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MSFC Holographic Correlation Techniques Facility Study

Contract No. NAS8-32362

FINAL REPORT

by

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June, 1977

Submitted to

National Aeronautics and Space Administration

George C. Marshall Space Flight Center

Huntsville, Alabama

by

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This report was prepared by Montevallo Research Associates under Contract No. NAS8-32362, MSFC Holographic Correlation Techniques Facility Study, for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under the technical direction of the Space Sciences Laboratory of the George C. Marshall Space Flight Center with Dr. Robert B. Owen acting as project manager.

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I. INTRODUCTION

This report presents the results of an analytical study of the MSFC Holographic Correlation Techniques Facility carried out under Contract No. NAS8-32362 by personnel of Montevallo Research Associates, Montevallo, Alabama during the period 12 January, 1977 to 12 June, 1977. Participating in the study were Dr. J. William Foreman, Jr. (Principal Investigator) and Mr. Joseph M. Cardone.

The objects of the study were originally stated as follows:

(1) To examine the existing MSFC experimental Holographic Correlation Techniques Facility and to become familiar with existing theory pertaining to its operation.

(2) To determine the theoretical effect, if any, on the correlation signal caused by insertion of a variable aperture into the object beam.

(3) To examine theoretically the existence of a secondary correlation peak.

(4) To determine theoretically the effect, if any, on the correlation signal caused by varying the surface roughness of the test object.

(5) To explain theoretically why the object can be

translated in the object plane over distances large compared to a wavelength of light with negligible loss of correlation signal.

It was anticipated from the beginning that it would probably not be possible to complete all five tasks during the present contract, and this turned out to be correct. A thorough theoretical understanding of the system, like a comprehensive experimental investigation of the system, will have to be developed as part of an ongoing study.

Task (3) above was eliminated from the list when Dr. Robert B. Owen, our technical contact during the study, advised us that the secondary correlation peaks which had been observed earlier were apparently spurious. Tasks (1) and (5) have been completed, and a variation of Task (5) involving translation of the object along the optical axis of the system was begun. Time did not allow us to undertake Tasks (2) or (4) during the present study.

It is suggested that the reader examine the Appendix before attempting to read the main body of this report, since many results which are used in developing the theory of the holographic correlation system are derived or summarized in the Appendix.

II. BASIC THEORY OF THE HOLOGRAPHIC CORRELATION SYSTEM

The basic theory of the holographic correlation system is developed in a number of commonly available references. However, the usual treatment is somewhat abbreviated, and therefore difficult to follow in detail. We present here a treatment of the basic theory which is worked out in sufficient detail so that it can be readily followed. Our treatment follows the general approach used by Goodman[1] in his discussion of the Vander Lugt filter[2].

Fig. 1 shows a schematic arrangement of a holographic correlation system. The operation of the system proceeds as follows:

(1) An object mask with a certain desired transmission function is placed in the object plane, one focal length f in front of the transform lens. A photographic plate is placed in the hologram plane, which is one focal length f behind the transform lens. The photographic plate is exposed simultaneously to the reference beam and the object beam. The exposure and development of the plate are controlled so that the transmission function of the developed plate is proportional to the total irradiance striking the plate during exposure.

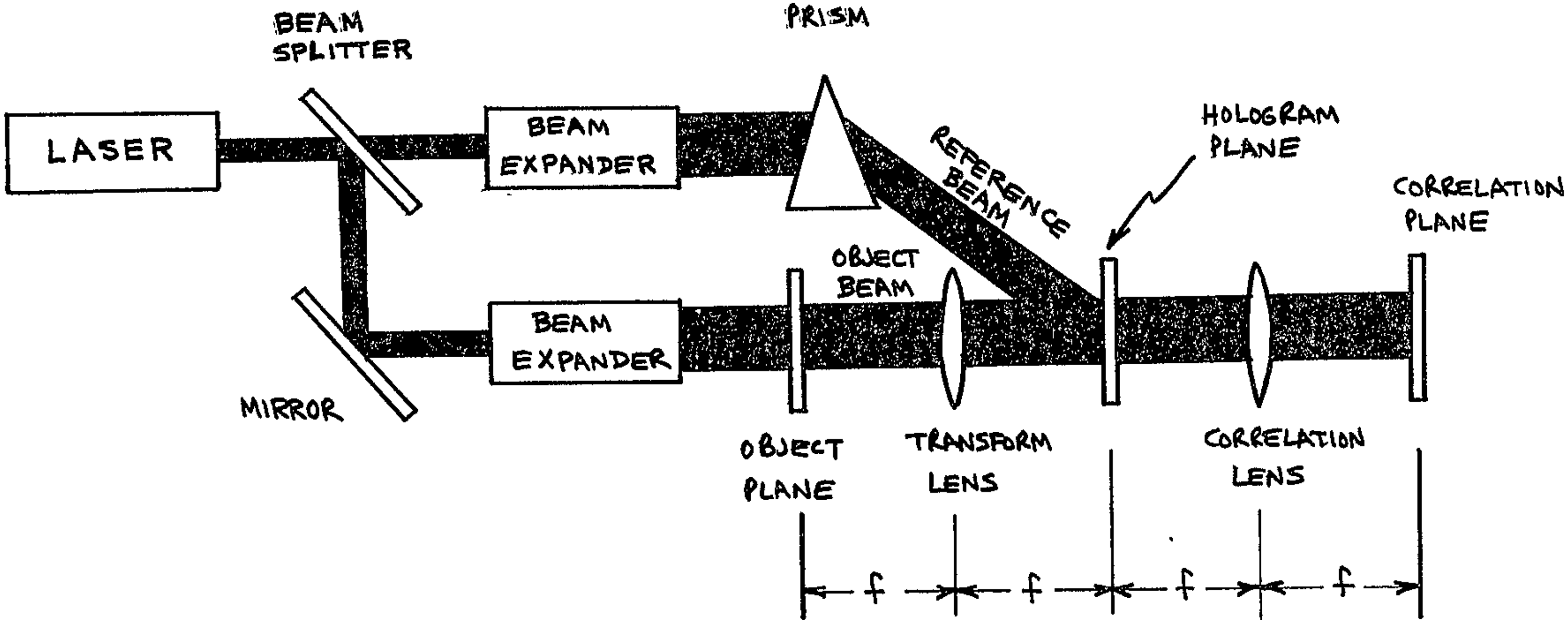


Figure 1. Schematic diagram of a holographic correlation system.

(2) The developed plate, or hologram, is then replaced in the hologram plane, the reference beam is blocked off, and the hologram is illuminated with a new object mask placed in the object plane. The light throughput from the hologram passes through a correlation lens of focal length f (identical to the focal length of the transform lens), and is examined in the correlation plane, which coincides with the back focal plane of the correlation lens. In general, four patches of light appear in the correlation plane, two of which overlap in the vicinity of the optical axis. One of the non-overlapping patches represents the cross-correlation of the original light amplitude distribution in the object plane (used to expose the plate) with the final light amplitude distribution in the object plane (used to illuminate the hologram). This is the term of interest here. Of course, if the hologram is illuminated with the same light amplitude distribution used to expose the photographic plate, the light patch represents the auto-correlation of this distribution with itself.

Incidentally, the light distributions in the object plane may not always be set up by passing a collimated light beam through a planar mask. They can also be set up by illuminating a planar reflector of some sort placed in the object plane in such a way that the reflected light is captured and processed by the transform lens. This is the way the MSFC holographic correlation setup operates at the present time.

In order to set up the basic theory of the system, use will be made of the coordinate systems shown in Fig. 2. Coordinates in the object plane are denoted by (x_1, y_1), in the hologram plane by (x_2, y_2), and in the correlation plane by (x_3, y_3). Note that an inverted coordinate system (x_3', y_3') has also been defined in the correlation plane. The reason for introducing this inverted coordinate system will be made clear later.

Making the Hologram

The first step in the theory is to describe the properties of the hologram. Suppose the original distribution of light amplitude in the object plane is defined by the complex function

$$p(x_1, y_1) = A_p(x_1, y_1)\, e^{j\varphi_p(x_1, y_1)}$$

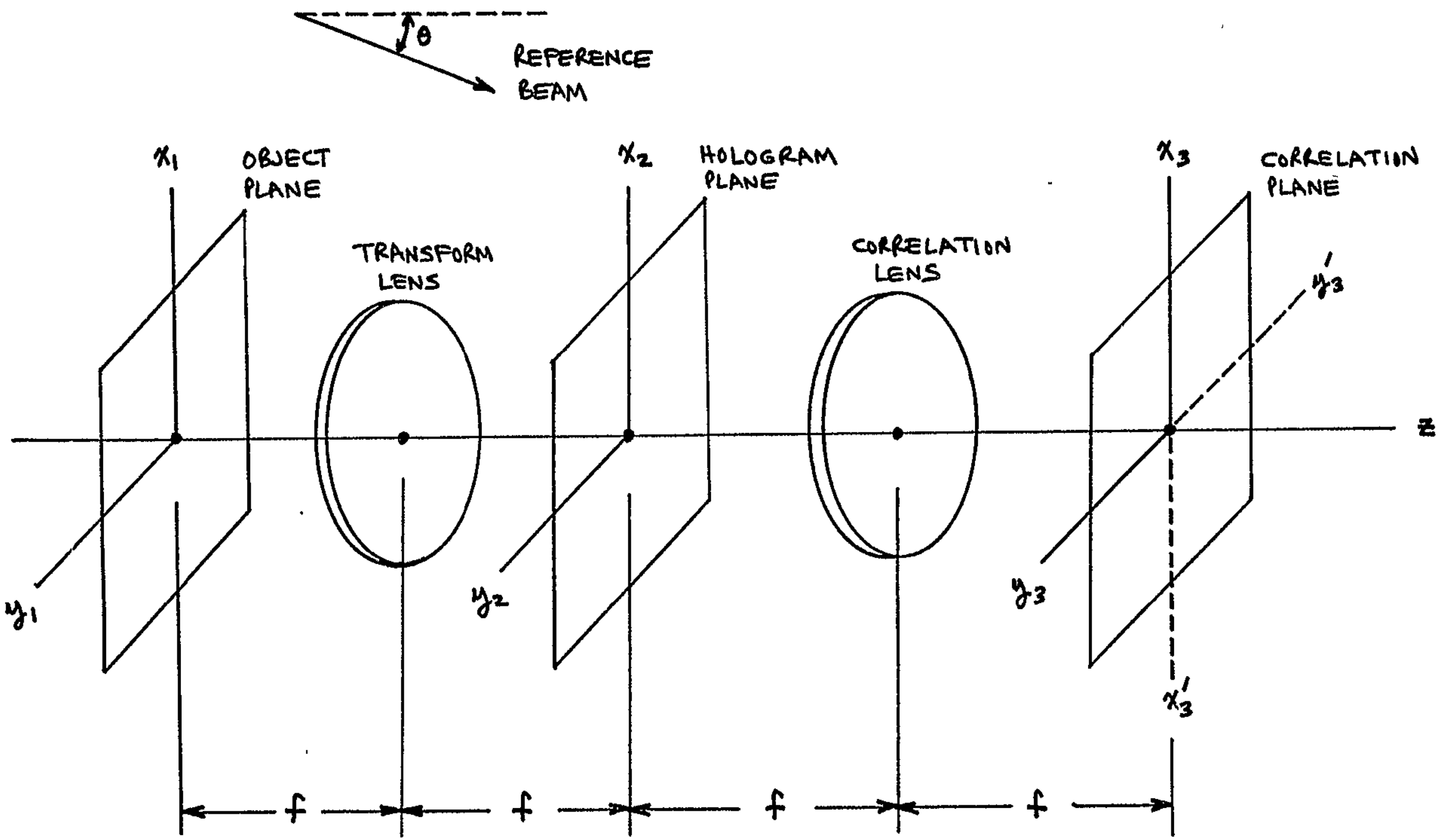


Figure 2. Coordinate systems for analysis of the holographic correlation system.

where $A_p(x_1, y_1)$ and $\varphi_p(x_1, y_1)$ are the amplitude and phase, respectively, of the light vibration at a point in the object plane with coordinates (x_1, y_1). The transform lens operates on the light leaving the object plane and, according to Eq. (A-17), produces a light amplitude distribution

$$\frac{1}{\lambda f} \mathcal{L}_{12} \left[\mathcal{F}(x_1, y_1) \right] = \frac{1}{\lambda f} \mathcal{F}(x_2, y_2)$$

in the hologram plane, where $\mathcal{L}_{12}$ is the "lens operator" defined in Eq. (A-16).

A plane-wave reference beam is also incident on the hologram plane. As indicated in Fig. 2, we will choose the direction of propagation of the reference beam to lie in the $x_2 z$-plane and to make an angle θ with the z-axis. A unit vector $\hat{\mu}$ in the direction of propagation of the reference beam would have a negative x_2-component, as can be seen from Fig. 2. According to Eq. (A-48), the light amplitude distribution in the hologram plane produced by the reference beam will be

$$r_0 \, e^{-j \frac{2\pi x_2 \sin\theta}{\lambda}}$$

where r_0 is the (constant) amplitude of the reference beam.

Since we are dealing here with coherent laser light, the light amplitudes are additive, so that the resultant light amplitude distribution in the hologram plane will be

$$r_0 e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} + \frac{1}{\lambda f} g(x_2, y_2) .$$

The irradiance distribution in the hologram plane, $I(x_2, y_2)$, is thus

$$I(x_2, y_2) = \left| r_0 e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} + \frac{1}{\lambda f} g(x_2, y_2) \right|^2$$

$$= r_0^2 + \frac{1}{(\lambda f)^2} \left| g(x_2, y_2) \right|^2$$

$$+ \frac{r_0}{\lambda f} g(x_2, y_2)\, e^{j\frac{2\pi x_2 \sin\theta}{\lambda}}$$

$$+ \frac{r_0}{\lambda f} g^*(x_2, y_2)\, e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} . \tag{1}$$

To make the hologram, a photographic plate with suitable sensitivity and spatial resolution is placed in the hologram plane, and its exposure and development are controlled in such a way that the transmission $\underset{\sim}{\tau}(x_2, y_2)$ of the developed plate is proportional to the irradiance $I(x_2, y_2)$ during exposure. We thus obtain a hologram with a transmission function

$$\underset{\sim}{\tau}(x_2, y_2) = \underset{\sim}{c}\, I(x_2, y_2) , \qquad (2)$$

where $\underset{\sim}{c}$ is a complex constant.

Illuminating the Hologram

After the hologram has been made, it is replaced in the hologram plane, the reference beam is blocked off, and a new light amplitude distribution $\underset{\sim}{r}(x_1, y_1)$ is established in the object plane. The transform lens operates on this light distribution, producing a light amplitude distribution

$$\frac{1}{\lambda f} \mathcal{L}_{12} \left[\underset{\sim}{r}(x_1, y_1) \right] = \frac{1}{\lambda f} \underset{\sim}{s}(x_2, y_2)$$

in the hologram plane. When the light wave producing this distribution passes through the hologram, the resulting light amplitude distribution in the hologram plane will be

$$\underset{\sim}{u}(x_2, y_2) = \frac{1}{\lambda f} \underset{\sim}{s}(x_2, y_2)\, \underset{\sim}{\tau}(x_2, y_2) .$$

Using Eqs. (1) and (2) we then have

$$\underset{\sim}{u}(x_2, y_2) = \frac{\underset{\sim}{c}}{\lambda f} \underset{\sim}{s}(x_2, y_2)\, I(x_2, y_2)$$

$$= \frac{\underset{\sim}{c}\, n_0^2}{\lambda f} \underset{\sim}{s}(x_2, y_2) + \frac{\underset{\sim}{c}}{(\lambda f)^3} \underset{\sim}{s}(x_2, y_2) \left| \underset{\sim}{q}(x_2, y_2) \right|^2$$

$$+ \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^2} \underset{\sim}{q}(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{j \frac{2\pi x_2 \sin\theta}{\lambda}}$$

$$+ \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^2} \underset{\sim}{q}^*(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{-j \frac{2\pi x_2 \sin\theta}{\lambda}} . \quad (3)$$

Finally, the correlation lens operates on this light amplitude distribution and produces a new light amplitude distribution $\underset{\sim}{w}(x_3, y_3)$ in the correlation plane given by

$$\underset{\sim}{w}(x_3, y_3) = \frac{1}{\lambda f} \mathcal{L}_{23} \left[\underset{\sim}{u}(x_2, y_2) \right] . \quad (4)$$

Putting Eq. (3) into Eq. (4) gives

$$\underset{\sim}{w}(x_3, y_3) = \frac{\underset{\sim}{c}\, n_0^2}{(\lambda f)^2} \mathcal{L}_{23} \left[\underset{\sim}{s}(x_2, y_2) \right]$$

$$+ \frac{\underset{\sim}{c}}{(\lambda f)^4} \mathcal{L}_{23} \left[\underset{\sim}{s}(x_2, y_2)\, \underset{\sim}{q}^*(x_2, y_2)\, \underset{\sim}{q}(x_2, y_2) \right]$$

$$+ \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^3} \mathcal{L}_{23} \left[\underset{\sim}{q}(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{j \frac{2\pi x_2 \sin\theta}{\lambda}} \right]$$

$$+ \frac{\underset{\sim}{c}\, \Lambda_0}{(\lambda f)^3}\, \mathcal{L}_{23}\left[\underset{\sim}{g}^{*}(x_2, y_2)\; \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} \right]. \tag{5}$$

For convenience, let us label the four terms in Eq. (5) as follows:

$$\underset{\sim}{W}_1(x_3, y_3) = \frac{\underset{\sim}{c}\, \Lambda_0^2}{(\lambda f)^2}\, \mathcal{L}_{23}\left[\underset{\sim}{s}(x_2, y_2) \right] \tag{6}$$

$$\underset{\sim}{W}_2(x_3, y_3) = \frac{\underset{\sim}{c}}{(\lambda f)^4}\, \mathcal{L}_{23}\left[\underset{\sim}{s}(x_2, y_2)\; \underset{\sim}{g}^{*}(x_2, y_2)\; \underset{\sim}{g}(x_2, y_2) \right] \tag{7}$$

$$\underset{\sim}{W}_3(x_3, y_3) = \frac{\underset{\sim}{c}\, \Lambda_0}{(\lambda f)^3}\, \mathcal{L}_{23}\left[\underset{\sim}{g}(x_2, y_2)\; \underset{\sim}{s}(x_2, y_2)\, e^{j\frac{2\pi x_2 \sin\theta}{\lambda}} \right] \tag{8}$$

$$\underset{\sim}{W}_4(x_3, y_3) = \frac{\underset{\sim}{c}\, \Lambda_0}{(\lambda f)^3}\, \mathcal{L}_{23}\left[\underset{\sim}{g}^{*}(x_2, y_2)\; \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} \right]. \tag{9}$$

Eq. (5) then becomes

$$\underset{\sim}{W}(x_3, y_3) = \underset{\sim}{W}_1(x_3, y_3) + \underset{\sim}{W}_2(x_3, y_3) + \underset{\sim}{W}_3(x_3, y_3) + \underset{\sim}{W}_4(x_3, y_3). \tag{10}$$

We will now investigate the individual terms in Eq.(10) in detail.

First Term, $\underset{\sim}{W}_1(x_3, y_3)$

From Eq. (6), this term is

$$\underset{\sim}{W}_1(x_3, y_3) = \frac{\underset{\sim}{C}\, n_0^2}{(\lambda f)^2}\, \mathcal{L}_{23}\left[\underset{\sim}{S}(x_2, y_2)\right]. \tag{6}$$

But

$$\underset{\sim}{S}(x_2, y_2) = \mathcal{L}_{12}\left[\underset{\sim}{r}(x_1, y_1)\right].$$

Eq. (6) therefore becomes

$$\underset{\sim}{W}_1(x_3, y_3) = \frac{\underset{\sim}{C}\, n_0^2}{(\lambda f)^2}\, \mathcal{L}_{23}\left[\mathcal{L}_{12}\left[\underset{\sim}{r}(x_1, y_1)\right]\right]. \tag{11}$$

But from Eq. (A-37) we have

$$\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[\underset{\sim}{r}(x_1, y_1)\right]\right] = (\lambda f)^2\, \underset{\sim}{r}(-x_3, -y_3). \tag{12}$$

Eq. (11) thus becomes

$$\underset{\sim}{W}_1(x_3, y_3) = \underset{\sim}{C}\, n_0^2\, \underset{\sim}{r}(-x_3, -y_3). \tag{13}$$

From here on, it will be convenient to work in terms of the inverted coordinate system (x_3', y_3') in the correlation plane in order to avoid negative arguments. Since $x_3' = -x_3$ and $y_3' = -y_3$, we can write Eq. (13) as

$$\underset{\sim}{W}_1(x_3, y_3) = \underset{\sim}{C}\, n_0^2\, \underset{\sim}{r}(x_3', y_3'). \tag{14}$$

The light amplitude distribution in the correlation plane represented by this term is centered at the origin and is proportional to an inverted version of the light amplitude distribution $\underset{\sim}{r}(x_1, y_1)$ used to illuminate the hologram.

This term is of no interest in the present context.

Second Term, $\underset{\sim}{W}_2(x_3, y_3)$

From Eq. (7) we have

$$\underset{\sim}{W}_2(x_3,y_3) = \frac{\underset{\sim}{c}}{(\lambda f)^4}\,\mathcal{L}_{23}\left[\underset{\sim}{s}(x_2,y_2)\,\underset{\sim}{g}^*(x_2,y_2)\,\underset{\sim}{g}(x_2,y_2)\right]. \quad (7)$$

Using the convolution theorem stated in Eq. (A-32) we can write

$$\mathcal{L}_{23}\left[\underset{\sim}{s}(x_2,y_2)\,\underset{\sim}{g}^*(x_2,y_2)\,\underset{\sim}{g}(x_2,y_2)\right]$$

$$= \frac{1}{(\lambda f)^2}\,\mathcal{L}_{23}\left[\underset{\sim}{s}(x_2,y_2)\right] * \mathcal{L}_{23}\left[\underset{\sim}{g}^*(x_2,y_2)\,\underset{\sim}{g}(x_2,y_2)\right]$$

$$= \frac{1}{(\lambda f)^4}\,\mathcal{L}_{23}\left[\underset{\sim}{s}(x_2,y_2)\right] * \mathcal{L}_{23}\left[\underset{\sim}{g}^*(x_2,y_2)\right] * \mathcal{L}_{23}\left[\underset{\sim}{g}(x_2,y_2)\right]. \quad (15)$$

But from Eq. (A-37) we have

$$\mathcal{L}_{23}\left[\underset{\sim}{g}(x_2,y_2)\right] = \mathcal{L}_{23}\left[\mathcal{L}_{12}\left[\underset{\sim}{p}(x_1,y_1)\right]\right]$$

$$= (\lambda f)^2\,\underset{\sim}{p}(-x_3,-y_3)$$

$$= (\lambda f)^2\,\underset{\sim}{p}(x_3',y_3'), \quad (16)$$

and from Eq. (A-43) we also have

$$\mathcal{L}_{23}\left[\mathfrak{g}^*(x_2,y_2)\right] = \mathcal{L}_{23}\left[\left\{\mathcal{L}_{12}\left[\wp(x_1,y_1)\right]\right\}^*\right]$$

$$= (\lambda f)^2\,\wp^*(x_3,y_3)$$

$$= (\lambda f)^2\,\wp^*(-x_3',-y_3'). \tag{17}$$

Putting the results of Eqs. (12), (16), and (17) into Eq. (15) yields

$$\mathcal{L}_{23}\left[\underline{s}(x_2,y_2)\,\mathfrak{g}^*(x_2,y_2)\,\mathfrak{g}(x_2,y_2)\right]$$
$$= (\lambda f)^2\,\underline{n}(x_3',y_3') * \wp^*(-x_3',-y_3') * \wp(x_3',y_3'). \tag{18}$$

We will now evaluate the convolution $\wp^*(-x_3',-y_3') * \wp(x_3',y_3')$ using the procedure outlined in Sec. B of the Appendix.

Change variable names: $\wp^*(-x_3',-y_3') \longrightarrow \wp^*(-\xi,-\eta)$

$\wp(x_3',y_3') \longrightarrow \wp(\xi,\eta)$.

Fold $\wp^*(-\xi,-\eta)$: $\wp^*(-\xi,-\eta) \longrightarrow \wp^*(\xi,\eta)$.

Translate $\wp^*(\xi,\eta)$ by amounts x_3' and y_3' in the ξ and η directions, respectively: $\wp^*(\xi,\eta) \longrightarrow \wp^*(\xi-x_3',\eta-y_3')$.

Multiply the functions

and integrate:

$$\iint_{-\infty}^{+\infty} p^*(\xi - x_3', \eta - y_3')\, p(\xi, \eta)\, d\xi\, d\eta .$$

Our final result is thus

$$p^*(-x_3', -y_3') * p(x_3', y_3') = \iint_{-\infty}^{+\infty} p^*(\xi - x_3', \eta - y_3')\, p(\xi, \eta)\, d\xi\, d\eta . \tag{19}$$

Comparing Eq. (19) with Eq. (A-4a) we see that the integral in Eq. (19) is just the auto-correlation of $p(x_3', y_3')$ with itself:

$$p^*(-x_3', -y_3') * p(x_3', y_3') = p(x_3', y_3') \otimes p(x_3', y_3') . \tag{20}$$

Putting Eq. (20) into Eq. (18) we then have

$$\mathcal{L}_{23}\left[\underset{\sim}{s}(x_2, y_2)\, g^*(x_2, y_2)\, g(x_2, y_2)\right]$$

$$= (\lambda f)^2\, \underset{\sim}{r}(x_3', y_3') * \left[p(x_3', y_3') \otimes p(x_3', y_3')\right] . \tag{21}$$

Finally, putting Eq. (21) into Eq. (7) gives

$$\underset{\sim}{W}_2(x_3, y_3) = \frac{\underset{\sim}{C}}{(\lambda f)^2}\, \underset{\sim}{r}(x_3', y_3') * \left[p(x_3', y_3') \otimes p(x_3', y_3')\right] . \tag{22}$$

The light amplitude distribution in the correlation plane represented by this term is centered at the origin and is proportional to the convolution of $\underset{\sim}{r}(x_3', y_3')$ with the auto-correlation function of $\mathfrak{f}(x_3', y_3')$. This term is not of direct interest in the present context.

Third Term, $\underset{\sim}{W}_3(x_3, y_3)$

From Eq. (8) we have

$$\underset{\sim}{W}_3(x_3, y_3) = \frac{\underset{\sim}{c}\,\Lambda_0}{(\lambda f)^3}\,\mathcal{L}_{23}\left[\underset{\sim}{f}(x_2, y_2)\,\underset{\sim}{S}(x_2, y_2)\,e^{j\frac{2\pi x_2 \sin\theta}{\lambda}}\right]. \quad (8)$$

The terms within the square brackets in Eq. (8) represent the light amplitude distribution in the (x_2, y_2) plane caused by the light wave leaving the (x_2, y_2) plane on its way to the correlation lens. It follows from the discussion in Sec. E of the Appendix that this light wave is travelling in the same general direction as a plane wave which would cause an amplitude distribution of the form $e^{j\frac{2\pi x_2 \sin\theta}{\lambda}}$ in the (x_2, y_2) plane. This direction, according to Sec. E of the Appendix, is along a unit vector $\hat{u}$ which lies in the x_2z-plane, makes an angle θ with the z- axis, and has a positive x_2 component. This information is summarized in Fig. 3. Since this light wave is travelling at an angle θ with the optical axis, we would expect the patch of light formed in the correlation plane by the correlation lens acting on this light wave to be centered off-axis in the correlation plane. This is the case, as the theory will show and as Fig. 3 indicates.

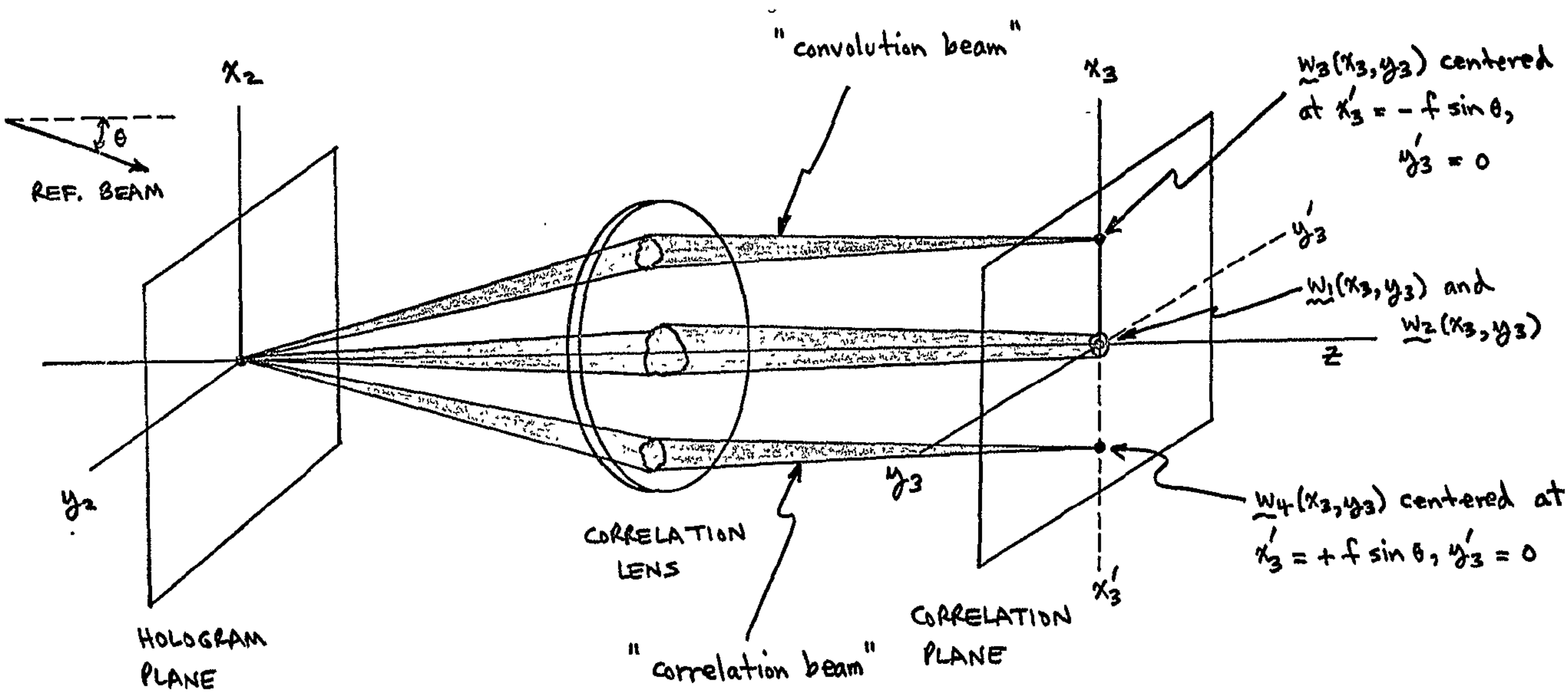


Figure 3. Geometry of the light beams forming the various light patches in the correlation plane.

Using the convolution theorem stated in Eq. (A-32) we have

$$\mathcal{L}_{23}\left[\underset{\sim}{g}(x_2,y_2)\,\underset{\sim}{s}(x_2,y_2)\,e^{j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\frac{1}{(\lambda f)^2}\,\mathcal{L}_{23}\left[\underset{\sim}{g}(x_2,y_2)\,\underset{\sim}{s}(x_2,y_2)\right]*\mathcal{L}_{23}\left[e^{j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\frac{1}{(\lambda f)^4}\,\mathcal{L}_{23}\left[\underset{\sim}{g}(x_2,y_2)\right]*\mathcal{L}_{23}\left[\underset{\sim}{s}(x_2,y_2)\right]*\mathcal{L}_{23}\left[e^{j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\frac{1}{(\lambda f)^4}\,\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[\underset{\sim}{p}(x_1,y_1)\right]\right]*\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[\underset{\sim}{n}(x_1,y_1)\right]\right]$$

$$*\,\mathcal{L}_{23}\left[e^{j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\underset{\sim}{p}(-x_3,-y_3)*\underset{\sim}{n}(-x_3,-y_3)*\mathcal{L}_{23}\left[e^{j\frac{2\pi x_2\sin\theta}{\lambda}}\right]. \qquad (23)$$

Putting Eq. (A-46) into Eq. (23) gives

$$\mathcal{L}_{23}\left[\underset{\sim}{g}(x_2,y_2)\,\underset{\sim}{s}(x_2,y_2)\,e^{j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=(\lambda f)^2\,\underset{\sim}{p}(-x_3,-y_3)*\underset{\sim}{n}(-x_3,-y_3)*\delta(-x_3+f\sin\theta,-y_3)$$

$$=(\lambda f)^2\,\underset{\sim}{p}(x_3',y_3')*\underset{\sim}{n}(x_3',y_3')*\delta(x_3'+f\sin\theta,y_3'). \qquad (24)$$

Putting Eq. (24) into Eq. (8) gives

$$\underset{\sim}{W}_3(x_3,y_3) = \frac{\underset{\sim}{c}\, r_0}{(\lambda f)}\, \underset{\sim}{f}(x_3',y_3') * \underset{\sim}{r}(x_3',y_3') * \delta(x_3'+f\sin\theta, y_3').$$

If we write the last equation in the form

$$\underset{\sim}{W}_3(x_3,y_3) = \frac{\underset{\sim}{c}\, r_0}{(\lambda f)} \left[\underset{\sim}{f}(x_3',y_3') * \underset{\sim}{r}(x_3',y_3')\right] * \delta(x_3'+f\sin\theta, y_3'), \tag{25}$$

then, in light of the discussion in Sec. C of the Appendix and Eq. (A-12) in particular, we can interpret Eq. (25) as the convolution function

$$\frac{\underset{\sim}{c}\, r_0}{(\lambda f)}\, \underset{\sim}{f}(x_3',y_3') * \underset{\sim}{r}(x_3',y_3')$$

centered at the location of the delta function $\delta(x_3'+f\sin\theta, y_3')$, which is at $x_3' = -f\sin\theta, y_3' = 0$. This term is sometimes of interest in optical filtering applications, but it is not of direct interest here.

<u>Fourth Term, $\underset{\sim}{W}_4(x_3,y_3)$</u>

From Eq. (9) we have

$$\underset{\sim}{W}_4(x_3,y_3) = \frac{\underset{\sim}{c}\, r_0}{(\lambda f)^3}\, \alpha_{23} \left[\underset{\sim}{g}^*(x_2,y_2)\, \underset{\sim}{s}(x_2,y_2)\, e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}}\right]. \tag{9}$$

Proceeding as usual, we have

$$\mathcal{L}_{23}\left[\underline{g}^*(x_2,y_2)\,\underline{s}(x_2,y_2)\,e^{-j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\frac{1}{(\lambda f)^2}\,\mathcal{L}_{23}\left[\underline{g}^*(x_2,y_2)\,\underline{s}(x_2,y_2)\right]*\mathcal{L}_{23}\left[e^{-j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\frac{1}{(\lambda f)^4}\,\mathcal{L}_{23}\left[\underline{g}^*(x_2,y_2)\right]*\mathcal{L}_{23}\left[\underline{s}(x_2,y_2)\right]*\mathcal{L}_{23}\left[e^{-j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\frac{1}{(\lambda f)^4}\,\mathcal{L}_{23}\left[\left\{\mathcal{L}_{12}\left[\underline{p}(x_1,y_1)\right]\right\}^*\right]*\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[\underline{\Delta}(x_1,y_1)\right]\right]$$

$$*\,\mathcal{L}_{23}\left[e^{-j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$

$$=\underline{p}^*(x_3,y_3)*\underline{\Delta}(-x_3,-y_3)*(\lambda f)^2\,\delta(-x_3-f\sin\theta,\,-y_3)\,,$$

where use has been made of Eq. (A-47) in the last step. Changing the arguments over to x_3' and y_3' we then have

$$\mathcal{L}_{23}\left[\underline{g}^*(x_2,y_2)\,\underline{s}(x_2,y_2)\,e^{-j\frac{2\pi x_2\sin\theta}{\lambda}}\right]$$
$$=(\lambda f)^2\,\underline{p}^*(-x_3',-y_3')*\underline{\Delta}(x_3',y_3')*\delta(x_3'-f\sin\theta,\,y_3')\,. \qquad (26)$$

Now from Eq. (20), we see that

$$\underline{p}^*(-x_3',-y_3')*\underline{\Delta}(x_3',y_3')=\underline{p}(x_3',y_3')\otimes\underline{\Delta}(x_3',y_3')\,. \qquad (27)$$

Putting Eq. (27) into Eq. (26) gives

$$\mathcal{L}_{23}\left[\underline{g}^*(x_2, y_2)\,\underline{s}(x_2, y_2)\,e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}}\right]$$

$$= (\lambda f)^2\left[\underline{p}(x_3', y_3') \otimes \underline{n}(x_3', y_3')\right] * \delta(x_3' - f\sin\theta, y_3'). \tag{28}$$

Putting Eq. (28) into Eq. (9) then gives

$$\underline{W}_4(x_3, y_3) = \frac{\underline{c}\,n_0}{(\lambda f)}\left[\underline{p}(x_3', y_3') \otimes \underline{n}(x_3', y_3')\right] * \delta(x_3' - f\sin\theta, y_3'). \tag{29}$$

The light amplitude distribution in Eq. (29) may be interpreted as the cross-correlation function

$$\frac{\underline{c}\,n_0}{(\lambda f)}\,\underline{p}(x_3', y_3') \otimes \underline{n}(x_3', y_3')$$

centered at the location of the delta function $\delta(x_3' - f\sin\theta, y_3')$, which is at $x_3' = f\sin\theta,\ y_3' = 0$. This is, of course, the term of interest in holographic correlation systems.

The term $e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}}$ in the square brackets in Eq. (9) implies that the light wave leaving the (x_2, y_2) plane on its way to the correlation lens, and which will eventually produce the light amplitude distribution in Eq. (29) in the correlation plane, is moving generally along a direction specified by a unit vector $\hat{\mu}$ which lies in the $x_2 z$-plane, makes an angle θ with the z-axis, and has a negative x_2-component. As shown in Fig. 3, this wave (called the "correlation beam") leaves the hologram plane travelling generally parallel to the original reference beam. The correlation lens processes this wave to produce the "correlation spot" $\underline{W}_4(x_3, y_3)$ centered at $x_3' = f \sin\theta$, $y_3' = 0$.

In order to separate the correlation spot $\underline{W}_4(x_3, y_3)$ from the two spots $\underline{W}_1(x_3, y_3)$ and $\underline{W}_2(x_3, y_3)$ centered at the origin, it is necessary that the angle θ at which the reference beam comes in with respect to the z-axis be made sufficiently large. In the present MSFC holographic correlation system, the functions $\underline{g}(x_1, y_1)$ and $\underline{h}(x_1, y_1)$ are very small patches of light in the object plane, so that the spots $\underline{W}_1(x_3, y_3)$, $\underline{W}_2(x_3, y_3)$, $\underline{W}_3(x_3, y_3)$, and $\underline{W}_4(x_3, y_3)$ are very small in size (less than one millimeter diameter). Consequently, if $f \sin\theta \gtrsim 1$ cm, there would be plenty of separation between the correlation spot and the two central spots which are of no interest. Since f is approximately 15 cm in the MSFC setup, the condition on θ would be

$$(15\text{ cm}) \sin\theta \gtrsim (1\text{ cm}),$$

which gives

$$\theta \gtrsim \sin^{-1}\left(\frac{1}{15}\right) = 3.8^\circ .$$

Conversely, the angle θ cannot be made too large, or else the correlation beam will miss the correlation lens and the system cannot operate properly. With reference to

Fig. 4 it will be seen that in order for the correlation beam to just strike the outer edge of the correlation lens, we must have

$$\tan\theta \leq \frac{r_{lens}}{f},$$

or

$$\theta \leq \tan^{-1}\left(\frac{r_{lens}}{f}\right).$$

In the MSFC system, r_{LENS} is about 10 cm, so we must require that

$$\theta \leq \tan^{-1}\left(\frac{10}{15}\right) = 33.7^{\circ}.$$

In the present MSFC setup, the reference beam angle therefore should lie in the approximate range

$$4^{\circ} \leq \theta \leq 30^{\circ}.$$

Finally, it should be mentioned that the photodetector which monitors the correlation spot in the correlation plane responds to the <u>irradiance</u> in the spot, not to the amplitude of the light vibration. The photodetector output current i will therefore be proportional to

$$\iint_{A} \left| \mathfrak{p}(x_3', y_3') \otimes \mathfrak{g}(x_3', y_3') \right|^2 dx_3' \, dy_3'$$

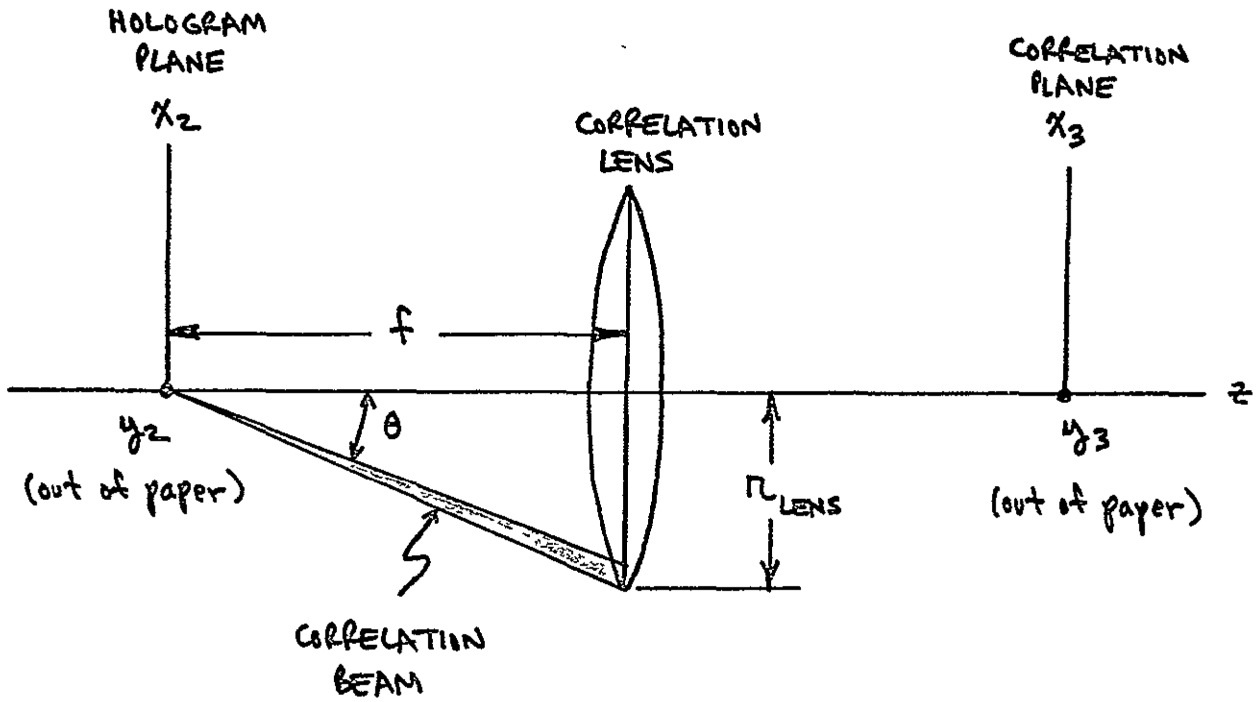


Figure 4. Geometry to determine the maximum possible value for the reference beam angle θ .

where A is the area of the photodetector aperture, understood to be centered at $x_3' = f \sin\theta$, $y_3' = 0$.

III. TRANSLATION OF THE OBJECT IN THE OBJECT PLANE

We now want to investigate the effect of translating the object distribution $\underset{\sim}{\Omega}(x_1,y_1)$ in the object plane when the hologram is illuminated. The plate is exposed using the distribution $\underset{\sim}{\varphi}(x_1,y_1)$, so that Eqs. (1) and (2) remain valid. Suppose now, however, that the hologram is illuminated with the function $\underset{\sim}{\Omega}(x_1-a, y_1-b)$ centered at $x_1=a, y_1=b$ instead of the function $\underset{\sim}{\Omega}(x_1,y_1)$ centered at the origin. When the transform lens operates on the distribution $\underset{\sim}{\Omega}(x_1-a,y_1-b)$ the resulting light amplitude distribution in the hologram plane will be

$$\frac{1}{\lambda f}\mathcal{L}_{12}\left[\underset{\sim}{\Omega}(x_1-a,\ y_1-b)\right] .$$

But using the shift theorem for the operator $\mathcal{L}$ expressed in Eq. (A-26), we have

$$\mathcal{L}_{12}\left[\underset{\sim}{\Omega}(x_1-a, y_1-b)\right] = e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)}\mathcal{L}_{12}\left[\underset{\sim}{\Omega}(x_1,y_1)\right] .$$

The light amplitude distribution in the hologram plane thus becomes

$$\frac{1}{\lambda f}e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)}\mathcal{L}_{12}\left[\underset{\sim}{\Omega}(x_1,y_1)\right] = \frac{1}{\lambda f}e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)}\underset{\sim}{\Sigma}(x_2,y_2)$$

When the light wave producing this distribution passes through the hologram, the resulting light amplitude distribution in the hologram plane will be given by Eq. (3) with $\underset{\sim}{s}(x_2, y_2)$ replaced by

$$e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)} \quad \underset{\sim}{s}(x_2, y_2)\,.$$

The result will thus be

$$\underset{\sim}{u}(x_2,y_2) = \frac{\underset{\sim}{c}\, r_0^2}{\lambda f}\, \underset{\sim}{s}(x_2,y_2)\, e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)}$$

$$+ \frac{\underset{\sim}{c}}{(\lambda f)^3}\, \underset{\sim}{s}(x_2,y_2) \left|\underset{\sim}{g}(x_2,y_2)\right|^2 e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)}$$

$$+ \frac{\underset{\sim}{c}\, r_0}{(\lambda f)^2}\, \underset{\sim}{g}(x_2,y_2)\, \underset{\sim}{s}(x_2,y_2)\; e^{j\frac{2\pi}{\lambda f}\left[x_2(f\sin\theta - a) - by_2\right]}$$

$$+ \frac{\underset{\sim}{c}\, r_0}{(\lambda f)^2}\, \underset{\sim}{g}^*(x_2,y_2)\, \underset{\sim}{s}(x_2,y_2)\; e^{-j\frac{2\pi}{\lambda f}\left[x_2(f\sin\theta + a) + by_2\right]} \,. \tag{30}$$

The fourth term of Eq. (30) leads to the correlation spot, so the only term of interest to us in Eq. (30) is

$$\underset{\sim}{u}_4(x_2,y_2) = \frac{\underset{\sim}{c}\, r_0}{(\lambda f)^2}\, \underset{\sim}{g}^*(x_2,y_2)\, \underset{\sim}{s}(x_2,y_2)$$

$$\times\; e^{-j\frac{2\pi}{\lambda f}\left[x_2(f\sin\theta + a) + by_2\right]} \,. \tag{31}$$

When the correlation lens operates on this term, the resulting light amplitude distribution in the correlation plane will be

$$\underline{w}_4(x_3,y_3) = \frac{\underline{c}\,n_0}{(\lambda f)^3}\,\mathcal{L}_{23}\Big[g^*(x_2,y_2)\,\underline{s}(x_2,y_2) \times e^{-j\frac{2\pi}{\lambda f}[x_2(f\sin\theta+a)+by_2]}\Big]. \quad (32)$$

Use of the convolution theorem [Eq. (A-32)] as usual then gives

$$\underline{w}_4(x_3,y_3) = \frac{\underline{c}\,n_0}{(\lambda f)^3}\,\mathcal{R}^*(-x_3',-y_3') * \underline{\Delta}(x_3',y_3') * \mathcal{L}_{23}\Big[e^{-j\frac{2\pi}{\lambda f}[x_2(f\sin\theta+a)+by_2]}\Big]. \quad (33)$$

From Eq. (27) we have

$$\mathcal{R}^*(-x_3',-y_3') * \underline{\Delta}(x_3',y_3') = \mathcal{R}(x_3',y_3') \otimes \underline{\Delta}(x_3',y_3'). \quad (34)$$

In order to handle the last factor in Eq. (33), we can make use of an obvious generalization of Eq. (A-47),

$$\mathcal{L}_{23}\Big[e^{-j\frac{2\pi}{\lambda}(cx_2+dy_2)}\Big] = (\lambda f)^2\,\delta(-x_3-fc,\,-y_3-fd). \quad (35)$$

Comparing Eqs. (35) and (33) we see that

$$c = \sin\theta + \frac{a}{f}$$

$$d = \frac{b}{f}$$

so that we have

$$\mathcal{L}_{23}\Big[e^{-j\frac{2\pi}{\lambda f}[x_2(f\sin\theta+a)+by_2]}\Big]$$

$$= (\lambda f)^2\,\delta(-x_3-f\sin\theta-a,\,-y_3-b)$$

$$= (\lambda f)^2 \delta(x_3' - f\sin\theta - a,\ y_3' - b) . \quad (36)$$

Putting Eqs. (34) and (36) into Eq. (33) gives

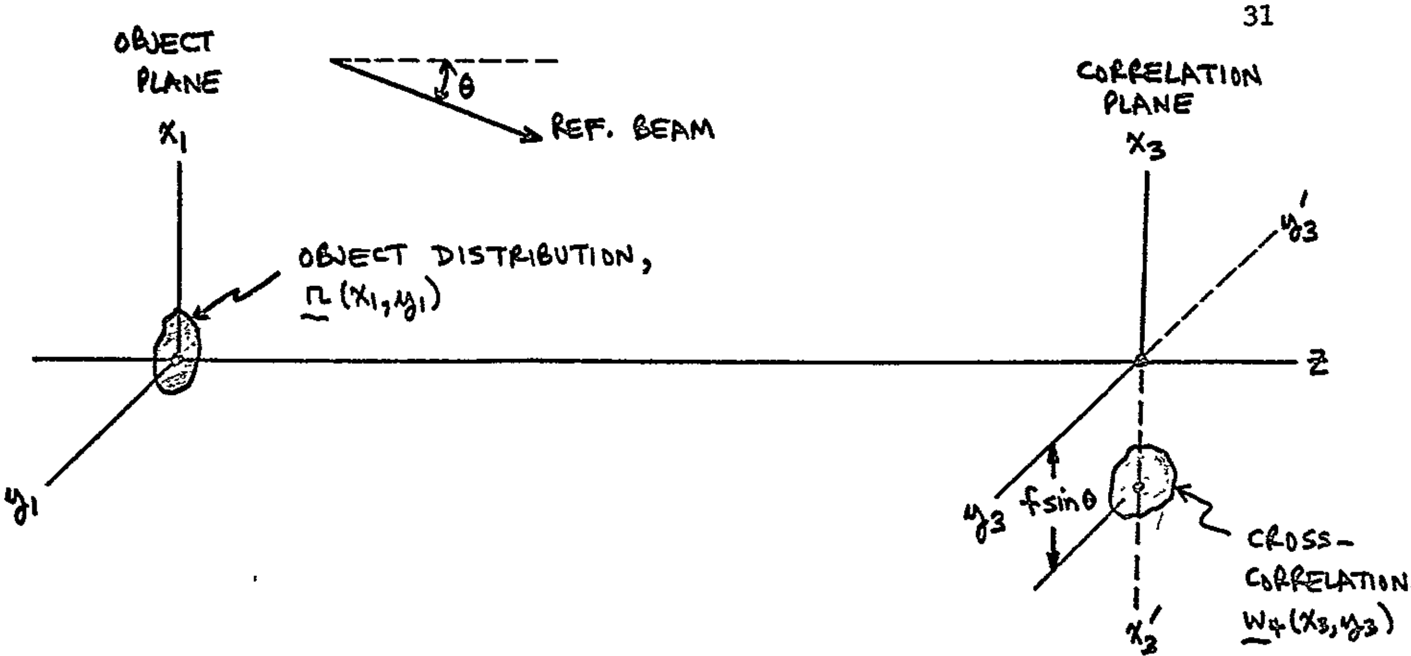
$$\underset{\sim}{W_4}(x_3, y_3) = \frac{\underset{\sim}{c}\, n_0}{(\lambda f)} \left[\mathscr{f}(x_3', y_3') \otimes \underset{\sim}{n}(x_3', y_3') \right] * \delta(x_3' - f\sin\theta - a, y_3' - b). \quad (37)$$

Eq. (37) states that the light amplitude distribution $\underset{\sim}{W_4}(x_3, y_3)$ consists of the cross-correlation function

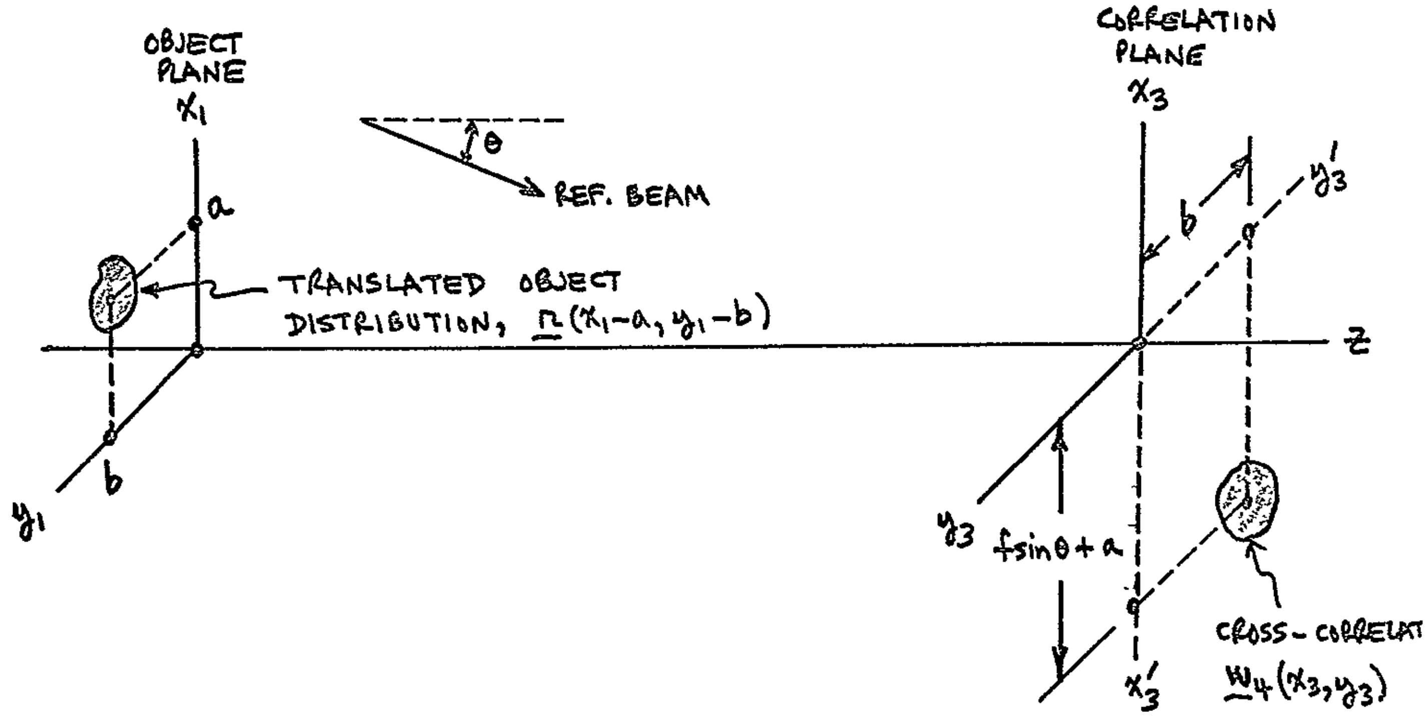
$$\frac{\underset{\sim}{c}\, n_0}{(\lambda f)} \mathscr{f}(x_3', y_3') \otimes \underset{\sim}{n}(x_3', y_3')$$

centered at the coordinates $x_3' = f\sin\theta + a, y_3' = b$. Comparing this result with our previous result in Eq. (29), we see that the only change has been a translation of the function $\underset{\sim}{W_4}$ by amounts a and b in the x_3' and y_3' directions, respectively. Thus, translation of the object distribution in the object plane merely translates the cross-correlation function in the correlation plane. Movement of the photodetector to the new location of the cross-correlation function should then produce the same output current.

The effect of translating the object distribution in the object plane is summarized in Fig. 5 for convenient reference.



(a) Situation when the hologram is illuminated with the object distribution $\underline{r}(x_1, y_1)$.



(b) Situation when the hologram is illuminated with the translated object distribution $\underline{r}(x_1-a, y_1-b)$.

Figure 5. Effect of translating the object distribution $\underline{r}(x_1, y_1)$ in the object plane.

IV. TRANSLATION OF THE OBJECT ALONG THE OPTICAL AXIS

As the final task in the present study, we started to look at the effect of translating the object distribution along the optical axis of the system (i.e., along the z-axis in Fig. 2). We were not able to complete this task, but we report here the partial results we have obtained.

Here again we suppose that the photographic plate is exposed using the distribution $\underset{\sim}{p}(x_1,y_1)$, so that Eqs. (1) and (2) are valid. Suppose, however, that when the hologram is illuminated with the distribution $\underset{\sim}{\Omega}(x_1,y_1)$, the object plane is at a distance f+Δz from the hologram plane, instead of the usual distance f. Then, according to Eqs. (A-13) and (A-16), the light amplitude distribution in the hologram plane will be

$$\frac{e^{j\frac{k}{2f}\left[1-\frac{(f+\Delta z)}{f}\right](x_2^2+y_2^2)}}{\lambda f}\,\mathcal{L}_{12}\left[\underset{\sim}{\Omega}(x_1,y_1)\right]$$

$$= \frac{e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}}{\lambda f}\,\underset{\sim}{S}(x_2,y_2),$$

where the constant phase shift $\frac{1}{j} = -j = e^{-j\frac{\pi}{2}}$ has been dropped in Eq. (A-13) as usual. When the hologram is illuminated with this light amplitude distribution, the resulting light amplitude distribution in the hologram plane will be given by Eq. (3) with $\underset{\sim}{s}(x_2, y_2)$ replaced by

$$e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}\ \underset{\sim}{s}(x_2, y_2) .$$

The result is

$$\begin{aligned}
\underset{\sim}{u}(x_2,y_2) = {} & \frac{\underset{\sim}{c}\, n_0^2}{\lambda f}\, \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} \\
& + \frac{\underset{\sim}{c}}{(\lambda f)^3}\, \underset{\sim}{s}(x_2, y_2) \left|\underset{\sim}{g}(x_2, y_2)\right|^2 e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} \\
& + \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^2}\, \underset{\sim}{g}(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}\, e^{j\frac{2\pi x_2 \sin\theta}{\lambda}} \\
& + \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^2}\, \underset{\sim}{g}^*(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}\, e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} .
\end{aligned}$$

The fourth term of this equation is the correlation term, so let us concentrate on this term:

$$\underset{\sim}{u}_4(x_2,y_2) = \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^2}\, \underset{\sim}{g}^*(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}\, e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} , \tag{38}$$

When the correlation lens operates on this term, the resulting light amplitude distribution in the correlation plane will be

$$\underset{\sim}{W}_4(x_3, y_3) = \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^3} \mathcal{L}_{23}\left[\underset{\sim}{g}^*(x_2, y_2)\, \underset{\sim}{s}(x_2, y_2)\, e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} \times e^{-j\frac{2\pi x_2 \sin\theta}{\lambda}} \right].$$

Application of Eqs. (A-32), (A-37), (A-43), and (A-47) then gives

$$\underset{\sim}{W}_4(x_3, y_3) = \frac{\underset{\sim}{c}\, n_0}{(\lambda f)^3} \mathcal{G}^*(x_3, y_3) * \underset{\sim}{\Omega}(-x_3, -y_3) * \mathcal{L}_{23}\left[e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} \right] * \delta(-x_3 - f\sin\theta, -y_3). \quad (39)$$

We have to pause now and calculate the third factor in Eq. (39),

$$\mathcal{L}_{23}\left[e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} \right].$$

By definition of the operator $\mathcal{L}$ given in Eq. (A-16), we have

$$\mathcal{L}_{23}\left[e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} \right] = \iint_{-\infty}^{+\infty} e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}\, e^{-j\frac{2\pi}{\lambda f}(x_2 x_3 + y_2 y_3)}\, dx_2 dy_2. \quad (40)$$

Now since $k = \frac{2\pi}{\lambda}$, we have

$$e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)} = e^{-j\cdot\left(\frac{2\pi}{\lambda}\right)\cdot\frac{\Delta z}{2f^2}(x_2^2+y_2^2)}$$

$$= e^{-j\frac{\pi\Delta z}{\lambda f^2}(x_2^2+y_2^2)} .$$

Thus Eq. (40) becomes

$$\mathcal{L}_{23}\left[e^{-j\frac{k\Delta z}{2f^2}(x_2^2+y_2^2)}\right] = \iint_{-\infty}^{+\infty} e^{-j\frac{\pi\Delta z}{\lambda f^2}(x_2^2+y_2^2)}\, e^{-j\frac{2\pi}{\lambda f}(x_2x_3+y_2y_3)}\, dx_2dy_2$$

$$= \iint_{-\infty}^{+\infty} e^{-j\frac{\pi\Delta z}{\lambda f^2}\left[x_2^2+y_2^2+\frac{2f}{\Delta z}(x_2x_3+y_2y_3)\right]}\, dx_2dy_2 . \quad (41)$$

Completing the square in the argument of the exponential term in Eq. (41) gives

$$e^{-j\frac{\pi\Delta z}{\lambda f^2}\left[x_2^2+y_2^2+\frac{2f}{\Delta z}(x_2x_3+y_2y_3)\right]}$$

$$= e^{-j\frac{\pi\Delta z}{\lambda f^2}\left[\left(x_2+\frac{f}{\Delta z}x_3\right)^2+\left(y_2+\frac{f}{\Delta z}y_3\right)^2\right]}$$

$$\times\ e^{\frac{j\pi}{\lambda\Delta z}(x_3^2+y_3^2)} .$$

Eq. (41) can thus be written as

$$\mathcal{L}_{23}\left[e^{-\frac{jk\Delta z}{2f^2}(x_2^2+y_2^2)}\right]$$

$$= e^{\frac{j\pi}{\lambda\Delta z}(x_3^2+y_3^2)} \iint_{-\infty}^{+\infty} e^{-\frac{j\pi\Delta z}{\lambda f^2}\left[(x_2+\frac{f}{\Delta z}x_3)^2+(y_2+\frac{f}{\Delta z}y_3)^2\right]} dx_2 dy_2 \tag{42}$$

The integral in Eq. (42) is just a constant, the value of which is obtained in Sec. F of the Appendix, Eq. (A-65). We thus have

$$\mathcal{L}_{23}\left[e^{-\frac{jk\Delta z}{2f^2}(x_2^2+y_2^2)}\right] = \frac{\lambda f^2}{\Delta z} e^{\frac{j\pi}{\lambda\Delta z}(x_3^2+y_3^2)} . \tag{43}$$

Putting Eq. (43) into Eq. (39) gives

$$\underset{\sim}{w}_4(x_3,y_3) = \frac{\underset{\sim}{c}\, n_0}{\lambda^2 f\,\Delta z}\, \mathscr{r}^*(x_3,y_3) * \underset{\sim}{\Omega}(-x_3,-y_3) * e^{\frac{j\pi}{\lambda\Delta z}(x_3^2+y_3^2)}$$

$$* \,\delta(-x_3-f\sin\theta, -y_3)$$

$$= \frac{\underset{\sim}{c}\, n_0}{\lambda^2 f\,\Delta z}\, \mathscr{r}^*(-x_3',-y_3') * \underset{\sim}{\Omega}(x_3',y_3') * e^{\frac{j\pi}{\lambda\Delta z}(x_3'^2+y_3'^2)}$$

$$* \,\delta(x_3'-f\sin\theta, y_3') . \tag{44}$$

Use of Eq. (27) then gives

$$\underset{\sim}{w}_4(x_3, y_3) = \frac{\underset{\sim}{c}\, r_0}{\lambda^2 f \Delta z} \left[\underset{\sim}{p}(x_3', y_3') \otimes \underset{\sim}{r}(x_3', y_3') \right] * e^{\frac{j\pi}{\lambda \Delta z}(x_3'^2 + y_3'^2)}$$

$$* \ \delta(x_3' - f \sin\theta, y_3'). \qquad (45)$$

The light amplitude distribution in Eq. (45) may be interpreted as the function

$$\frac{\underset{\sim}{c}\, r_0}{\lambda^2 f \Delta z} \left[\underset{\sim}{p}(x_3', y_3') \otimes \underset{\sim}{r}(x_3', y_3') \right] * e^{\frac{j\pi}{\lambda \Delta z}(x_3'^2 + y_3'^2)}$$

centered at $x_3' = f \sin\theta$, $y_3' = 0$. This function represents the usual cross-correlation function, $\left[\underset{\sim}{p}(x_3', y_3') \otimes \underset{\sim}{r}(x_3', y_3') \right]$, convolved with the spherical phase factor

$$e^{\frac{j\pi}{\lambda \Delta z}(x_3'^2 + y_3'^2)}$$

Unfortunately, this is as far as we have been able to develop the theory within the time frame of the present contract.

V. CONCLUSIONS AND RECOMMENDATIONS

We regret that the study of the effect of translating the object along the optical axis reported in Sec. IV could not be completed during the present contract. In practical holographic correlation systems used to examine three-dimensional objects, it is certainly possible for the objects to change size, or to undergo small translations, along the optical axis. We would recommend that this study be brought to completion eventually.

An observation which strikes us as being very important is the fact that while the theory of the holographic correlation system is always developed on the assumption that the object plane, the hologram plane, and the correlation plane are all parallel to each other, in practical setups this is usually not the case. The effect of having these three planes slanted with respect to each other has not been analyzed in the literature, as far as we have been able to determine. We feel that this is an important study which ought to be given highest priority.

APPENDIX

The purpose of this appendix is to summarize some important results which will be needed in developing the theory of the holographic correlation system.

A. The Fourier Transform

Let $\underline{p}(x,y)$ be a complex-valued function of two independent real variables, x and y. (NOTE: In this report, complex-valued functions will be indicated by an underline.) The two-dimensional Fourier transform of $\underline{p}(x,y)$ is defined as

$$\mathcal{F}\left[\underline{p}(x,y)\right] = \underline{P}(f_x,f_y) = \iint_{-\infty}^{+\infty} \underline{p}(x,y)\, e^{-j2\pi(xf_x+yf_y)}\, dx\, dy, \quad \text{(A-1)}$$

where f_x and f_y are spatial frequencies with dimensions of $\frac{1}{x}$ and $\frac{1}{y}$, respectively, and $j=\sqrt{-1}$. The inverse Fourier transform of $\underline{P}(f_x,f_y)$ is defined as

$$\mathcal{F}^{-1}\left[\underline{P}(f_x,f_y)\right] = \underline{p}(x,y) = \iint_{-\infty}^{+\infty} \underline{P}(f_x,f_y)\, e^{j2\pi(xf_x+yf_y)}\, df_x\, df_y. \quad \text{(A-2)}$$

B. The Convolution and Correlation Integrals

The convolution integral of two functions $\underline{p}(x,y)$ and $\underline{q}(x,y)$ is denoted by $\underline{p}(x,y) * \underline{q}(x,y)$, and is defined as follows:

$$\mathfrak{p}(x,y) * \mathfrak{q}(x,y) = \iint_{-\infty}^{+\infty} \mathfrak{p}(\xi,\eta)\, \mathfrak{q}(x-\xi, y-\eta)\, d\xi\, d\eta. \quad \text{(A-3)}$$

It will be observed that the result of the convolution operation is a function of the original variables x and y. This function is called the convolution function of $\mathfrak{p}$ and $\mathfrak{q}$. The variables ξ and η in Eq. (A-3) are just dummy variables of integration.

The convolution operation is both commutative and associative. That is,

$$\mathfrak{p}(x,y) * \mathfrak{q}(x,y) = \mathfrak{q}(x,y) * \mathfrak{p}(x,y)$$

and

$$\mathfrak{p}(x,y) * \left[\mathfrak{q}(x,y) * \mathfrak{r}(x,y)\right] = \left[\mathfrak{p}(x,y) * \mathfrak{q}(x,y)\right] * \mathfrak{r}(x,y).$$

Since the convolution operation is both commutative and associative, it follows that the individual operations in a string of convolutions can be performed in any order whatever.

In order to set up the convolution integral of two functions $\mathfrak{p}(x,y)$ and $\mathfrak{q}(x,y)$, the following steps are taken:

(1) Change the names of the variables from (x,y) to (ξ,η). The functions then become $\mathfrak{p}(\xi,\eta)$ and $\mathfrak{q}(\xi,\eta)$.

(2) Fold $\mathfrak{q}(\xi,\eta)$ about the origin to get $\mathfrak{q}(-\xi,-\eta)$.

(3) Translate $\mathfrak{q}(-\xi,-\eta)$ by amounts x and y in the ξ and η directions, respectively, to get $\mathfrak{q}(x-\xi, y-\eta)$.

(NOTE: Since convolution is commutative, steps (2) and (3) could equally well have been applied to the function $f(x,y)$. The final results would be identical either way.)

(4) Multiply the folded and translated function $g(x-\xi, y-\eta)$ by the other function $f(\xi,\eta)$, and integrate over all space to get the final result,

$$f(x,y) * g(x,y) = \iint_{-\infty}^{+\infty} f(\xi,\eta)\, g(x-\xi, y-\eta)\, d\xi\, d\eta .$$

The <u>cross-correlation integral</u> of two functions $f(x,y)$ and $g(x,y)$ is denoted by $f(x,y) \otimes g(x,y)$, and is defined as follows:

$$f(x,y) \otimes g(x,y) = \iint_{-\infty}^{+\infty} f^{*}(\xi-x, \eta-y)\, g(\xi,\eta)\, d\xi\, d\eta . \quad \text{(A-4)}$$

Notice that the resulting function, called the cross-correlation function of f and g , is a function of the original variables x and y.

The cross-correlation operation is not commutative. That is,

$$f(x,y) \otimes g(x,y) \neq g(x,y) \otimes f(x,y) .$$

However, the translation operation which has been applied to the function f^{*} in Eq. (A-4) can be applied instead to the function g in the opposite sense without changing the value of the integral. That is,

$$\mathcal{f}(x,y) \otimes \mathcal{g}(x,y) = \iint_{-\infty}^{+\infty} \mathcal{f}^*(\xi,\eta)\, \mathcal{g}(\xi+x,\eta+y)\, d\xi\, d\eta . \quad \text{(A-5)}$$

(NOTE: If the functions $\mathcal{f}(x,y)$ and $\mathcal{g}(x,y)$ are identical, then the operation

$$\mathcal{f}(x,y) \otimes \mathcal{f}(x,y) = \iint_{-\infty}^{+\infty} \mathcal{f}^*(\xi-x,\eta-y)\, \mathcal{f}(\xi,\eta)\, d\xi\, d\eta \quad \text{(A-4a)}$$

is called the <u>auto-correlation</u> function of $\mathcal{f}$ with itself.)

The cross-correlation integral of two functions $\mathcal{f}(x,y)$ and $\mathcal{g}(x,y)$ can be set up by use of the following steps:

(1) Change the names of the variables from (x,y) to (ξ,η), obtaining the functions $\mathcal{f}(\xi,\eta)$ and $\mathcal{g}(\xi,\eta)$.

(2) Take the complex conjugate of the first function $\mathcal{f}(\xi,\eta)$ named in the operation, obtaining $\mathcal{f}^*(\xi,\eta)$.

(3) Translate the function $\mathcal{f}^*(\xi,\eta)$ by amounts x and y in the ξ and η directions, respectively, to get $\mathcal{f}^*(\xi-x,\eta-y)$.

(4) Multiply the conjugated and translated function $\mathcal{f}^*(\xi-x,\eta-y)$ by the second function $\mathcal{g}(\xi,\eta)$, and integrate over all space to obtain the final result,

$$\mathcal{f}(x,y) \otimes \mathcal{g}(x,y) = \iint_{-\infty}^{+\infty} \mathcal{f}^*(\xi-x,\eta-y)\, \mathcal{g}(\xi,\eta)\, d\xi\, d\eta .$$

Notice that convolution and cross-correlation differ in two respects, folding (done in convolution but not in cross-correlation) and complex conjugation (done in cross-correlation but not in convolution).

C. The Dirac Delta Function

The well known Dirac delta function has several important properties which we wish to summarize here for convenient reference. For purposes of this report we are interested in the two-dimensional version of the delta function. In the equations which follow, a and b represent arbitrary real constants.

Basic Definition

$$\delta(x,y) = \begin{cases} \infty & \text{if } x=y=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\iint_{-\infty}^{+\infty} \delta(a-x, b-y)\, dx\, dy = 1 \,. \quad \text{(A-6)}$$

Sifting Property

$$\iint_{-\infty}^{+\infty} f(x,y)\, \delta(a-x, b-y)\, dx\, dy = f(a,b)\,, \quad \text{(A-7)}$$

where $f(x,y)$ is an arbitrary function of x and y.

Scaling Property

$$\delta(ax, by) = \frac{1}{|ab|}\,\delta(x, y)\,. \quad \text{(A-8)}$$

NOTE: It follows from the scaling property expressed in Eq. (A-8) that

$$\delta(x-a, y-b) = \delta\left[(-1)(a-x),\ (-1)(b-y)\right]$$

$$= \frac{1}{|(-1)(-1)|}\,\delta(a-x, b-y) = \delta(a-x, b-y),$$

so that the delta function is a symmetrical function of its arguments, and Eqs. (A-6) and (A-7) can equally well be written

$$\iint_{-\infty}^{+\infty} \delta(x-a, y-b)\,dx\,dy = 1 \quad \text{(A-9)}$$

$$\iint_{-\infty}^{+\infty} f(x,y)\,\delta(x-a, y-b)\,dx\,dy = f(a, b)\,. \quad \text{(A-10)}$$

Integral Representation

$$\iint_{-\infty}^{+\infty} e^{j2\pi(xf_x+yf_y)}\,dx\,dy = \iint_{-\infty}^{+\infty} e^{-j2\pi(xf_x+yf_y)}\,dx\,dy$$

$$= \delta(f_x, f_y)\,. \quad \text{(A-11)}$$

Convolution of a Given Function With a Delta Function

Suppose a function $f(x,y)$ is convolved with a delta function $\delta(x-a, y-b)$ located at $x=a, y=b$. Following the four steps listed earlier for setting up a convolution integral, we have:

(1) Functions become $f(\xi,\eta)$ and $\delta(\xi-a, \eta-b)$.

(2) Either function can be folded and translated; we choose to fold and translate $f(\xi,\eta)$. Folding produces $f(-\xi,-\eta)$.

(3) Translation produces $f(x-\xi, y-\eta)$.

(4) Multiplication of the functions and integration over all space gives

$$f(x,y) * \delta(x-a, y-b) = \iint_{-\infty}^{+\infty} f(x-\xi, y-\eta)\, \delta(\xi-a, \eta-b)\, d\xi\, d\eta .$$

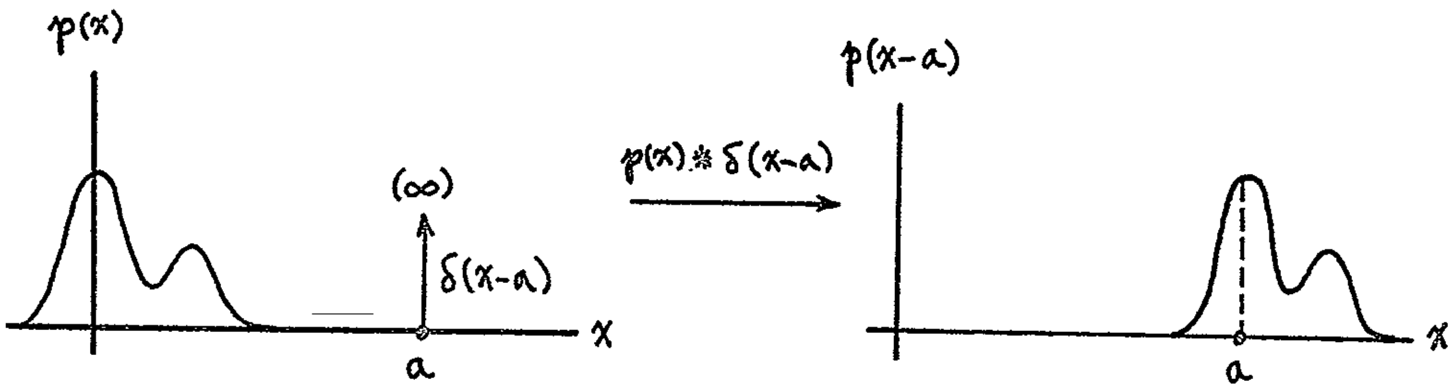
Using the sifting property of the delta function expressed in Eq. (A-10), we now have

$$f(x,y) * \delta(x-a, y-b) = f(x-\xi, y-\eta)\Big|_{\substack{\xi=a\\ \eta=b}}$$

$$= f(x-a, y-b) . \quad \text{(A-12)}$$

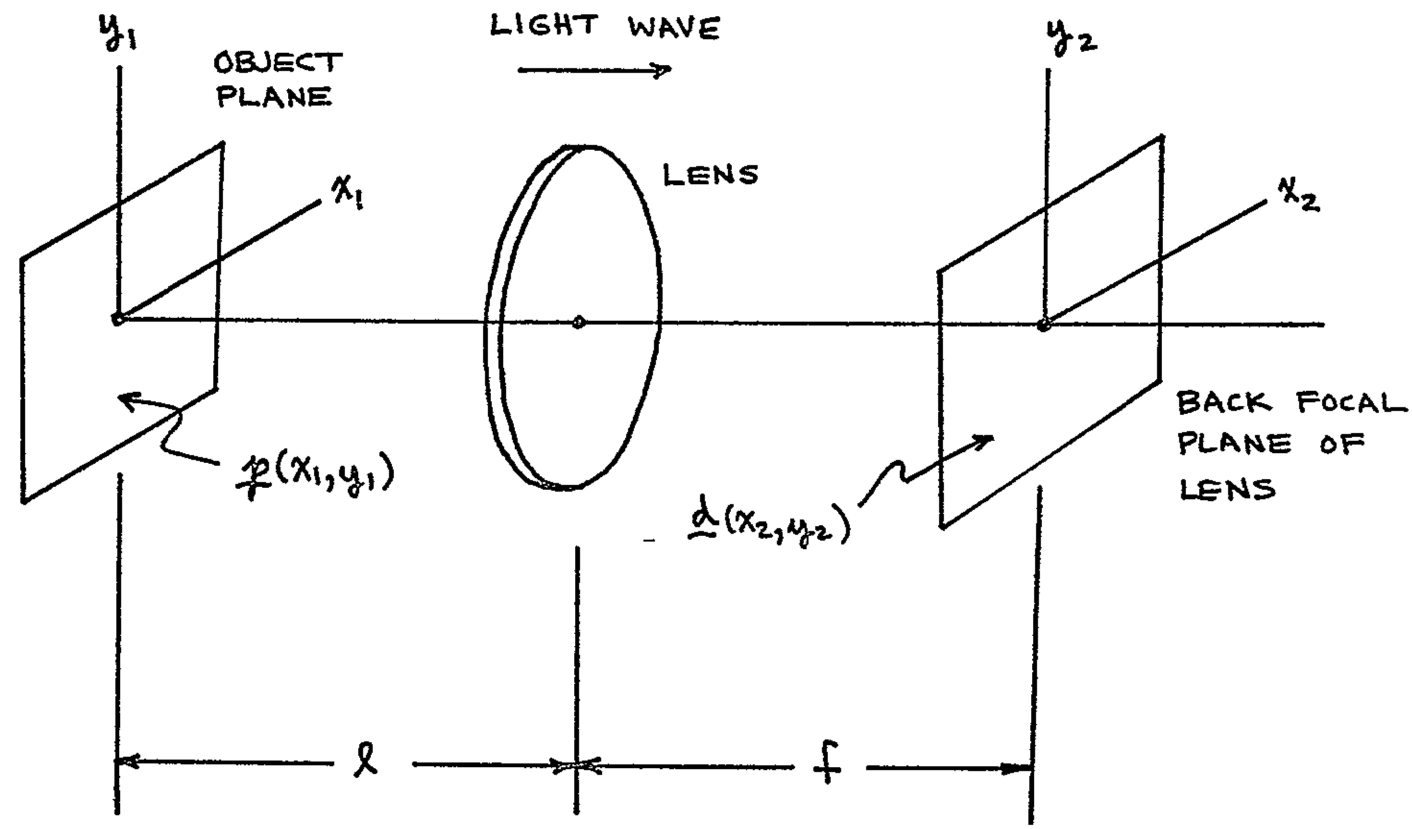
The result, according to Eq. (A-12), is a translation of the function $p(x,y)$ by an amount a in the x-direction and an amount b in the y-direction. Stated another way, we can say that the function has been translated to the location of the delta function, x=a and y=b.

It is difficult to indicate graphically the effect of this operation in two dimensions, but a one-dimensional picture is feasible. The diagram shown below indicates graphically the result of convolving a real function $p(x)$ with the delta function $\delta(x-a)$ located at x=a.



D. <u>The Transformation Properties of a Lens</u>

Suppose a light wave advancing along the z-axis gives rise to a light amplitude distribution $p(x_1, y_1)$ in an object plane at a distance ℓ in front of a lens of focal length f, as shown below. When the light wave is acted upon by the lens,



the resulting light amplitude distribution $\underset{\sim}{d}(x_2,y_2)$ in the back focal plane of the lens is

$$\underset{\sim}{d}(x_2,y_2) = \frac{e^{\frac{jk}{2f}\left(1-\frac{\ell}{f}\right)\left(x_2^2+y_2^2\right)}}{j\lambda f} \iint_{-\infty}^{+\infty} \underset{\sim}{p}(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}\left(x_1x_2+y_1y_2\right)}\, dx_1 dy_1, \tag{A-13}$$

where $k = 2\pi/\lambda$.

In the special case where the object plane coincides with the front focal plane of the lens, so that $\ell = f$, the factor $\left(1-\frac{\ell}{f}\right)$ in the exponential in front of the integral in Eq. (A-13) vanishes, so that the exponential becomes equal to unity, and Eq. (A-13) becomes

$$\underset{\sim}{d}(x_2,y_2) = \frac{1}{j\lambda f}\iint_{-\infty}^{+\infty} \underset{\sim}{f}(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1\, dy_1 \,. \qquad \text{(A-14)}$$

Now the factor $\frac{1}{j} = -j = e^{-j\frac{\pi}{2}}$ in front of the integral in Eq. (A-14) represents a constant phase shift of $-\frac{\pi}{2}$, which is of no practical importance. We can thus drop this factor and write

$$\underset{\sim}{d}(x_2,y_2) = \frac{1}{\lambda f}\iint_{-\infty}^{+\infty} \underset{\sim}{f}(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1\, dy_1 \,. \qquad \text{(A-15)}$$

The integral in Eq. (A-15) can, of course, be written in the form of a Fourier transform by defining spatial frequencies $f_x = x_2/\lambda f$ and $f_y = y_2/\lambda f$. However, it is our opinion that when this is done and the theory of the holographic correlation system is developed using Fourier transform notation, considerable confusion can result about the arguments of the functions involved, and constant factors of λf tend to be dropped indiscriminantly.

The theory of the holographic correlation system can also be carried through using the integral operation in Eq. (A-15) directly. This is the approach used by Smith[3]. However, this procedure becomes very cumbersome, and in the process of using this approach one essentially derives anew many of the commonly known properties of the Fourier transform.

Because of these problems, we will use a somewhat different approach here. Let us define an integral operator $\mathcal{L}$, based on Eq. (A-15), as follows:

$$g(x_2,y_2) = \mathcal{L}_{12}\left[p(x_1,y_1)\right] = \iint_{-\infty}^{+\infty} p(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1 dy_1. \tag{A-16}$$

The symbol "$\mathcal{L}$" stands for "lens operator," and the subscript "12" indicates that this operator transforms a light amplitude distribution $p(x_1,y_1)$ in a plane (x_1,y_1) at a distance f in front of the lens into a light amplitude distribution $d(x_2,y_2)$ in a plane (x_2,y_2) at a distance f behind the lens. From Eqs. (A-15) and (A-16), we can write the distribution $d(x_2,y_2)$ in the back focal plane of the lens as

$$d(x_2,y_2) = \frac{1}{\lambda f}\mathcal{L}_{12}\left[p(x_1,y_1)\right] = \frac{1}{\lambda f}\, g(x_2,y_2). \tag{A-17}$$

We will now proceed to establish those properties of the operator $\mathcal{L}_{12}$ which will be needed in developing the theory of the optical correlation system. These properties, of course, are analogous to the corresponding properties of the standard Fourier transform.

The Inverse Operator, $\mathcal{L}_{12}^{-1}$

To establish the form of the inverse operator $\mathcal{L}_{12}^{-1}$, we will first convert Eq. (A-16) over into a standard Fourier transform. If, as indicated earlier, we define spatial frequencies

$$f_x = \frac{x_2}{\lambda f} \tag{A-18}$$

$$f_y = \frac{y_2}{\lambda f}, \tag{A-19}$$

then Eq. (A-16) can be written in the form of a standard Fourier transform,

$$\mathcal{L}_{12}\left[p(x_1,y_1)\right] = q(x_2,y_2) = \mathcal{F}\left[p(x_1,y_1)\right] = \underline{P}(f_x,f_y) = \iint\limits_{-\infty}^{+\infty} p(x_1,y_1)\, e^{-j2\pi(x_1f_x+y_1f_y)}\, dx_1\,dy_1 \tag{A-20}$$

From Eq. (A-2), the inverse operator would be

$$\mathcal{L}_{12}^{-1}\left[q(x_2,y_2)\right] = p(x_1,y_1) = \mathcal{F}^{-1}\left[\underline{P}(f_x,f_y)\right] = \iint\limits_{-\infty}^{+\infty} \underline{P}(f_x,f_y)\, e^{j2\pi(x_1f_x+y_1f_y)}\, df_x\,df_y\,. \tag{A-21}$$

Now from Eqs. (A-18) and (A-19) we have

$$df_x = d\left(\frac{x_2}{\lambda f}\right)$$

$$df_y = d\left(\frac{y_2}{\lambda f}\right),$$

and we also observe from Eq. (A-20) that $g(x_2,y_2)$ and $\underline{F}(f_x,f_y)$ are just different names for the same function. It then follows that Eq. (A-21) can be rewritten in the form

$$\mathcal{L}_{12}^{-1}\left[g(x_2,y_2)\right] = p(x_1,y_1) = \iint_{-\infty}^{+\infty} g(x_2,y_2)\, e^{j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, d\left(\frac{x_2}{\lambda f}\right) d\left(\frac{y_2}{\lambda f}\right)$$

$$= \frac{1}{(\lambda f)^2}\iint_{-\infty}^{+\infty} g(x_2,y_2)\, e^{j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_2\, dy_2 . \quad \text{(A-22)}$$

Eq. (A-22) establishes the form of the inverse operator $\mathcal{L}_{12}^{-1}$.

Shift Theorem

Suppose that

$$\mathcal{L}_{12}\left[p(x_1,y_1)\right] = g(x_2,y_2) = \iint_{-\infty}^{+\infty} p(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1\, dy_1 . \quad \text{(A-23)}$$

Then if $\mathcal{L}_{12}$ operates on the translated function $\not{p}(x_1-a, y_1-b)$ we have

$$\mathcal{L}_{12}\left[\not{p}(x_1-a, y_1-b)\right] = \iint\limits_{-\infty}^{+\infty} \not{p}(x_1-a, y_1-b)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1\, dy_1 . \tag{A-24}$$

If we introduce new variables

$$\alpha = x_1 - a$$
$$\beta = y_1 - b ,$$

then we have

$$x_1 = \alpha + a$$
$$y_1 = \beta + b$$
$$dx_1 = d\alpha$$
$$dy_1 = d\beta ,$$

and Eq. (A-24) becomes

$$\mathcal{L}_{12}\left[\not{p}(x_1-a, y_1-b)\right] = \iint\limits_{-\infty}^{+\infty} \not{p}(\alpha, \beta)\, e^{-j\frac{2\pi}{\lambda f}\left[(\alpha+a)x_2+(\beta+b)y_2\right]}\, d\alpha\, d\beta$$

$$= e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)} \iint\limits_{-\infty}^{+\infty} \not{p}(\alpha, \beta)\, e^{-j\frac{2\pi}{\lambda f}(\alpha x_2+\beta y_2)}\, d\alpha\, d\beta . \tag{A-25}$$

Comparing Eqs. (A-25) and (A-23), we see that

$$\iint_{-\infty}^{+\infty} f(\alpha,\beta)\, e^{-j\frac{2\pi}{\lambda f}(\alpha x_2+\beta y_2)}\, d\alpha\, d\beta = g(x_2,y_2) = \mathcal{L}_{12}\left[f(x_1,y_1)\right],$$

since the name of the dummy variable of integration is irrelevant. Eq. (A-25) therefore becomes

$$\mathcal{L}_{12}\left[f(x_1-a,y_1-b)\right] = e^{-j\frac{2\pi}{\lambda f}(ax_2+by_2)}\, \mathcal{L}_{12}\left[f(x_1,y_1)\right]. \quad \text{(A-26)}$$

Convolution Theorem

Suppose that $\mathcal{L}_{12}\left[f(x_1,y_1)\right] = g(x_2,y_2)$ and $\mathcal{L}_{12}\left[h(x_1,y_1)\right] = s(x_2,y_2)$. Then consider what happens when $\mathcal{L}_{12}$ operates on the product function $f(x_1,y_1)\, h(x_1,y_1)$. By definition of the $\mathcal{L}_{12}$ operator in Eq. (A-16), we have

$$\mathcal{L}_{12}\left[f(x_1,y_1)h(x_1,y_1)\right] = \iint_{-\infty}^{+\infty} f(x_1,y_1)h(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1\, dy_1. \quad \text{(A-27)}$$

But from Eq. (A-22) we can write

$$h(x_1,y_1) = \mathcal{L}_{12}^{-1}\left[s(x_2,y_2)\right] = \frac{1}{(\lambda f)^2}\iint_{-\infty}^{+\infty} s(\alpha,\beta)\, e^{j\frac{2\pi}{\lambda f}(x_1\alpha+y_1\beta)}\, d\alpha\, d\beta, \quad \text{(A-28)}$$

where we have written the dummy variables of integration as α, β instead of x_2, y_2 to avoid confusion with the variables x_2, y_2 in Eq. (A-27). Putting Eq. (A-28) into Eq. (A-27) we have

$$\mathcal{L}_{12}\left[\mathfrak{p}(x_1,y_1)\,\underset{\sim}{\Omega}(x_1,y_1)\right] = \iint\limits_{-\infty}^{+\infty} \mathfrak{p}(x_1,y_1)\left[\frac{1}{(\lambda f)^2}\iint\limits_{-\infty}^{+\infty} \underset{\sim}{s}(\alpha,\beta)\, e^{j\frac{2\pi}{\lambda f}(x_1\alpha+y_1\beta)}\, d\alpha\, d\beta\right]$$

$$\times\; e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1\, dy_1 \,.$$

Interchanging the order of integration gives

$$\mathcal{L}_{12}\left[\mathfrak{p}(x_1,y_1)\,\underset{\sim}{\Omega}(x_1,y_1)\right] = \frac{1}{(\lambda f)^2}\iint\limits_{-\infty}^{+\infty} \underset{\sim}{s}(\alpha,\beta)\, d\alpha\, d\beta$$

$$\times \iint\limits_{-\infty}^{+\infty} \mathfrak{p}(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}\left[x_1(x_2-\alpha)+y_1(y_2-\beta)\right]}\, dx_1 dy_1 \,. \tag{A-29}$$

Comparing the final integration in Eq. (A-29) with Eq. (A-16) we see that

$$\iint\limits_{-\infty}^{+\infty} \mathfrak{p}(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}\left[x_1(x_2-\alpha)+y_1(y_2-\beta)\right]}\, dx_1 dy_1 = \mathfrak{P}(x_2-\alpha,\, y_2-\beta).$$

Putting this result into Eq. (A-29) then yields

$$\mathcal{L}_{12}\left[\mathfrak{p}(x_1,y_1)\,\underset{\sim}{\Omega}(x_1,y_1)\right] = \frac{1}{(\lambda f)^2}\iint\limits_{-\infty}^{+\infty} \underset{\sim}{s}(\alpha,\beta)\, \mathfrak{P}(x_2-\alpha,\, y_2-\beta)\, d\alpha\, d\beta. \tag{A-30}$$

Comparing Eq. (A-30) with Eq. (A-3) we observe that

$$\mathcal{L}_{12}\left[p(x_1,y_1)\,r(x_1,y_1)\right] = \frac{1}{(\lambda f)^2}\,\underline{s}(x_2,y_2) * g(x_2,y_2)$$

$$= \frac{1}{(\lambda f)^2}\,g(x_2,y_2) * \underline{s}(x_2,y_2), \quad \text{(A-31)}$$

where the interchange in order is legal because the convolution operation is commutative. But, by our original assumption,

$$g(x_2,y_2) = \mathcal{L}_{12}\left[p(x_1,y_1)\right]$$

and

$$\underline{s}(x_2,y_2) = \mathcal{L}_{12}\left[r(x_1,y_1)\right].$$

Eq. (A-31) can therefore be written in the form

$$\mathcal{L}_{12}\left[p(x_1,y_1)\,r(x_1,y_1)\right] = \frac{1}{(\lambda f)^2}\,\mathcal{L}_{12}\left[p(x_1,y_1)\right] * \mathcal{L}_{12}\left[r(x_1,y_1)\right]. \quad \text{(A-32)}$$

<u>Double-Transform Theorem</u>

Suppose again that $\mathcal{L}_{12}\left[p(x_1,y_1)\right] = g(x_2,y_2)$.

What is $\mathcal{L}_{23}\left[g(x_2,y_2)\right] = \mathcal{L}_{23}\left[\mathcal{L}_{12}\left[p(x_1,y_1)\right]\right]$?

From Eq. (A-16) we have

$$g(x_2,y_2) = \mathcal{L}_{12}\left[p(x_1,y_1)\right] = \iint_{-\infty}^{+\infty} p(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1dy_1, \tag{A-33}$$

and

$$\mathcal{L}_{23}\left[g(x_2,y_2)\right] = \mathcal{L}_{23}\left[\mathcal{L}_{12}\left[p(x_1,y_1)\right]\right] = \iint_{-\infty}^{+\infty} g(x_2,y_2)\, e^{-j\frac{2\pi}{\lambda f}(x_2x_3+y_2y_3)}\, dx_2dy_2. \tag{A-34}$$

Putting Eq. (A-34) into Eq. (A-33) then gives

$$\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[p(x_1,y_1)\right]\right] = \iint_{-\infty}^{+\infty} e^{-j\frac{2\pi}{\lambda f}(x_2x_3+y_2y_3)}\left[\iint_{-\infty}^{+\infty} p(x_1,y_1) \times e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1dy_1\right] dx_2dy_2.$$

Interchanging the order of integration gives

$$\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[p(x_1,y_1)\right]\right] = \iint_{-\infty}^{+\infty} p(x_1,y_1)\left[\iint_{-\infty}^{+\infty} e^{-j\frac{2\pi}{\lambda f}\left[x_2(x_1+x_3)+y_2(y_1+y_3)\right]} \times dx_2dy_2\right] dx_1dy_1. \tag{A-35}$$

Now upon comparing the inner integral in Eq. (A-35) with Eq. (A-11) we see that

$$\iint\limits_{-\infty}^{+\infty} e^{-j\frac{2\pi}{\lambda f}\left[x_2(x_1+x_3)+y_2(y_1+y_3)\right]} \, dx_2 \, dy_2$$

$$= \delta\left[\frac{x_1+x_3}{\lambda f}, \frac{y_1+y_3}{\lambda f}\right]$$

$$= (\lambda f)^2 \, \delta\left(x_1+x_3, y_1+y_3\right), \quad \text{(A-36)}$$

where the second step follows from the scaling property of the delta function expressed in Eq. (A-8). Putting Eq. (A-36) into Eq. (A-35) we obtain

$$\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[p(x_1,y_1)\right]\right] = (\lambda f)^2 \iint\limits_{-\infty}^{+\infty} p(x_1,y_1)\, \delta(x_1+x_3, y_1+y_3)\, dx_1 \, dy_1 .$$

Using the sifting property of the delta function in Eq. (A-10), we finally obtain

$$\mathcal{L}_{23}\left[\mathcal{L}_{12}\left[p(x_1,y_1)\right]\right] = (\lambda f)^2 \, p(-x_3,-y_3). \quad \text{(A-37)}$$

We will also have need for the double transform in a somewhat different case. In particular, if

$$g(x_2,y_2) = \mathcal{L}_{12}\left[p(x_1,y_1)\right],$$

we need to evaluate $\mathcal{L}_{23}\left[g^{*}(x_2,y_2)\right]$. To begin with, from Eq. (A-16) we have

$$g(x_2,y_2) = \iint\limits_{-\infty}^{+\infty} f(x_1,y_1)\, e^{-j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1 dy_1 . \quad \text{(A-38)}$$

It follows from Eq. (A-38) that

$$g^{*}(x_2,y_2) = \iint\limits_{-\infty}^{+\infty} f^{*}(x_1,y_1)\, e^{j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1 dy_1 . \quad \text{(A-39)}$$

From Eq. (A-16) we then have

$$\mathcal{L}_{23}\left[g^{*}(x_2,y_2)\right] = \iint\limits_{-\infty}^{+\infty} g^{*}(x_2,y_2)\, e^{-j\frac{2\pi}{\lambda f}(x_2x_3+y_2y_3)}\, dx_2 dy_2 . \quad \text{(A-40)}$$

Putting Eq. (A-39) into Eq. (A-40) gives

$$\mathcal{L}_{23}\left[g^{*}(x_2,y_2)\right] = \iint\limits_{-\infty}^{+\infty} e^{-j\frac{2\pi}{\lambda f}(x_2x_3+y_2y_3)} \left[\iint\limits_{-\infty}^{+\infty} f^{*}(x_1,y_1) \times e^{j\frac{2\pi}{\lambda f}(x_1x_2+y_1y_2)}\, dx_1 dy_1\right] dx_2 dy_2 .$$

Interchanging the order of integration gives

$$\mathcal{L}_{23}\left[g^{*}(x_2,y_2)\right] = \iint\limits_{-\infty}^{+\infty} f^{*}(x_1,y_1) \left[\iint\limits_{-\infty}^{+\infty} e^{-j\frac{2\pi}{\lambda f}\left[x_2(x_3-x_1)+y_2(y_3-y_1)\right]}\right.$$

$$\times \quad dx_2 dy_2 \Bigg] \, dx_1 dy_1 \ . \qquad \text{(A-41)}$$

By use of Eqs. (A-11) and (A-8) we then have

$$\iint_{-\infty}^{+\infty} e^{-j \frac{2\pi}{\lambda f}\left[x_2(x_3-x_1)+y_2(y_3-y_1)\right]} \, dx_2 \, dy_2$$

$$= \delta\left[\frac{x_3-x_1}{\lambda f}, \frac{y_3-y_1}{\lambda f}\right]$$

$$= (\lambda f)^2 \, \delta(x_3-x_1, y_3-y_1) \ . \qquad \text{(A-42)}$$

Putting Eq. (A-42) into Eq. (A-41) we obtain

$$\mathcal{L}_{23}\left[\mathcal{g}^*(x_2,y_2)\right] = (\lambda f)^2 \iint_{-\infty}^{+\infty} \mathcal{f}^*(x_1,y_1) \, \delta(x_3-x_1, y_3-y_1) \, dx_1 dy_1 \ .$$

Finally, use of Eq. (A-7) gives

$$\mathcal{L}_{23}\left[\mathcal{g}^*(x_2,y_2)\right] = (\lambda f)^2 \, \mathcal{f}^*(x_3,y_3) \, .$$

Going back to the definition of $\mathcal{g}(x_2,y_2)$, we can then write this result in the form

$$\mathcal{L}_{23}\left[\left\{\mathcal{L}_{12}\left[\mathcal{f}(x_1,y_1)\right]\right\}^*\right] = (\lambda f)^2 \, \mathcal{f}^*(x_3,y_3) \ . \qquad \text{(A-43)}$$

Effect on Complex Exponentials

Consider the operation $\mathcal{L}_{23}\left[e^{j\frac{2\pi c x_2}{\lambda}}\right]$, where c is a real constant. From Eq. (A-16) we have

$$\mathcal{L}_{23}\left[e^{j\frac{2\pi c x_2}{\lambda}}\right] = \iint_{-\infty}^{+\infty} e^{j\frac{2\pi c x_2}{\lambda}} e^{-j\frac{2\pi}{\lambda f}(x_2 x_3 + y_2 y_3)} dx_2 dy_2$$

$$= \iint_{-\infty}^{+\infty} e^{j\frac{2\pi}{\lambda f}\left[x_2(-x_3+fc) + y_2(-y_3)\right]} dx_2 dy_2 . \tag{A-44}$$

Comparing Eq. (A-44) with Eq. (A-11) we see that

$$\iint_{-\infty}^{+\infty} e^{j\frac{2\pi}{\lambda f}\left[x_2(-x_3+fc) + y_2(-y_3)\right]} dx_2 dy_2 = \delta\left[\frac{-x_3+fc}{\lambda f}, \frac{-y_3}{\lambda f}\right]$$

$$= (\lambda f)^2 \delta(-x_3+fc, -y_3), \tag{A-45}$$

where the last step follows from Eq. (A-8). Putting Eq. (A-45) into Eq. (A-44) gives

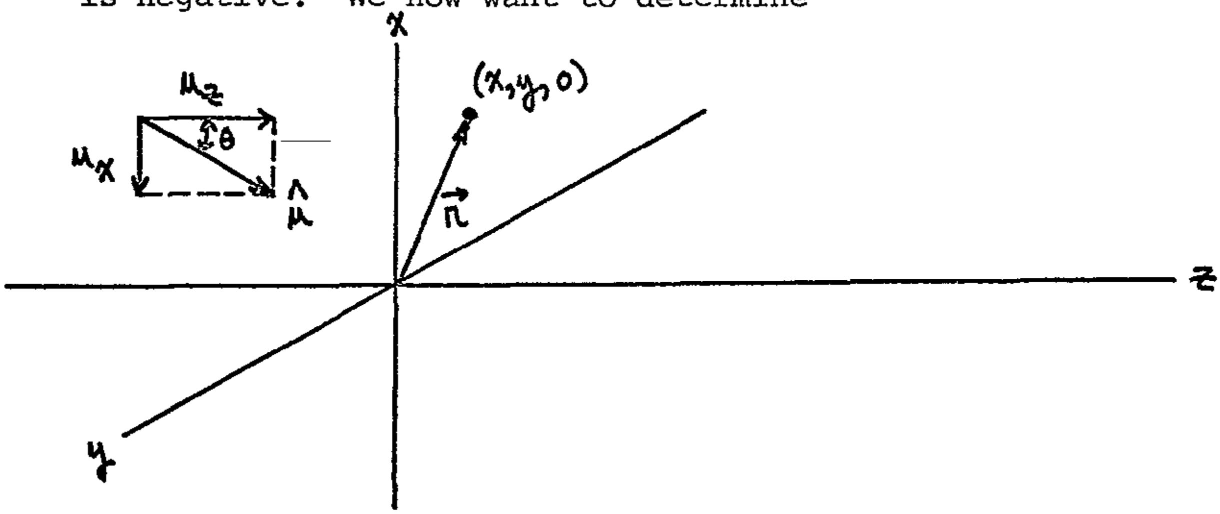
$$\mathcal{L}_{23}\left[e^{j\frac{2\pi c x_2}{\lambda}}\right] = (\lambda f)^2 \delta(-x_3+fc, -y_3). \tag{A-46}$$

To obtain the result for $\mathcal{L}_{23}$ operating on $e^{-j\frac{2\pi c x_2}{\lambda}}$ it is only necessary to change the sign of c on both sides of Eq. (A-46). We then have

$$\mathcal{L}_{23}\left[e^{-j\frac{2\pi c x_2}{\lambda}} \right] = (\lambda f)^2 \delta(-x_3 - fc, -y_3). \quad \text{(A-47)}$$

E. Some Properties of Plane Waves

Consider a plane wave propagating in a direction defined by a unit vector $\hat{u}$. We will take $\hat{u}$ to lie in the xz-plane, as shown below, and we will suppose that $\hat{u}$ makes an angle θ with the z-axis and that the x-component of $\hat{u}$ is negative. We now want to determine



the light amplitude distribution $\underline{a}(x,y)$ in the xy-plane due to the incident plane wave moving in the direction $\hat{u}$.

The equation of a general plane wave is

$$r_o e^{j \vec{k} \cdot \vec{r}}$$

where r_o is a constant amplitude and

$$\vec{k} = \frac{2\pi}{\lambda} \hat{u}$$

where λ is the wavelength of the light (assumed monochromatic). From the figure above it will be seen that

$$\hat{u} = (-\sin\theta, 0, \cos\theta)$$

$$\vec{r} = (x, y, 0),$$

so that for any point (x,y) in the xy-plane we have

$$\vec{k} \cdot \vec{r} = \frac{2\pi}{\lambda} \hat{u} \cdot \vec{r} = \frac{2\pi}{\lambda}(-x \sin\theta) = -\frac{2\pi x \sin\theta}{\lambda}.$$

The light amplitude distribution in the xy-plane due to the plane wave in question is thus

$$\underline{d}(x,y) = r_o e^{-j \frac{2\pi x \sin\theta}{\lambda}} . \qquad \text{(A-48)}$$

Conversely, if a plane wave sets up a light amplitude distribution $\underline{d}(x,y)$ in the xy-plane given by Eq. (A-48), it follows that the unit vector $\hat{u}$ which describes the direction of propagation of the plane wave makes an angle θ with the z-axis, lies in the xz-plane, and has a negative x-component.

Following through the same reasoning for the case where $\hat{u}$ lies in the xz-plane, makes an angle θ with the z-axis, but has a positive x-component, one finds the light amplitude distribution set up in the xy-plane to be

$$\underset{\sim}{d}(x,y) = r_0\, e^{j\frac{2\pi x \sin\theta}{\lambda}} . \tag{A-49}$$

Conversely, if a plane wave sets up a light amplitude distribution $\underset{\sim}{d}(x,y)$ in the xy-plane given by Eq. (A-49), it follows that the unit vector $\hat{u}$ describing the direction of propagation of the wave makes an angle θ with the z-axis, lies in the xz-plane, and has a positive x-component.

By way of generalization of these ideas, Smith[4] shows that if a light wave sets up a light amplitude distribution

$$\underset{\sim}{d}(x,y) = \underset{\sim}{g}(x,y)\, e^{-j\frac{2\pi x \sin\theta}{\lambda}} \tag{A-50}$$

in the xy-plane, where $\underset{\sim}{g}(x,y)$ is a general complex function, then the light wave causing this amplitude distribution is travelling generally in the same direction as the plane wave which caused the distribution in Eq. (A-48), except that the wavefronts in the case of Eq. (A-50) are not planes, but rather curved surfaces of some type. Similarly, if a light wave sets up a light amplitude distribution

$$\underline{d}(x,y) = \underline{g}(x,y)\, e^{j\frac{2\pi x \sin\theta}{\lambda}}$$

in the xy-plane, then this wave is travelling generally in the same direction as the plane wave which caused the distribution in Eq. (A-49), but the wavefronts are curved rather than planar.

F. <u>Evaluation of the Integral in Eq. (42)</u>

Let us denote the integral in Eq. (42) by $\underline{\gamma}$:

$$\underline{\gamma} = \iint_{-\infty}^{+\infty} e^{-j\frac{\pi\Delta z}{\lambda f^2}\left[\left(x_2+\frac{f}{\Delta z}x_3\right)^2+\left(y_2+\frac{f}{\Delta z}y_3\right)^2\right]}\, dx_2 dy_2 . \quad \text{(A-51)}$$

Let us also introduce a new constant

$$Q = \frac{\pi\Delta z}{\lambda f^2} \quad \text{(A-52)}$$

and new variables of integration

$$\alpha = x_2 + \frac{f}{\Delta z}x_3 \quad \text{(A-53)}$$

$$\beta = y_2 + \frac{f}{\Delta z}y_3 . \quad \text{(A-54)}$$

Since x_3 and y_3 are constants as far as the integral in Eq. (A-51) is concerned, we have from Eqs. (A-53) and (A-54) that

$$d\alpha = dx_2$$

$$d\beta = dy_2 .$$

Furthermore, the limits of integration in terms of α and β are the same as the limits of integration in terms of x_2 and y_2. Eq. (A-51) therefore becomes

$$\underline{\gamma} = \iint_{-\infty}^{+\infty} e^{-jQ(\alpha^2+\beta^2)}\, d\alpha\, d\beta$$

$$= \iint_{-\infty}^{+\infty} \left[\cos Q(\alpha^2+\beta^2) - j \sin Q(\alpha^2+\beta^2) \right] d\alpha\, d\beta$$

$$= \iint_{-\infty}^{+\infty} \cos Q(\alpha^2+\beta^2)\, d\alpha\, d\beta - j \iint_{-\infty}^{+\infty} \sin Q(\alpha^2+\beta^2)\, d\alpha\, d\beta$$

$$= \iint_{-\infty}^{+\infty} \left[\cos Q\alpha^2 \cos Q\beta^2 - \sin Q\alpha^2 \sin Q\beta^2 \right] d\alpha\, d\beta$$

$$- j \iint_{-\infty}^{+\infty} \left[\sin Q\alpha^2 \cos Q\beta^2 + \cos Q\alpha^2 \sin Q\beta^2 \right] d\alpha\, d\beta . \quad \text{(A-55)}$$

Now let us write, for convenience,

$$\underline{\gamma} = \gamma_1 - j\gamma_2 \,, \tag{A-56}$$

where

$$\gamma_1 = \iint\limits_{-\infty}^{+\infty} \left[\cos Q\alpha^2 \cos Q\beta^2 - \sin Q\alpha^2 \sin Q\beta^2\right] d\alpha\, d\beta \tag{A-57}$$

and

$$\gamma_2 = \iint\limits_{-\infty}^{+\infty} \left[\sin Q\alpha^2 \cos Q\beta^2 + \cos Q\alpha^2 \sin Q\beta^2\right] d\alpha\, d\beta . \tag{A-58}$$

We then have

$$\gamma_1 = \int_{-\infty}^{+\infty} d\beta \left[\cos Q\beta^2 \int_{-\infty}^{+\infty} \cos Q\alpha^2 d\alpha - \sin Q\beta^2 \int_{-\infty}^{+\infty} \sin Q\alpha^2 d\alpha\right] . \tag{A-59}$$

Now consider the integrals

$$I_1 = \int_{-\infty}^{+\infty} \cos Q\alpha^2\, d\alpha = 2\int_0^{+\infty} \cos Q\alpha^2\, d\alpha$$

and

$$I_2 = \int_{-\infty}^{+\infty} \sin Q\alpha^2\, d\alpha = 2\int_0^{+\infty} \sin Q\alpha^2\, d\alpha .$$

Introduce a new variable $\eta^2 = Q\alpha^2$. Then $\eta = \sqrt{Q}\,\alpha$, $d\alpha = \frac{1}{\sqrt{Q}}\, d\eta$, and we have

$$I_1 = \frac{2}{\sqrt{Q}} \int_0^{+\infty} \cos \eta^2 \, d\eta$$

$$I_2 = \frac{2}{\sqrt{Q}} \int_0^{+\infty} \sin \eta^2 \, d\eta .$$

But, we have[5]

$$\int_0^{+\infty} \cos \eta^2 \, d\eta = \int_0^{+\infty} \sin \eta^2 \, d\eta = \frac{1}{2}\sqrt{\frac{\pi}{2}} .$$

Thus

$$I_1 = \int_{-\infty}^{+\infty} \cos Q\alpha^2 \, d\alpha = I_2 = \int_{-\infty}^{+\infty} \sin Q\alpha^2 \, d\alpha = \sqrt{\frac{\pi}{2Q}} . \qquad \text{(A-60)}$$

Putting Eq. (A-60) into Eq. (A-59) we have

$$\gamma_1 = \int_{-\infty}^{+\infty} d\beta \left[\cos Q\beta^2 \cdot \sqrt{\frac{\pi}{2Q}} - \sin Q\beta^2 \cdot \sqrt{\frac{\pi}{2Q}} \right]$$

$$= \sqrt{\frac{\pi}{2Q}} \left\{ \int_{-\infty}^{+\infty} \cos Q\beta^2 \, d\beta - \int_{-\infty}^{+\infty} \sin Q\beta^2 \, d\beta \right\} .$$

Eq. (A-60) then gives

$$\gamma_1 = \sqrt{\frac{\pi}{2Q}} \left\{ \sqrt{\frac{\pi}{2Q}} - \sqrt{\frac{\pi}{2Q}} \right\} = 0. \tag{A-61}$$

Going back to Eq. (A-58), we have in a similar manner

$$\gamma_2 = \int_{-\infty}^{+\infty} d\beta \left[\cos Q\beta^2 \int_{-\infty}^{+\infty} \sin Q\alpha^2 \, d\alpha + \sin Q\beta^2 \int_{-\infty}^{+\infty} \cos Q\alpha^2 \, d\alpha \right].$$

Invoking Eq. (A-60) gives

$$\gamma_2 = \int_{-\infty}^{+\infty} d\beta \left[\cos Q\beta^2 \cdot \sqrt{\frac{\pi}{2Q}} + \sin Q\beta^2 \cdot \sqrt{\frac{\pi}{2Q}} \right]$$

$$= \sqrt{\frac{\pi}{2Q}} \left\{ \int_{-\infty}^{+\infty} \cos Q\beta^2 d\beta + \int_{-\infty}^{+\infty} \sin Q\beta^2 \, d\beta \right\}$$

$$= \sqrt{\frac{\pi}{2Q}} \left\{ \sqrt{\frac{\pi}{2Q}} + \sqrt{\frac{\pi}{2Q}} \right\} = \frac{\pi}{Q}. \tag{A-62}$$

Putting Eqs. (A-61) and (A-62) into Eq. (A-56) gives

$$\underset{\sim}{\gamma} = -j \frac{\pi}{Q}. \tag{A-63}$$

The result in Eq. (A-63) assumes that $Q>0$ since otherwise the change of variable $\eta^2 = Q\alpha^2$ would not be valid. From Eq. (A-52), in turn, this requires that $\Delta z > 0$.

If, on the other hand, $\Delta z < 0$, then $Q < 0$. Let us then define a positive real number Q' such that

$$Q' = -Q. \tag{A-64}$$

We would then have

$$\underset{\sim}{I} = \iint_{-\infty}^{+\infty} e^{jQ'(\alpha^2+\beta^2)} \, d\alpha \, d\beta ,$$

which following the same steps as before, would lead to the result

$$\underset{\sim}{\gamma} = j\frac{\pi}{Q'} .$$

But in view of Eq. (A-64), this becomes

$$\underset{\sim}{\gamma} = -j\frac{\pi}{Q} .$$

which is our previous result. In other words, Eq. (A-63) is valid whether $Q>0$ or $Q<0$. We thus have as our general result

$$\underset{\sim}{\gamma} = -j\frac{\pi}{Q} = -j\frac{\pi}{\left(\frac{\pi \Delta z}{\lambda f^2}\right)} = -j\frac{\lambda f^2}{\Delta z} .$$

Now $-j = e^{-j\frac{\pi}{2}}$ is a constant phase factor of no interest, so we will drop it and write

$$\underset{\sim}{\gamma} = \frac{\lambda f^2}{\Delta z} . \tag{A-65}$$

REFERENCES

1. J. W. Goodman, Introduction to Fourier Optics, McGraw-Hill Book Co., New York (1968), pp. 171-177.

2. A. Vander Lugt, "Signal Detection by Complex Spatial Filtering," IEEE Transactions on Information Theory, Vol. 10 (April, 1964), pp. 139-145.

3. H. M. Smith, Principles of Holography, Second Edition, John Wiley and Sons, New York (1969), pp. 242-246.

4. Reference 3, pp. 19-20. See especially Fig. 2.8, p. 20.

5. See, for example, Standard Mathematical Tables, 19th Edition, Chemical Rubber Co., Cleveland (1971), p. 447, #636.