T = \frac{1}{\tau}\int_0^\tau ds\ \overline{f}(\mu, s) - \frac{1}{\tau}\int_0^T ds\ \overline{f}(\mu, s)$$

III.25

$$+ \frac{T}{\tau(\tau - T)}\int_0^\tau ds\ \overline{f}(\mu, s) - \frac{T}{\tau(\tau - T)}\int_0^T ds\ \overline{f}(\mu, s)$$

Therefore,

$$<f>_T \underset{\tau\uparrow}{\sim} f_0(0) \qquad \text{III.26}$$

for any T less than infinity. Thus, we conclude that $<f>_T$ must asymptote to an isotropic state, with increasing $\tau$, which is actually the isotropic component of the initial probability distribution function at $\tau = 0$.

c). <u>The First Derivative ($\mu = 0$)</u>

We introduce the notation,

$$\overline{Q}(\tau) \equiv \frac{1}{\tau}\int_0^\tau ds\ Q(s) \qquad \text{III.27}$$

for any quantity, Q, and note that if the integral on the right side of this expression is bounded for any finite $\tau$, then $\overline{Q}(\tau) \sim <Q>_T$, $(\tau \to \infty)$.

We also reintroduce the functions $g(\mu, \tau)$ and $h(\mu, \tau)$ which were originally defined by equations I.47 and I.48 in paper I. We adopt these functions here with the exponential correlation function replaced by an arbitrary functional dependence on spatial separation. With $g(\mu, \tau)$, the quasi-linear diabatic equation for $N(\mu, \tau)$, equation I.24 of paper I., can be written,

$$\frac{\partial N}{\partial \tau} = \eta^2 g \qquad \text{III.27}$$

On differentiating $g(\mu, \tau)$ and $h(\mu, \tau)$ with respect to time we find,

$$\frac{\partial g}{\partial \tau} = (1 - \mu^2) \frac{\partial^2 N}{\partial \mu^2} - h + (1 - \mu^2) \int_0^\tau d\lambda \; \frac{\partial R(\mu(\tau - \lambda))}{\partial \tau} \cos(\tau - \lambda) \frac{\partial^2 N(\lambda)}{\partial \mu^2} \tag{III.28}$$

and,

$$\frac{\partial h}{\partial \tau} = g + (1 - \mu^2) \int_0^\tau d\lambda \; \frac{\partial R\big(\mu(\tau - \lambda)\big)}{\partial \tau} \sin(\tau - \lambda) \frac{\partial^2 N(\lambda)}{\partial \mu^2} \tag{III.29}$$

At $\mu = 0$, these equations reduce to

$$\frac{\partial N(0, \tau)}{\partial \tau} = \eta^2 g(0, \tau) \tag{III.30}$$

$$\frac{\partial g(0, \tau)}{\partial \tau} = \frac{\partial^2 N(0, \tau)}{\partial \mu^2} - h(0, \tau) \tag{III.31}$$

and

$$\frac{\partial h(0, \tau)}{\partial \tau} = g(0, \tau) \tag{III.32}$$

because,

$$\frac{\partial R\big(\mu(\tau - \lambda)\big)}{\partial \tau} = \mu R'(\zeta)\Big|_{\zeta = \mu(\tau - \lambda)} \xrightarrow[\mu \to 0]{} 0 \tag{III.33}$$

for bounded R' (0) which we will assume.

On applying the integral operation given by equation III.27 to equations III.30 through III.32 we obtain,

$$\frac{N(0, \tau) - N(0, 0)}{\tau} = \eta^2 \overline{g}(0, \tau) \tag{III.34}$$

$$\frac{g(0, \tau)}{\tau} = \overline{\frac{\partial^2 N(0, \tau)}{\partial \mu^2}} - \overline{h}(0, \tau) \tag{III.35}$$

and,

$$\frac{h(0, \tau)}{\tau} = \bar{g}(0, \tau) \qquad \text{III.36}$$

Through the substitution of equation III.36 into equation III.34 we find,

$$h(0, \tau) = \frac{N(0, \tau) - N(0, 0)}{\eta^2} \qquad \text{III.37}$$

from which we conclude that $h(0, \tau)$ is a bounded function of time in view of equation III.4. From equation III.37 we determine $\bar{h}(0, \tau)$ and substitute the result into equation III.35 to obtain,

$$\frac{g(0, \tau)}{\tau} = \frac{\overline{\partial^2 N(0, \tau)}}{\partial \mu^2} - \left(\frac{\bar{N}(0, \tau) - N(0, 0)}{\eta^2}\right) \qquad \text{III.38}$$

Our evaluation of the asymptotic behavior of this expression depends on the behavior of $g(0, \tau)$ as $\tau \to \infty$.

From equation III.32 we see that,

$$h(0, \tau) = \int_0^\tau ds\ g(0, s) \qquad \text{III.39}$$

Since $h(0, \tau)$ is a bounded function of time we conclude that $g(0, \tau)$ must also be a bounded function of time except that it could contain integrable singularities. However, it is possible to show that if we assume the existence of an integrable singularity in the time dependence of $g(0, \tau)$ we obtain a contradiction with the boundedness of $h(0, \tau)$ through its definition which is,

$$h(0, \tau) \equiv \int_0^\tau d\lambda\ \sin(\tau - \lambda)\ \frac{\partial^2 N(0, \lambda)}{\partial \mu^2} \qquad \text{III.40}$$

If $g(0, \tau)$ contains an integrable singularity, then from equation III.39, $h(0, \tau)$ remains bounded at the position of that singularity. However, if $g(0, \tau)$ contains an integrable singularity then $\dot{g}(0, \tau)$ contains a non-integrable singularity at the same position in $\tau$. Since $h(0, \tau)$ is a bounded function of time we see from equation III.31 that in this situation $\partial^2 N(0, \tau)/\partial\mu^2$ must also contain a non-integrable singularity in time at the position of the singularity in $\dot{g}(0, \tau)$. But, from equation III.40 we see that then $h(0, \tau)$ must become unbounded at the position of the singularity leading to a contradiction with the assumed integrability of the singularity in equation III.39. Thus, we conclude that $g(0, \tau)$ must in fact be a bounded function in time. Therefore,

$$<f'>_T \underset{\tau\uparrow}{\sim} \frac{1/2 - N(0, 0)}{\eta^2} \qquad \text{III.41}$$

follows from equation III.38 as $\tau \to \infty$.

d). The First Derivative ($\mu \neq 0$)

The Laplace transform of equation I.24 (paper I.) is,

$$p\tilde{N}(\mu, p) - N(\mu, 0) = \eta^2 \tilde{\mathfrak{D}}(\mu, p) \frac{\partial \tilde{f}(\mu, p)}{\partial \mu} \qquad \text{III.42}$$

Therefore,

$$p \frac{\partial \tilde{f}(\mu, p)}{\partial \mu} = \frac{p\left[p\tilde{N}(\mu, p) - N(\mu, 0)\right]}{\eta^2 \tilde{\mathfrak{T}}(\mu, p)} \qquad \text{III.43}$$

Division by $\tilde{\mathfrak{D}}(\mu, p)$ is allowable since at $\mu = \pm 1$, $p N(\mu, p) - N(\mu, 0) = 0$, and the right side of this equation remains bounded. In addition we exclude $\mu = 0$ from consideration where $\tilde{\mathfrak{T}}(0, p)$ has a zero when $p = 0$. From equation III.10 we see that we expect the right side of this equation to $\sim 0$ as $p \to 0$ unless the quantity on the right side in the square bracket becomes unbounded as $p \to 0$. In the following we first show that this quantity in fact remains bounded as $p \to 0$. We then use this result to deduce the behavior of $< f' >_T$ as $\tau \to \infty$ when $\mu \neq 0$.

Since the infinite expansion given by equation III.21 is known to converge uniformly in $\mu$, we can integrate that expansion term by term to find $p\tilde{N}(\mu, p)$. We find,

$$p\tilde{N}(\mu, p) = (\mu + 1) f_0(0) + \sum_{m=1}^{\infty} \frac{\left(\frac{p}{\lambda_m(p)}\right) \chi_m(\mu, p) \int_{-1}^{1} d\mu' \psi_m(\mu', p) \bar{f}(\mu', 0)}{E_m^2(p) \left[\eta^2 + \left(\frac{p}{\lambda_m(p)}\right)\right]} \tag{III.44}$$

where,

$$\chi_m(\mu, p) \equiv \int_{-1}^{\mu} d\mu' \psi_m(\mu', p) \tag{III.45}$$

But, from equation III.7,

$$\chi_m(\mu, p) = -\left(\frac{\tilde{\mathfrak{D}}(\mu, p)}{\lambda_m(p)}\right) \frac{\partial \psi_m(\mu, p)}{\partial \mu} \tag{III.46}$$

Therefore,

$$p\tilde{N}(\mu, p) = (\mu + 1) f_0(0) - \sum_{m=1}^{\infty} \frac{\left(\frac{p}{\lambda_m(p)}\right) \frac{\partial \psi_m(\mu, p)}{\partial \mu} \left(\frac{\tilde{\mathfrak{D}}(\mu, p)}{\lambda_m(p)}\right) \int_{-1}^{1} d\mu' \psi_m(\mu', p) \bar{f}(\mu', 0)}{E_m^2(p) \left[\eta^2 + \frac{p}{\lambda_m(p)}\right]} \tag{III.47}$$

and, from equation III.20, we see that,

$$p\tilde{N}(\mu, p) \underset{p\downarrow}{\sim} (\mu + 1)\ f_0(0) \qquad \text{III.48}$$

which is certainly bounded. Since $N(\mu, 0)$ is also bounded we conclude that

$$p\ \frac{\partial\ \tilde{\bar{f}}(\mu, p)}{\partial \mu} \underset{p\downarrow}{\sim} 0 \qquad \text{III.49}$$

We now assume that $\partial\,\bar{f}(\mu, \tau)/\partial\mu$ is a bounded function of time. Therefore, we can apply Wiener's generalized Tauberian theorem to Laplace transforms, to obtain (see Appendix A),

$$\frac{1}{\tau}\int_0^{\tau} ds\ \frac{\partial\,\bar{f}(\mu, s)}{\partial \mu} \underset{\tau\uparrow}{\sim} 0 \qquad \text{III.50}$$

and then, we can conclude,

$$\langle f' \rangle_T \underset{\tau\uparrow}{\sim} 0 \qquad (\mu \neq 0) \qquad \text{III.51}$$

Before proceeding, we note that as a corollary of the above results we can conclude that if the quasi-linear diabatic probability distribution function actually approaches a final steady state given by,

$$\bar{f}(\mu, \tau) \underset{\tau\uparrow}{\sim} S(\mu) \qquad \text{III.52}$$

and

$$\frac{\partial\,\bar{f}(\mu, \tau)}{\partial \mu} \underset{\tau\uparrow}{\sim} S'(\mu) \qquad \text{III.53}$$

then,

$$S(\mu) = f_0(0) \qquad \text{III.54}$$

and,

$$S'(\mu) = \begin{cases} \dfrac{1/2 - N(0,0)}{\eta^2} & (\mu = 0) \\ 0 & (\mu \neq 0) \end{cases} \qquad \text{III.55}$$

Otherwise, as we showed above, this final steady state is the time average of the final evolution of the quasi-linear diatatic solutions. One of the main conclusions of the next section will be a proof that the first local approximation to the quasi-linear diabatic solutions must approach this final steady state as $\tau \to \infty$ (no time averaging is necessary).

## III. THE FIRST LOCAL APPROXIMATION TO THE QUASI-LINEAR DIABATIC SOLUTIONS

In this section we consider the first local approximation to the quasi-linear diabatic solutions whose large time properties were discussed in the preceding section. The central issue here is an H-theorem which we will construct for the first local approximation and which we will use to show that in very general circumstances the approximate solutions must asymptote to the steady state characterized by $S(\mu)$ and $S'(\mu)$ above. Thus, in this section we show that the approximate and exact solutions must asymptote to exactly the same final steady state if the exact solutions have a final steady state. If the exact solutions do not have a final steady state, then what we show in this section is that the first local approximations and the time average of the exact solutions must still asymptote to that final state given by $S(\mu)$ and $S'(\mu)$.

We divide this section into four parts also. In the first part we construct the H-theorem and from it deduce that $\bar{f}_1(\mu, \tau) \sim$ isqtropy, $(\tau \to \infty)$ and $\bar{f}_1'(\mu, \tau) \sim 0$ $(\tau \to \infty)$ almost everywhere in $\mu$ $\left(\bar{f}_1'(\mu, \tau) = \partial \bar{f}_1(\mu, \tau)/\partial\mu\right)$. The amplitude of the isotropic final state is not given by the H-theorem. However, we restrict our consideration to solutions for which $\bar{f}_1'(\pm 1, \tau)$ and $\partial \bar{f}_1'(\pm 1, \tau)/\partial\tau$ are bounded, so that total probability is conserved, and then the amplitude is given by $S(\mu) = f_0(0)$. The H-theorem does not determine the asymptotic behavior of $\bar{f}_1'(\mu, \tau)$ at those points where $\bar{f}_1'(\mu, \tau) \not\to 0 \, (\tau \to \infty)$, it does not tell where those points are, and it also tells nothing of the asymptotic behavior of $\bar{f}_1'(0, \tau)$. The remaining three parts of this section are directed to a fuller understanding of these issues.

In the second part of this section we restrict our attention to those solutions for which $\bar{f}_1(\mu, \tau)$ remains integrable in $\mu$ so that $N_1(\mu, \tau)$ exists and is finite for all $(\mu, \tau)$. Since this property is an essential ingredient in the definition of a probability distribution function we feel that this restriction is reasonable. For these solutions we show that $\bar{f}_1'(\mu, \tau)$ is a bounded function of time pointwise in $\mu$. From this result it follows that both $\bar{f}_1(\mu, \tau)$ and $N_1(\mu, \tau)$ are continuous functions in the closed interval $-1 \leq \mu \leq 1$, and therefore that $\bar{f}_1(\mu, \tau)$ is a bounded function of $\mu$ for any $\tau$. Thus, these solutions are normalizable in the sense of section II. In the third part of this section we show that because these solutions are normalizable,

$$\bar{f}_1'(0, \tau) \underset{\tau\uparrow}{\sim} \frac{1/2 - N(0, 0)}{\eta^2} \qquad \text{III.56}$$

and in the fourth part we show that $\bar{f}_1'(\mu, \tau)$ is continuous in $\mu$ for any finite $\tau$, is continuous in $\mu$ at $\tau = \infty$ and $\mu \neq 0$, and is discontinuous in $\mu$ at $\tau = \infty$ and $\mu = 0$. To prove this result we must make the further assumption that both $\mathfrak{D}_0(\mu)$ and $\mathfrak{D}_1(\mu)$ are continuous functions of $\mu$. The significance of this result is that we can then conclude that $\bar{f}_1'(\mu,\tau) \sim 0\,(\tau \to \infty)$ for all $\mu \neq 0$. Therefore, we conclude that $\bar{f}_1'(\mu,\tau) \sim S'(\mu)\,(\tau \to \infty)$.

a). <u>The H-theorem</u>

In order to construct the H-theorem we require,

$$\mathfrak{M}_1(\mu) \equiv \int_0^\infty d\tau\, R(\mu\tau)\, \tau \cos\tau \leq 0 \qquad \text{III.57}$$

with $\mathfrak{M}_1(\mu = \pm 1)$ bounded. This requirement is met by the exponential correlation function used in paper I. and II. to obtain the numerical results given there. However, we have not been able to devise any simple criterion for the correlation function or its associated power spectrum in order that equation III.57 be satisfied. For the remainder of this section we do assume that this condition on the magnetostatic turbulence is met, and in Appendix B we list a few alternative expressions for equation III.57 none of which, unfortunately, are particularly transparent. We also continue to assume a simple power spectrum as defined in the previous section.

Equation II.62 (paper II.) generates the first local approximation to the quasi-linear diabatic probability distribution function. We reproduce this equation here for $\bar{f}_1(\mu,\tau)$ instead of $N_1(\mu,\tau)$:

$$\frac{\partial}{\partial\tau}\left[\bar{f}_1 + \eta^2 \frac{\partial}{\partial\mu}\, \mathfrak{D}_1(\mu)\, \frac{\partial \bar{f}_1}{\partial\mu}\right] = \eta^2 \frac{\partial}{\partial\mu}\, \mathfrak{D}_0(\mu)\, \frac{\partial \bar{f}_1}{\partial\mu} \qquad \text{III.58}$$

subject to the reinitialized condition which is given in terms of the initial condition by,

$$\bar{f}_1(\mu, 0^+) + \eta^2 \frac{\partial}{\partial\mu}\, \mathfrak{D}_1(\mu)\, \frac{\partial \bar{f}_1(\mu, 0^+)}{\partial\mu} = \bar{f}(\mu, 0) \qquad \text{III.59}$$

We further introduce the quantity, $H(\tau)$, through,

$$H(\tau) \equiv \frac{1}{2}\int_{-1}^{1} d\mu \left[\bar{f}_1^2(\mu,\tau) - \eta^2 \mathfrak{D}_1(\mu)\left(\bar{f}_1'(\mu,\tau)\right)^2\right] \qquad \text{III.60}$$

and note that through its definition and the assumed properties of $\mathfrak{D}_1(\mu)$, $H(\tau) \geq 0$. Through integration by parts we find that the time derivative of $H(\tau)$ is given by,

$$\dot{H}(\tau) = \int_{-1}^{1} d\mu\, \bar{f}_1 \frac{\partial}{\partial\tau}\left[\bar{f}_1 + \eta^2 \frac{\partial}{\partial\mu}\mathfrak{D}_1(\mu)\frac{\partial \bar{f}_1}{\partial\mu}\right]$$

$$= -\eta^2 \int_{-1}^{1} d\mu\, \mathfrak{D}_0(\mu)\left(\frac{\partial \bar{f}_1}{\partial\mu}\right)^2 \qquad \text{III.61}$$

and thus, $\dot{H}(\tau) \leq 0$. It is impossible for $H(\tau)$ to remain greater than zero unless $\dot{H}(\tau) \sim 0\,(\tau \to \infty)$. Thus, we conclude that $\bar{f}_1'(\mu,\tau) \sim 0\,(\tau \to \infty)$ almost everywhere in $\mu$. Since $\mathfrak{D}_0(0) = 0$, it is clear that $\bar{f}_1'(0,\tau)$ does not have to asymptote to zero. It is also possible that $\bar{f}_1'(\mu,\tau)$ does not asymptote to zero at other points in $\mu$ so long as these points do not contribute to the integral in equation III.61. Nevertheless, we must have,

$$\bar{f}_1(\mu,\tau) = \bar{f}_1(\mu_0,\tau) + \int_{\mu_0}^{\mu} d\mu'\, \bar{f}_1'(\mu',\tau) \underset{\tau\uparrow}{\sim} \bar{f}_1(\mu_0,\tau) \qquad \text{III.62}$$

where $\mu_0$ is any point in $\mu$ at which $\bar{f}_1'(\mu_0,\tau)$ does asymptote to zero. Thus, we conclude that $\bar{f}_1(\mu,\tau)$ must asymptote to an isotropic function of $\tau$. But, it is also easy to see that this function must be constant in $\tau$. Because $\dot{H}(\tau) \sim 0\,(\tau \to \infty)$, we must have $H(\tau) \sim H_f\,(\tau \to \infty)$ where $H_f$ is indeed a constant number which is independent of $\tau$. But, from equation III.60, we see that $H(\tau) \sim \bar{f}_1^2(\mu_0,\tau)\,(\tau \to \infty)$.

Therefore, $\bar{f}_1^2(\mu_0, \tau) \sim H_f\ (\tau \to \infty)$. We conclude that $\bar{f}_1(\mu, \tau)$ must asymptote to an isotropic steady state as $\tau \to \infty$. We further restrict our attention to those solutions for which $\bar{f}_1'(\pm 1, \tau)$ and $\partial \bar{f}_1'(\pm 1, \tau)/\partial\tau$ are bounded functions of $\tau$. Then we see from equation II.62 (paper II.) that $N_1(\pm 1, \tau)$ is constant in time; i.e., these solutions conserve total probability. In this case it is clear that $\bar{f}_1(\mu, \tau) \sim f_0(0)\ (\tau \to \infty)$.

In the remainder of this section we determine the sufficient conditions to ensure that $\bar{f}_1'(\mu, \tau) \sim 0\ (\tau \to \infty)$ everywhere in $\mu$ except at $\mu = 0$.

b). $\bar{f}_1'(\mu, \tau)$ Is Bounded Pointwise in $\mu$

We have already assumed that $\bar{f}_1'(\pm 1, \tau)$ is bounded in $\tau$. Here, we show that $\bar{f}_1'(\mu, \tau)$ is bounded pointwise in $\mu$ in the open interval, $-1 < \mu < 1$. To proceed, we define,

$$q(\mu, \tau) \equiv -\eta^2 \mathcal{D}_1(\mu)\, \bar{f}_1'(\mu, \tau) \qquad \text{III.63}$$

and,

$$\Delta(\mu) \equiv -\frac{\mathcal{D}_0(\mu)}{\mathcal{D}_1(\mu)} \geq 0 \qquad (-1 < \mu < 1) \qquad \text{III.64}$$

Here, we are assuming that $\mathfrak{M}_1(\mu)$ has no zeroes in the open interval. Then, equation II.62 (paper II.) can be rewritten for positive $\tau$ as,

$$\frac{\partial q(\mu, \tau)}{\partial \tau} + \Delta(\mu)\, q(\mu, \tau) = \frac{\partial N_1(\mu, \tau)}{\partial \tau} \qquad \text{III.65}$$

and integrated.

$$q(\mu, \tau) = e^{-\Delta(\mu)\tau}\left[q(\mu, 0^+) + \int_0^\tau d\lambda\, e^{\Delta(\mu)\lambda}\, \frac{\partial N_1(\mu, \lambda)}{\partial \lambda}\right] \qquad \text{III.66}$$

With the use of integration by parts and reinitialization we find,

$$q(\mu, \tau) = N_1(\mu, \tau) - e^{-\Delta(\mu)\tau} N(\mu, 0) - \Delta(\mu)\, e^{-\Delta(\mu)\tau} \int_0^\tau d\lambda\, e^{\Delta(\mu)\lambda} N_1(\mu, \lambda) \qquad \text{III.67}$$

Thus,

$$|q(\mu, \tau)| \leq |N_1(\mu, \tau)| + |e^{-\Delta(\mu)\tau} N(\mu, 0)| + \left|\Delta(\mu)\, e^{-\Delta(\mu)\tau} \int_0^\tau d\lambda\, e^{\Delta(\mu)\lambda} N_1(\mu, \lambda)\right| \qquad \text{III.68}$$

We have already assumed that the initial condition is normalizable and that $N_1(\mu, \tau)$ is uniformly bounded in $\mu$ and $\tau$; i.e., $|N_1(\mu, \tau)| < M_1$ where $M_1$ is finite. Therefore,

$$\begin{aligned} |q(\mu, \tau)| &\leq M_1 + 1 + M_1\, |1 - e^{-\Delta(\mu)\tau}| \\ &\leq 2M_1 + 1 \end{aligned} \qquad \text{III.69}$$

From equation III.63 we see then that $\bar{f}_1'(\mu, \tau)$ is bounded on the open interval in $\mu$. We can further conclude that both $\bar{f}_1(\mu, \tau)$ and $N_1(\mu, \tau)$ are continuous on the closed interval, $-1 \leq \mu \leq 1$. Therefore, both of these quantities are uniformly bounded in $\mu$. We are thus able to assume that these solutions are normalizable without loss of generality, and that,

$$0 \leq N_1(\mu, \tau) \leq 1 \qquad (-1 \leq \mu \leq 1) \qquad \text{III.70}$$

for any $\tau$.

c). $\bar{f}_1'(0, \tau)$

Since $\mathfrak{D}_0(0) = 0$, we see from equation II.62 (paper II.) that,

$$\frac{\partial}{\partial \tau}\left[N_1(0, \tau) + \eta^2 \mathfrak{D}_1(0) \frac{\partial \bar{f}_1(0, \tau)}{\partial \mu}\right] = 0 \qquad (\tau > 0) \qquad \text{III.71}$$

Therefore, the quantity in brackets in this equation is a constant in time which, by integrating equation III.59 in $\mu$, we can find is $N(0, 0)$. Thus,

$$N_1(0, \tau) - \eta^2 \frac{\partial \bar{f}_1(0, \tau)}{\partial \mu} = N(0, 0) \qquad \text{III.72}$$

since $\mathfrak{D}_1(0) = -1$. Given that these solutions are normalizable, we have $N_1(0, \tau) \sim 1/2 \, (\tau \to \infty)$. Therefore,

$$\bar{f}_1'(0, \tau) \underset{\tau\uparrow}{\sim} \frac{1/2 - N(0, 0)}{\eta^2} \qquad \text{III.73}$$

which is exactly the same value that this derivative must take on in a time averaged sense for the exact quasi-linear diabatic solutions.

d). $\bar{f}_1'(\mu, \tau)$, $\mu \neq 0$

It is relatively easy to see from equation III.67 that $q(\mu, \tau)$, and therefore $\bar{f}_1'(\mu, \tau)$, is a continuous function of $\mu$ for any finite $\tau$, and when $\tau = \infty$ that $q(\mu, \tau)$ remains continuous in $\mu$ for $\mu \neq 0$. We examine $|q(\mu_1, \tau) - q(\mu_0, \tau)|$ where $\mu_1$ and $\mu_0$ are two different values of $\mu$, and show that this difference can be made as small as we like if $\mu_1 - \mu_0$ is made small enough except when $\mu_0 = 0$ and $\tau = \infty$ (we hold $\mu_0$ fixed and let $\mu_1 \to \mu_0$).

We define,

$$\epsilon \equiv N_1(\mu_1, \tau) - N_1(\mu_0, \tau) \qquad \text{III.74}$$

and,

$$\delta \equiv \Delta(\mu_1) - \Delta(\mu_0) \qquad \text{III.75}$$

We have already shown that $N_1(\mu, \tau)$ is a continuous function of $\mu$, and we have assumed that $\Delta(\mu)$ is also continuous. Therefore, both $\epsilon$ and $\delta$ can be made as small as we like for small enough $\mu_1 - \mu_0$. In general $\epsilon$ should be considered a function of time, however, in light of equation III.70 a bound on this difference can be found which is uniform in $\tau$. Thus,

$$\begin{aligned} |q(\mu_1, \tau) - q(\mu_0, \tau)| \leq\ & |N_1(\mu_1, \tau) - N_1(\mu_0, \tau)| \\ & + \left|e^{-\Delta(\mu_0)\tau}(e^{-\delta\tau} - 1)\, N(\mu_0, 0) + \epsilon\, e^{-\Delta(\mu_1)\tau}\right| \\ & + \left|\Delta(\mu_0) \int_0^\tau d\lambda\; e^{-\Delta(\mu_0)}(e^{-\delta\lambda} - 1)\, N_1(\mu_0, \tau - \lambda)\right| \\ & + |\epsilon| \left|\Delta(\mu_0) \int_0^\tau d\lambda\; e^{-\Delta(\mu_0)\lambda}\right| + |\delta| \left|\int_0^\tau d\lambda\; e^{-\Delta(\mu_1)\lambda}\, N(\mu_1, \tau - \lambda)\right| \end{aligned} \qquad \text{III.76}$$

and then, with the use of equation III.70,

$$|q(\mu_1, \tau) - q(\mu_0, \tau)| \leq |\epsilon| + e^{-\Delta(\mu_0)\tau}(1 - e^{-\delta\tau})$$

$$+ |\epsilon| e^{-\Delta(\mu_1)\tau} + \left|\Delta(\mu_0)\left(\frac{1 - e^{-(\Delta(\mu_0)+\delta)\tau}}{\Delta(\mu_0) + \delta}\right) - \left(1 - e^{-\Delta(\mu_0)\tau}\right)\right|$$

$$+ |\epsilon|(1 - e^{-\Delta(\mu_0)\tau}) + |\delta|\left(\frac{1 - e^{-\Delta(\mu_1)\tau}}{\Delta(\mu_1)}\right) \qquad \text{III.77}$$

It is clear that for finite $\tau$, this expression can be made as small as is necessary by making $\epsilon$ and $\delta$ small enough. When $\tau = \infty$ we consider the two cases, $\mu_0 \neq 0$ and $\mu_0 = 0$, separately. If $\mu_0 \neq 0$, then (we assume $\mu_1 \neq 0$ also),

$$|q(\mu_1, \infty) - q(\mu_0, \infty)| \leq 2|\epsilon| + 2\left|\frac{\delta}{\Delta(\mu_1)}\right| \qquad \text{III.78}$$

which can be made as small as we like. However, if $\mu_0 = 0$, then,

$$|q(\mu_1, \infty) - q(0, \infty)| \leq 3 + |\epsilon| \qquad \text{III.79}$$

In this case we cannot make the difference as small as we like by reducing $\epsilon$ and $\delta$. Thus, we conclude that $q(\mu, \tau)$ is continuous in $\mu$ for any finite $\tau$, and that $q(\mu, \tau)$ is continuous in $\mu$ for $\tau = \infty$ when $\mu \neq 0$; $q(\mu, \tau)$ is not continuous at $\mu = 0$ and $\tau = \infty$. Since $\mathfrak{D}_1(\mu)$ has been assumed continuous in $\mu$, these same conclusions hold for $\bar{f}_1'(\mu, \tau)$.

Since in a). we have learned from the H-theorem that $\bar{f}_1'(\mu, \tau)$ is zero almost everywhere at $\tau = \infty$, we can now conclude from the continuity of $\bar{f}_1'(\mu, \tau)$ that in fact $\bar{f}_1'(\mu, \tau) \sim 0\ (\tau \to \infty)$ for all $\mu$ except $\mu = 0$. We have seen in c). that

$\bar{f}_1'(\mu, \tau)$ does not asymptote to zero at $\mu = 0$; this is consistent with equation III.79.

We conclude, finally, that under the conditions outlined in this section, the first local approximation to the quasi-linear diabatic solutions must asymptote to the final steady state characterized by $S(\mu)$ and $S'(\mu)$ in the previous section. As peculiar as this final state is, we see that the first local approximation yields exactly the time averaged behavior of the quasi-linear diabatic solutions as $\tau \to \infty$.

In Figure 1, we have plotted $\bar{f}_1'(\mu, \tau)$ at various $\tau$ for the broad parallel beam injection considered earlier in papers I. and II. in order to give some indication of how the final state, $S(\mu)$ and $S'(\mu)$, develops; figure 20 of paper II. is also useful for this purpose. As $\tau$ increases, we can see from figure 20 of paper II. that $\bar{f}_1(\mu, \tau)$ tends to the isotropic state, $S(\mu) = f_0(0)$, but with a narrow region near $\mu = 0$ in which a steep gradient persists. From Figure 1 of this paper, we see that the width of this region decreases with increasing $\tau$ while the amplitude of the gradient at $\mu = 0$ increases. We suppose that as $\tau \to \infty$, the amplitude of the gradient saturates at the value given by equation III.73, the width of the peak in the gradient shown in Figure 1 goes to zero, and $\bar{f}_1(\mu, \tau)$ reaches isotropy. The amplitude of the peak in the gradient is plotted in Figure 2 as a function of $\tau$. From this figure, the approach of this peak value to the maximum $S'(0)$ seems reasonable. We also point out that the approach to $S'(\mu) = 0 \ (\mu \neq 0)$ is consistent with Figure 1 for $\mu \neq 0$.

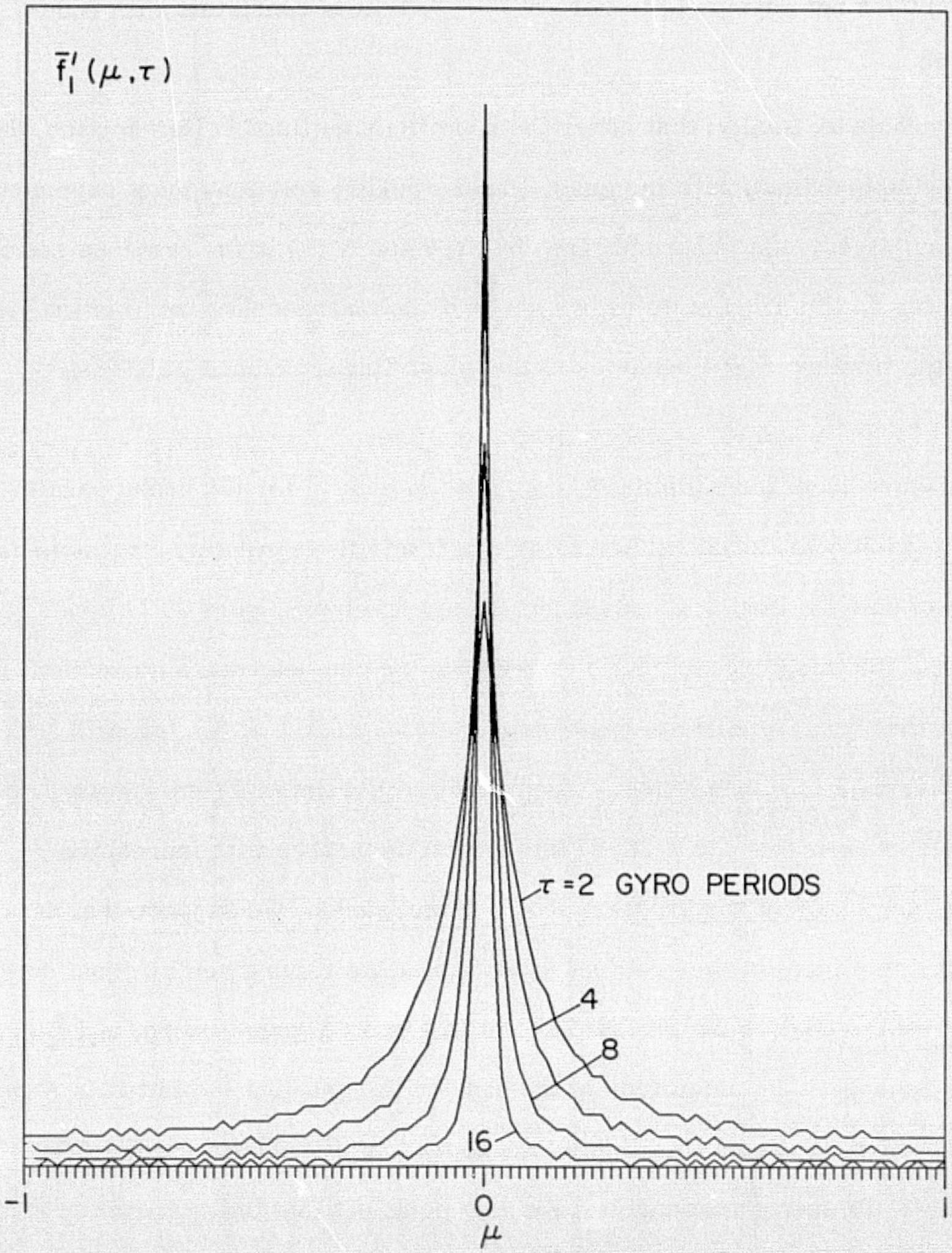


Figure 1. The first derivative with respect to $\mu$ of the first local approximation to the broad parallel beam injection solution computed earlier in papers I. and II. The derivative is plotted as a function of $\mu$ for several values of time in units of gyro-periods. The irregularity of these curves is due to the presence of numerical noise which has been amplified by the differentiation process.

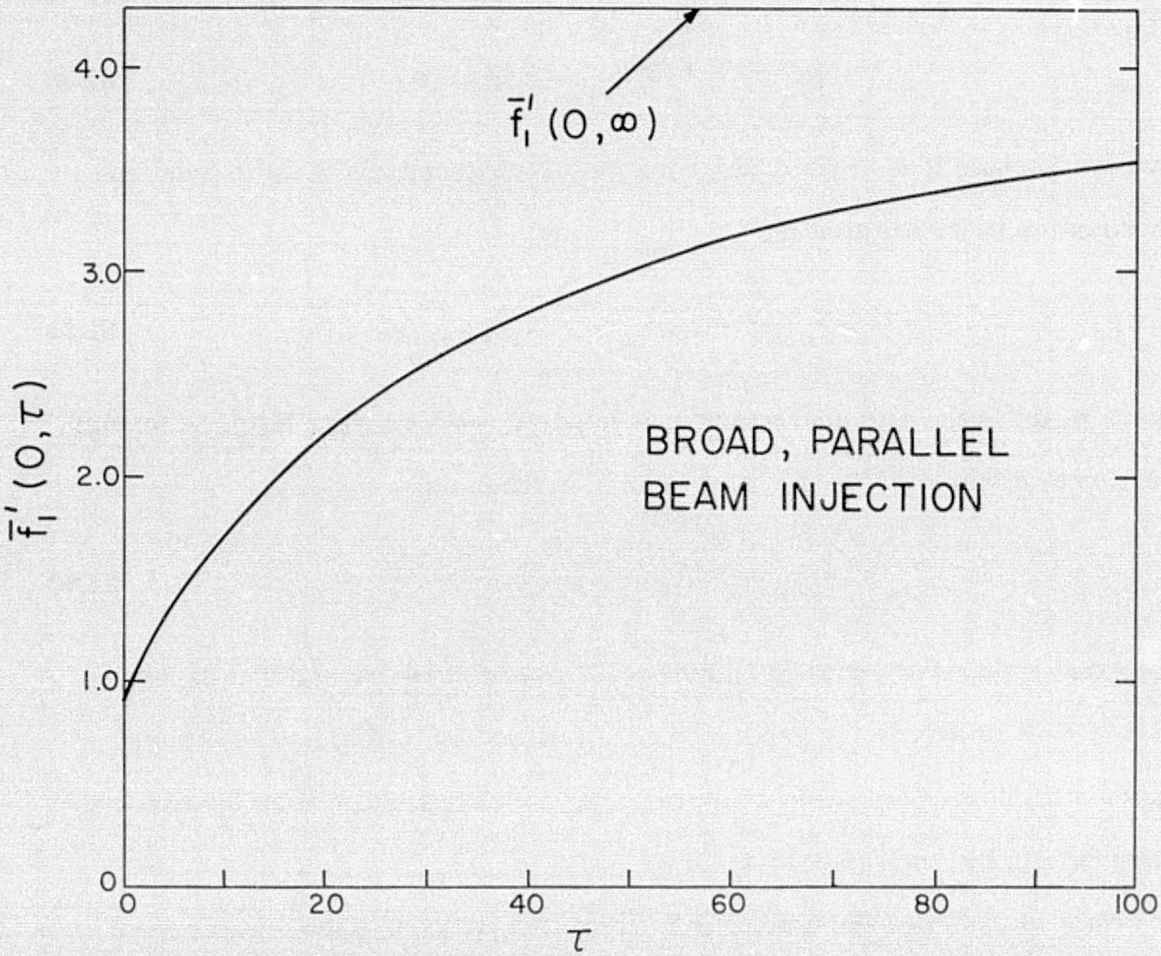


Figure 2. The first derivative with respect to $\mu$, evaluated at $\mu = 0$, of the first local approximation to the broad parallel beam injection solution computed earlier in papers I. and II.

## IV. THE GENERAL N'TH LOCAL APPROXIMATION

In paper II we constructed the local approximation method as it applies to the integrodifferential equation defined by equation II.1. The Laplace transform of that equation is,

$$p\tilde{f}(p) - f(0) = \epsilon \tilde{K}(p) \tilde{f}(p) \qquad \text{III.80}$$

and, the Laplace transform of the equation which generates the n'th local approximation to its solution is,

$$p\tilde{f}_n(p) - f(0) = \epsilon \tilde{K}_n(p) \tilde{f}_n(p) \qquad \text{III.81}$$

where $\tilde{K}_n(p)$ is the n'th convergent (see paper II, section II) to $\tilde{K}(p)$. A formal solution of either of these equations can be written as,

$$\tilde{f}_n(p) = \left[p - \epsilon \tilde{K}_n(p)\right]^{-1} f(0) \qquad \text{III.82}$$

The exact solution of equation III.80 is obtained in some exceptional cases for all $n \geq N$ (N finite), but in general can be obtained by letting $n \to \infty$. Thus, equation III.82 contains all of the local approximations, as well as the exact solution, within the entire set of n-values.

We have retained the notion, and associated notation, here that the local approximations are generated by those entries in the Pade' table (see paper II.) which are near the diagonal and are the convergents to K (p). However, the following discussion holds true for those approximations to the solution of equation II.1 which are generated by <u>any</u> of the Pade' approximants. Thus, for the discussion in this section we can generalize the definition of the local approximations to include all approximations which are generated by any Pade' table entry.

From equation III.82 we see that generally,

$$p\tilde{f}_n(p) = \left[1 - \epsilon\left(\frac{\tilde{K}_n(p)}{p}\right)\right]^{-1} f(0) \underset{p\downarrow}{\sim} \left[1 - \epsilon\left(\frac{\mathfrak{M}_0}{p}\right)\right]^{-1} f(0) \qquad \text{III.83}$$

for any n. Thus, through the application of the Tauberian theorems which we discussed in the previous two sections we can come to the very important conclusion that <u>the time averages (as defined through equation III.1) of all of the local approximations, including the exact solution, must approach a single final state given by,</u>

$$\lim_{p \to 0} \left[1 - \epsilon\left(\frac{\mathfrak{M}_0}{P}\right)\right]^{-1} f(0) \qquad \text{III.84}$$

There are exceptions to this very general statement; the failure of the adiabatic approximation is due to one of them. In the following we will discuss these exceptions briefly.

a). <u>The Exactly Soluble Model Equation of Paper II</u>

The exactly soluble model equation has a C-number kernel whose moments are given by equation II.48. In this case, from equation III.84, we expect that the time average of all of the local approximations, including the exact solution, to asymptote to zero as $\tau \to \infty$. From figures 2 through 5 of paper II we see that this expected behavior is obtained except in some special situations. First, from figure 4 we see that the third local approximation does not fit in to the general pattern predicted above. Indeed, as was discussed in paper II, $f_3(\tau) \sim -\infty\,(\tau \to \infty)$ for $0 < \mu \lesssim 0.6$. Since this local approximation is not bounded from below, we simply cannot apply the Tauberian theorems which lead to equation III.84. Thus, there is no contradiction here; we cannot apply equation III.84 to this case in the first place.

The remaining exceptions to the general statement made in the previous paragraph occur in figure 5 at $\mu = 0$ in the exactly soluble model equation. In this case we see that the time averages of the first, second, and third local approximations agree exactly with the time average of the exact solution, but, none of these time averages are zero. In addition, the time average of the zero'th local approximation (the adiabatic approximation) obviously does not agree with any of the others. This peculiar behavior occurs because $M_0(0) = 0$, and $M_1(0) = -1 \neq 0$. The time average given by equation III.84 must be replaced by,

$$\lim_{p \to 0} \left[1 - \epsilon\, M_1(0)\right]^{-1} f(0) \qquad \text{III.85}$$

in this situation. Since $M_1(0) = -1$ and $\epsilon = 1$ in the example being discussed here, we see that application of equation III.85 does lead to the prediction of the correct time averages (= 0.5) for the first through fourth local approximations. On the other hand, we see that the zero'th local approximation must fail to produce the correct time average at $\mu = 0$ because the dominant term in the continued fraction expansion of $K(p)$, namely $M_1(0)$, is apriori neglected in this case. Thus, we see that equation III.84 does yield the asymptotic state of the time averages of all the local approximations so long as $M_0 \neq 0$. If $M_0 = 0$, then other considerations must be made which lead to replacements for equation III.84 . . . for example, when $M_1 \neq 0$ then equation III.85 follows . . . and the adiabatic approximation must be abandoned.

b). <u>The Quasi-Linear Diabatic Equation</u>

We have seen in paper I (equations I.65 through I.67) that the Laplace transform of the exact solution of the quasi-linear diabatic equation can be written,

$$\tilde{f}(\mu, p) = \sum_{m=0}^{\infty} f_m(p)\,\psi_m(\mu, p) \qquad \text{III.86}$$

where the $\psi_m(\mu, p)$ are the Laplace modes and where,

$$p f_m(p) = \frac{\left(\frac{1}{E_m^2(p)}\right) \int_{-1}^{1} d\mu\, \psi_m(\mu, p)\, \overline{f}(\mu, 0)}{\left[1 + \eta^2\left(\frac{\lambda_m(p)}{p}\right)\right]} \qquad \text{III.87}$$

In addition, we found there that $\lambda_0(p) = 0$, and that $\lambda_1(0) = 0$, but, from part a). of section II in this paper we found that $p/\lambda_m(p) \sim 0\ (p \to 0)$ for all $m > 0$ including $m = 1$. Therefore, $p f_m(p) \sim 0\ (p \to 0)$ for all $m > 0$ and

$$p f_0(p) \underset{p\downarrow}{\sim} \frac{1}{2} \int_{-1}^{1} d\mu\, \overline{f}(\mu, 0) \qquad \text{III.88}$$

Thus, through the use of the Tauberian theorems we conclude (as we have earlier in this paper) that the time average of the quasi-linear diabatic solution must asymptote to the final isotropic state,

$$\frac{1}{2} \int_{-1}^{1} d\mu\, \overline{f}(\mu, 0) \qquad \text{III.89}$$

Equation III.89 gives the projection of the initial state onto that Laplace mode which is the only one outside the range of $\tilde{K}(p)$. The zero'th Laplace mode is given by $\psi_0(\mu, p) = 1$, and the projection of the initial state onto $\psi_0(\mu, p)$ is given by equation III.89. We say that $\psi_0(\mu, p)$ is outside the range of $\tilde{K}(p)$ because,

$$\tilde{K}(p)\,\psi_0(\mu, p) = \left[\frac{\partial}{\partial\mu}\,(1-\mu^2)\,\tilde{K}(\mu, p)\,\frac{\partial}{\partial\mu}\right]\psi_0(\mu, p) = 0 \qquad \text{III.90}$$

It is that projection of the initial state which is outside the range of $\tilde{K}(p)$ which survives as $\tau \to \infty$ and becomes the final state which the time average of the solution must approach. The first Laplace mode is not outside of the range of $\tilde{K}(p)$ since,

$$\tilde{K}(p)\,\psi_1(\mu, p) = -\lambda_1(p)\,\psi_1(\mu, p) \neq 0 \qquad (p > 0) \qquad \text{III.91}$$

But, in the zero'th local or adiabatic approximation we replace this relationship with,

$$\tilde{K}_0(p)\,\psi_1(\mu, 0) = -\lambda_1(0)\,\psi_1(\mu, 0) = 0 \qquad \text{III.92}$$

with the hope that $\psi_1(\mu, 0)$ and $\lambda_1(0)$ are reasonable approximations to $\psi_1(\mu, p)$ and $\lambda_1(p)$ when p is small. Unfortunately, in this attempt we have introduced a convergent to $\tilde{K}(p)$ which, crudely speaking, "contains a zero" which the original $\tilde{K}(p)$ did not. (This feature is analogous to that of the simple model equation discussed above in which $M_0(0) = 0$.) Speaking more precisely, we have introduced a convergent to $\tilde{K}(p)$ which has a reduced range compared to $\tilde{K}(p)$. This reduction in range then allows additional projections of the initial state outside the range of the kernel transform which survive as $\tau \to \infty$ and falsely modify the final state of the time averaged solution. More specifically, we have seen above that $p f_1(p) \sim 0$ $(p \to 0)$ and therefore the projection of the initial state onto $\psi_1(\mu, 0)$ cannot contribute to the final state. However, if we replace $\lambda_1(p)$ by its zero'th local approximation, $\lambda_1(0) = 0$, before taking the limit, $p \to 0$, we find erroneously that,

$$p f_1(p) \underset{p\downarrow}{\sim} \frac{1}{2} \int_{-1}^{1} d\mu \, \psi_1(\mu, 0) \, \bar{f}(\mu, 0) \qquad \text{III.93}$$

and then the erroneous final state for the time average of the solution contains a step in it at $\mu = 0$ due to the presence of $\psi_1(\mu, 0)$ (see figures 2 and 6 of paper I).

At this point we are able to make the statement preceding equation III.84 more precise. We can say that the time averages of all those local approximations, including the exact solution, which are bounded functions of time must approach a single final state. This statement does not apply to those local approximations which are generated by a convergent, $\tilde{K}_n(p)$, which has a reduced range compared to the range of the exact operator kernel transform, $\tilde{K}(p)$. We feel that this general principle should be considered in future statistical mechanical studies of the approach to equilibrium.

## V. CONCLUSION

In this series of three papers we have addressed two separate issues. In order to clarify these individual issues we present here a brief summary of our work.

We begin our summary with the equation which Klimas and Sandri derived (see equation I.8) from first principles, using the method of Kaufman, which governs the exact evolution of the ensemble averaged test particle probability distribution function in magnetostatic turbulence. This "master equation" . . . an unfortunate name, since it has nothing to do with the well known master equation formalism . . . follows directly from the Liouville equation through averaging over an ensemble of stochastic magnetic fields. In principle this master equation yields an exact solution; in practice, it must be simplified in order to extract any information from it at all.

Two steps must be taken in order to reduce the master equation to the Fokker-Planck equation. First, if either the magnetostatic turbulence is weak or if the energy of the test particle is high, a small parameter enters the master equation which allows its expansion. If a power series expansion of the master equation is constructed and then truncated beyond the leading non-trivial term, then the quasi-linear approximation results. The product of this truncation is an integro-differential equation in time which, in paper I., we named the quasi-linear diabatic equation (see equation I.20 for a specific example in a simple model of the magnetostatic turbulence and also following gyro-phase averaging). In the second step, we apply the adiabatic approximation which reduces the quasi-linear diabatic equation to the quasi-linear adiabatic equation that is also known as the Fokker-Planck equation. In this step we have reduced the quasi-linear diabatic equation, which is integro-differential or non-local in time, to its Markovian approximation. The end result is a velocity space diffusion equation which is first order partial differential in time and which governs the evolution of the probability distribution function in the spirit of the Markov chain; i.e., changes in the probability distribution function depend only on its instantaneous state, and are independent of the manner in which this state was achieved.

As we have discussed in paper I., solutions of the Fokker-Planck equation are so blatantly unreasonable that even in the absence of any exact solutions of the master equation we are able to conclude that something went wrong in our two step simplification which lead to the Fokker-Planck equation. The standard response to this failure has been various attempts, labeled non-linear, or orbit modification, or diffusing orbit, or partially averaged theories which are all attempts to renormalize the power series expansion of the master equation in such a way that, on further applying the adiabatic approximation, the result is a velocity space diffusion equation much like the Fokker-Planck equation but with a diffusion coefficient which is considered more reasonable. These attempts are all based upon the apriori prejudice that a reasonable approximation to the unavailable solutions of the paster equation must be governed by a Markov chain of events as expressed through a velocity space diffusion equation similar to the Fokker-Planck equation. However, as we have just mentioned, exact solutions of the master equation are not available, and furthermore, to our knowledge, solutions of the quasi-linear diabatic equation, which follows after step one mentioned above, and before the imposition of the Markovian approximation, have also never been calculated prior to this work. Thus, the predictions of the non-Markovian theory have been unknown. In addition, as we showed in paper I., the imposition of step two, the adiabatic approximation which leads to the Markov chain, clearly leads to an incorrect description of the quasi-linear diabatic solutions. In fact, although it is only step two of the simplification procedure which leads to the Fokker-Planck equation which is clearly in error, it is only step one which has been modified in all previous attempts to correct the Fokker-Planck predictions. Apparently, the idea of a Markov chain is so imbedded that its abandonment has not been considered even in situations where it clearly introduces serious errors.

Thus, we come to the two separate issues which we have considered in these papers. First, we ask, "What are the predictions of the non-Markovian quasi-linear diabatic equation, and do its solutions indicate that a renormalization of the power series expansion is indicated?" and second, regardless of the necessity or not of the renormalization, we ask, "Is there a modification, or replacement, for the adiabatic approximation which then allows us to produce a successful approximation to the solutions of integro-differential equations like the quasi-linear diabatic equation?" After all, if modifications of the quasi-linear diabatic solutions are necessary, then it makes sense to start with the quasi-linear diabatic solutions and see what direction these modifications must take. In addition, since solutions of the quasi-linear diabatic equation are very difficult to obtain, even numerically, a simpler method for obtaining good approximations to these solutions cannot hurt in this endeavor. If this method happens to be powerful enough to be applicable more generally, then all the better.

In paper II. of this series we introduced the new local approximation method. In that paper we showed that this approximation method can be applied to the fundamental equations of statistical mechanics if they are cast in the integro-differential form of the quasi-linear diabatic equation (the master equation is included). In addition to showing the wide applicability of this method we also demonstrated its application to a very simple exactly soluble problem and furthermore, to the quasi-linear diabatic equation which we constructed in paper I. In general we found this new approximation method to be remarkably successful. However, in the case of the quasi-linear diabatic equation, since both the exact and the approximate solutions could only be obtained through numerical integrations of the respective equations, the comparisons we have of these solutions extend only over a little more than two Larmor periods of elapsed time. Thus, in this paper we have presented an analytic study of the long time behavior of both the quasi-linear diabatic solutions and their local approximations.

We have found in this paper that the first local approximation to the quasi-linear diabatic solutions probably remains remarkably accurate for all time. More specifically, we have found that an H-theorem can be constructed for the first local approximation which, under fairly general conditions, allows us to conclude that the first local approximate solutions must asymptote, with increasing time, to exactly that final state which the time average of the quasi-linear diabatic solutions must also approach. Given the remarkable agreement between these solutions during early times, and also this manner of agreement for very large times, we feel reasonably confident in the accuracy of the first local approximation for all time. Thus, we feel that the second of the two questions posed above has been answered. The non-Markovian local approximation method does successfully replace the adiabatic approximation in applications to integro-differential equations like the quasi-linear diabatic equation.

From the results of this paper, the first question posed above can now be answered as well. From paper I. (see discussion leading to equation I.10) we can see that the power series expansion of the master equation which we discussed above is actually in powers of $\eta\, G_0\, \mathfrak{L}'$ where $\eta$ is our small parameter and $G_0$ and $\mathfrak{L}'$ are linear operators which are defined fully in paper I. For our purposes here, it is sufficient to realize that $\mathfrak{L}'$ is actually a first order differential operator with respect to the velocity phase space variables. Thus, if the quasi-linear diabatic solutions are to be considered a leading approximation to the solutions of the master equation in a perturbation expansion in $\eta$, it is clear that the quasi-linear diabatic solutions should not contain velocity space gradients which grow large with decreasing $\eta$. But this kind of behavior is exactly what we have discovered here. We have found that as $\tau \to \infty$ the first derivative with respect to $\mu$ of the time average of the quasi-linear diabatic solutions approaches a value which, at $\mu = 0$, is proportional to $\eta^{-2}$. Thus, at $\mu = 0$, $\eta G_0 \mathfrak{L}' \sim O(\eta^{-1})$ $(\tau \to \infty)$ when operating on the quasi-linear diabatic solutions. We must conclude that we have no basis for believing that the power series expansion converges in the vicinity of $\mu = 0$; in fact the evidence seems to indicate just the opposite. It seems clear that, in the vicinity of $\mu = 0$, some presently unknown renormalization of the power series expansion must be constructed. However, this renormalization cannot be based upon the Fokker-Planck approximation since we have shown here that this approximation does not accurately represent the leading term in the power series expansion. We point out the possibility, which we feel as a result of this work is quite likely, that the renormalized result may not be Markovian nor governed by a velocity space diffusion equation.

REFERENCES


1. A. J. Klimas, G. Sandri, J. D. Scudder and D. R. Howell, Test Particle Propagation in Magnetostatic Turbulence I. Failure of the Diffusion Approximation, NASA X-Document # X-692-76-207, Sept., 1976.
2. M. L. Goldstein, A. J. Klimas and G. Sandri, Ap. J. 195, 787 (1975).
3. G. Sandri, Ann. Phys. 24, 332 (1963).
4. A. J. Klimas, G. Sandri, J. D. Scudder and D. R. Howell, Test Particle Propagation in Magnetostatic Turbulence II. The Local Approximation Method, NASA X-Document # X-692-76-252, Nov., 1976.
5. D. V. Widder, The Laplace Transform, Princeton University Press, Princeton, N. J. (1946), p. 192.
6. D. V. Widder, The Laplace Transform, Princeton University Press, Princeton, N. J. (1946), p. 221.

## APPENDIX A

In this appendix we restate Wiener's generalized Tauberian theorem as given by theorem 13b., Chapter V of Widder, but in our notation. We then apply Wiener's theorem to Laplace transforms, much in the same way as Widder does to obtain his theorem 14 in Chapter V, but with a slight modification which allows us to then conclude equation III.50.

a). Statement of Theorem 13b

Let

1). $g_1(\tau) \geq 0 \qquad g_1(\tau) \in L \qquad (0 \leq \tau < \infty)$

(functions in L can be represented by absolutely convergent Fourier integrals)

2). $$\int_0^\infty d\tau\, g_1(\tau)\, \tau^{i\chi} \neq 0 \qquad (-\infty \leq \chi < \infty)$$

3). $F(\tau) \geq -c$ (c finite and positive)

4). $g_2(\tau)$ be continuous almost everywhere on $(0, \infty)$

5). $$\sum_{n=-\infty}^{\infty} \underset{e^n < \tau \leq e^{n+1}}{\text{u. b.}} \tau\, |g_2(\tau)| < \infty$$

6). $$h_1(p) = p \int_0^\infty d\tau\, g_1(p\tau)\, F(\tau)$$

exist and be bounded for $(0 < p < \infty)$.

Then if $h_1(0)$ exists,

$$\lim_{p \to 0} p \int_0^\infty d\tau\, g_2(p\tau)\, F(\tau) = h_1(0) \frac{\int_0^\infty d\tau\, g_2(\tau)}{\int_0^\infty d\tau\, g_1(\tau)} \qquad \text{A.1}$$

b). <u>Application to Laplace Transforms</u>

We introduce the notation, $\bar{f}'(\mu, \tau) = \partial \bar{f}(\mu, \tau)/\partial\mu$ and use the theorem stated above to prove the following: If $\bar{f}'(\mu, \tau) \geq -c$ for $0 \leq \tau < \infty$ and is such that the integral,

$$\int_0^\infty d\tau \; e^{-p\tau} \, \bar{f}'(\mu, \tau) \qquad \text{A.2}$$

converges for all $p > 0$ (i.e., $\bar{f}'(\mu, \tau)$ has a Laplace transform for $p > 0$), and if,

$$\int_0^\infty d\tau \; e^{-p\tau} \, \bar{f}'(\mu, \tau) \underset{p\downarrow}{\sim} \frac{A(\mu)}{p} \qquad \text{A.3}$$

then,

$$\frac{1}{\tau} \int_0^\tau ds \; \bar{f}'(\mu, s) \underset{\tau\uparrow}{\sim} A(\mu) \qquad \text{A.4}$$

Proof:

First notice that,

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^1 ds \; \bar{f}'(\mu, s) = 0 \qquad \text{A.5}$$

and since,

$$\left| p \int_0^1 d\tau \; e^{-p\tau} \, \bar{f}'(\mu, \tau) \right| \leq p \int_0^1 d\tau \, |\bar{f}'(\mu, \tau)| \qquad \text{A.6}$$

then,

$$\lim_{p \to 0} p \int_0^1 d\tau \; e^{-p\tau} \, \bar{f}'(\mu, \tau) = 0 \qquad \text{A.7}$$

Thus, if we introduce,

$$F(\mu, \tau) = \begin{cases} 0 & (0 \leq \tau \leq 1) \\ \bar{f}'(\mu, \tau) & (1 < \tau < \infty) \end{cases} \qquad \text{A.8}$$

and complete this proof for $F(\mu, \tau)$, then we will have completed this proof for $\bar{f}'(\mu, \tau)$ as well.

Following Widder, we chose

$$g_1(\tau) = e^{-\tau} \qquad (0 \leq \tau < \infty) \qquad \text{A.9}$$

and,

$$g_2(\tau) = \begin{cases} 1 & (0 \leq \tau < 1) \\ 0 & (1 < \tau < \infty) \end{cases} \qquad \text{A.10}$$

Then conditions 1)., 3). and 4). of section a). are obviously satisfied. Condition 2). holds since

$$\int_0^\infty d\tau \; \tau^{i\chi} e^{-\tau} = \Gamma(1 + i\chi) \neq 0 \qquad (-\infty < \chi < \infty) \qquad \text{A.11}$$

Condition 5). becomes

$$\sum_{n=-\infty}^{\infty} \underset{e^n < \chi \leq e^{n+1}}{\text{u. b.}} \tau \left| g_2(\tau) \right| = \sum_{n=-\infty}^{-1} e^{n+1} = \frac{e}{e - 1} \qquad \text{A.12}$$

Finally 6). is satisfied since

$$h_1(p) = p \int_1^\infty d\tau \; e^{-p\tau} F(\mu, \tau) \qquad \text{A.13}$$

is clearly continuous in the interval $(0 < p < \infty)$ and approaches finite limits when p approaches zero or becomes infinite. By equation A.3, $h_1(p)$ approaches $A(\mu)$ as $p \to 0$. For $\tau \geq 1$, and $1 < p < \infty$,

$$p\tau = \tau + \left(1 - \frac{1}{p}\right) p\tau \geq \tau + \left(1 - \frac{1}{p}\right) p \qquad \text{A.14}$$

Therefore,

$$\begin{aligned} |h_1(p)| &\leq p \int_0^\infty d\tau \; e^{-p\tau} \, |F(\mu, \tau)| \\ &\leq p \; e^{-(p-1)} \int_1^\infty d\tau \; e^{-\tau} \, |F(\mu, \tau)| \qquad \text{A.15} \\ &\underset{p\uparrow}{\sim} 0 \end{aligned}$$

All conditions of section a). are satisfied and when we apply equation A.1 we obtain the desired result, equation A.4. This result has been used to obtain equation III.50 in section II.

## APPENDIX B

In order to construct the H-theorem in section III., in addition to assuming a "simple" power spectrum we have assumed that $\mathfrak{M}_1(\mu) \leq 0$ where $\mathfrak{M}_1(\mu)$ is defined by equation III.57. In addition we have assumed that $\mathfrak{M}_1(\mu = \pm 1)$ is bounded. Since the correlation function is normally assumed to have a finite spatial range such that its moments exist, it is natural to assume $\mathfrak{M}_1(\mu = \pm 1)$ is bounded. We have not found a clear characterization of the magnetostatic turbulence which would guarantee that $\mathfrak{M}_1(\mu) \leq 0$ however. In the following we simply list a number of expressions for $\mathfrak{M}_1(\mu)$ with the hope that at least one of these may prove useful in future studies of this issue.

Following immediately from its definition, it is easy to see that,

$$\mathfrak{M}_1(\mu) = \frac{1}{\mu^2} \int_0^\infty ds \; s \; R(s) \cos\left(\frac{s}{\mu}\right) \qquad \text{B.1}$$

and therefore,

$$\mathfrak{M}_1(\mu) = \frac{\partial}{\partial |\mu|} \int_0^\infty ds \; R(s) \sin\left(\frac{s}{|\mu|}\right) \qquad \text{B.2}$$

With the introduction of the Fourier transform of the correlation function,

$$\bar{R}(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty d\zeta \; e^{-iK\zeta} R(\zeta) \qquad \text{B.3}$$

we also find,

$$\mathfrak{M}_1(\mu) = \lim_{p \to 0} \frac{-1}{\sqrt{2\pi}} \int_0^\infty dK\, \bar{R}(K) \left[ \frac{1 - 2p^2}{\left[p^2 + (K\mu - 1)^2\right]^2} + \frac{1 - 2p^2}{\left[p^2 + (K\mu + 1)^2\right]^2} \right] \tag{B.3}$$

which can be expressed as a Cauchy principle value

$$\mathfrak{M}_1(\mu) = - \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\mu^2}\right) \int_{-\infty}^{\infty} dK\, \bar{R}'\left(K + \frac{1}{|\mu|}\right) \frac{\mathcal{P}}{K} \tag{B.4}$$

More explicitly,

$$\mathfrak{M}_1(\mu) = - \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\mu^2}\right) \int_\epsilon^\infty \frac{dK}{K} \left[ \bar{R}'\left(K + \frac{1}{|\mu|}\right) - \bar{R}'\left(-K + \frac{1}{|\mu|}\right) \right] \tag{B.5}$$

and through integration by parts,

$$\mathfrak{M}_1(\mu) = - \lim_{\epsilon \to 0} \left\{ \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\mu^2}\right) \int_\epsilon^\infty dK \left(\frac{1}{K}\right)^2 \left[ \bar{R}\left(K + \frac{1}{|\mu|}\right) + \bar{R}\left(K - \frac{1}{|\mu|}\right) \right] - \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\mu^2}\right) \left[ \frac{\bar{R}\left(\epsilon + \frac{1}{|\mu|}\right) + \bar{R}\left(\epsilon - \frac{1}{|\mu|}\right)}{\epsilon} \right] \right\} \tag{B.6}$$

All of these expressions yield,

$$\mathfrak{M}_1(\mu) = - \frac{1 - \mu^2}{(1 + \mu^2)^2} \tag{B.7}$$

for the exponential correlation function used in the analysis contained in papers I. and II.