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16. Abstract

Linear least squares estimation and regression analyses continue to play a major role in orbit determination and related areas. In this report we document a library of FORTRAN subroutines that have been developed to facilitate analyses of a variety of parameter estimation problems. Our purpose is to present an easy to use multi-purpose set of algorithms that are reasonably efficient and which use a minimal amount of computer storage. Subroutine inputs, outputs, usage and listings are given, along with examples of how these routine can be used. The following outline indicates the scope of this report: Section I, introduction with reference to background materials; Section II, examples and applications; Section III, a subroutine directory summary; Section IV, the subroutine directory user description with input, output and usage explained; and Section V, subroutine FORTRAN listings. The routines are compact and efficient and are far superior to the normal equation data processing algorithms that are often used for least squares analyses.

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A Parameter Estimation Subroutine Package

G. J. Bierman
M. W. Nead

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California Institute of Technology
Pasadena, California 91103

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PREFACE

The work described in this report was performed by the Systems Division of the Jet Propulsion Laboratory.

AKNOWLEDGEMENT

The construction of this estimation subroutine package (ESP) was motivated by an involvement with a particular problem; construction of fast, efficient and simple least squares data processing algorithms to be used for determining ephemeris corrections. Discussions with T. Duxbury led to the proposal of a subroutine strategy which would have great flexibility. The general utility of such a subroutine package was made evident by H. Koble and N. Mottinger who had a different but related problem that involved combining estimates from different missions. Thanks and credit are also due to J. Ellis, N. Hamata, and F. Peters for contributing to and experimenting with this package of subroutines.

ABSTRACT

Linear least squares estimation and regression analyses continue to play a major role in orbit determination and related areas. In this report we document a library of FORTRAN subroutines that have been developed to facilitate analyses of a variety of parameter estimation problems. Our purpose is to present an easy to use multi-purpose set of algorithms that are reasonably efficient and which use a minimal amount of computer storage. Subroutine inputs, outputs, usage and listings are given, along with examples of how these routines can be used. The following outline indicates the scope of this report: Section I, introduction with reference to background material; Section II, examples and applications; Section III, a subroutine directory summary; Section IV, the subroutine directory user description with input, output and usage explained; and Section V, subroutine FORTRAN listings. The routines are compact and efficient and are far superior to the normal equation data processing algorithms that are often used for least squares analyses.

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I. Introduction

Techniques related to least squares parameter estimation play a prominent role in orbit determination and related analyses. Numerical and algorithmic aspects of least squares computation are documented in the excellent reference work by Lawson and Hanson, Ref. [1]. Their algorithms, available from the JPL subroutine library, Ref. [2], are very reliable and general. Experience has, however, shown that in reasonably well posed problems one can streamline the least squares algorithm codes and reduce both storage and computer times. In this report, we document a collection of subroutines most of which we have written that can be used to solve a variety of parameter estimation problems.

The algorithms for the most part involve triangular and/or symmetric matrices and to reduce storage requirements these are stored in vector form, e.g., an upper triangular matrix U is written as

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & \\ & U_{22} & U_{23} & U_{24} & \text{etc.} \\ & & U_{33} & U_{34} & \\ 0 & & & U_{44} & \end{bmatrix} = \begin{bmatrix} U(1) & U(2) & U(4) & U(7) & \\ & U(3) & U(5) & U(8) & \text{etc.} \\ & & U(6) & U(9) & \\ 0 & & & U(10) & \end{bmatrix}$$

Thus, the element from row i and column j of U, $i \le j$, is stored in vector component $j(j-1)/2 + i$. We hasten to point out that the engineer, with few exceptions, need have no direct contact with the vector subscripting. By this we mean that the vector subscript related operations are internal to the subroutines, vector arrays transmitted from one subroutine to another are compatible, and vector arrays displayed using the print subroutine TRIMAT appear in a triangular matrix format.

Aside: The most notable exception is that matrix problems are generally formulated using doubly subscripted arrays. Transforming a double subscripted symmetric or upper triangular matrix A(·,·) to a vector stored form, U(·) is quite simply accomplished in FORTRAN via

```
      IJ = 0
      DO 1 J = 1,N
      DO 1 I = 1,J
      IJ = IJ+1
    1 U(IJ) = A(I,J)
```

Similarly, transforming an initial vector D(·) of diagonal positions of a vector stored form, U(·), is accomplished using

```
      JJ = 0                      JJ = N*(N+1)/2
      DO 1 J = 1,N        or      DO 1 J = N,1,-1
      JJ = JJ+J                   U(JJ) = D(J)
    1 U(JJ) = D(J)              1 JJ = JJ-J
```

The conversion on the right has the modest advantage that D and U can share common storage (i.e., U can overwrite D). These conversions are too brief to be efficiently used as subroutines. It seems that when such conversions are needed one can readily include them as in line code.

End of Aside

Although this package of subroutines is designed in the main, for the analysis of parameter estimation problems one can use it to solve problems that involve process noise. With modest amounts of additional programming one can even apply our package to filtering problems that involve colored noise and mapping. In the latter case, however, reductions gained from our use of vector storage are for the most part lost.

Mathematical background regarding Householder orthogonal transformations for least squares analyses and U-D matrix factorization for covariance matrix analyses are discussed in references [1] and [3]. Our plan is to illustrate, in Section II, with examples how one can use the basic algorithms and matrix manipulation to solve a variety of important problems. The subroutines which comprise our estimation subroutine package are described in Section III, and detailed input/output descriptions are presented in Section IV.

Section V contains FORTRAN listings of the subroutines. There are several reasons for including such listings. Making these listings available to the engineer analyst allows him to assess algorithm complexity for himself; and to appreciate the simplicity of the routines he tends otherwise to use as a black box. The routines are not truly portable, and users can, when necessary make modifications so that the subroutine package can operate on systems other than the UNIVAC 1108. When estimation problems arise to which our package does not directly apply (or which can be made to apply by an awkward concatenation of the routines) one may be able to modify the codes and widen still further the class of problems that can be efficiently solved.

II. APPLICATIONS AND EXAMPLES

Our purpose in this section is to illustrate, with a number of examples, some of the problems that can be solved using this ESP. The examples, in addition, serve to catalogue certain estimation techniques that are quite useful.

To begin, let us catalogue the subroutines that comprise the ESP:

1)	AGTRN	(A G Turner)	Agee-Turner rank 1 update
2)	A2A1	(A to A one)	Matrix A to matrix A1
3)	COMBO	(combo)	Combine R and A namelists
4)	COV2RI	(cov to R I)	Covariance to R inverse
5)	COV2UD	(cov to U D)	Covariance to U-D factors
6)	C2C	(C to C)	Permute the rows and columns of matrix C
7)	INF2R	(inf to R)	Information matrix to (triangular) R
8)	PERMUT	(permute)	Permute the columns of matrix A
9)	RINCON	(rin con)	R inverse with condition number bound
10)	RI2COV	(R I to cov)	R inverse to covariance
11)	R2A	(R to A)	Triangular R to matrix A
12)	R2RA	(R to R A)	Transfer a triangular block of R to triangular RA
13)	RUDR	(rudder)	SRIF R to U-D factors or vice versa
14)	THH	(T H H)	Triangular Householder data processing
15)	TRIMAT	(tri mat)	Triangular matrix print
16)	TTHH	(T T H H)	Two triangular matrix Householder processing
17)	TZERO	(T zero)	Zero a horizontal segment of a triangular matrix

18) UDMES (U D measurement) U-D measurement updating

19) UD2COV (U D to cov) U-D factors to covariance

20) UD2SIG (U D to sig) U-D factors to sigmas

21) UTINV (U T inverse) Upper triangular matrix inverse

22) UTIROW Upper triangular inverse, inverting only the upper rows

23) WGS (W G-S) Weighted Gram-Schmidt triangular reduction

These routines are described in succeedingly more detail in sections III, IV, and V. The examples to follow are chosen to demonstrate how these various subroutines can be used to solve orbit determination and other parameter estimation problems. It is important to keep in mind that these examples are not by any means all inclusive, and that this package of subroutines has a wide scope of applicability.

II.1 Simple Least Squares

Given data in the form of an overdetermined systems of linear equations one may want a) the least squares solution; b) the estimate error covariance, assuming that the data has normalized errors; and c) the sum of squares of the residuals. The solution to this problem, using the ESP can be symbolically depicted as

- $[A\ z] \xrightarrow{\text{THH}} [\hat{R}\ \hat{z}],\ e$

Remarks: The array $[A\ z]$ corresponds to the equations $Ax = z-\nu$, $\nu \varepsilon N(0,I)$; the array $[\hat{R}\ \hat{z}]$ corresponds to the triangular data equation $\hat{R}x = \hat{z}-\hat{\nu}$, $\nu \varepsilon N(0,I)$ and $e = ||z-A\hat{x}||$

- $[\hat{R}\ \hat{z}] \xrightarrow{\text{UTINV}} [\hat{R}^{-1}\ \hat{x}]$

Remark: $\hat{x} = \hat{R}^{-1}\ \hat{z}$

One may be concerned with the integrity of the computed inverse and the estimate. If one uses subroutine RINCON instead of UTINV then in addition one obtains an estimate (lower and upper bounds) for the condition number R. If this condition number estimate is large the computed inverse and estimate are to be regarded with suspicion. By large, we mean considerable with the machine accuracy (viz. on an 18 decimal digit machine numbers larger than 10^{15}). Note that the condition number estimate is obtained with negligible additional computation and storage.

- $[\hat{R}^{-1}] \xrightarrow{\text{RI2COV}} [C]$

Remarks: $C = \hat{R}^{-1}\hat{R}^{-T}$ = estimate error covariance. Some computation can be avoided in RI2COV if only some (or all) of the standard deviations are wanted.

II.2 Least Squares With A Priori

If a priori information is given, it can be included as additional equations (in the A array) or used to initialize the R array in subroutine THH (see the subroutine argument description given in section IV). One is sometimes interested in seeing how the estimate and/or the formal statistics change corresponding to the use of different a priori conditions. In this case one should compute $[\hat{R}\ \hat{z}]$ as in case II.1, and then include the a priori $[R_o\ z_o]$ using either subroutine THH, or subroutine TTHH when the a priori is diagonal or triangular, e.g.,

- $\left.\begin{matrix}[\hat{R}\ \hat{z}] \\ [R_o z_o]\end{matrix}\right\} \xrightarrow{\text{TTHH}} [\hat{R}\ \hat{z}]^*$

*The new result overwrites the old.

It is often good practice to process the data and form $[\hat{R}\ \hat{z}]$ before including the effects of a priori. When this is done one can analyze the effect of different a priori, $[R_o\ z_o]$ without reprocessing the data.

If a priori is given in the form of an information matrix, Λ, (as for example would be the case if the problem is being initialized with data processed using normal equation data accumulation*) then one can obtain R_o from Λ using INF2R;

$$\Lambda \xrightarrow{\text{INF2R}} R_o$$

If there were a normal equation estimate $z = A^T b$, then $z_o = R_o^{-T} z$.

II.3 Batch Sequential Data Processing

Prime reasons for batch sequential data processing are that many problems are too large to fit in core, are too expensive in terms of core cost, and for certain problems it is desirable to be able to incorporate new data as it becomes available. Subroutines TTH and UDMES are specially designed for this kind of problem. Both of these subroutines overwrite the a priori with the result which then acts as a priori for the next batch of data. If the data is stored on a file or tape as A_1, z_1, A_2, z_2,... then the sequential process can be represented as follows:

SRIF Processing**

a) Initialize [R z] with a-priori values or zero

b) Read the next [A z] from the file

*i.e., solving $Ax = b-\nu$ with normal equations, $A^T A\hat{x}_o = A^T b$; $\Lambda = A^T A$ is the information matrix.

**The acronym SRIF represents Square Root Information Filter. The SRIF is discussed at length in reference [3].

c) $\left.\begin{matrix}[\hat{R}\ \hat{z}] \\ [A\ z]\end{matrix}\right\} \xrightarrow{\text{THH}} [\hat{R}\ \hat{z}]^*$

d) If there is more data go back to b)

e) Compute estimates and/or covariances using UTINV and RI2COV (as in example II.1)

U-D** Processing

a´) Initialize $[\hat{U}\text{-}\hat{D}\ \hat{x}]$ with a priori U-D information and estimate

b´) Read the next [A z] scalar measurement from the file

c´) $\left.\begin{matrix}[\hat{U}\text{-}\hat{D}\ \hat{x}] \\ [A\ z]\end{matrix}\right\} \xrightarrow{\text{UDMES}} [\hat{U}\text{-}\hat{D}\ \hat{x}]^*$

d´) If there is more data go back to b´)

e´) Compute standard deviations or covariances using UD2SIG or UD2COV.

Note that subroutine THH is best (most efficiently) used with data batches of substantial size (say 5 or more) and that UDMES processes measurement vectors one component at a time. If the dimension of the state is small the cost of using either method is generally negligible. The UDMES subroutine is best used in problems where estimates are wanted with great frequency or where one wishes to monitor the effects of each update. In a given application one might choose to process data in batches for awhile and during critical periods it may be

*The new result overwrites the old.

**U-D processing is a numerically stable algorithmic formulation of the Kalman filter measurement update algorithm, cf reference [3]. The estimate error covariance is used in its UDU^T factored form, where U is unit upper triangular and D is diagonal.

desirable to monitor the updating process on a point by point basis. In cases such as this, one may use RUDR to convert a SRIF array to U-D form or vice-versa.

Remarks: Another case where an R to U-D conversion can be useful occurs in large order problems (with say 100 or more parameters) where after data has been SRIF processed one wants to examine estimate and/or covariance sensitivity to the a priori variances of only a few of the variables. Here it may be more convenient to update using the UDMES subroutine.

II.4 Reduced State Estimates and/or Covariances From a SRIF Array

Suppose, for example, that data has been processed and that we have a triangular SRIF array $[\hat{R}\ \hat{z}]$ corresponding to the 14 parameter names, a_r, a_x, a_y, x, y, z, v_x, v_y, v_z, GM, CU41, LO41, CU43, LO43 (constant spacecraft accelerations, position and velocity, target body gravitational constant, and spin axis and longitude station location errors).

Let us ask first what would the computed error covariance be of a model containing only the first 10 variables, i.e., by ignoring the effect of the station location errors. One would apply UTINV and RI2COV just as in example II.1, except here we would use N (the dimension of the filter) = 10, instead of N=14.

Next, suppose that we want the solution and associated covariance of the model without the 3 acceleration errors. One ESP solution is to use

- $[\hat{R}\ \hat{z}] \xrightarrow{\text{R2A}} [A]$

  NAME ORDER OF A

  $x,\ y,\ z,\ v_x,\ v_y,\ v_z,$

  GM, CU41, LO41, CU43, LO43,

  RHS*, $a_r,\ a_x,\ a_y,$

Remark: One could also have used subroutine COMBO, with the desired namelist as simply a_r, a_x, a_y. This would achieve the same A matrix form.

- $[A] \xrightarrow{\text{THH}} [R]$

Remark: R here can replace the original $\hat{R}$ and $\hat{z}$.

- $[R] \xrightarrow{\text{UTINV}} [R^{-1}\ x_{est}] \xrightarrow{\text{RI2COV}} [\text{COV}\ x_{est}]$

Remarks: Here, use only N=11, i.e., 11 variables and the RHS. x_{est} is the 11 state estimate based on a model that does not contain acceleration errors a_r, a_x, or a_y.

Note how triangularizing the rearranged R matrix produces the desired lower dimensional SRIF array; and this is the same result one would obtain if the original data had been fit using the 11 state model.

As the last subcase of this example suppose that one is only interested in the SRIF array corresponding to the position and velocity variables. The difference between this example and the one above is that here we want to include the effects due to the other variables.

---

*z is often given the label RHS (right hand side)

One might want this sub-array to combine with a position-velocity SRIF array obtained from, say, optical data. One method to use would be,

- $[\hat{R}\ \hat{z}] \xrightarrow{\text{R2RA}} [R_A\ z_A]$

INPUT NAMES:	OUTPUT NAMES:
a_r, a_x, a_y, x, y, z, v_x, v_y, v_z, GM	x, y, z, v_x, v_y, v_z, GM
CU41, LO41, CU43, LO43, RHS	CU41, LO41, CU43, LO43, RHS

Remark: The lower triangle starting with x is copied into R_A.

- $[R_A\ z_A] \xrightarrow{\text{R2A}} [A,\ z_A]$ (Reordering)

  NAMES: GM, CU41, LO41, CU43, LO43,
  x, y, z, v_x, v_y, v_z, RHS

- $[A,\ z_A] \xrightarrow{\text{THH}} [\hat{R}_A\ \hat{z}_A]$ (Triangularizing)
- $[\hat{R}_A\ \hat{z}_A] \xrightarrow{\text{R2RA}} [R_x\ z_x]$ (Shifting array)

  NAMES: x, y, z, v_x, v_y, v_z, RHS

Remark: The lower right triangle starting with x is copied into R_x. We note that one could have elected to use COMBO in place of the first R2RA usage and R2A; this would have involved slightly more storage, but a lesser number of inputs. The sequence of operations is in this case,

- $[\hat{R}\ \hat{z}] \xrightarrow{\text{COMBO}} [A\ z]$

  ORIGINAL NAMES DESIRED NAMES: x, y, z, v_x, v_y, v_z, RHS

Remark: By using COMBO the columns of $[\hat{R}\ \hat{z}]$ are ordered corresponding to the names a_r, a_x, a_y, GM, CU41, LO41, CU43, and LO43, followed by the desired names list.

- $[A\ z] \xrightarrow{\text{THH}} [\hat{R}\ \hat{z}]$

<u>Remark</u>: The $[\hat{R}\ \hat{z}]$ array that is output from this procedure is equivalent but different from the $[\hat{R}\ \hat{z}]$ array that we began with.

- $[\hat{R}\ \hat{z}] \xrightarrow{\text{R2RA}} [R_x\ z_x]$

<u>Remark</u>: As before, the lower right triangle starting with x is copied into R_x.

To delete the last k parameters from a SRIF array, it is not necessary to use subroutines R2A and THH. The first $N - k = \overline{N}$ columns of the array already correspond to a square root information matrix of the reduced system. If estimates are involved one can simply move the z column left using:

$$R\ (\overline{N}*(\overline{N} + 1)/2 + i) = R(N*(N+1)/2 + i),\ i = 1,\ldots,k.$$

<u>Remark</u>: We mention in passing that if one is only interested in estimates and/or covariances corresponding to the last k parameters then one can use R2RA to transform the lower right triangle of the SRIF array to an upper left triangle after which UTINV and RI2COV can be applied.

II.5 <u>Sensitivity, Perturbation, Computed Covariance and Consider Covariance Matrix Computation</u>

Suppose that one is given a SRIF array

$$\begin{array}{cccc} \underbrace{N_x} & \underbrace{N_y} & \underbrace{1} & \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} R_x & R_{xy} & z_x \\ 0 & R_y & z_y \end{array}\right]} & \begin{array}{l} \}N_x \\ \}N_y \end{array} \end{array} \qquad \text{(II.5a)}$$

in which the N_y variables are to be considered. (One can, of course, using subroutines R2A and THH reorder and retriangularize an arbitrarily arranged SRIF array so that a given set of variables fall at the end.) For various reasons one may choose to ignore the y variables in the equation

$$R_x x + R_{xy} y = z_x - \nu_x \ , \quad \nu_x \varepsilon N(0,I) \tag{II.5b}$$

and take as the estimate $x_c = R_x^{-1} z_x$. It then follows that

$$x - x_c = -R_x^{-1} R_{xy} \, y - R_x^{-1} \nu_x \ , \tag{II.5c}$$

and from this one obtains

$$\text{Sen} \equiv \frac{\partial (x-x_c)}{\partial y} = -R_x^{-1} R_{xy} \tag{II.5d}$$

(sensitivity of the estimate error to the unmodeled y parameters)

$$\text{Pert} = \text{Sen Diag } (\sigma_y(1),\ldots,\sigma_y(N_y)) \tag{II.5e}$$

where $\sigma_y(1),\ldots,\sigma_y(N_y)$ are a priori y parameter uncertainties.

(The perturbations are a measure of how much the estimate error could be expected to change due to the unmodeled y parameters.)

$$P_{con} = R_x^{-1} R_x^{-T} + \text{Sen } P_y \text{ Sen}^T \tag{II.5f}$$

$$= P_c + (\text{Pert})(\text{Pert})^T \text{ if } P_y \text{ is diagonal*}$$

where P_c is the estimate error covariance of the reduced model.

An easy way to compute P_c, Pert and P_{con} is as follows: Use subroutine R2RA to place the y variable a priori $[P_y^{\frac{1}{2}}(0) \ \ \hat{y}_o]$** into the lower right

*Pert = Sen $P_y^{\frac{1}{2}}$

**The a priori estimate y_o of consider parameters is generally zero.

corner of (II.5a), replacing R_y and z_y, i.e.,

$$\left.\begin{matrix}[\tilde{R} & z] \\ [P_y^{1/2}(0) & \hat{y}_o]\end{matrix}\right\} \xrightarrow{\text{R2RA}} \begin{bmatrix} R_x & R_{xy} & z_x \\ 0 & P_y^{1/2}(0) & \hat{y}_o \end{bmatrix}$$

Now apply subroutine UTIROW to this system (with a -1 set in the lower right corner*)

$$\begin{bmatrix} R_x & R_{xy} & z_x \\ \hline 0 & P_y^{1/2}(0) & \hat{y}_o \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{UTIROW}} \begin{bmatrix} R_x^{-1} & \text{Pert}^{**} & x_c \\ \hline 0 & P_y^{1/2}(0) & \hat{y}_o \\ 0 & 0 & -1 \end{bmatrix}$$

Note that the lower portion of the matrix is left unaltered, i.e., the purpose of UTIROW is to invert a triangular matrix, given that the lower rows have already been inverted. From this array one can, using subroutine RI2COV, get both P_c and P_{con}

$$[R_x^{-1}] \xrightarrow{\text{RI2COV}} [P_c] \qquad \text{computed covariance}$$

$$[R_x^{-1} \quad \text{Pert}] \xrightarrow{\text{RI2COV}} [P_{con}] \qquad \text{consider covariance}$$

Suppose now that one is dealing with a U-D factored Kalman filter formulation. In this case estimate error sensitivities can be sequentially

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* To have estimates from the triangular inversion routines one sets a -1 in the last column (below the right hand side).

** Strictly speaking this is not what we call the perturbation unless $R_y(0)$ is diagonal.

calculated as each scalar measurement ($z = a_x^T x + a_y^T y + \nu$) is processed.

$$Sen_j = Sen_{j-1} - K_j (a_x^T \, Sen_{j-1} + a_y^T)$$

where Sen_{j-1} is the sensitivity prior to processing this (j-th) measurement, and K_j is the Kalman gain vector. In this formulation one computes P_{con} in a manner analogous to that described in section II.7;

Let $\bar{U}_1 = U_j$, $\bar{D}_1 = D_j$ (filter U-D factors)

$[s_1, \ldots, s_{n_y}] = Sen_j$ (estimate error sensitivities)

then compute

$$\bar{U}_k\text{-}\bar{D}_k \, , \, \sigma_k^2, \, s_k \xrightarrow{\text{AGTRN}} \bar{U}_{k+1}\text{-}\bar{D}_{k+1} \qquad k = 1, \ldots, n_y$$

For the final $\bar{U}$-$\bar{D}$ we have

$$U_{j+1}^{con} = \bar{U}_{n_y+1} \; , \; D_{j+1}^{con} = D_{n_y+1}$$

If $P_y(0) = U_y D_y U_y^T$, instead of $P_y(0) = \text{Diag}\,(\sigma_1^2, \ldots, \sigma_{n_y}^2)$, then in the U-D recursion one should replace the Sen_j columns by those of $Sen_j U_j$ and σ_j^2 should be replaced by the corresponding diagonal elements of D_y.

II.6 Combining Various Data Sets

In this example we collect several related problems involving data sets with different parameter lists.

Suppose that the parameter namelist of the current data does not correspond to that of the a priori SRIF array. If the new data involves a permutation or a subset of the SRIF namelist then an application of subroutine PERMUT will create the desired data rearrangement. If the data involves parameters not present in the SRIF namelist then one could use subroutine R2A to modify the SRIF array to include the new names and then if necessary use PERMUT on the data, to rearrange it compatibly.

Suppose now that two data sets are to be combined and that each contains parameters peculiar to it (and of course there are common parameters). For example let data set 1 contain names ABC and data set 2 contain names DEB. One could handle such a problem by noting that the list ABCDE contains both name lists. Thus one could use subroutine PERMUT on each data set comparing it to the master list, ABCDE, and then the results could be combined using subroutine THH An alternative automated method for handling this problem is to use subroutine COMBO with data set 1 (assuming it is in triangular form) and namelist 2. The result would be data set 1 in double subscripted form and arranged to the namelist ACDEB (names A and C are peculiar to data set 1 and are put first). Having determined the namelist one could apply subroutine PERMUT to data set 2 and give it a compatible namelist ordering.

The process of increasing the namelist size to accommodate new variables can lead to problems with excessively long namelists, i.e., with high dimension. If it is known that a certain set of variables will not occur in future data sets then these variables can be eliminated and the problem dimension reduced. To eliminate a vector y from a SRIF array, first use subroutine R2A to put the y names first in the namelist; then use subroutine THH to retriangularize and finally use subroutine R2RA to put the y independent subarray in position for further use; viz.

$$[R] \xrightarrow{R2A} [A] \xrightarrow{THH} \begin{bmatrix} R_y & R_{yx} & z_y \\ 0 & R_x & z_x \end{bmatrix} \xrightarrow{R2RA} [R_x \; z_x]$$

The rows $[R_y \; R_{yx} \; z_y]$ can be used to recover a y estimate (and its covariance) when an estimate for x (and its covariance) are determined. (See example II.4).

Still another application related to the combining of data sets involves the combining of SRIF triangular data arrays. One might encounter such problems when combining data from different space missions (that involve common parameters) or one might choose to process data of each type* or tracking station separately and then combine the resulting SRIF arrays. Triangular arrays can be combined using subroutine TTHH, assuming that subroutines R2A, THH and R2RA have been used previously to formulate a common parameter set for each of the sub problems.

II.7 Batch Sequential White Noise

It is not uncommon to have a problem where each data set contains a set of parameters that apply only to that set and not to any other, viz. the data is of the form

$$A_j x + B_j y_j = z_j - \nu_j \qquad j = 1,\ldots,N$$

where there is generally a priori information on the vector y_j variables. Rather than form a concatenated state vector composed of x, $y_1,\ldots,y_N$ which might create a problem involving exhorbitant amounts of storage and computation we solve the problem as follows. Apply subroutine THH to $[B_1 \; A_1 \; z_1]$, with the corresponding R initialized with the y_1 a priori. The resulting SRIF array is of the form

*viz. range, doppler, optical, etc.

$$N_{y_1}\left\{\begin{bmatrix} R_{y_1} & R_{y_1x} & z_{y_1} \\ 0 & R_{x_1} & z_{x_1} \end{bmatrix}\right.$$

Copy the top N_{y_1} rows if one will later want an estimate or covariance of the y_1 parameters. Apply subroutine TZERO to zero the top N_{y_1} rows and using subroutine R2RA set in the y_2 a priori*. This SRIF array is now ready to be combined with the second set of data $[B_2\ A_2\ Z_2]$ and the procedure repeated.

A somewhat analogous situation is represented by the class of problems that involve noisy model variations, i.e., the state at step j+1 satisfies

$$x_{j+1} = x_j + G_j w_j$$

where matrix G_j is defined so that w_j is independent of x_j and $w_j \epsilon N(0,Q_j)$. Models of this type are used to reflect that the problem at hand is not truly one of parameter estimation, and that some (or all) of the components vary in a random (or at least unknown) manner that is statistically bounded. To solve this problem in a SRIF formulation suppose that a priori for x_j and w_j are written in data equation form (cf ref. [3]),

$$R_j x_j = z_j - \nu_j \quad ; \quad \nu_j \epsilon N(0, I_{n_w})$$

$$Q_j^{-\frac{1}{2}} w_j = 0 - \nu_j^{(w)} \quad ; \quad \nu_j^{(w)} \epsilon N(0,I)$$

where $Q_j^{\frac{1}{2}}$ is a Cholesky factor of Q_j that is obtainable from COV2RI. Combining these two equations with the one for x_{j+1} gives

*In this example it is assumed that all of the y_j variables have the same dimension. This assumption, though not essential, simplifies our description of the procedure.

$$\begin{bmatrix} I_{n_w} & 0 \\ -R_j G_j Q_j^{1/2} & R_j \end{bmatrix} \begin{bmatrix} \hat{w}_j \\ x_{j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ z_j \end{bmatrix} - \begin{bmatrix} \nu_j^{(w)} \\ \nu_j \end{bmatrix}$$

where $Q_j^{1/2} w_j = w_j$. This is the equation to be triangularized with subroutine THH, i.e.,

$$\begin{array}{c} \\ \text{Dim } w \{ \\ \text{Dim } x \{ \end{array} \begin{array}{c} \begin{array}{ccc} \overbrace{}^{\text{Dim } w} & \overbrace{}^{\text{Dim } x} & \overbrace{}^{1} \end{array} \\ \begin{bmatrix} I_{n_w} & 0 & 0 \\ -R_j G_j Q_j^{1/2} & R_j & z_j \end{bmatrix} \end{array} \xrightarrow{\text{THH}} \begin{bmatrix} R_j^{(w)} & R_j^{(wx)} & z_j^w \\ 0 & R_{j+1} & z_{j+1} \end{bmatrix}$$

If the problem is arranged so that Q_j is diagonal one can reduce storage and computation. The form of this algorithm is designed to allow the use of singular Q_j matrices.

When the a priori for x_j and Q_j are given in U-D factored form, one can obtain the U-D factors for x_{j+1} as follows:

Let $\quad Q_j = U^{(q)} D^{(q)} (U^{(q)})^T \qquad$ (use COV2UD if necessary)

Set $\quad \bar{G} = G_j U^{(q)} = [g_1, \ldots, g_{n_w}] \quad , D^{(q)} = \mathrm{Diag}(d_1, \ldots, d_{n_w})$

Apply subroutine AGTRN n_w times, with $\bar{U}_1 = \bar{U}_j$, $\bar{D}_1 = D_j$

$$\left.\begin{array}{ll} & (\bar{U}\text{-}\bar{D})_k \; ; \; d_k, \; g_k \xrightarrow{\text{AGTRN}} (\bar{U}\text{-}\bar{D})_{k+1} \\ \text{i.e.} & (\bar{U}_k \bar{D}_k \bar{U}_k^T + d_k g_k g_k^T = \bar{U}_{k+1} \bar{D}_{k+1} \bar{U}_{k+1}^T) \end{array}\right\} \quad k = 1, \ldots, n_w$$

Then $U_{j+1} = \bar{U}_{n_w}$, $D_{j+1} = \bar{D}_{n_w}$.

Certain filtering problems involve dynamic models of the form

$$x_{j+1} = \Phi_j x_j + G_j w_j$$

Given an estimate for x_j, $\hat{x}_j$, the predicted estimate for x_{j+1}, denoted $\tilde{x}_{j+1}$ is simply*

$$\tilde{x}_{j+1} = \Phi_j \hat{x}_j$$

The U-D factors of the estimate error corresponding to the estimate $\tilde{x}_{j+1}$ can be obtained using the weighted Gram-Schmidt triangularization subroutine

$$\left[\Phi_j U_j \,\vdots\, \bar{G}\right], \text{Diag}\,(D_j, D^{(q)}) \xrightarrow{W\ G\ S} (\tilde{U}_{j+1} - \tilde{D}_{j+1})$$

II.8 Miscellaneous Uses of the Various ESP Subroutines

In certain parameter analyses we may want to reprocess a set of data suppressing different subsets of variables. In this case the original data should be left unaltered and subroutine A2A1 used to copy A into A_1, which then can be modified as dictated by the analysis.

Covariance analyses sometimes are initialized using a covariance matrix from a different problem (or a different phase of the same problem). In such cases it may be necessary to permute, delete or insert rows and columns into the covariance matrix; and that can be achieved using subroutine C2C.

If a priori for the problem at hand is given as a covariance matrix then one can compute the corresponding SRIF or U-D initialization using

*In statistical notation that is commonly used, one writes

$$x(j+1|j) = \Phi_j\, x(j|j)$$

subroutines COV2RI or COV2UD. Of course, if the covariance is diagonal the appropriate R and U-D factors can be obtained more simply. To convert a priori given in the form of an information matrix to a corresponding SRIF matrix one applies subroutine INF2R. To display covariance results corresponding to the SRIF or U-D filter one can use subroutines UTINV, R12COV and UD2COV. The vector stored covariance results are displayed in a triangular format using subroutine TRIMAT.

Aside: After careful consideration it was decided that subroutines to multiply matrices would not be included in our ESP. Our reasons are that parameter estimation does not, in the main, involve matrix multiplication; and when such products occur they generally involve matrices with special structures (viz. rectangle x triangle, triangle x rectangle, diagonal x triangle, etc). To see that these computations are not lengthy or complicated we illustrate how to compute z = Rx where R is a triangular vector stored matrix and x is an N vector,

```
      II=0
      DO 2 I=1,N
      SUM=0.
      II=II+I                    @II=(I,I)
      IK=II
      DO 1 K=I,N
      SUM=SUM+R(IK)*x(K)         @IK=(I,K)
1     IK=IK+K
2     z(I)=SUM                   @z can overwrite x if desired.
```

Note that the II and IK incremental recursions are used to circumvent the N(N+1)/2 calculations of IK=K(K-1)/2+I.

A later more encyclopedic subroutine directory may include the various matrix products that occur in linear algebra applications.

End of Aside

III. SUBROUTINE DIRECTORY SUMMARY

1. AGTRN - (Agee-Turner)

Computes updated U-D factors corresponding to a rank 1 matrix modification; i.e., given U-D, a scalar c, and vector v, $\bar{U}$ and $\bar{D}$ are computed so that $\bar{U}\ \bar{D}\ \bar{U}^T = U\ D\ U^T + c\ v\ v^T$. Both c and v are destroyed during the computation, and the resultant (vector stored) U-D array replaces the original one. Uses for this routine include (a) adding process noise effects to a U-D factored Kalman filter; (b) computing consider covariances (cf Section II.5); (c) computing "actual" covariance factors resulting from the use of suboptimal Kalman filter gains; and (d) adding measurements to a U-D factored information matrix.

2. A2A1 - (A to A1)

Reorders the columns of a rectangular matrix A, storing the result in matrix A1. Columns can be deleted and new columns added. Zero columns are inserted which correspond to new column name entries. Matrices A and A1 cannot share common storage.

Example III.1

$$\underset{A}{\begin{array}{ccc} \alpha & B & C \\ \end{array} \atop \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}} \xrightarrow{\text{A2A1}} \underset{A1}{\begin{array}{ccccc} B & F & G & C & H \\ \end{array} \atop \begin{bmatrix} 5 & 0 & 0 & 9 & 0 \\ 6 & 0 & 0 & 10 & 0 \\ 7 & 0 & 0 & 11 & 0 \\ 8 & 0 & 0 & 12 & 0 \end{bmatrix}}$$

The new namelist (BFGCH) contains F, G and H as new columns and deletes the column corresponding to name α.

Example III.2

Suppose one is given an observation data file with regression coefficients corresponding to a state vector with components say, x, y, z, v_x, v_y, v_z and station location errors. Suppose further, that the vector being estimated has components a_r^*, a_x^*, a_y^*, x, y, z, v_x, v_y, v_z, GM and station location errors. A2A1 can be used to reorder the matrix of regression coefficients to correspond to the state being estimated. Zero coefficients are set in place for the accelerations and GM which are not present in the original file.

3. COMBO - (combine R and A namelists)

The upper triangular vector stored matrix R has its columns permuted and is copied into matrix A. The names associated with R are to be combined with a second namelist.

The namelist for A is arranged so that R names not contained in the second list appear first (left most). These are then followed by the second list. Names in the second list that do not appear in the R namelist have columns of zeros associated with them.

Example III.3

$$\begin{array}{cccc} \alpha & B & C & D \end{array} \qquad \qquad \begin{array}{cccccc} & \multicolumn{5}{c}{\overbrace{\hphantom{B \quad E \quad \alpha \quad F \quad D}}^{\text{NAM2 list}}} \\ C & B & E & \alpha & F & D \end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 0 & 1 & 0 & 7 \\ 5 & 3 & 0 & 0 & 0 & 8 \\ 6 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

R-Vector stored A-Double subscripted

* in track and cross track accelerations

A principal application of this subroutine is to the problem of combining equation sets containing different variables, and automating the process of combining name lists.

4. COV2RI - (Covariance to R inverse)

An input positive semi-definite vector stored matrix P is replaced by its upper triangular vector stored Cholesky factor U, $P = UU^T$. The name RI is used because when the input covariance is positive definite, $U = R^{-1}$.

5. COV2UD - (Covariance to U-D factors)

An input positive semi-definite vector stored matrix P is replaced by its upper triangular vector stored U-D factors. $P = UDU^T$.

6. C2C - (C to C)

Reorders the rows and columns of a square (double subscripted) matrix C and stores the result back in C. Rows and columns of zeros are added when new column entries are added.

Example III.4

$$\begin{array}{c} \\ A \\ B \\ \Gamma \end{array}\begin{array}{c} \begin{array}{ccc} A & B & \Gamma \end{array} \\ \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \end{array} \xrightarrow{C2C} \begin{array}{c} \\ \Gamma \\ P \\ B \\ Q \end{array}\begin{array}{c} \begin{array}{cccc} \Gamma & P & B & Q \end{array} \\ \begin{bmatrix} 9 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Names P and Q have been added and name A deleted. An important application of this subroutine is to the rearranging of covariance matrices.

7. INF2R - (Information matrix to R)

Replaces a vector stored positive semi-definite information matrix Λ by its lower triangular Cholesky factor R^T; $\Lambda = R^TR$. The upper triangular matrix R is in the form utilized by the SRIF algorithms. The algorithm is designed to handle singular matrices because it is a common practice to omit a priori information on parameters that are either poorly known or which will be well determined by the data.

8. PERMUT

Reorders the columns of matrix A, storing the result back in A. This routine differs from A2A1 principally in that here the result overwrites A. PERMUT is especially useful in applications where storage is at a premium or where the problem is of a recursive nature.

9. RINCON - (R inverse with condition number bound, CNB)

Computes the inverse of an upper triangular vector stored matrix R using subroutine UTINV. A Frobenius bound (CNB) for the condition number of R is computed too. This bound acts as both an upper and a lower bound, because $CNB/N \leq$ condition number $\leq CNB$. When this bound is within several orders of magnitude of the machine accuracy the computed inverse is not to be trusted, (viz if $CNB \geq 10^{15}$ on an 18 decimal digit machine R is ill-conditioned).

10. RI2COV - (RI to covariance)

This subroutine computes sigmas (standard deviations) and/or the covariance of a vector stored upper triangular square root covariance matrix, RINV (SRIF inverse). The result, stored in COVOUT (covariance output) is also vector stored. COVOUT can overwrite RINV.

11. R2A - (R to A)

The columns of a vector stored upper triangular matrix R are permuted and variables are added and/or deleted. The result is stored in the double subscripted matrix A. In other respects the subroutine is like A2A1.

Example III.5

$$\begin{array}{c} \begin{array}{ccccc} \alpha & B & C & D & E \end{array} \\ \begin{bmatrix} 2 & 4 & 8 & 14 & 22 \\ 0 & 6 & 10 & 16 & 24 \\ 0 & 0 & 12 & 18 & 26 \\ 0 & 0 & 0 & 20 & 28 \\ 0 & 0 & 0 & 0 & 30 \end{bmatrix} \\ R \end{array} \xrightarrow{R2A} \begin{array}{c} \begin{array}{cccc} E & F & C & B \end{array} \\ \begin{bmatrix} 22 & 0 & 8 & 4 \\ 24 & 0 & 10 & 6 \\ 26 & 0 & 12 & 0 \\ 28 & 0 & 0 & 0 \\ 30 & 0 & 0 & 0 \end{bmatrix} \\ A \end{array}$$

R is vector stored as R = (2,4,6,8,10,12,14,16,18,20,22,24,26,28,30) with namelist (α,B,C,D,E) associated with it. Names α and D are not included in matrix A, and a column of zeros corresponding to name F is added.

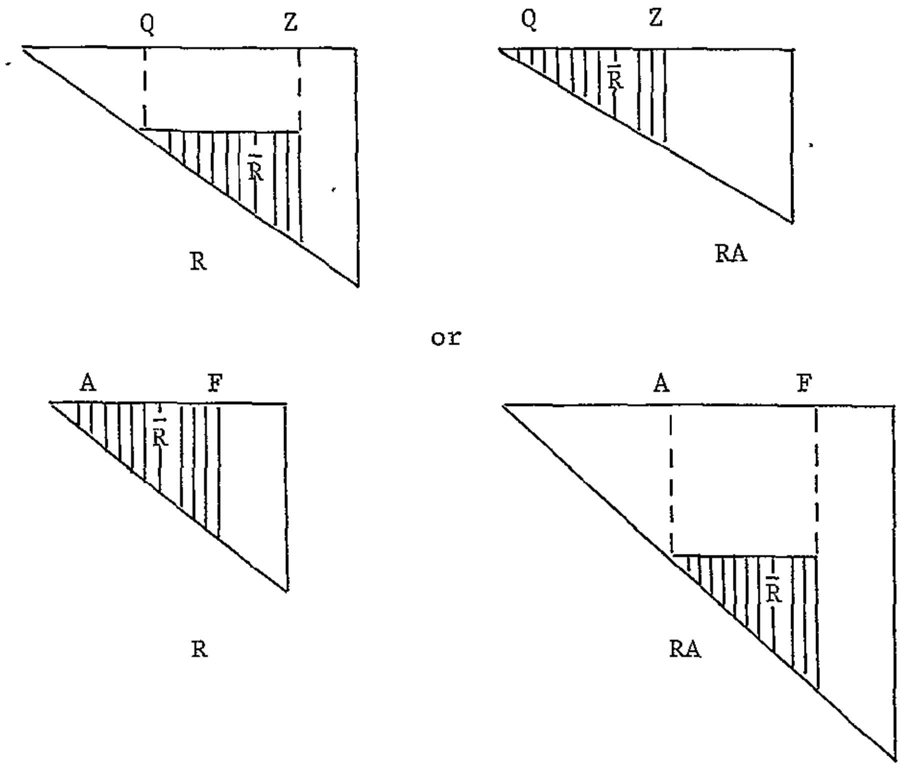
One trivial, but perhaps useful, application is to convert a vector stored matrix to a double subscripted form.* R2A is used most often when one wants to rearrange the columns of a SRIF array so that reduced order estimates, sensitivites, etc. can be obtained; or so that data sets containing different parameters can be combined.

12. R2RA - (Triangular block of R to triangular block of RA)

A triangular portion of the vector stored upper triangular matrix R is put into a triangular portion of the vector stored matrix RA. The names corresponding to the relocated block are also moved. R can coincide with RA.

*see also the aside in the introduction

Examples III.6



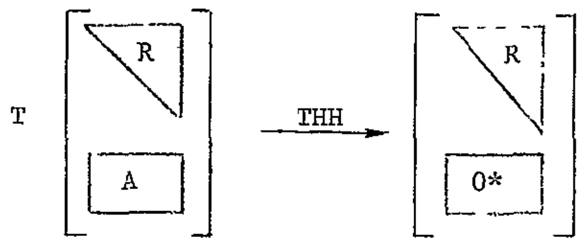
Note that an upper left triangular submatrix can slide to any lower position along the diagonal, but that a submatrix moving up must go to the upper leftmost corner. Upper shifting is used when one is interested in that subsystem; and the lower shifting is used, for example, when inserting a priori information for consider analyses.

13. RUDR - (SRIF R converted to U-D form or vice versa)

A vector stored SRIF array is replaced by a vector stored U-D form or conversely. A point to be noted is that when data is involved the right side of the SRIF data equation transforms to the estimate in the U-D array.

14. THH - (Triangular Householder data packing)

An upper triangular vector stored matrix R is combined with a rectangular doubly subscripted matrix A by means of Householder orthogonal transformations. The result overwrites R, and A is destroyed in the process.



15. TRIMAT - (Triangular matrix print)

Prints a vector stored upper triangular matrix, using a matrix format.

Example III.7

R(10) = (2,4,6,8,10,12,14,16,18,20) with associated namelist (A,B,C,D) is printed as

	A	B	C	D
A	2	4	8	14
B		6	10	16
C			12	18
D				20

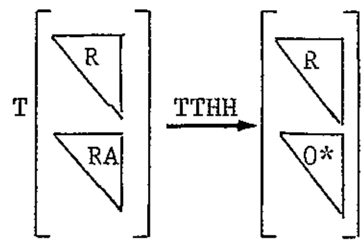
(The numbers are printed to 8 significant floating point digits).

To appreciate the importance of this subroutine compare the vector R(10) with the double subscript representation.

16. TTHH - (Two triangular arrays are combined using Householder orthogonal transformations)

This subroutine combines two single subscripted upper triangular SRIF arrays, R and RA using Householder orthogonal transformations. The result overwrites R.

*The elements are not explicitly set to zero.



17. TZERO - (Zero a horizontal segment of a vector stored upper triangular matrix)

Upper triangular vector stored matrix R has its rows between ISTART and IFINAL set to zero.

Example III.8

To zero row 2 and 3 of R(15), in the example of subroutine 11.

R(15) = (2,4,6,8,10,12,14,16,18,20,22,24,26,28,30)

R(15) = (2,4,0,8,0,0,14,0,0,20,22,0,0,28,30)

i.e.,

$$\underset{\text{R-vector stored}}{\begin{bmatrix} 2 & 4 & 8 & 14 & 22 \\ 0 & 6 & 10 & 16 & 24 \\ 0 & 0 & 12 & 18 & 26 \\ 0 & 0 & 0 & 20 & 28 \\ 0 & 0 & 0 & 0 & 30 \end{bmatrix}} \xrightarrow{\text{TZERO}} \underset{\text{R-vector stored}}{\begin{bmatrix} 2 & 4 & 8 & 14 & 22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 28 \\ 0 & 0 & 0 & 0 & 30 \end{bmatrix}}$$

*The elements are not explicitly set to zero.

18. UDMES - (U-D measurement update)

Given the U-D factors of the a priori estimate error covariance and the measurement, $z = Ax + \nu$ this routine computes the updated estimate and U-D covariance factors, the predicted residual, the predicted residual variance, and the normalized Kalman gain. This is Bierman's U-D measurement update algorithm.

19. UD2COV - (U-D factors to covariance)

The input vector stored U-D matrix (diagonal D elements are stored as the diagonal entries of U) is replaced by the covariance P, also vector stored. $P = UDU^T$. P can overwrite U to economize on storage.

20. UD2SIG - (U-D factors to sigmas)

Standard deviations corresponding to the diagonal elements of the covariance are computed from the U-D factors. This subroutine, a restricted version of UD2COV can print out the resulting sigmas and a title. The input U-D matrix is unaltered.

21. UTINV - (Upper triangular matrix inversion)

An upper triangular vector stored matrix RIN(R in) is inverted and the result, vector stored, is put in ROUT(R out). ROUT can overwrite RIN to economize on storage. If a right hand side is included and the bottommost tip of RIN has a -1 set in then ROUT will have the solution in the place of the right hand side.

22. UTIROW - (Upper triangular inversion, inverting only the upper rows)

$$\begin{array}{c} \\ n_y \end{array} \begin{array}{c} \text{INPUT} \\ \left[\begin{array}{cc} R_x & R_{xy} \\ \hline 0 & R_y^{-1} \end{array}\right] \end{array} \xrightarrow{\text{UTIROW}} \begin{array}{c} \text{OUTPUT} \\ \left[\begin{array}{cc} R_x^{-1} & -R_x^{-1} R_{xy} R_y^{-1} \\ \hline 0 & R_y^{-1} \end{array}\right] \end{array}$$

An input vector stored R matrix with its lower left triangle assumed to have been already inverted is used to construct the upper rows of the matrix inverse of the result. The result, vector stored, can overwrite the input to economize on storage.

If the columns comprising R_{xy} represent consider terms then taking R_y^{-1} as the identity gives the sensitivity on the upper right portion of the result. If $R_y^{-1} = \text{Diag}(\sigma_y, \ldots, \sigma_{n_y})$ then the upper right portion of the result represents the perturbation. Note that if z (the right hand side of the data equation) is included in R_{xy} then taking the corresponding R_y^{-1} diagonal as -1 results in the filter estimate appearing as the corresponding column of the output array. When n_y is zero this subroutine is equivalent to UTINV.

23. WGS - (Weighted Gram Schmidt matrix triangularization)

An input rectangular (possibly square) matrix W and a diagonal weight matrix, D_w, are transformed to (U-D) form; i.e.,

$$S\ D_w\ W^T = UDU^T$$

where U is unit upper triangular and D is diagonal. The weights D_w are assumed nonnegative, and this characteristic is inherited by the resulting D.

IV. SUBROUTINE DIRECTORY USER DESCRIPTION

1. AGTRN (Agee-Turner U-D rank one modification)

Purpose

To compute the (updated) U-D factors of $UDU^T + cVV^T$.

```
CALL AGTRN (UIN,UOUT,N,C,V)
```

Argument Definitions

UIN(N*(N+1)/2)	Input vector stored positive semi-definite U-D array (with the D entries stored on the diagonal of U)
UOUT(N*(N+1)/2)	Output vector stored result UOUT=UIN is allowed
N	Matrix dimension
C	Input scalar, destroyed by the algorithm
V(N)	Input vector, destroyed by the algorithm

Remarks and Restrictions

If C negative is used the algorithm is numerically unstable, and the result may be numerically unreliable. Singular U matrices are allowed, and these can result in singular output U matrices.

Functional Description

This rank one modification is based on a result published by Agee and Turner (1972), White Sands Missile Range Tech. Report No. 38. See also Ref. [3] where the algorithm is derived using geometric arguments.

2. A2A1 (A to A1)

Purpose

To rearrange the columns of a namelist indexed matrix to conform to a desired namelist.

CALL A2A1(A,IA,IR,LA,NAMA,A1,IA1,LA1,NAMA1)

Argument Definitions

A(IR,LA)	Input rectangular matrix
IA	Row dimension of A, IA.GE.IR
IR	Number of rows of A that are to be arranged
LA	Number of columns in A; this also represents the number of parameter names associated with A
NAMA(LA)	Parameter names associated with A
A1(IR,LA1)	Output rectangular matrix
IA1	Row dimension of A1, IA1.GE.IR
LA1	Number of columns in A1; this also represents the number of parameter names associated with A1
NAMA1(LA1)	Input list of parameter names to be associated with the output matrix A1

Remarks and Restrictions

A1 cannot overwrite A. This subroutine can be used to add on columns corresponding to new names and/or to delete variables from an array.

Functional Description

The columns of A are copied into A1 in an order corresponding to the NAMA1 parameter namelist. Columns of zeros are inserted in those A1 columns which do not correspond to names in the input parameter namelist NAMA.

3. COMBO (Combine parameter namelists)

Purpose

To rearrange a vector stored triangular matrix and store the result in matrix A. The difference between this subroutine and R2A is that there the namelist for A is input; here it is determined by combining the list for R with a list of desired names.

```
CALL COMBO  (R,L1,NAM1,L2,NAM2,A,IA,LA,NAMA)
```

Argument Definitions

R(L1*(L1+1)/2)	Input vector stored upper triangular matrix
L1	No. of parameters in R (and in NAM1)
NAM1(L1)	Names associated with R
L2	No. of parameters in NAM2
NAM2(L2)	Parameter names that are to be combined with R (NAM1 list); these names may or may not be in NAM1
A(L1,LA)	Output array containing the rearranged R matrix L1.LE.IA
IA	Row dimension of A
LA	No. of parameter names in NAMA, and the column dimension of A. LA = L1 + L2 - No. names common to NAM1 and NAM2; LA is computed and output
NAMA(LA)	Parameter names associated with the output A matrix ; consists of names in NAM1 not in NAM2 followed by NAM2

Remarks and Restrictions

The column dimension of A is a result of this subroutine. To avoid having A overwrite neighboring arrays one can bound the column dimension of A by L1 + L2.

Functional Description

First the NAM1 and NAM2 lists are compared and the names appearing in NAM1 only have their corresponding R column entries stored in A (e.g. if NAM1(2) and NAM1(6) are the only names not appearing in the NAM2 list then columns 2 and 6 of R are copied into columns 1 and 2 of A). The remaining columns of A are labeled with NAM2. The A namelist is recorded in NAMA. The NAM1 list is compared with NAM2 and matching names have their R column entries copied into the appropriate columns of A. NAM2 entries not appearing in NAM1 have columns of zero placed in A.

4. COV2RI (Covariance to Cholesky Square Root, RI)

Purpose

To construct the upper triangular Cholesky factors of a positive semi-definite matrix. Both the input covariance and the output Cholesky factor (square root) are vector stored. The output overwrites the input. Covariance (input) = U*U**T (output U = Rinverse).

$\boxed{\text{CALL COV2RI(U,N)}}$

Argument Definitions

U(N*(N+1)/2)	Contains the input vector stored covariance matrix (assumed positive definite)and on output it contains the upper triangular square root factor
N	Dimension of the matrices involved

Remarks and Restrictions

No check is made that the input matrix is positive semi-definite. Singular factors (with zero columns) are obtained if the input is (a) in fact singular, (b) ill-conditioned, or (c) in fact indefinite; and the latter two situations are cause for alarm. Case (c) and possibly (b) can be identified by using RI2COV to reconstruct the input matrix.

Functional Description

An upper triangular Cholesky reduction of the input matrix is implemented using a geometric algorithm described in Ref. [3].

$$U(\text{input}) = U(\text{output}) * U(\text{output})^T$$

At each step of the reduction diagonal testing is used and negative terms are set to zero.

5. COV2UD (Covariance to UD factors)

Purpose

To obtain the U-D factors of a positive semi-definite matrix. The input vector stored matrix is overwritten by the output U-D factors which are also vector stored.

$\boxed{\text{CALL COV2UD(U,N)}}$

Argument Definitions

U(N*(N+1)/2)	Contains the input vector stored covariance matrix; on output it contains the vector stored U-D covariance factors.
N	Matrix dimension

Remarks and Restrictions

No checks are made in this routine to test that the input U matrix is positive semi-definite. Singular results (with zero columns) are obtained if the input is (a) in fact singular, (b) ill-conditioned, or (c) in fact indefinite; and the latter two situations are cause for alarm. Case (c) and possibly case (b) can be identified by using UD2-COV to reconstruct the input matrix. Note that although indefinite matrices have U-D factorizations, the algorithm here applies only to matrices with non-negative eigenvalues.

Functional Description

An upper triangular U-D Cholesky factorization of the input matrix is implemented using a geometric algorithm described in Ref. [3].

$$U(\text{input}) = U*D*U^T , \qquad \text{U-D stored in U on output}$$

at each step of the reduction diagonal testing is used to zero negative terms.

6. C2C (C to C)

Purpose

To rearrange the rows and columns of C, from NAM1 order to NAM2 order. Zero rows and columns are associated with output defined names that are not contained in NAM1.

`CALL C2C(C,IC,L1,NAM1,L2,NAM2)`

Argument Definitions

C(L1,L1)	Input matrix
IC	Row dimension of C IC.GE.L = MAX(L1,L2)
L1	No. of parameter names associated with the input C
NAM1(L)	Parameter names associated with C on input. (Only the first L1 entries apply to the input C)
L2	No. of parameter names associated with the output C
NAM2(L2)	Parameter names associated with the output C

Remarks and Restrictions

The NAM2 list need not contain all the original NAM1 names and L1 can be .GE. or .LE. L2. The NAM1 list is used for scratch and appears permuted on output. If L2.GT.L1 the user must be sure that NAM1 has L2 entries available for scratch purposes.

Functional Description

The rows and columns of C and NAM1 are permuted pairwise to get the names common to NAM1 and NAM2 to coalesce. Then the remaining rows and columns of C(L2,L2) are set to zero.

7. INF2R (Information matrix to R)

Purpose

To compute a lower triangular Cholesky factorization of the input positive semi-definite matrix. The result transposed, is vector stored; this is the form of an upper triangular SRIF matrix.

```
CALL INF2R(P,N)
```

Argument Definitions

P(N*(N+1)/2)	Input vector stored positive semi-definite (information) matrix; on output it represents the transposed lower triangular Cholesky factor (i.e. the SRIF R matrix)
N	Matrix dimension

Remarks and Restrictions

No checks are made on the input matrix to guard against negative eigenvalues of the input, or to detect ill-conditioning. Singular output matrices have one or more rows of zeros.

Functional Description

A Cholesky type lower triangular factorization of the input matrix is implemented using the geometric formulation described in Ref. [3].

$$U(\text{input}) = [U(\text{output})]^T * [U(\text{output})]$$

At each step of the factorization diagonal testing is used to zero columns corresponding to negative entries. The result is vector stored in the form of a square root information matrix as it would be used for SRIF analyses.

8. PERMUT (Permute A)

Purpose

To rearrange the columns of a namelist indexed matrix to conform to a desired namelist. The resulting matrix is to overwrite the input.

```
CALL PERMUT(A,IA,IR,L1,NAM1,L2,NAM2)
```

Argument Definitions

A(IR,L)	Input rectangular matrix, L = max(L1,L2)
IA	Row dimension of A, IA.GE.IR
IR	Number of rows of A that are to be rearranged
L1	Number of parameter names associated with the input A matrix
NAM1(L)	Parameter names associated with A on input (only the first L1 entries apply to the input A)
L2	Number of parameter names associated with the output A matrix
NAM2	Parameter names associated with the output A

Remarks and Restrictions

This subroutine is similar to A2A1; but because the output matrix in this case overwrites the input there are several differences. The NAM1 vector is used for scratch, and on output it contains a permutation of the input NAM1 list. The user must allocate L = max(L1,L2) elements of storage to NAM1. The extra entries, when L2 > L1, are used for scratch.

Functional Description

The columns of A are rearranged, a pair at a time, to match the NAM2 parameter namelist. The NAM1 entries are permuted along with the columns, and this is why dim (NAM1) must be larger than L1 (when L2>L1). Columns of zeroes are inserted in A which correspond to output names that do not appear in NAM1.

9. RINCON (R inverse with condition number bound)

Purpose

To compute the inverse of an upper triangular vector stored triangular matrix, and an estimate of its condition number.

```
CALL RINCON(RIN,N,ROUT,CNB)
```

Argument Definitions

RIN(N*(N+1)/2)	Input vector stored upper triangular matrix
N	Matrix dimension
ROUT(N*(N+1)/2)	Output vector stored matrix inverse (RIN = ROUT is permitted)
CNB	Condition number bound. If κ is the condition number of RIN, then CNB/N.LE.κ.LE CNB

Remarks and Restrictions

The condition number bound, CNB serves as an estimate of the actual condition number. When it is large the problem is ill-conditioned. The matrix inversion is computed using subroutine UTINV.

Functional Description

The matrix inversion, a triangular back substitution, is accomplished via subroutine UTINV. If any diagonal element of the input R matrix is zero the inversion is not attempted; instead a message is printed. The condition number bound is computed as follows:

$$\text{F.NORM } R = \sum_{J=1}^{NTOT} R(J)^2$$

$$\text{F.NORM } R^{-1} = \sum_{J=1}^{NTOT} R^{-1}(J)^2$$

where NTOT = N*(N+1)/2 is the number of elements in the vector stored triangular matrix. The condition number bound, CNB, is given by

$$CNB = (F.NORM\ R * F.NORM\ R^{-1})^{1/2}$$

F.NORM is the Frobenius norm, squared. The inequality

$$CNB/N \leq \text{condition number } R \leq CNB$$

is a simple consequence of the Frobenius norm inequalities given in Lawson-Hanson "Solving Least Squares," page 234.

10. RI2COV (RI Triangular to covariance)

Purpose

To compute the covariance matrix and/or the standard deviation of a vector stored upper triangular square root covariance matrix. The output covariance matrix, also vector stored, may overwrite the input.

```
CALL RI2COV(RINV,N,SIG,COVOUT,KOV)
```

Argument Definitions

RINV(N*(N+1)/2)		Input vector stored upper triangular covariance square root (RINV=R inverse is the inverse of the SRIF matrix).
N		Dimension of the RINV matrix
SIG(N)		Output vector of standard deviations
COVOUT(N*(N+1)/2)		Output vector stored covariance matrix (COVOUT = RINV is allowed)
KOV	.GT.0	Compute covariance and sigmas using the first KOV rows of RINV
	.LT.0	Compute only the sigmas using the first KOV rows of RINV
	.EQ.0	No covariance, but all sigmas (e.g. use all N rows of RINV)

Remarks and Restrictions

Replacing N by |KOV| corresponds to computing the covariance of a lower dimensional system.

Functional Description

COVOUT=RINV*RINV**T.

11. R2A (R to A)

Purpose

To place the upper triangular vector stored matrix R into the matrix A and to arrange the columns to match the desired NAMA parameter list. Names in the NAMA list that do not correspond to any name in NAMR have zero entries in the corresponding A columns.

```
CALL R2A(R,LR,NAMR,A,IA,LA,NAMA)
```

Argument Definitions

R(LR*(LR+1)/2)	Input upper triangular vector stored array
LR	Row dimension of vector stored R
NAMR(LR)	Parameter names associated with R
A(LR,LA)	Matrix to house the rearranged R matrix
IA	Row dimension of A, IA.GE.LR.
LA	No. of parameter names associated with the output A matrix.
NAMA(LA)	Parameter names for the output A matrix.

Functional Description

The matrix A is set to zero and then the columns of R are copied into A.

12. R2RA (Permute a subportion R_A of a vector stored triangular matrix)

Purpose

To copy the upper left (lower right) portion of a vector stored upper triangular matrix R into the lower right (upper left) portion of a vector stored triangular matrix RA.

```
CALL R2RA(R,NR,NAM,RA,NRA,NAMA)
```

Argument Definitions

R(NR*(NR+1)/2)	Input vector stored upper triangular matrix
NR	Dimension of vector stored R matrix†
NAM(NR)	Names associated with R.
RA(NRA*(NRA+1)/2)	Output vector stored upper triangular matrix
NRA	If NRA = 0 on input, then NAMA(1) should have the first name of the output namelist. In this case the number of names in NAMA, NRA, will be computed. The lower right block of R will be the upper left block of RA. If NRA = last name of the upper left block that is to be moved then this upper block is to be moved to the lower right corner of RA. When used in this mode NRA=NR on output†.
NAMA(NRA)	Names associated with RA. Note that NRA used here denotes the output value of NRA.

Remarks and Restrictions

RA and NAMA can overwrite R and NAM. The meaning of the NRA = 0 option is clarified by the following example:

$$
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\left[\begin{array}{cc|ccc}
2 & 4 & 8 & 14 & 22 \\
 & 6 & 10 & 16 & 24 \\
\hline
 & & 12 & 18 & 26 \\
 & & & 20 & 28 \\
 & & & & 30
\end{array}\right] \\
R
\end{array}
\longrightarrow
\begin{array}{c}
\begin{array}{ccc} C & D & E \end{array} \\
\left[\begin{array}{ccc}
12 & 18 & 26 \\
 & 20 & 28 \\
 & & 30
\end{array}\right] \\
RA
\end{array}
$$

INPUT
NR = 5
NAM = 'A','B','C','D','E'
NRA = 0
NAMA(1) = 'C'
R

OUTPUT
NAMA = 'C', 'D', 'E'

†see the concluding paragraph of Remarks and Restrictions

When NRA = 0 and NAMA(1) = 'C' we are asking that the lower triangular portion of R, beginning at the column labeled C, be moved to form the first (in this case 3) columns of RA. Incidentally, RA could have additional columns; these columns and their names would be unaltered by the subroutine.

The meaning of the other NRA option is illustrated by the following example;

$$\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccc|cc} 2 & 4 & 8 & 14 & 22 \\ & 6 & 10 & 16 & 24 \\ & & 12 & 18 & 26 \\ \hline & & & 20 & 28 \\ & & & & 30 \end{array}\right] \\ R \end{array} \longrightarrow \begin{array}{c} \begin{array}{ccccc} A & B & A & B & C \end{array} \\ \left[\begin{array}{cc|ccc} 2 & 4 & 8 & 14 & 22 \\ & 6 & 10 & 16 & 24 \\ \hline & & 2 & 4 & 8 \\ & & & 6 & 10 \\ & & & & 12 \end{array}\right] \\ R \end{array}$$

<u>INPUT</u>
NR = 5
NAM = 'A','B','C','D','E'
NRA = 'C'
R

<u>OUTPUT</u>
NRA = 5
NAMA(3-5) = 'A','B','C'
RA

When NRA = 'C' we are asking that the upper left block of <u>R,</u> up to the column labeled C, be moved to the lower left portion of RA and the corresponding names be moved too. If RA overwrites R, as in the example, then the first two rows of R remain unchanged and since NAMA overwrites NAM, the labels of the first two columns remain unaltered.

The remark that NRA=NR on output means, in this example, that the column with name C in R is moved over to column 5. If one wanted to slide the upper left triangle corresponding to names ABC of R to columns 7-9 of an RA matrix (of unspecified dimension, ≥ 9), then one should set NR=9 in the subroutine call. Thus NR, when used in this sliding down the diagonal mode, does not represent the dimension of R; but indicates how far the slide will be.

13. RUDR (R to U-D or U-D to R)

Purpose

To transform an upper triangular vector stored SRIF array to U-D form or vice versa.

CALL RUDR(RIN,N,ROUT,IS)

Argument Definitions

RIN(NBAR*(NBAR+1)/2)	Input upper triangular vector stored SRIF or U-D array; NBAR = ABS(N) + 1
ROUT(NBAR*(NBAR+1)/2)	Output upper triangular vector stored U-D or SRIF array (RIN = ROUT is permitted)
N	Matrix dimension, N.GT.0 represents an R to U-D conversion and N.LT.0 represents a U-D to R conversion.
IS	If IS = 0 the input array is assumed not to contain a right side (or an estimate), and IS = 1 means an appropriate additional column is included. In the IS = 0 case the last column of RIN is ignored and NBAR = ABS(N) is used.

Subroutine used: UTINV

Functional Description

Consider the N>0 case. RIN = R is transformed to ROUT = R inverse using subroutine UTINV with dimension N+IS. If IS = 1 the subroutine sets RIN((N+1)(N+2))/2) = -1. so that the N+1st column of ROUT will be the X estimate followed by -1. $R^{-1} = UD^{1/2}$ so that the diagonals are square root scaled U columns. This information is used to construct the U-D array which overwrites ROUT.

If N<0 the input is assumed to be a U-D array. This array is converted to ROUT=$UD^{\frac{1}{2}}$ and then using UTINV, R is computed and stored in ROUT. If IS = 1 the U-D matrix is assumed augmented by X (estimate), and on output the right side term of the SRIF array is obtained.

14. THH (Triangular Householder Orthogonalization)

Purpose

To compute [R z] such that

$$T \begin{bmatrix} \tilde{R} & \tilde{z} \\ A & z \end{bmatrix} = \begin{bmatrix} \hat{R} & \hat{z} \\ 0 & e \end{bmatrix} \quad T \text{ - orthogonal}$$

This is the key algorithm used in the square root information batch sequential filter.

```
CALL THH(R,N,A,IA,M,SOS,NSTRT)
```

Argument Definitions

R(N*(N+3)/2)	Input upper triangular vector stored square root information matrix. If estimates are involved SOS.GE.0 and R is augmented with the right hand side (stored in the last N locations of R). If SOS.LT.0 only the first N*(N+1)/2 locations of R are used. The result of the subroutine overwrites the input R
N	No. of parameters
A(M,N+1)	Input measurement matrix. The N+1st column is only used if SOS.GE.0, in which case it represents the right side of the equation $\nu + AX = z$. A is destroyed by the algorithm, but it is not explicitly set to zero.
IA	Row dimension of A
M	The number of rows of A that are to be combined with R
SOS	Accumulated residual sum of squares corresponding to the data processed prior to this time. On exit SOS represents the updated sum of squares of the residuals $\sum_i \lvert\lvert z_i - A_i X_{est} \rvert\rvert^2$, summed over the old and new data. It also includes the a priori term $\lvert\lvert R_o X_{est} - z_o \rvert\rvert^2$. Because SOS cannot be used if data, z, is not included we use SOS.LT.0 to indicate when data is

not included.

NSTRT — First column of the input A matrix that has a nonzero entry. In certain problems, especially those involving the inclusion of a priori statistics, it is known that the first NSTRT-1 columns of A all have zero entries. This knowledge can be used to reduce computation. If nothing is known about A then NSTRT.LE.1 gives a default value of 1, i.e. it is assumed that A may have nonzero entries in the very first column.

Remarks and Restrictions

It is trivial to arrange the code so that R output need not overwrite the input R. This was not done because, in the author's opinion, there are too few times when one desires to have ROUT ≠ RIN.

Functional Description

Assume for simplicity that NSTRT = 1. Then at step j, j = 1,...,N (or N+1 if data is present) the algorithm implicitly determines an elementary Householder orthogonal transformation which updates row j of R and all the columns of A to the right of the jth. At the completion of this step column j of A is in theory zero, but it is not explicitly set to zero. The orthogonalization process is discussed at length in the books by Lawson and Hanson, [1] and Bierman [3].

15. TRIMAT (Triangular matrix print)

Purpose

To display a vector stored upper triangular matrix in a two dimensional 8-digit triangular format.

```
CALL TRIMAT(A,N,CAR,TEXT,NCHAR,NAMES)
```

Argument Definitions

A(N*N+1)/2)	Vector stored upper triangular matrix
N	Dimension of A
CAR(N)	Parameter names (alphanumeric) associated with A
TEXT(NCHAR)	An array of field data characters to be printed as a title preceding the matrix
NCHAR	No. of characters (including spaces) that are to be printed in text( ) ABS(NCHAR).LE.126.NCHAR negative is used to avoid skipping to a new page to start printing
NAMES	A logical flag. If NAMES=.F. the CAR namelist is ignored and the columns and rows of A on output appear with numerical column heads

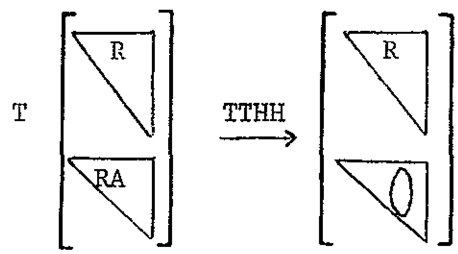
Remarks and Restrictions

Using NCHAR nonnegative, and starting the print at the top of a new page makes it easier to locate the printed result and is especially recommended when dealing with large dimensioned arrays. Page economy can, however, be achieved using the NCHAR negative option. In this case the print begins on the next line.

16. TTHH (Two triangular matrix Householder reduction)

Purpose

To combine two vector stored upper triangular matrices, R and RA by applying Householder orthogonal transformations. The result overwrites R.



CALL TTHH (R,RA,N)

Argument Definitions

R(N*(N+1)/2)	Input vector stored upper triangular matrix, which also houses the result
RA(N*(N+1)/2)	Second input vector stored upper triangular matrix. This matrix is destroyed by the computation.
N	Matrix dimension N less than zero is used to indicate that R and RA have right sides (|N|+1 columns) and have dimension |N|*(|N|+3)/2).

Remarks and Restrictions

RA is theoretically zero on output, but is not set to zero.

17. TZERO (Triangular matrix zero)

Purpose

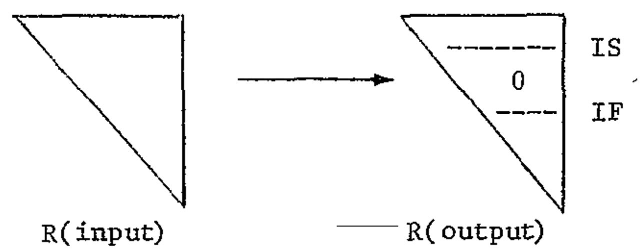
To zero out rows IS(Istart) to IF(Ifinal) of the vector stored upper triangular matrix R.

```
CALL TZERO(R,N,IS,IF)
```

Argument Definition

R(N*(N+1)/2)	Input vector stored upper triangular matrix
N	Row dimension of vector stored matrix
IS	First row of R that is to be set to zero
IF	Last row of R that is to be set to zero

Functional Description



18. UDMES (U-D measurement update)

Purpose

Kalman filter measurement updating using Bierman's U-D measurement update algorithm, cf 1975 CONF. DEC. CONTROL paper. A scalar measurement $z = A^T x + \nu$ is processed, the covariance U-D factors and estimate (if included) are updated, and the Kalman gain and innovations variance are computed.

```
CALL UDMES(U,N,R,A,G,ALPHA)
```

Argument Definitions

INPUTS

U(N*(N+1)/2)	Upper triangular vector stored input matrix. D elements are stored on the diagonal. The U vector corresponds to an a priori covariance. If state estimates are involved the last column of U contains X. In this case Dim U = (N+1)*(N+2)/2 and on output (U((N+1)*(N+2)/2) = z-A**T*X(a priori est).
N	Dimension of the state vector
R	Measurement variance
A(N)	Vector of Measurement coefficients; if data then A(N+1) = z
ALPHA	If ALPHA.LT.zero no estimates are computed (and X and z need not be included)

OUTPUTS

U	Updated vector stored U-D factors. When ALPHA (input) is nonnegative the (N+1)st column contains the updated estimate and the predicted residual.
ALPHA	Innovations variance of the measurement residual.
A	Contains U**T*A(input) and when ALPHA (input) is nonnegative A(N+1) = z-A**T*X(a priori est)/ALPHA.

G(N) Vector of unweighted Kalman gains, K = G/ALPHA.

Remarks and Restrictions

One can use this algorithm with R negative to delete a previously processed data point. One should, however, note that data deletion sometimes introduces numerical errors.

The algorithm holds for R = 0 (a perfect measurement) but the code may fail (zero divides occur) if any of the ALPHA terms appearing in the code vanish. Changes in the code which remove the zero divide problems are commented in the code.

Functional Description

The algorithm updates the columns of U, from left to right, using Bierman's algorithm, cf Proc. 1975 Conf. Dec. Control, Houston, Texas, pp 337-346.

19. UD2COV (U-D factor to covariance)

Purpose

To obtain a covariance from its U-D factorization. Both matrices are vector stored and the output covariance can overwrite the input U-D array. U-D and P are related via $P = UDU^T$.

```
CALL UD2COV(UIN,N,POUT)
```

Argument Definitions

UIN(N*(N+1)/2)	Input vector stored U-D factors, with D entries stored on the diagonal.
POUT(N*(N+1)/2)	Output vector stored covariance matrix (POUT = UIN is permitted).
N	Dimension of the matrices involved.

20. UD2SIG (U-D factors to sigmas)

Purpose

To compute variances from the U-D factors of a matrix.

```
CALL UD2SIG(U,N,SIG,TEXT,NCT)
```

Argument Definitions

U(N*(N+1)/2)	Input vector stored array containing the U-D factors. The D (diagonal) elements are stored on the diagonal of U.
N	Dimension of the U matrix
SIG(N)	Output vector of standard deviations
TEXT ( )	Output label of field data characters, which precedes the printed vector of standard deviations.
NCT	Number of characters of text, 0.LE.NCT.LE.126. If NCT = 0, no sigmas are printed, i.e. nothing is printed.

Functional Description

If U and D are written as doubly subscripted matrices then

$$\mathrm{SIG(J)} = \left(\mathrm{D(J,J)} + \sum_{K=J+1}^{N} \mathrm{D(K,K)}[\mathrm{U(J,K)}]^2 \right)^{\frac{1}{2}}$$

If NCT.GT.0 a title is printed, followed by the sigmas.

21. UTINV (Upper triangular matrix inverse)

Purpose

To invert an upper triangular vector stored matrix and store the result in vector form. The algorithm is so arranged that the result can overwrite the input.

```
CALL UTINV(RIN,N,ROUT)
```

Argument Definitions

RIN(N*(N+1)/2)	Input vector stored upper triangular matrix
N	Matrix dimension
ROUT(N*(N+1)/2)	Output vector stored upper triangular matrix inverse (ROUT = RIN is permitted

Remarks and Restrictions

Ill conditioning is not tested, but for nonsingular systems the result is as accurate as is the full rank singular value decomposition inverse. Singularity occurs if a diagonal is zero. The subroutine terminates when it reaches a zero diagonal. The columns to the left of the zero diagonal are, however, inverted and the result stored in ROUT.

This routine can also be used to produce the solution to RX = Z. Place Z in column N+1 (viz. RIN(N*(N+1)/2+1) = Z(1), etc.), define RIN((N+1)(N+2)/2) = -1 and call the subroutine using N+1 instead of N. On return the first N entries of column N+1 contain the solution (e.g. ROUT(N*(N+1)/2+1) = X(1), etc.).

Because matrix inversion is numerically sensitive we recommend using this subroutine only in double precision.

Functional Description

The matrix inversion is accomplished using the standard back substitution method for inverting triangular matrices, cf. the book references by Lawson and Hanson, [1] or Bierman [3].

22. UTIROW (Upper triangular inverse, inverting only the upper rows)

Purpose

To compute the inverse of a vector stored upper triangular matrix, when the lower right corner triangular inverse is given.

```
CALL UTIROW(RIN,N,ROUT,NRY)
```

Argument Definitions

RIN(N*(N+1)/2)	Input vector stored upper triangular matrix. Only the first N - NRY rows are altered by the algorithm.
N	Matrix dimension.
ROUT(N*(N+1)/2)	Output vector stored upper triangular matrix inverse. On input the lower NRY dimensional right corner contains the given (known) inverse. This lower right corner matrix is left unchanged. (ROUT = RIN is permitted.)
NRY	Number of rows, starting at the bottom, that are assumed already inverted.

Remarks and Restrictions

The purpose of this subroutine is to complete the computation of an upper triangular matrix inverse, given that the lower right corner has already been inverted. Part of the input, the rows to be inverted, are inserted via the matrix RIN. The portion of the matrix that has already been inverted is entered via the matrix ROUT. It may seem odd that part of the input matrix is put into RIN and part into ROUT. The reasoning behind this decision is that RIN represents the input matrix to be inverted (it just happens that we do not make use of the lower right triangular entries); ROUT represents the inversion result, and therefore that portion of the inversion that is given should be entered in this array.

Ill conditioning is not tested, but for nonsingular systems the result is accurate. Singularity halts the algorithm if any of the first N-NRY diagonal elements is zero. If the first zero encountered moving up the diagonal (starting at N-NRY) is at diagonal j then the rows below this element will be correctly represented in ROUT.

To generate estimates do the following: put N+1 into the matrix dimension argument; in the first N-NRY rows of the last column of RIN put the right hand side elements of the equation $R_x x + R_{xy} y = z_x$ (i.e., R_x, R_{xy}, and z_x make up the first N-NRY rows of RIN); in the next NRY entries of ROUT, beginning in the (N-NRY+1)st element, put y_{est} (i.e., R_y^{-1} and y_{est} make up rows N-NRY+1,...,N of ROUT); and ROUT((N+1)(N+2)/2) = -1. On output, the last column of ROUT will contain x_{est}, y_{est} and -1.

When NRY = 0 this algorithm is equivalent to subroutine UTINV.

Functional Description

The matrix inversion is accomplished using the standard back substitution method. The computations are arranged row-wise, starting at the bottom (from row N-NRY, since it is assumed that the last NRY rows have already been inverted).

23. WGS (Weighted Gram-Schmidt matrix triangularization)

Purpose

To compute a vector stored U-D array from an input rectangular matrix W, and a diagonal matrix D_w so that $W D_w W^T = UDU^T$.

```
CALL WGS(W,IMAXW,IW,JW,DW,U,V)
```

Argument Definitions

W(IW,JW)	Input rectangular matrix, destroyed by the computations
IMAXW	Row dimension of input W matrix, IMAXW.GE.IW
DW(JW)	Diagonal input matrix; the entries are assumed to be nonnegative. This vector is unaltered by the computations
U(IW*(IW+1)/2)	Vector stored output U-D array
V(JW)	Work vector in the computation

Remarks and Restrictions

The algorithm is not numerically stable when negative DW weights are used; negative weights are, however, allowed. If JW is less than IW (more rows than columns), the output U-D array is singular; with IW-JW zero diagonal entries in the output U array.

Functional Description

A D_w-orthogonal set of row vectors, ϕ_1, $\phi_2, \ldots$, ϕ_{IW}, are constructed from the input rows of the W matrix, i.e., $W = U \phi$, , $\phi D_w \phi^T = D$. The construction is accomplished using the modified Gram-Schmidt orthogonal construction (see refs. [1] or [3]). This algorithm is reputed to have excellent numerical properties. Note that the ϕ vectors are not of interest in this routine, and they are overwritten; The V vector used in the program houses vector IW-j+1 of ϕ at step j of algorithm. The fact that the computed ϕ vectors may not be D orthogonal is of no import in regard to the U and D computed results.

V. FORTRAN Subroutine Listings

```
      SUBROUTINE AGTRN (UIN,UOUT,N,C,V)
C
C         AGEE-TURNER U-D FACTOR RANK 1 UPDATE
C
C              (UOUT)*DOUT*(UOUT)**T=(UIN)*DIN*(UIN)**T+C*V*V**T
C
C     UIN(N*(N+1)/2)  INPUT VECTOR STORED POSITIVE SEMI-DEFINITE U-D
C                     ARRAY, WITH D ELEMENTS STORED ON THE DIAGONAL
C     UOUT(N*(N+1)/2) OUTPUT VECTOR STORED POSITIVE (POSSIBLY) SEMI-
C                     DEFINITE U-D RESULT. UOUT=UIN IS PERMITTED
C     N               DIMENSION OF THE STATE
C     C               SCALAR. SHOULD BE NON-NEGATIVE
C                     C IS DESTROYED DURING THE PROCESS
C     V(N)            INPUT VECTOR FOR RANK ONE MODIFICATION. V IS
C                     DESTROYED DURING THE PROCESS
C
C     COGNIZANT PERSONS: G.J.BIERMAN/M.W.NEAD   (JPL,FEB.1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION     UIN(1),   UOUT(1),  V(1)
C
      Z=0.0
      IF (C.EQ.Z) RETURN
C
      JJ=N*(N+1)/2
      DO 50 J=N,2,-1
        S=V(J)
        D=UIN(JJ)+C*S*S
        IF (D) 5,10,30
    5   WRITE (6,100)
        RETURN
   10   JJ=JJ-J
        WRITE (6,110)
        DO 20 K=1,J
   20   UOUT(JJ+K)=Z
        GO TO 50
   30   B=C/D
        BETA=S*B
        C=B*UIN(JJ)
        UOUT(JJ)=D
        JJ=JJ-J
        JM1=J-1
        DO 40 I=1,JM1
          V(I)=V(I)-S*UIN(JJ+I)
   40     UOUT(JJ+I)=UIN(JJ+I)+BETA*V(I)
   50   CONTINUE
C
      UOUT(1)=UIN(1)+C*V(1)**2
      RETURN
C
  100 FORMAT (1H0,10X,'* * * ERROR RETURN DUE TO A COMPUTED NEGATIVE COM
     1PUTED DIAGONAL IN AGTRN * * *')
  110 FORMAT (1H0,10X,'* * * NOTE: U-D RESULT IS SINGULAR * * *')
      END
```

```
      SUBROUTINE A2A1 (A,IA,IR,LA,NAMA,A1,IA1,LA1,NAMA1)
C
C        SUBROUTINE TO REARRANGE THE COLUMNS OF A(IR,LA), IN NAMA ORDER
C        AND PUT THE RESULT IN A1(IR,LA1) IN NAMA1 ORDER.  ZERO COLUMNS
C        ARE INSERTED IN A1 CORRESPONDING TO THE NEWLY DEFINED NAMES.
C
C        A(IR,LA)      INPUT RECTANGULAR MATRIX
C        IA            ROW DIMENSION OF A, IR.LE.IA
C        IR            NO. OF ROWS OF A THAT ARE TO BE REARRANGED
C        LA            NO. OF PARAMETER NAMES ASSOCIATED WITH A
C        NAMA(LA)      PARAMETER NAMES ASSOCIATED WITH A
C        A1(IR,LA1)    OUTPUT RECTANGULAR MATRIX
C                      A AND A1 CANNOT SHARE COMMON STORAGE
C        IA1           ROW DIMENSION OF A1, IR.LE.IA1
C        LA1           NO. OF PARAMETER NAMES ASSOCIATED WITH A1
C        NAMA1(LA1)    INPUT LIST OF PARAMETER NAMES TO BE ASSOCIATED
C                      WITH THE OUTPUT MATRIX A1
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      DIMENSION A(IA,1), NAMA(1), A1(IA1,1),NAMA1(1)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
      ZERO=0.
      DO 100 J=1,LA1
        DO 60 I=1,LA
          IF (NAMA(I).EQ.NAMA1(J)) GO TO 80
   60     CONTINUE
        DO 70 K=1,IR
   70     A1(K,J)=ZERO        @ ZERO COL. CORRES. TO NEW NAME
        GO TO 100
   80   DO 90 K=1,IR
   90     A1(K,J)=A(K,I)      @ COPY COL. ASSOC. WITH OLD NAME
  100   CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE COMBO (R,L1,NAM1,L2,NAM2,A,IA,LA,NAMA)
C
C          TO REARRANGE A VECTOR STORED TRIANGULAR MATRIX AND STORE
C          THE RESULT IN MATRIX A.  THE DIFFERENCE BETWEEN THIS SUB-
C          ROUTINE AND R2A IS THAT THERE THE NAMELIST FOR A IS INPUT.
C          HERE IT IS DETERMINED BY COMBINING THE LIST FOR R WITH
C          A LIST OF DESIRED NAMES.
C
C        R(L1*(L1+1)/2)  INPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C        L1              NO. OF PARAMETERS IN R (AND IN NAM1)
C        NAM1(L1)        NAMES ASSOCIATED WITH R
C        L2              NO. OF PARAMETERS IN NAM2
C        NAM2(L2)        PARAMETER NAMES THAT ARE TO BE COMBINED WITH R
C                        (NAM1 LIST). THESE NAMES MAY OR MAY NOT BE IN
C                        NAM1.
C        A(L1,LA)        OUTPUT ARRAY CONTAINING THE REARRANGED
C                        R MATRIX, L1.LE.IA.
C        IA              ROW DIMENSION OF A
C        LA              NO. OF PARAMETER NAMES IN NAMA, AND THE
C                        COLUMN DIMENSION OF A.  LA=L1+L2-NO. NAMES
C                        COMMON TO NAM1 AND NAM2.  LA IS COMPUTED AND
C                        OUTPUT.
C        NAMA(LA)        PARAMETER NAMES ASSOCIATED WITH THE OUTPUT A
C                        MATRIX.  CONSISTS OF NAMES IN NAM1 NOT IN
C                        NAM2 FOLLOWED BY NAM2.
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION  R(1), A(IA,1), NAM1(1), NAM2(1), NAMA(1)
C
      ZERO=0.0
      K=1
      DO 100 I=1,L1
        DO 50 J=1,L2
          IF (NAM1(I).EQ.NAM2(J)) GO TO 100
   50     CONTINUE
        NAMA(K)=NAM1(I)
      JJ=I*(I-1)/2
        DO 60 L=1,I
   60     A(L,K)=R(JJ+L)
      IF (I.EQ.L1) GO TO 80
      IP1 = I+1
      DO 70 L=IP1,L1
   70 A(L,K) = ZERO
   80   K=K+1
  100   CONTINUE
C               NAMES UNIQUE TO NAM1 ARE NOW IN NAMA
      DO 200 J=1,L2
        DO 150 I=1,L1
          IF (NAM2(J).EQ.NAM1(I)) GO TO 170
  150     CONTINUE
        NAMA(K)=NAM2(J)
        DO 160 L=1,L1
  160     A(L,K)=ZERO
```

```
C               NAMES UNIQUE TO NAM2 ARE NOW IN NAMA
        GO TO 190
  170   NAMA(K)=NAM2(J)
C             LOCATE DIAGONAL OF PRECEDING COLUMN
      JJ=I*(I-1)/2
        DO 180 L=1,I
  180     A(L,K)=R(JJ+L)
      IF (I.EQ.L1) GO TO 190
      IP1=I+1
      DO 185 L=IP1,L1
  185 A(L,K)=ZERO
  190   K=K+1
  200   CONTINUE
      LA=K-1
C               NAMES MUTUAL TO NAM1 AND NAM2 ARE NOW IN NAMA
      RETURN
      END
```

```
      SUBROUTINE COV2RI(U,N)
C
C        TO CONSTRUCT THE UPPER TRIANGULAR CHOLESKY FACTOR OF A
C        POSITIVE SEMI-DEFINITE MATRIX. BOTH THE INPUT COVARIANCE
C        AND THE OUTPUT CHOLESKY FACTOR (SQUARE ROOT) ARE VECTOR
C        STORED. THE OUTPUT OVERWRITFS THE INPUT.
C        COVARIANCE(INPUT)=U*U**T  (U IS OUTPUT).
C
C        IF THE INPUT COVARIANCE IS SINGULAR THE OUTPUT FACTOR HAS
C        ZERO COLUMNS.
C
C        U(N*(N+1)/2)  CONTAINS THE INPUT VECTOR STORED COVARIANCE
C                      MATRIX (ASSUMED POSITIVE DEFINITE) AND ON OUTPUT
C                      IT CONTAINS THE UPPER TRIANGULAR SQUARE ROOT
C                      FACTOR.
C        N             DIMFNSION OF THE MATRICES INVOLVED
C
C        COGNIZANT PERSONS:  G.J.BIFRMAN/M.W.NEAD (JPL, FEB. 1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION U(1)
C
      ZERO=0.0
      ONE=1.
      JJ=N*(N+1)/2
      JJN=JJ
C
      DO 5 J=N,2,-1
         IF (U(JJ).LT.ZERO) U(JJ)=7FRO
         U(JJ)= SQRT(U(JJ))
         IF (U(JJ).GT.ZERO) ALPHA=ONE/U(JJ)
C
         KK=0
         JJN=JJ-J                           @ NEXT DIAGONAL
         JM1=J-1
           DO 4 K=1,JM1
             U(JJN+K)=ALPHA*U(JJN+K)        @ JJN+K=(K,J)
             S=U(JJN+K)
             DO 3 I=1,K
    3          U(KK+I)=U(KK+I)-S*U(JJN+I)   @ KK+I=(I,K)
    4        KK=KK+K
    5 JJ=JJN
      IF (U(1).LT.ZERO) U(1)=ZERO
      U(1)= SQRT(U(1))
C
      RETURN
      END
```

```
      SUBROUTINE COV2UD (U,N)
C
C        TO OBTAIN THE U-D FACTORS OF A POSITIVE SEMI-DEFINITE MATRIX.
C        THE INPUT MATRIX VECTOR STORED IS OVERWRITTEN BY THE OUTPUT
C        U-D FACTORS WHICH ARE ALSO VECTOR STORED.
C
C        U(N*(N+1)/2)  CONTAINS INPUT VECTOR STORED COVARIANCE MATRIX.
C                      ON OUTPUT IT CONTAINS THE VECTOR STORED U-D
C                      COVARIANCE FACTORS.
C        N             MATRIX DIMENSION
C
C        SINGULAR INPUT COVARIANCES RESULT IN OUTPUT MATRICES WITH ZERO
C        COLUMNS
C
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/R.A.JACOBSON (JPL, FEB. 1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
      DIMENSION U(1)
C
      Z=0.0
      ONE=1.0
C
      JJ=N*(N+1)/2
      DO 50 J=N,2,-1
        ALPHA=Z
        IF (U(JJ).LT.Z) U(JJ)=Z
        IF (U(JJ).GT.Z) ALPHA=ONE/U(JJ)
        JJ=JJ-J
        KK=0
        KJ=JJ
        JM1=J-1
        DO 40 K=1,JM1
          KJ=KJ+1
          BETA=U(KJ)
          U(KJ)=ALPHA*U(KJ)
          IJ=JJ
          IK=KK
          DO 30 I=1,K
            IK=IK+1
            IJ=IJ+1
   30       U(IK)=U(IK)-BETA*U(IJ)
   40     KK=KK+K
   50   CONTINUE
      IF (U(1).LT.7) U(1)=7
      RETURN
      END
```

```fortran
      SUBROUTINE C2C (C,IC,L1,NAM1,L2,NAM2)
C
C        SUBROUTINE TO REARRANGE THE ROWS AND COLUMNS OF MATRIX
C        C(L1,L1) IN NAM1 ORDER AND PUT THE RESULT IN
C        C(L2,L2) IN NAM2 ORDER. ZERO COLUMNS AND ROWS ARE
C        ASSOCIATED WITH OUTPUT DEFINED NAMES THAT ARE NOT CONTAINED
C        IN NAM1.
C
C        C(L1,L1)     INPUT MATRIX
C        IC           ROW DIMENSION OF C, IC.GE.L=MAX(L1,L2)
C        L1           NO. OF PARAMETER NAMES ASSOCIATED WITH THE INPUT C
C        NAM1(L)      PARAMETER NAMES ASSOCIATED WITH C ON INPUT. (ONLY
C                     THE FIRST L1 ENTRIES APPLY TO THE INPUT C)
C        L2           NO. OF PARAMETER NAMES ASSOCIATED WITH THE OUTPUT C
C        NAM2(L2)     PARAMETER NAMES ASSOCIATED WITH THE OUTPUT C
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION  C(IC,1), NAM1(1), NAM2(1)
C
      ZERO=0.
      L=MAX(L1,L2)
      IF (L.LE.L1) GO TO 5
      NM=L1+1
      DO 1 K=NM,L
    1   NAM1(K)= ZERO         @ ZERO REMAINING NAM1 LOCNS
    5 DO 90 J=1,L2
        DO 10 I=1,L
          IF (NAM1(I).EQ.NAM2(J)) GO TO 30
   10     CONTINUE
        GO TO 90
   30    IF (I.EQ.J) GO TO 90
        DO 40 K=1,L
          H=C(K,J)            @ INTERCHANGE COLUMNS I AND J
          C(K,J)=C(K,I)
   40     C(K,I)=H
        DO 80 K=1,L
          H=C(J,K)            @ INTERCHANGE ROWS I AND J
          C(J,K)=C(I,K)
   80     C(I,K)=H
          NM=NAM1(I)          @ INTERCHANGE LABELS I AND J
          NAM1(I)=NAM1(J)
          NAM1(J)=NM
   90   CONTINUE
C
C         FIND NAM2 NAMES NOT IN NAM1 AND SET CORRESPONDING ROWS AND
C         COLUMNS TO ZERO
C
      DO 120 J=1,L2
        DO 100 I=1,L
          IF (NAM1(I).EQ.NAM2(J)) GO TO 120
  100     CONTINUE
        DO 110 K=1,L2
          C(J,K)=ZERO
  110     C(K,J)=ZERO
  120   CONTINUE
C
      RETURN
      END
```

```
C
      SUBROUTINE INF2R (P,N)
C
C         TO CHOLESKY FACTOR AN INFORMATION MATRIX
C
C     COMPUTES A LOWER TRIANGULAR VECTOR STORED CHOLESKY FACTORIZATION
C     OF A POSITIVE SEMI-DEFINITE MATRIX. P=R(**T)R, R UPPER TRIANGULAR.
C     BOTH MATRICES ARE VECTOR STORED AND THE RESULTS OVERWRITES
C     THE INPUT
C
C     P(N*(N+1)/2)  ON INPUT THIS IS A POSITIVE SEMI-DEFINITE MATRIX,
C                   AND ON OUTPUT IT IS A TRIANGULAR FACTOR. IF THE
C                   INPUT MATRIX IS SINGULAR THE OUTPUT MATRIX WILL
C                   HAVE ZERO DIAGONAL ENTRIES
C     N             DIMENSION OF MATRICES INVOLVED
C
C     COGNIZANT PERSON: G.J.BIERMAN/M.W.NEAD     (JPL,FEB.1977)
C
      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
C
      DIMENSION P(1)
C
      Z=0.0
      ONE=1.0
      JJ=0
      NN=N*(N+1)/2
      NM1=N-1
      DO 10 J=1,NM1
        JJ=JJ+J                               @ JJ=(J,J)
        IF (P(JJ).LT.Z)  P(JJ)=Z
        P(JJ)=SQRT(P(JJ))
        ALPHA=Z
        IF (P(JJ).GT.Z) ALPHA=ONE/P(JJ)
        JK=NN+J                               @ JK=(J,K)
        JP1=J+1
        JIS=JK                                @ JIS=(J,I) START
        DO 10 K=N,JP1,-1
          JK=JK-K
          P(JK)=ALPHA*P(JK)
          BETA=P(JK)
          KI=NN+K
          JI=JIS
          DO 10 I=N,K,-1
            KI=KI-I
            JI=JI-I
   10       P(KI)=P(KI)-P(JI)*BETA
C
      IF (P(NN).LT.Z) P(NN)=Z
      P(NN)=SQRT(P(NN))
      RETURN
      END
```

```
      SUBROUTINE PERMUT (A,IA,IR,L1,NAM1,L2,NAM2)
C
C        SUBROUTINE TO REARRANGE PARAMETERS OF A(IR,L1), NAM1 ORDER
C        TO A(IR,L2), NAM2 ORDER.  ZERO COLUMNS ARE INSERTED
C        CORRESPONDING TO THE NEWLY DEFINED NAMES.
C
C        A(IR,L)    INPUT RECTANGULAR MATRIX, L=MAX(L1,L2)
C        IA         ROW DIMENSION OF A, IA.GE.IR
C        IR         NUMBER OF ROWS OF A THAT ARE TO BE REARRANGED
C        L1         NUMBER OF PARAMETER NAMES ASSOCIATED WITH THE INPUT
C                   A MATRIX
C        NAM1(L)    PARAMETER NAMES ASSOCIATED WITH A ON INPUT
C                   (ONLY THE FIRST L1 ENTRIES APPLY TO THE INPUT A)
C        L2         NUMBER OF PARAMETER NAMES ASSOCIATED WITH THE OUTPUT
C                   A MATRIX
C        NAM2       PARAMETER NAMES ASSOCIATED WITH THE OUTPUT A
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(IA,1), NAM1(1), NAM2(1)
C
      ZERO=0.
      L=MAX(L1,L2)
      IF (L.LE.L1) GO TO 50
      NM=L1+1
      DO 40 K=NM,L
   40   NAM1(K)=0           @ ZERO REMAINING NAM1 LOCS
   50 DO 100 J=1,L2
        DO 60 I=1,L
          IF (NAM1(I).EQ.NAM2(J)) GO TO 65
   60     CONTINUE
      GO TO 100
   65 CONTINUE
        IF (I.EQ.J) GO TO 100
        DO 70 K=1,IR        @ INTERCHANGE COLS I AND J
          W=A(K,J)
          A(K,J)=A(K,I)
   70     A(K,I)=W
          NM=NAM1(I)        @ INTERCHANGE I AND J COL. LABELS
          NAM1(I)=NAM1(J)
          NAM1(J)=NM
  100   CONTINUE
C                       REPEAT TO FILL NEW COLS
        DO 200 J=1,L2
          DO 160 I=1,L
            IF (NAM1(I).EQ.NAM2(J)) GO TO 200
  160      CONTINUE
          DO 170 K=1,IR
  170       A(K,J)=ZERO
  200    CONTINUE
C
      RETURN
      END
```

```
C
      SUBROUTINE RINCON (RIN,N,ROUT,CNB)
C
C     TO COMPUTE THE INVERSE OF THE UPPER TRIANGULAR VECTOR STORED
C     INPUT MATRIX RIN AND STORE THE RESULT IN ROUT. (RIN=ROUT IS
C     PERMITTED) AND TO COMPUTE A CONDITION NUMBER ESTIMATE.
C     CNB=FROB.NORM(R)*FROB.NORM(R**-1).
C     THE FROBENIUS NORM IS THE SQUARE ROOT OF THE SUM OF SQUARES
C     OF THE ELEMENTS. THIS CONDITION NUMBER BOUND IS USED AS
C     AN UPPER BOUND AND IT ACTS AS A LOWER BOUND ON THE ACTUAL
C     CONDITION NUMBER OF THE PROBLEM. (SEE THE BOOK 'SOLVING LEAST
C     SQUARES', BY LAWSON AND HANSON)
C
C     RIN(N*(N+1)/2)  INPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C     N               DIMENSION OF R MATRICES
C     ROUT(N*(N+1)/2) OUTPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C                     INVERSE (RIN=ROUT IS PERMITTED)
C     CNB             CONDITION NUMBER BOUND
C
C     COGNIZANT PERSONS: G.J.BIERMAN/M.W.NEAD   (JPL,FEB.1977)
C
C     SUBROUTINES REQUIRED:  UTINV
C
      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
      DIMENSION RIN(1), ROUT(1)
C
      Z=0.0
      NTOT=N*(N+1)/2
C
      RNM=Z
      DO 10 J=1,NTOT
   10   RNM=RNM+RIN(J)**2
C
      CALL UTINV (RIN,N,ROUT)
      RNMOUT=Z
      DO 20 J=1,NTOT
   20   RNMOUT=RNMOUT+ROUT(J)**2
C
      CNB=SQRT(RNM*RNMOUT)
C
      WRITE (6,30) CNB
      RETURN
C
   30 FORMAT(1H0,5X,'CONDITION NUMBER BOUND=',D18.10,2X,'CNB/N.LE.CONDIT
     1ION NUMBER.LE.CNB',/)
      END
```

```
      SUBROUTINE RI2COV (RINV,N,SIG,COVOUT,KOV)
C
C        TO COMPUTE THE COVARIANCE MATRIX AND/OR THE STANDARD DEVIATIONS
C        OF A VECTOR STORED UPPER TRIANGULAR SQUARE ROOT COVARIANCE
C        MATRIX.  THE OUTPUT COVARIANCE MATRIX IS ALSO VECTOR STORED.
C
C        RINV(N*(N+1)/2)    INPUT VECTOR STORED UPPER TRIANGULAR COVARI-
C                           ANCE SQUARE ROOT. (RINV=R INVERSE IS THE
C                           INVERSE OF THE SRIF MATRIX)
C        N                  DIMENSION OF THE RINV MATRIX
C        SIG(N)             OUTPUT VECTOR OF STANDARD DEVIATIONS
C        COVOUT(N*(N+1)/2)  OUTPUT VECTOR STORED COVARIANCE MATRIX
C                           (COVOUT = RINV IS ALLOWED)
C        KOV   .GT.0        COMPUTE COVARIANCE AND SIGMAS USING KOV ROWS
C                           OF RINV.
C              .LT.0        COMPUTE ONLY THE SIGMAS USING KOV ROWS OF
C                           RINV.
C              .EQ.0        NO COVARIANCE, BUT ALL SIGMAS (E.G. USE
C                           N ROWS OF RINV).
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
      DIMENSION RINV(1), SIG(1), COVOUT(1)
C
      ZERO=0.0
      LIM=N
      IF (KOV.NE.0) LIM=IABS(KOV)
C          *** COMPUTE SIGMAS
      IKS=0
      DO 2 J=1,LIM
        IKS=IKS+J
        SUM=ZERO
        IK=IKS
        DO 1 K=J,N
          SUM=SUM+RINV(IK)**2
    1     IK=IK+K
    2   SIG(J)=SQRT(SUM)
C
      IF (KOV.LE.0) RETURN
C          *** COMPUTE COVARIANCE
      JJ=0
      NM1=LIM-1
      DO 10 J=1,NM1
        JJ=JJ+J
        COVOUT(JJ)=SIG(J)**2
        IJS=JJ+J
        JP1=J+1
        DO 10 I=JP1,N
          IK=IJS
          IMJ=I-J
          SUM=ZERO
          DO 5 K=I,N
            IJK=IK+IMJ
            SUM=SUM+RINV(IK)*RINV(IJK)
    5       IK=IK+K
          COVOUT(IJS)=SUM
   10     IJS=IJS+I
      IF (KOV.EQ.N) COVOUT(JJ+N)=SIG(N)**2
C
      RETURN
      END
```

```
      SUBROUTINE R2A(R,LR,NAMR,A,IA,LA,NAMA)
C
C        TO PLACE THE TRIANGULAR VECTOR STORED MATRIX R INTO THE
C        MATRIX A AND TO ARRANGE THE COLUMNS TO MATCH THE DESIRED
C        NAMA PARAMETER LIST.  NAMES IN THE NAMA LIST THAT DO NOT
C        CORRESPOND TO ANY NAME IN NAMR HAVE ZERO ENTRIES IN THE
C        CORRESPONDING A COLUMN.
C
C        R(LR*(LR+1)/2)  INPUT UPPER TRIANGULAR VECTOR STORED ARRAY
C        LR              DIMENSION OF R
C        NAMR(L)         PARAMETER NAMES ASSOCIATED WITH R.  ONLY THE
C                        FIRST LR ENTRIES APPLY TO R, L=MAX(LR,LA).
C        A(IR,LA)        MATRIX TO HOUSE THE REARRANGED R MATRIX
C        IA              ROW DIMENSION OF A, IA.GE.LR
C        LA              NO. OF PARAMETER NAMES ASSOCIATED WITH THE
C                        OUTPUT A MATRIX
C        NAMA(LA)        PARAMETER NAMES FOR THE OUTPUT A MATRIX
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION R(1),NAMR(1),A(IA,1),NAMA(1)
C
       ZERO=0.
      DO 5 J=1,LA
      DO 5 K=1,LR
    5 A(K,J)=ZERO               @ ZERO A(LR,LA)
      DO 40 J=1,LA
        DO 10 I=1,LR
          IF (NAMR(I).EQ.NAMA(J)) GO TO 20
   10     CONTINUE
        GO TO 40
   20 JJ=I*(I-1)/2
        DO 30 K=1,I
   30     A(K,J)=R(JJ+K)
   40   CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE R2RA (R,NR,NAM,RA,NRA,NAMA)
C
C        TO COPY THE UPPER LEFT (LOWER RIGHT) PORTION OF A VECTOR
C        STORED UPPER TRIANGULAR MATRIX R INTO THE LOWER RIGHT
C        (UPPER LEFT) PORTION OF A VECTOR STORED TRIANGULAR
C        MATRIX RA.
C
C        R(NR*(NR+1)/2)      INPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C        NR                  DIMENSION OF R
C        NAM(NR)             NAMES ASSOCIATED WITH R
C        RA(NRA*(NRA+1)/2) OUTPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C        NRA                 DIMENSION ASSOCIATED WITH RA
C        NAMA(NRA)           NAMES ASSOCIATED WITH RA
C
C     IF NRA=0 ON INPUT, THEN NAMA(1) SHOULD HAVE THE FIRST NAME OF THE
C     OUTPUT NAMELIST AND THE NUMBER OF NAMES IN NAMA IS COMPUTED.
C     THE LOWER RIGHT BLOCK OF R WILL BE THE UPPER LEFT BLOCK OF RA.
C
C     IF NRA=LAST NAME OF THE UPPER LEFT BLOCK THAT IS TO BE MOVED,
C     THEN THE UPPER BLOCK IS TO BE MOVED TO THE LOWER RIGHT POSITION.
C     WHEN USED IN THIS MODE NRA=NR ON OUTPUT.
C
C     THE NAMES OF THE RELOCATED BLOCK ARE ALSO MOVED.  THE RESULT
C     CAN COINCIDE WITH R AND NAMA WITH NAM.
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, SEPT. 1976)
C
      IMPLICIT DOUBLE PRECISION   (A-H,O-Z)
      DIMENSION      R(1),RA(1), NAM(1), NAMA(1)
      LOGICAL        IS
C
      IS=.FALSE.
      LOCN=NAMA(1)
C        IS=FALSE CORRESPONDS TO MOVING UPPER LFT. CORNER OF R TO
C        LOWER RT. CORNER OF RA
      IF (NRA.EQ.0) GO TO 1
      LOCN=NRA
      IS=.TRUE.
C        IS=TRUE CORRESPONDS TO MOVING LOWER LFT. CORNER OF R TO
C        UPPER RT. CORNER OF RA
    1 DO 3 I=1,NR
      IF (NAM(I).EQ.LOCN) GO TO 4
    3 CONTINUE
      WRITE (6,100)
  100 FORMAT (1H0,20X,'NAMA(1) NOT IN NAMELIST OF R MATRIX')
      RETURN
C
    4 K=I
      KM1=K-1
      IF (IS) GO TO 15
C
      IJS=K*(K+1)/2-1
      NRA=NR-K+1
      IJA=0
      KOLA=0
```

```
      DO 10 KOL=K,NR                                                    R2RA0550
        KOLA=KOLA+1                                                     R2RA0560
      NAMA(KOL-KM1)=NAM(KOL)                                            R2RA0570
        DO 5 IR=1,KOLA                                                  R2RA0580
          IJA=IJA+1                                                     R2RA0590
    5     RA(IJA)=R(IJS+IR)                                             R2RA0600
   10   IJS=IJS+KOL                                                     R2RA0610
      RETURN                                                            R2RA0620
C                                                                       R2RA0630
   15  IJ=K*(K+1)/2                                                     R2RA0640
       IJA=NR*(NR+1)/2                                                  R2RA0650
       L=NR-KM1                                                         R2RA0660
       KOL=K                                                            R2RA0670
          DO 25 KOLA=NR,L,-1                                            R2RA0680
          IJS=IJA                                                       R2RA0690
          NAMA(KOLA)=NAM(KOL)                                           R2RA0700
            DO 20 IR=KOLA,L,-1                                          R2RA0710
              RA(IJS)=R(IJ)                                             R2RA0720
              IJS=IJS-1                                                 R2RA0730
   20         IJ=IJ-1                                                   R2RA0740
          IJA=IJA-KOLA                                                  R2RA0750
   25     KOL=KOL-1                                                     R2RA0760
       NRA=NR                                                           R2RA0770
C                                                                       R2RA0780
      RETURN                                                            R2RA0790
      END                                                               R2RA0800
```

```
      SUBROUTINE RUDR(RIN,N,ROUT,IS)
C
C     FOR N.GT.0 THIS SUBROUTINE TRANSFORMS AN UPPER TRIANGULAR VECTOR
C     STORED SRIF MATRIX TO U-D FORM, AND WHEN N.LT.0 THE U-D VECTOR
C     STORED ARRAY IS TRANSFORMED TO A VECTOR STORED SRIF ARRAY
C
C     RIN((N+1)*(N+2)/2)    INPUT VECTOR STORED SRIF OR U-D ARRAY
C     ROUT((N+1)*(N+2)/2)   OUTPUT IS THE CORRESPONDING U-C OR SRIF
C                           ARRAY (RIN=ROUT IS PERMITTED)
C     N                     ABS(N)= MATRIX DIMENSION
C             N.GT.0        THE (INPUT) SRIF ARRAY IS OUTPUT IN U-D FORM
C             N.LT.0        THE (INPUT) U-D ARRAY IS OUTPUT IN SRIF FORM
C     IS =    0             THERE IS NO RT. SIDE OR ESTIMATE STORED IN
C                           COLUMN N+1, AND RIN NEED HAVE ONLY
C                           N COLUMNS, I.E. RIN(N*(N+1)/2)
C     IS =    1             THERE IS A RT. SIDE INPUT TO THE SRIF AND
C                           AN ESTIMATE FOR THE U-D ARRAY. THESE RESIDE
C                           IN COLUMN N+1.
C
C     THIS SUBROUTINE USES SUBROUTINE UTINV
C
C     COGIZANT PERSONS G.J.BIERMAN/M.W.NEAD  (JPL, FEB.1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION RIN(1), ROUT(1)
C
      ONE= 1.0
      NP1= IS + ABS(N)
      JJ=1                                @ INITIALIZE DIAGONAL INDEX
      IDIMR= NP1*(NP1 +1)/2
      IF (IS.EQ.1) RIN(IDIMR)= - ONE
C
      IF (N.LT.0) GO TO 30
      CALL UTINV(RIN,NP1,ROUT)
      ROUT(1)= ROUT(1)**2
      IF (N.EQ.1) RETURN
      DO 20 J=2,N
      S=ONE/ROUT(JJ+J)
      ROUT(JJ+J)= ROUT(JJ+J)**2
      JM1=J-1
      DO 10 I=1,JM1
   10 ROUT(JJ+I)= ROUT(JJ+I)*S
   20 JJ=JJ+ J
      RETURN
C
   30 N=-N
      ROUT(1)= SQRT(RIN(1))
      IF(N.EQ.1) GO TO 60
      DO 50 J=2,N
      ROUT(JJ+J)= SQRT(RIN(JJ+J))
      S=ROUT(JJ+J)
      JM1=J-1
      DO 40 I=1,JM1
   40 ROUT(JJ+I)= RIN(JJ+I)*S
   50 JJ=JJ+J
   60 CALL UTINV(ROUT,NP1,ROUT)
C

      RETURN
      END
```

```
      SUBROUTINE THH(R,N,A,IA,M,SOS,NSTRT)
C
C        THIS SUBROUTINE PERFORMS A DOUBLE PRECISION TRIANGULARIZATION
C        OF A RECTANGULAR MATRIX INTO A SINGLY-SUBSCRIPTED ARRAY BY
C        APPLICATION OF HOUSEHOLDER ORTHONORMAL TRANSFORMATIONS.
C
C        R(N*(N+3)/2) VECTOR STORED SQUARE ROOT INFORMATION MATRIX
C                     (LAST N LOCATIONS MAY CONTAIN A RIGHT HAND SIDE)
C        N            NUMBER OF PARAMETERS
C        A(IA,N+1)    MEASUREMENT MATRIX
C        IA           ROW DIMENSION OF A
C        M            NUMBER OF OBSERVATIONS IN THIS BATCH
C        SOS          ACCUMULATED SUM OF SQUARES OF THE RESIDUALS
C                     (Z-A*X(EST)**2), INCLUDES A PRIORI
C        NSTRT        FIRST COL OF THE INPUT A MATRIX THAT HAS A NONZERO
C                     ENTRY. IF NSTRT.LE.1, IT IS SET TO 1. THIS OPTION
C                     IS CONVENIENT WHEN PACKING A PRIORI BY BATCHES AND
C                     THE A MATRIX HAS LEADING COLUMNS OF ZEROS.
C
C     ON ENTRY R CONTAINS A PRIORI SQUARE ROOT INFORMATION FILTER (SRIF)
C        ARRAY, AND ON EXIT IT CONTAINS THE A POSTERIORI (PACKED) ARRAY.
C     ON ENTRY A CONTAINS OBSERVATIONS WHICH ARE DESTROYED BY THE
C        INTERNAL COMPUTATIONS.
C     ON ENTRY IF SOS IS .LT. ZERO ,PROGRAM WILL ASSUME THERE IS NO
C        RIGHT HAND SIDE DATA AND WILL NOT COMPUTE SOS OR USE LAST N
C        LOCATIONS OF VECTOR R.
C
C           COGNIZANT  PERSONS G.J.BIERMAN/N.HAMATA (JPL, OCT.1975)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(IA,1),R(1)
      DOUBLE PRECISION SUM
      DATA ZERO/0.D0/, ONE/1.D0/
C
      IF (NSTRT.LE.0) NSTRT=1
      NP1=N+1                      @ NO. COLUMNS OF R
      IF(SOS.LT.ZERO) NP1=N        @ NO COLS. = N IF SOS.LT.0
      KK=NSTRT*(NSTRT-1)/2
      DO 100 J=NSTRT,N             @ J-TH STEP OF HOUSEHOLDER REDUCTION
      KK=KK+J
      SUM=ZERO
      DO 20 I=1,M
   20 SUM=SUM+A(I,J)**2
      IF(SUM.LE.ZERO) GO TO 100    @ IF J-TH COL. OF A.EQ.0 GO TO STEP J+1
      SUM=SUM+R(KK)**2
      SUM=DSQRT(SUM)
      IF(R(KK).GT.ZERO) SUM=-SUM
      DELTA=R(KK)-SUM
      R(KK)=SUM
      BETA=ONE/(SUM*DELTA)
      JJ=KK
      L=J
      J1=J+1
C        ** READY TO APPLY J-TH HOUSEHOLDER TRANS.
      DO 40 K=J1,NP1
```

```
      JJ=JJ+L
      L=L+1
      SUM=DELTA*R(JJ)
      DO 30 I=1,M
   30 SUM=SUM+A(I,J)*A(I,K)
      IF(SUM.EQ.ZERO) GO TO 40
      SUM=SUM*BETA
      R(JJ)=R(JJ)+SUM*DELTA
      DO 35 I=1,M
   35 A(I,K)=A(I,K)+SUM*A(I,J)
   40 CONTINUE
  100 CONTINUE
      IF(SOS.LT.ZERO) RETURN
C
C     CALCULATE SOS
C
      SUM=ZERO
      DO 110 I=1,M
  110 SUM=SUM+A(I,NP1)**2
      SOS=DSQRT(SOS**2+SUM)
C
      RETURN
      END
```

```
      SUBROUTINE TRIMAT (A,N,CAR,TEXT,NCHAR,NAMES)
C
C        TO DISPLAY A VECTOR STORED UPPER TRIANGULAR MATRIX IN A
C        TWO-DIMENSIONAL TRIANGULAR FORMAT
C
C        A(N*(N+1)/2) VECTOR CONTAINING UPPER TRIANGULAR MATRIX    (DP)
C        N            DIMENSION OF MATRIX                          (I)
C        CAR(N)       PARAMETER NAMES                              (I)
C        TEXT( )      AN ARRAY OF FIELDATA CHARACTERS TO BE PRINTED AS
C                     A TITLE PRECEDING THE MATRIX
C        NCHAR        NUMBER OF CHARACTERS, INCLUDING SPACES, THAT
C                     ARE TO BE PRINTED IN TEXT( )
C                     ABS(NCHAR).LE.126. NCHAR NEGATIVE IS USED
C                     TO AVOID SKIPPING TO A NEW PAGE TO START
C                     PRINTING
C        NAMES        TRUE TO PRINT PARAMETER NAMES
C
C         COGNIZANT PERSONS:  G.J.BIERMAN/M.W.NEAD (JPL, OCT.1975)
C
      DOUBLE PRECISION  A(N)
      INTEGER CAR(N), TEXT(1), L(7), LIST(7)
      LOGICAL NAMES
      INTEGER  V(4),VFMT(7)
      DATA  V/'(2X,','A6,1X,',' ','D17.8)'/,
     1 VFMT/'7','017X,6','034X,5','051X,4','068X,3','085X,2','102X,1'/
C
C        M1,M2     ROW LIMITS FOR EACH PRINT SEQUENCE
C        N1,M2     COL LIMITS FOR EACH LINE OF PRINT
C        L(I)      LOC OF EACH COLUMN IN A ROW
C        KT        ROW COUNTER
C        KP        PRINT COUNTER
C
C  * * * * *   INITIALIZE COUNTERS
C
      M1=1
      M2=7
      N1=1
      KT=0
      KP=0
      IF (.NOT.NAMES) V(2)='I5,2X'
C
      NC=IABS(NCHAR)/6
      IF (MOD(NCHAR,6).NE.0) NC=NC+1
      IF (NCHAR.GE.0) WRITE (6,200) (TEXT(I),I=1,NC)
      IF (NCHAR.LT.0) WRITE (6,205) (TEXT(I),I=1,NC)
   10 IF (M2.GT.N) M2=N
      IF (.NOT.NAMES) GO TO 20
      WRITE (6,210) (CAR(I),I=N1,M2)
      GO TO 40
   20 M=N1
      L2=M2-N1+1
      DO 30 I=1,L2
        LIST(I)=M
   30   M=M+1
      WRITE (6,220) (LIST(I),I=1,L2)
```

```
   40 CONTINUE
C    * * * * * *
      DO 190 IC=M1,M2
        K=1
        IF (IC.LE.(KT*7)) GO TO 60
        JJ=0
        DO 50 J=1,IC
   50     JJ=JJ+J
        L(K)=JJ
        I1=IC-KT*7
        IF (I1.EQ.7) GO TO 90
        GO TO 70
   60   CONTINUE
C
        I1=1
        L(K)=L(K)+1
   70   CONTINUE
        DO 80 I=I1,6
          K=K+1
          II=I+KT*7
   80     L(K)=L(K-1)+II                  @ OBTAIN COL INDEX FOR ROW
   90   CONTINUE
C
        I2=MIN0(8,(M2+1-KT*7))-I1
        V(3)=VFMT(I1)
        IF (.NOT.NAMES) GO TO 180
        WRITE (6,V) CAR(IC),(A(L(I)),I=1,I2)
        GO TO 190
  180   WRITE (6,V) IC,(A(L(I)),I=1,I2)
  190   CONTINUE
      IF (M2.EQ.N) RETURN
      N1=M2+1
      M2=M2+7
      KT=KT+1
      KP=KP+1
      IF (KP.LT.3) GO TO 10
      WRITE (6,200) (TEXT(I),I=1,NC)
      GO TO 10
C
  200 FORMAT (1H1,2X,21A6)              @ TITLE
  205 FORMAT (1H0,2X,21A6)              @ TITLE
  210 FORMAT (1H0,5X,7(11X,A6))          @ HORIZONTAL NAMES
  220 FORMAT (1H0,3X,7(11X,I6))
C
      END
```

```
      SUBROUTINE TTHH(R,RA,N)
C
C        THIS SUBROUTINE COMBINES TWO SINGLE SUBSCRIPTED SRIF ARRAYS
C        USING HOUSEHOLDER ORTHOGONAL TRANSFORMATIONS
C
C       R(N*(N+1)/2)    VECTOR STORED SRIF ARRAY.
C       RA(N*(N+1)/2)   THE SECOND VECTOR STORED SRIF ARRAY
C       N               DIMENSION OF THE ESTIMATED PARAMETER VECTOR.
C                       A NEGATIVE VALUE FOR N IS USED TO NOTE THAT
C                       R AND RA HAVE RT. HAND SIDES INCLUDED AND
C                       HAVE DIM=ABSN*(ABSN+3)/2.
C
C        ON EXIT RA IS CHANGED AND R CONTAINS THE RESULTING SRIF ARRAY
C
C          COGNIZANT PERSONS G.J.BIERMAN/M.W.NEAD (JPL, JAN.1976)
C
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION RA(1), R(1)
C     DOUBLE PRECISION SUM  @ FOR USE IN SINGLE PRECISION VERSION
C
      ZERO=0.
      ONE=1.
      NP1=N
      IF (N.GT.0) GO TO 10
      N=-N
      NP1=N+1
   10  IJS=1                @ IJ(START)
      KK=0
       DO 100 J=1,N         @ J-TH STEP OF HOUSEHOLDER REDUCTION
         KK=KK+J
       SUM=R(KK)**2
       DO 20 I=IJS,KK
   20 SUM=SUM+RA(I)**2
      IF (SUM.LE.ZERO) GO TO 100
      SUM=SQRT(SUM)
      IF (R(KK).GT.ZERO) SUM=-SUM
      DELTA=R(KK)-SUM
      R(KK)=SUM
      BETA=ONE/(SUM*DELTA)
      JJ=KK
      L=J
      JP1=J+1
      IKS=KK+1
C         * * * J-TH HOUSEHOLDER TRANS. DEFINED
C               40 LOOP APPLIES TRANSFORM. TO COLS. J+1 TO NP1
      DO 40 K=JP1,NP1
      JJ=JJ+L
      L=L+1
      IK=IKS
      SUM=DELTA*R(JJ)
      DO 30 I=IJS,KK
      SUM=SUM+RA(IK)*RA(I)
   30 IK=IK+1
      IF (SUM.EQ.ZERO) GO TO 40
      SUM=SUM*BETA
```

```
      R(JJ)=R(JJ)+SUM*DELTA                                             TTHH0550
      IK=IKS                                                            TTHH0560
      DO 35 I=IJS,KK                                                    TTHH0570
      RA(IK)=RA(IK)+SUM*RA(I)                                           TTHH0580
   35 IK=IK+1                                                           TTHH0590
   40  IKS=IKS+K                                                        TTHH0600
  100 IJS=KK+1                                                          TTHH0610
C                                                                       TTHH0620
      RETURN                                                            TTHH0630
      END                                                               TTHH0640
```

```
      SUBROUTINE TZERO (R,N,IS,IF)
C
C        TO ZERO OUT ROWS IS (ISTART) TO IF (IFINAL) OF A VECTOR
C        STORED UPPER TRIANGULAR MATRIX
C
C        R(N*(N+1)/2)  INPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C        N             DIMENSION OF R
C        IS            FIRST ROW OF R THAT IS TO BE SET TO ZERO
C        IR            LAST ROW OF R THAT IS TO BE SET TO ZERO
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/C.F.PETERS (JPL, NOV. 1975)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION R(1)
C
      ZERO=0.0
        IJS=IS*(IS-1)/2
        DO 10 I=IS,IF
        IJS=IJS+I
        IJ=IJS
        DO 10 J=I,N
        R(IJ)=ZERO
        IJ=IJ+J
   10 CONTINUE
C
        RETURN
        END
```

```
      SUBROUTINE UDMES (U,N,R,A,G,ALPHA)
C
C        COMPUTES ESTIMATE AND U-D MEASUREMENT UPDATED
C        COVARIANCE, P=UDU**T
C
C        *** INPUTS ***
C
C        U      UPPER TRIANGULAR MATRIX, WITH D ELEMENTS STORED AS THE
C               DIAGONAL. U IS VECTOR STORED AND CORRESPONDS TO THE
C               A PRIORI COVARIANCE. IF STATE ESTIMATES ARE COMPUTED,
C               THE LAST COLUMN OF U CONTAINS X.
C        N      DIMENSION OF THE STATE ESTIMATE.
C        R      MEASUREMENT VARIANCE
C        A      VECTOR OF MEASUREMENT COEFFICIENTS, IF DATA THEN A(N+1)=Z
C        ALPHA  IF ALPHA LESS THAN ZERO NO ESTIMATES ARE COMPUTED
C               (AND X AND Z NEED NOT BE INCLUDED)
C
C        *** OUTPUTS ***
C
C        U       UPDATED, VECTOR STORED FACTORS AND ESTIMATE AND
C                U((N+1)(N+2)/2) CONTAINS (Z-A**T*X)
C
C        ALPHA   INNOVATIONS VARIANCE OF THE MEASUREMENT RESIDUAL
C        G       VECTOR OF UNWEIGHTED KALMAN GAINS, K=G/ALPHA
C        A       CONTAINS U**TA AND (Z-A**T*X)/ALPHA
C
C           COGNIZANT PERSONS: G.J. BIERMAN/M.W. NEAD (JPL, SEPT.1976)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION U(1), A(1), G(1)
      DOUBLE PRECISION SUM
      LOGICAL IEST
C
      ZERO=0.0
      IEST=.FALSE.
      ONE=1.
      NP1=N+1
      NTOT=N*NP1/2
      IF (ALPHA.LT.ZERO) GO TO 3
      SUM=A(NP1)
      DO 1 J=1,N
    1    SUM=SUM-A(J)*U(NTOT+J)
      U(NTOT+NP1)=SUM                         @ Z=Z-A**T*X
      IEST=.TRUE.
C
    3 KJ=NTOT
      DO 10 J=N,2,-1
         SUM=A(J)
         JM1=J-1
         DO 5 K=JM1,1,-1
           KJ=KJ-1
    5      SUM=SUM+U(KJ)*A(K)
         A(J)=SUM
         KJ=KJ-1
   10 G(J)=SUM*U(KJ+J)
```

```
      G(1)=U(1)*A(1)
C         A=U**T*A AND G=D*(U**T*A)
C
      SUM=R+G(1)*A(1)                             @ SUM(1)
C     GAMMA=0                                       @ FOR R=0 CASE
C     IF (G(1).EQ.ZERO) GO TO 11                    @ FOR R=0 CASE
      GAMMA=ONE/SUM
      U(1)=U(1)*R*GAMMA                           @ D(1)
C
   11 KJ=2
      DO 20 J=2,N
          BETA=SUM                                @ BETA=SUM(J-1)
          SUM=SUM+G(J)*A(J)                       @ SUM(J)
          P=-A(J)*GAMMA                           @ P=-F(J)*(1/SUM(J-1)) EQN(21)
          JM1=J-1
          DO 15 K=1,JM1
            S=U(KJ)
            U(KJ)=S+P*G(K)                        @ EQN(22)
            G(K)=G(K)+G(J)*S                      @ EQN(23)
   15            KJ=KJ+1
C     IF (G(J).EQ.ZERO) GO TO 20                    @ FOR R=0 CASE
          GAMMA=ONE/SUM                           @ GAMMA=1/SUM(J)
          U(KJ)=U(KJ)*BETA*GAMMA                  @ D(J)       EQN(19)
   20 KJ=KJ+1
      ALPHA=SUM
C
C         EQN. NOS. REFER TO BIERMAN'S 1975 CDC PAPER, PP. 337-346.
C
      IF (.NOT.IEST) RETURN
      A(NP1)=U(NTOT+NP1)*GAMMA
      DO 30 J=1,N
   30   U(NTOT+J)=U(NTOT+J)+G(J)*A(NP1)
C
      RETURN
      END
```

```
C
      SUBROUTINE UD2COV (UIN,POUT,N)
C
C     TO OBTAIN A COVARIANCE FROM ITS U-D FACTORIZATION. BOTH MATRICES
C     ARE VECTOR STORED AND THE OUTPUT COVARIANCE CAN OVERWRITE THE
C     INPUT U-D ARRAY. UIN=U-D IS RELATED TO POUT VIA POUT=UDU(**T)
C
C     UIN(N*(N+1)/2)  INPUT U-D FACTORS, VECTOR STORED WITH THE D
C                     ENTRIES STORED ON THE DIAGONAL OF UIN
C     POUT(N*(N+1)/2) OUTPUT COVARIANCE, VECTOR STORED.
C                     (POUT=UIN IS PERMITTED)
C     N               DIMENSION OF THE MATRICES INVOLVED
C
C     COGNIZANT PERSONS: G.J.BIERMAN/M.W.NEAD  (JPL, FEB. 1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
      DIMENSION   UIN(1),   POUT(1)
C
      POUT(1)=UIN(1)
      JJ=1
      DO 20 J=2,N
        JJL=JJ                       @ (J-1,J-1)
        JJ=JJ+J
        POUT(JJ)=UIN(JJ)
        S=POUT(JJ)
        II=0
        JM1=J-1
        DO 20 I=1,JM1
          II=II+I
          ALPHA=S*UIN(JJL+I)              @ JJL+I=(I,J)
          IK=II
          DO 10 K=I,JM1
            POUT(IK)=POUT(IK)+ALPHA*UIN(JJL+K)     @ JJL+K=(K,J)
   10       IK=IK+K
   20   POUT(JJL+I)=ALPHA
C
      RETURN
      END
```

```
C
      SUBROUTINE UD2SIG(U,N,SIG,TEXT,NCT)
C
C     COMPUTE STANDARD DEVIATIONS (SIGMAS) FROM U-D COVARIANCE FACTORS
C
C     U(N*(N+1)/2)  INPUT VECTOR STORED ARRAY CONTAINING THE U-D
C                   FACTORS. THE D (DIAGONAL) ELEMENTS ARE STORED
C                   ON THE DIAGONAL
C     SIG(N)        VECTOR OF OUTPUT STANDARD DEVIATIONS
C     TEXT( )       ARRAY OF FIELDATA CHARACTERS TO BE PRINTED
C                   PRECEDING THE VECTOR OF SIGMAS
C     NCT       .   NUMBER OF CHARACTERS IN TEXT, 0.LE.NCT.LE.126
C                   IF NCT=0, NO SIGMAS ARE PRINTED
C
C     COGNIZANT PERSONS: G.J.BIERMAN/M.W.NEAD   (JPL, FEB. 1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      INTEGER TEXT(1)
      DIMENSION U(1), SIG(1)
C
      JJ=1
      SIG(1)=U(1)
      DO 10 J=2,N
        JJL=JJ                        @ (J-1,J-1)
        JJ=JJ+J
        S=U(JJ)
        SIG(J)=S
        JM1=J-1
        DO 10 I=1,JM1
   10     SIG(I)=SIG(I)+S*U(JJL+I)**2
C
C          WE NOW HAVE VARIANCES
C
      DO 20 J=1,N
   20   SIG(J)=SQRT(SIG(J))
      IF (NCT.EQ.0) GO TO 30
      NC=NCT/6
      IF (MOD(NC,6).NE.0) NC=NC+1
      WRITE (6,40) (TEXT(I),I=1,NC)
      WRITE (6,50) (SIG(I),I=1,N)
   30 RETURN
C
   40 FORMAT (1H0,2X,21A6)
   50 FORMAT (1H0,(6D18.10))
      END
```

```
      SUBROUTINE UTINV(RIN,N,ROUT)
C
C        TO INVERT AN UPPER TRIANGULAR VECTOR STORED MATRIX AND STORE
C        THE RESULT IN VECTOR FORM. THE ALGORITHM IS SO ARRANGED THAT
C        THE RESULT CAN OVERWRITE THE INPUT.
C        IN ADDITION TO  SOLVE RX=Z, SET RIN(N*(N+1)/2+1)=Z(1), ETC.,
C        AND SET RIN((N+1)*(N+2)/2)=-1. CALL THE SUBROUTINE USING N+1
C        INSTEAD OF N. ON RETURN THE FIRST N ENTRIES OF COLUMN N+1
C        WILL CONTAIN X.
C
C        RIN(N*(N+1)/2)  INPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C        N               MATRIX DIMENSION
C        ROUT(N*(N+1)/2) OUTPUT VECTOR STORED UPPER TRIANGULAR MATRIX
C                        INVERSE
C
C        COGNIZANT PERSONS:  G.J.BIERMAN/J.ELLIS (JPL, SEPT. 1976)
C
      DOUBLE PRECISION  RIN(1), ROUT(1), WORK, ONE, ZERO ,DIN
      DATA  ONE/1.0D0/,ZERO/ 0.0D0/
      IPV =   N*(N+1)/2
      IN  = IPV
      DO 6    I=1,N
      IF (RIN(IPV).NE.ZERO) GO TO 1
      WRITE (6,10) I
      RETURN
    1 DIN    =         ONE/ RIN(IPV)
      ROUT( IPV )  = DIN
      MIN =N
      KEND = I-1
      LANF  =    N - KEND
      IF (I.EQ.1) GO TO 5
    2 J= IN
C
C             INITIALIZE  ROW LOOP
C
      DO 4    K=1,KEND
      WORK =ZERO
      MIN= MIN -  1
      LIN= IPV
      LOT= J
C
C            START  INNER LOOP
C
      DO 3    L=LANF, MIN
      LIN= LIN+L
      LOT= LOT+1
    3 WORK = WORK   + RIN(LIN)* ROUT(LOT)
      ROUT(J)  =    -WORK* DIN
    4 J=  J- MIN
    5 IPV =  IPV -MIN
    6 IN= IN -1
      RETURN
   10 FORMAT (1H0,10X,'UTINV DIAGONAL',I4,'IS ZERO')
      END
```

```
C
      SUBROUTINE  UTIROW (RIN,N,ROUT,NRY)
C
C     TO COMPUTE THE INVERSE OF AN UPPER TRIANGULAR (VECTOR STORED)
C     MATRIX WHEN THE LOWER PORTION OF THE INVERSE IS GIVEN
C
C         ON INPUT:
C
C           RX    RXY              *      *                RX    RXY
C     RIN=                  ROUT=               WHERE    R=
C           *     *                0    RY**-1              0     RY
C
C         ON OUTPUT: RIN IS UNCHANGED AND ROUT=R**-1
C         THE RESULT CAN OVER-WRITE THE INPUT (I.E. RIN=ROUT)
C
C     RIN(N*(N+1)/2)     INPUT VECTOR STORED TRIANGULAR MATRIX
C                        THE BOTTOM NRY ROWS ARE IGNORED
C     N                  MATRIX DIMENSION
C     ROUT(N*(N+1)/2)    OUTPUT VECTOR STORED MATRIX. ON INPUT THE
C                        BOTTOM NRY ROWS CONTAIN THE LOWER PORTION
C                        OF R**-1. ON OUTPUT ROUT=R**-1
C     NRY                DIMENSION OF LOWER (ALREADY INVERTED)
C                        TRIANGULAR R. IF NRY=0, ORDINARY MATRIX
C                        INVERSION RESULTS.
C
C     COGNIZANT PERSONS:   G.J.BIERMAN/M.W.NEAD   (JPL MARCH 1977)
C
      DOUBLE PRECISION   RIN(1), ROUT(1), SUM, ZERO, ONE, DINV
      DATA   ONE/1.D0/, ZERO/0.D0/
C
C         INITIALIZATION
C
      NR=N*(N+1)/2                   @ NO. ELEMENTS IN R
      ISTRT=N-NRY                    @ FIRST ROW TO BE INVERTED
      IRLST=ISTRT+1                  @ IRLST=PREVIOUS IROW INDEX
      II=ISTRT*IRLST/2               @ II=DIAGONAL
      DO 40 IROW=ISTRT,1,-1
        IF (RIN(II).NE.ZERO) GO TO 10
        WRITE (6,50) IROW
        RETURN
   10   DINV=ONE/RIN(II)
        ROUT(II)=DINV
        KJS=NR+IROW                  @ KJ(START)
        IKS=II+IROW                  @ IK(START)
C
        IF (IRLST.GT.N) GO TO 35
        DO 30 J=N,IRLST,-1
          KJS=KJS-J
          SUM=ZERO
          IK=IKS
          KJ=KJS
C
          DO 20 K=IRLST,J
            KJ=KJ+1
            SUM=SUM+RIN(IK)*ROUT(KJ)
```

```
   20         IK=IK+K                                                    UTIR0550
C                                                                        UTIR0560
   30       ROUT(KJS)=-SUM*DINV                                          UTIR0570
   35     IRLST=IROW                                                     UTIR0580
   40     II=II-IROW                                                     UTIR0590
      RETURN                                                             UTIR0600
   50 FORMAT (1H0,10X,'RIN DIAGONAL',I4,'IS ZERO')                       UTIR0610
      END                                                                UTIR0620
```

```
C
      SUBROUTINE WGS (W,IMAXW,IW,JW,D,U,V)
C     MODIFIED GRAMM-SCHMIDT ALGORITHM FOR REDUCING WDW(**T) TO UDU(**T)
C     FORM WHERE U IS A VECTOR STORED TRIANGULAR MATRIX WITH THE
C     RESULTING D ELEMENTS STORE ON THE DIAGONAL
C
C     W(IW,JW)           INPUT MATRIX TO BE REDUCED TO TRIANGULAR FORM.
C                        THIS MATRIX IS DESTROYED BY THE CALCULATION
C                        IW.LE.IMAXW.
C     D(IW)              VECTOR OF NON-NEGATIVE WEIGHTS FOR THE
C                        ORTHOGONALIZATION PROCESS. THE D'S ARE UNCHANGED
C                        BY THE CALCULATION.
C     U(IW*(IW+1)/2)     OUTPUT UPPER TRIANGULAR VECTOR STORED OUTPUT
C     V(JW)              WORK VECTOR
C
C                        (SEE BOOK:
C     ' FACTORIZATION METHODS FOR DISCRETE SEQUENTIAL ESTIMATION ',
C                           BY G.J.BIERMAN)
C      ESTIMATION
C
C     COGNIZANT PERSONS: G.J.BIERMAN/M.W.NEAD    (JPL, MARCH 1977)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION W(IMAXW,1), D(1), U(1), V(1)
C
      Z=0.0
      ONE=1.0
      DO 100 J=IW,1,-1
        SUM=Z
        DO 40 K=1,JW
          V(K)=W(J,K)
          U(K)=D(K)*V(K)                 @U HERE IS USED AS A WORK VECTOR
   40     SUM=V(K)*U(K)+SUM
        W(J,J)=SUM                        @ EQ.(4.9) OF BOOK, NEW D(J)
        IF (J.EQ.1) GO TO 100
        DINV=Z
        IF (SUM.GT.Z) DINV=ONE/SUM
        JM1=J-1
        DO 70 K=1,JM1
          SUM=Z
          DO 50 I=1,JW
   50       SUM=W(K,I)*U(I)+SUM
          SUM=SUM*DINV
C
          DO 60 I=1,JW
   60       W(K,I)=W(K,I)-SUM*V(I)
   70     W(J,K)=SUM                      @ EQ.(4.10) OF BOOK
  100   CONTINUE                          @ U(K,J) STORED IN W(J,K)
C
C         THE LOWER PART OF W IS U TRANSPOSE
C
      IJ=0
      DO 110 J=1,IW
        DO 110 I=1,J
          IJ=IJ+1
  110     U(IJ)=W(J,I)
C
      RETURN
      END
```

References

[1] Lawson, C. L., Hanson, R. J., Solving Least Squares Problems, Prentice Hall, Englewood Cliffs, N. J. (1974).

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[3] Bierman, G. J., Factorization Methods for Discrete Sequential Estimation, Academic Press, New York (1977).