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ALTKAL - AN OPTIMUM LINEAR FILTER FOR GEOS-3 ALTIMETER DATA

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16. Abstract

ALTKAL is a computer program designed to smooth sea surface height data obtained from the GEOS-3 altimeter, and to produce minimum variance estimates of sea surface height and sea surface slopes, along with their standard derivations. The program operates by processing the data through a Kalman filter in both the forward and backward directions, and optimally combining the results. The sea surface height signal is considered to have a geoid signal, modeled by a third order Gauss-Markov process, corrupted by additive white noise. The governing parameters for the signal and noise processes are the signal correlation length and the signal-to-noise ratio. Detailed mathematical derivations of the filtering and smoothing algorithms are presented. The smoother characteristics are illustrated by giving the frequency response, the data weighting sequence and the transfer function of a realistic steady-state smoother example. Based on nominal estimates for geoidal undulation amplitude and correlation length, standard deviations for the estimated sea surface height and slope are 12 cm and 3 arc seconds, respectively.

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INTRODUCTION

A filter is desired to separate the geoid undulation information from measurement noise in the GEOS-3 altimeter data. Obviously, for any separation to be possible at all, the geoid undulation and the noise must possess some distinct characteristics. The synthesis of the filter and the resulting filter performance depend on the available information about these characteristics. Frequently the availability and reliability of this information are the weakest link in the data filtering process.

In this analysis, it is assumed the geoid undulation and the altimeter noise are uncorrelated random processes with known second order statistical properties (i.e., means and correlations). The modeling of the signal (geoid undulation) as a random process is comparatively recent and its conceptual basis was clarified in the fundamental work of Shannon (Reference 1). The restriction to second order statistics arises out of necessity because of a lack of knowledge of the underlying physical processes. It is expected the analysis of GEOS-3 altimeter data will yield improved information about the spatial auto-correlation of the geoid undulation. Before such analysis has been completed, however, the TASC model (Reference 2) seems to be the best available, and will be used. The altimeter noise shall be assumed correlated exponentially in time. At low altimeter sampling rates when the sampling period is long in comparison with the noise correlation time, the noise may be considered uncorrelated from sample to sample.

As long as higher order statistical information is not available, an optimum linear filter may be considered the optimum filter. The proposed filter will consist of a set of computational algorithms easily implemented on a digital computer. The derivation of the optimum filter algorithms is given in subsequent sections. A realistic numerical example showing the convergence characteristics of the filter and accuracies to be expected from the filtering operation is given in Section 5.2.

SECTION 2.0
THE OPTIMUM LINEAR FILTER

The GEOS-3 altimeter measurements, after the orbit information and other known corrections have been subtracted, may be written as

$$y_i = h_i + n_i \qquad (i=0,1,2,3...N-1) \tag{2.1}$$

where

h = geoid undulation

n = measurement noise

i = index for measurement time.

The objective of the filter is to determine an optimum estimate of the geoid undulation $h^*(t)$ from the N measurements $y_i (i=0,1,2,3,...N-1)$. A general linear estimate may be written as

$$\hat{h}(t) = \sum_{i=0}^{N-1} \omega_i(t)\, y_i \tag{2.2}$$

with an estimate error

$$\hat{e}(t) \triangleq \hat{h}(t) - h(t) = \sum_{i=0}^{N-1} \omega_i(t)\, y_i - h(t) \tag{2.3}$$

The weighting factors $\omega_i(t)$ are the filter coefficients which define the filter. The optimum filter is obtained if one may impose the following condition on the weighting factors

$$E(\hat{e}(t)y_\ell) = \sum_{i=0}^{N-1} \omega_i(t)\, E(y_i y_\ell) - E(h(t)y_\ell) = 0 \tag{2.4}$$

$(\ell = 0, 1, 2 \ldots N-1)$

Equation (2.4) states there exists no correlation between the estimate error and the measurements, and therefore there is no way (within second order statistics) of extracting more information out of the measurements. Since the measurement autocorrelation matrix $E(y_i y_\ell)$ may generally be taken to be positive definite, Equation (2.4) may be inverted to obtain the optimum weighting factors as,

$$\underset{N\times 1}{[\omega^*(t)]} = \underset{N\times N}{E([y][y]^T)^{-1}}\ \underset{N\times 1}{E(h(t)[y])} \tag{2.5}$$

where

$$[\omega^*(t)]^T = [\omega_0^* \quad \omega_1^* \quad \omega_2^* \ldots \omega_{N-1}^*]$$

$$[y]^T = [y_0 \quad y_1 \quad y_2 \ldots y_{N-1}]$$

The meaning of the weighting factor as given by Equation (2.5) is obvious and is intuitively reasonable. Notice that these optimum weighting factors depend only on the correlation functions and do not presume the existence of a functional relation between the signal and the measurement such as Equation (2.1).

Under the assumption that the geoid undulation and the altimeter noise are uncorrelated, one has the further simplification

$$E([y][y]^T) = E([h][h]^T) + E([n][n]^T) \quad (2.6)$$

$$E(h(t)[y]) = E(h(t)[h]) \quad (2.7)$$

$$[h]^T = [h_0 \ h_1 \ h_2 \ \ldots \ h_{N-1}]$$

Equations (2.2), (2.5), (2.6) and (2.7) show that formally the optimum filter can be constructed when the auto-correlation function of the geoid undulation and the altimeter noise are given. However, the algorithm given by these equations as they stand involves a great deal of computation and is impractical for processing a large amount of data*. To overcome this difficulty one needs to postulate simple structures for the underlying signal and noise processes. Since all one knows about these processes are second order statistics, it is only important that the postulated processes have the same second order statistics. Wiener (Reference 3) and Kalman (Reference 4) showed us that if such processes may be generated as the outputs of finite-dimensional linear dynamic systems (e.g., ordinary differential or difference equations or the corresponding transfer functions) driven by white noises, computationally efficient algorithms may be readily devised to implement the optimum filter. In other words, one looks for finite-order Markov process representation of the signal and the noise. Being Markovian means only a small number of parameters need be kept, resulting in vastly reduced computational and memory requirements.

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*If the correlation functions are stationary, the Levinson recursion may be used to lessen the computational requirement (Ref. 3).

From the above discussion one sees that given the geoid undulation correlation function, the key to the construction of the optimum filter for the GEOS-3 altimeter data is the modeling of the geoid undulation as the output of a linear ordinary differential equation system driven by white noise.

SECTION 3.0
MARKOV MODELS FOR GEOID UNDULATION AND ALTIMETER NOISE

3.1 WHITE-NOISE-DRIVEN DIFFERENTIAL EQUATION MODEL FOR TASC GEOID UNDULATION CORRELATION FUNCTION

The Analytical Sciences Corporation (TASC) introduced the following model for the correlation of the geoid undulation with surface distance (Ref. 2).

$$E(h(\ell+d)\ h(d)) = \sigma^2(1+\lambda\ell+ \frac{\lambda^2}{3}\ell^2)\ e^{-\lambda\ell} \qquad (3.1)$$

$$(\ell\geq 0)$$

Since GEOS-3 is in a circular orbit with "constant" ground speed, one may convert this to a correlation in time as

$$E(h(\tau+t)\ h(t)) = \sigma^2(1+\beta\tau+ \frac{\beta^2}{3}\tau^2)\ e^{-\beta\tau} \qquad (3.2)$$

$$(\tau\geq 0)$$

First of all, notice this correlation function is stationary; i.e., it depends on the time separation τ only. This means the correlation function may be the output of a differential equation with constant coefficients. Secondly, the correlation function depends on three linearly independent functions of time, implying it needs a 3rd order system to describe it. Thirdly, the forms of these linearly independent functions $e^{-\beta\tau}$, $\tau e^{-\beta\tau}$ and $\tau^2 e^{-\beta\tau}$ are reminiscent of a linear differential equation with triple roots $-\beta$, or the geoid undulation may possibly be modeled by the system

$$\left(\frac{d}{dt} + \beta\right)^3 h = w \tag{3.3}$$

where w is a zero-mean white noise with correlation function

$$E(w(t)w(\tau)) = q^2\delta(t-\tau) \tag{3.4}$$

It will be shown that indeed the above statement is correct. With knowledge of the differential Equation (3.3) or the corresponding transfer function

$$H(s) = 1/(s+\beta)^3$$

the classical procedure would be to design a Wiener Filter implemented by hardware. The Wiener Filter tacitly assumes the filtering process has reached a "statistical steady state" and may involve some errors for short GEOS-3 passes. Since the present filter is to be implemented on a digital computer, no such sacrifices are necessary and we shall adopt the Kalman formulation in state space. To proceed, let us define a convenient set* of state variables as

*Different choices may be made.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (\frac{d}{dt} + \beta)^2 \; h \\ (\frac{d}{dt} + \beta) \; h \\ h \end{bmatrix} \tag{3.5}$$

which satisfy the state equation

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\beta & 0 & 0 \\ 1 & -\beta & 0 \\ 0 & 1 & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w \tag{3.6}$$

It may be shown by substitution that these state variables have the state transition matrix

$$[\phi(t,0)] = e^{-\beta t} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{t^2}{2} & t & 1 \end{bmatrix} \tag{3.7}$$

The general solution of Equation (3.6) is

$$[x(t)] = [\phi(t,0)][x(0)] + \int_0^t e^{-\beta(t-\tau)} \begin{bmatrix} 1 \\ t-\tau \\ \frac{1}{2}(t-\tau)^2 \end{bmatrix} w(\tau)\,d\tau \tag{3.8}$$

From Equation (3.8) and the fact that future input is uncorrelated with past state, one obtains the correlation matrix

$$E([x(t)][x(0)]^T) = [\phi(t,0)] \; E([x(0)][x(0)]^T)$$

In particular,

$$E(x_3(t)x_3(0)) = e^{-\beta t}\{E(x_3(0)^2)+t\,E(x_2(0)x_3(0)) + \frac{t^2}{2}E(x_1(0)x_3(0))\} \tag{3.9}$$

Since the above expression is nothing but the geoid undulation correlation function, by comparing it with Equation (3.2), one must have

$$\begin{aligned} E(x_3(0)^2) &= \sigma^2 \\ E(x_1(0)x_3(0)) &= \frac{2\beta^2}{3}\sigma^2 \\ E(x_2(0)x_3(0)) &= \beta\sigma^2 \end{aligned} \tag{3.10}$$

Alternatively, one may obtain the covariance matrix of the state variables from Equation (3.8) as

$$E([x(t)][x(t)]^T) = [\phi(t,0)]\; E([x(0)][x(0)]^T)[\phi(t,0)]^T + [Q(t,0)] \tag{3.11}$$

where

$$[Q(t,0)] \triangleq \begin{bmatrix} q_{11} & \text{symmetric} & \\ q_{12} & q_{22} & \\ q_{13} & q_{23} & q_{33} \end{bmatrix} = q^2\int_0^t e^{-2\beta(t-\tau)} \begin{bmatrix} 1 \\ t-\tau \\ \frac{1}{2}(t-\tau)^2 \end{bmatrix} [1 \mid t-\tau \mid \frac{(t-\tau)^2}{2}]\, d\tau \tag{3.12}$$

is the increment in the covariance matrix due to white-noise excitation. The evaluation of the integral gives us the elements of [Q] as

$$
\begin{aligned}
q_{11} &= \gamma(1-e^{-2\beta t}) \\
q_{12} &= \alpha q_{11}-\gamma t e^{-2\beta t} \\
q_{13} &= \alpha q_{12}-0.5\ \gamma t^2 e^{-2\beta t} \\
q_{22} &= 2q_{13} \\
q_{23} &= 3\alpha q_{13}-0.5\ \gamma t^3 e^{-2\beta t} \\
q_{33} &= 2\alpha q_{23}-0.25\ \gamma t^4 e^{-2\beta t} \\
&(\gamma = q^2/2\beta\ ,\quad \alpha = 1/2\beta)
\end{aligned}
\tag{3.13}
$$

which consist of a constant part and a damped transient part. For the system to be stationary, the white noise excited time-dependent transient part must be balanced by the free transients of the system. It is straightforward to show from Equation (3.11) that this is indeed so, provided $q^2 = \frac{16}{3}\sigma^2\beta^5$, and then one has,

$$
E([x(t)][x(t)]^T) = \lim_{t\to\infty} [Q(t,0)] \tag{3.14}
$$

Notice this result is in agreement with those in Equation (3.10).

It has been shown that the state variable x_3 of the linear system Equations (3.5) and (3.6), with the a priori covariance, Equation (3.14), models the geoid undulation with the correlation function, Equation (3.2). One may give the following loose but physically meaningful interpretation to the state variables defined in Equation (3.5), and to Equation (3.6) governing these variables. x_3, x_2 and x_1 are related to the geoid undulation, its "slope" and "curvature", respectively. Equation (3.6) says the rate of change of the curvature of geoid undulation, although having similar mean square values, is uncorrelated from one location to another.

3.2 ALTIMETER NOISE MODEL

Two models of the altimeter noise sequence n_i are considered. For sampling rates at 10 samples/sec or lower, the altimeter noise will be considered uncorrelated from sample to sample; i.e.,

$$E(n_i n_j) = R_i \delta_{ij} = \begin{cases} 0 & j \neq i \\ R_i & j = i \end{cases} \tag{3.15}$$

For higher sampling rates the noise sequence will be considered to be correlated exponentially with some correlation time 1/b; i.e.,

$$E(n_i n_j) = R_i e^{-b(t_i - t_j)} \qquad (i \geq j) \tag{3.16}$$

The exponentially correlated noise may be modeled, similar to x_1 in the preceding section, by the following Markov sequence,

$$\begin{aligned} n_i &= e^{-b(t_i - t_{i-1})} n_{i-1} + u_{i-1} \\ E(u_\ell u_k) &= R_\ell (1 - e^{-2b(t_\ell - t_k)}) \\ &\quad \ell \geq k \end{aligned} \tag{3.17}$$

When the correlation time approaches zero; i.e., $b \to \infty$, the exponentially correlated noise approaches the white noise.

SECTION 4.0

OPTIMUM ZERO-LAG RECURSIVE FILTER (KALMAN) AT LOW ALTIMETER SAMPLING RATES (10 samples/sec)

The measurements, the state variables governing the geoid undulation and the altimeter noise are described by equations in the preceding sections in the standard forms for an immediate construction of the celebrated Kalman filter (Reference 4). When no time lag exists; i.e., estimates of geoid undulation at any time t are to be based on past measurements only, this filter is a recursive realization of the optimum filter, Equation (2.5). When all data are considered, the zero-lag filter is not optimum because it does not make use of later measurements. However, it is known that an optimum "smoother", making use of all the data, may be constructed by properly combining a forward and backward pass of the data through a zero-lag filter (Reference 5), and this will be discussed in Section 5.

A simple derivation of the zero-lag recursive filter for altimeter data is given in the following for the benefit of those who may not be familiar with Kalman filter and for the physical insight the derivation provides. In the derivation we shall make constant use of the following:

1. White noise is "purely random", or unpredictable.

2. The optimum estimator, Equation (2.5), is reduced to the following for a single measurement,

   - optimum estimate of signal

$$= \frac{\text{Cross-correlation of signal and measurement}}{\text{Auto-correlation of measurement}} * (\text{measurement}) \tag{4.0}$$

4.1 A PRIORI ESTIMATE, INTERPOLATION BETWEEN AND EXTRAPOLATION BEYOND MEASUREMENTS

Prior to any measurements, the geoid undulation and the other related state variables are considered zero-mean random processes with covariance matrix Equation (3.14). Therefore one has

$$[\hat{x}(0^-)] = [0] \tag{4.1}$$

with

$$E([\hat{x}(0^-)-x(0)][\hat{x}(0^-)-x(0)]^T)=[P(0^-)] = \gamma \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 2\alpha^2 & 3\alpha^3 \\ \alpha^2 & 3\alpha^3 & 6\alpha^4 \end{bmatrix} \tag{4.2}$$

The geoid undulation state at two time instants, t and $t_i^+ \leq t$ are related by

$$[x(t)] = [\phi(t,t_i)][x(t_i^+)] + \int_{t_i}^{t} e^{-\beta(t-\tau)} \begin{bmatrix} 1 \\ t-\tau \\ \frac{1}{2}(t-\tau)^2 \end{bmatrix} w(\tau)\, d\tau \tag{4.3}$$

Since $w(\tau)$ is a white noise, past measurements convey no information about it. Therefore, if no measurement is made between t_i^+ and t, given an optimum estimate $[\hat{x}(t_i^+)]$, the best estimate one can make about $[x(t)]$ is

$$[\hat{x}(t)] = [\phi(t,t_i)]\ [\hat{x}(t_i^+)] \tag{4.4}$$

$$(t_{i-1} > t > t_i)$$

The natural "damping" of the system tends to decrease the uncertainty in this estimate over that of the past estimate $[\hat{x}(t_i^+)]$. However, the unknown white-noise excitation over this time period degrades the confidence one has in this new estimate. These two effects are uncorrelated and additive, resulting in a covariance matrix of the new estimate error given by

$$[P(t)]=E(([\hat{x}(t)]-[x(t)])([\hat{x}(t)]-[x(t)])^T)$$

$$=[\phi(t,t_i)]E([\hat{x}(t_i^+)-x(t)][\hat{x}(t_i^+)-x(t)]^T)[\phi(t,t_i)]^T+[Q(t,t_i)]$$

$$\triangleq[\phi(t,t_i)][P(t_i^+)][\phi(t,t_i)]^T+[Q(t,t_i)] \tag{4.5}$$

where the elements of $[\phi(t,t_i)]$ and $[Q(t,t_i)]$ are given by Equations (3.7) and (3.13) with the quantity t there replaced by $t-t_i$.

4.2 IMPROVEMENT IN ESTIMATE RESULTING FROM A MEASUREMENT

Consider an a priori unbiased estimate $[\hat{x}(t_j^-)]$, together with its error covariance $[P(t_j^-)]=E([x(t_j^-)-x(t_j)][x(t_j^-)-x(t_j)]^T)$ are available prior to the next measurement. An altimeter measurement

$$\begin{aligned} y_j &= x_3(t_j)+n_j \\ E(n_j) &= 0 \ , \quad E(n_j^2) = R_j \end{aligned} \qquad (4.6)$$

is now made. Consider first, the altimeter is operating at low sampling rates. The measurement noise n may then be considered uncorrelated with past measurement noises and therefore uncorrelated with the a priori estimate error. One obtains immediately an improved a posteriori estimate of the geoid undulation $x_3(t_j)$, based on the measurement y_j and the a priori estimate $\hat{x}_3(t_j^-)$, as

$$\hat{x}_3(t_j^+) = \hat{x}_3(t_j^-) + \frac{P_{33}(t_j^-)}{P_{33}(t_j^-)+R_j}(y_j-\hat{x}_3(t_j^-)) \qquad (4.7)$$

Equation (4.7) follows from an application of Equation (4.0) and may also be considered the result of optimally combining two pieces of information (a priori and measurement) with uncorrelated errors. One may also rewrite this equation as,

$$\hat{x}_3(t_j^+)-\hat{x}_3(t_j^-) \triangleq \text{"estimated signal"} = \frac{(S/N)}{(S/N)+1}(y_j-\hat{x}_3(t_j^-))$$

$$\triangleq \frac{(S/N)}{(S/N)+1}(\text{"received signal"})$$

where $(S/N) \triangleq P_{33}(t_j^-)/R_j \triangleq$ a priori signal-to-noise power ratio. Knowledge of an improved $\hat{x}_3(t_j^+)$ also enables us to obtain improved estimates of $x_1(t_j)$ and $x_2(t_j)$ based on the a priori correlation $P_{13}(t_j^-)$ and $P_{23}(t_j^-)$. Another application of Equation (4.0) with $\hat{x}_3(t_j^+)-\hat{x}_3(t_j^-)$ playing the role of known quantity gives the a posteriori estimates

$$\hat{x}_1(t_j^+)-\hat{x}_1(t_j^-) = \frac{P_{13}(t_j^-)}{P_{33}(t_j^-)+R_j} (y_j-\hat{x}_3(t_j)) \tag{4.8}$$

$$\hat{x}_2(t_j^+)-\hat{x}_2(t_j^-) = \frac{P_{23}(t_j^-)}{P_{33}(t_j^-)+R_j} (y_j-\hat{x}_3(t_j^-)) \tag{4.9}$$

Equations (4.7), (4.8) and (4.9) for the a posteriori estimate may be rewritten in the following compact matrix form

$$[\hat{x}(t_j^+)] = [\hat{x}(t_j^-)] + [K(t_j)] (y_j-\hat{x}_3(t_j^-)) \tag{4.10}$$

$$[K(t_j)] \triangleq \frac{1}{P_{33}(t_j^-)+R_j} \begin{bmatrix} P_{13}(t_j^-) \\ P_{23}(t_j^-) \\ P_{33}(t_j^-) \end{bmatrix} \triangleq \text{optimum gain} \tag{4.11}$$

The corresponding estimate errors may be written as

$$[\hat{x}(t_j^+)-x(t_j)] = [\hat{x}(t_j^-)-x(t_j)] + [K(t_j)] \{n_j-(\hat{x}_3(t_j^-)-x_3(t_j))\}.$$

From this expression and noting that present measurement errors are uncorrelated with past estimates, one obtains the a posteriori estimate error covariance matrix

$$[P(t_j^+)] \overset{\Delta}{=} E([\hat{x}(t_j^+)-x(t_j)][\hat{x}(t_j^+)-x(t_j)]^T)$$

$$= [P(t_j^-)] - [K(t_j)] \; [P_{13}(t_j^-) \vdots P_{23}(t_j^-) \vdots P_{33}(t_j^-)] \quad (4.12)$$

The optimum zero-lag filter may now be summarized as follows:

1. Equations (4.4) and (4.10) are the recursive filter equations.

2. Equations (4.5) and (4.12) are the recursive equation for the estimate error covariance.

3. Equations (4.1) and (4.2) are initial conditions for starting these recursive equations.

4. Equations (4.10) and (4.12) govern the interpolation between and extrapolation beyond measurements. If interpolation and extrapolation are not of interest, one uses $t=t_{i+1}^-$ in these equations to obtain a priori estimates just prior to the next measurements.

5. In Equation (4.10) the difference between the new measurement and the a priori estimate is multiplied by a gain to obtain an optimum correction to the

a priori estimate. The optimum gain is computed from Equation (4.11), which shows its dependence on the a priori error covariance and the measurement noise level. Equation (4.12) describes the decrease in the estimate error covariance as a result of measurement.

SECTION 5.0
FIXED-INTERVAL FILTER

5.1 OPTIMUM FIXED-INTERVAL FILTER* (SMOOTHER)

The optimum fixed interval filter, Equation (2.5), makes use of all the altimeter data y_i, $i=0,1,\ldots N-1$. A zero-lag filter, on the other hand, uses only part of the data y_j, $j=0,1,\ldots q$, ($t_q \le t$=filter output time). However, once the optimum recursive zero-lag filter is constructed, an optimum smoother realizing Equation (2.5) may be obtained by properly combining a forward and a backward pass of the data through the zero-lag filter. Although this forward-backward-filter formulation of the optimum smoother is relatively recent (Reference 5), the underlying idea is deceptively simple. The altimeter data $y_0, y_1, \ldots y_{N-1}$ are first fed through the zero-lag optimal filter described in Section 4. The output of the filter gives us an estimate $[\hat{x}(t)]$ together with the error covariance $[p(t)]$. Since the geoid undulation is considered a stationary random process, which terminal of the data pass is the beginning and which in the end really does not matter. Therefore, one could feed the altimeter data in reverse order; i.e., $y_{N-1}, y_{N-2}, \ldots y_0$ through the same filter† and obtain an estimate $[\hat{x}_b(\tau)]$ and error covariance $[p_b(\tau)]$. At any time $t=t_q$ there exists two estimates, $[\hat{x}(t_q^+)]$ and $[\hat{x}_b(\tau^- = T_{N-1} - t_q^-)]$ obtained by optimally processing altimeter data y_i, $i=0,1,\ldots q$ and y_j, $j=N-1, N-2,\ldots q+1$, respectively. Since these estimates are based on different data which are corrupted only by unpredictable white noises, the estimate errors are uncorrelated. Therefore the over-all optimum estimate based on all the data may be obtained by optimally combining these two uncorrelated estimates.

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*"Fixed-interval" refers to the fact that estimates of interest are to be derived from a common data set spanning a time interval of fixed duration. For conciseness, the terminology "smoother" will be used interchangeably.

†For non-stationary processes, a minor modification of the filter, reflecting the backward propagation of dynamics, is needed.

A word of caution is needed in combining the forward and backward filtered estimates because the forward and the backward state variables have different physical meanings. Although $\hat{x}_{3a}$ and $\hat{x}_{3b}$ represent estimates of the geoid undulations h, it is evident from Eq. (3.5) defining the state variables that $\hat{x}_{2a}$ is an estimate of $(\frac{d}{dt}+\beta)h$ while $\hat{x}_{2b}$ is an estimate of $(-\frac{d}{dt}+\beta)h$, if t is reckoned as the forward time. It is straight forward to show that the transformation

$$[x_b^*] = [H][\hat{x}_b]$$

$$[p_b^*] = [H][p_b][H]^T$$

with

$$[H] = \begin{bmatrix} 1 & -4\beta & 4\beta^2 \\ 0 & -1 & 2\beta \\ 0 & 0 & 1 \end{bmatrix}$$

gives the backward estimates $[x_b^*]$ and covariance $[p_b^*]$ which represent the same quantities as the forward filter output. From the result given in the Appendix one obtains immediately the smoothed estimate by optimally combining the forward and backward estimates as

$$[\hat{x}(t)] = [p(t)]_s \left([p(t^+)]^{-1}[\hat{x}(t^+)]+[p_b^*(\tau^-)]^{-1}[x_b^*(\tau^-)]\right) ,$$

$$(\tau^- = T_{N-1}-t^-) \qquad (5.1)^*$$

with an error covariance

$$[p(t)]_s = \left([p(t^+)]^{-1} + [p_b^*(\tau^-)]^{-1}\right)^{-1} \qquad (5.2)^*$$

Other theoretically equivalent but computationally more efficient algorithms are given in the Appendix. Notice the smoother is symmetrical with respect to forward and backward filtering. It is immaterial whether forward or backward filtering is done first. If backward filtering is done first, the filter output $[\hat{x}_b(\tau^+)]$ and $[p_b(\tau^+)]$ will be obtained to be stored in reverse order (normal order in t). The quantities $[\hat{x}(t^-)]$ and $[p(t^-)]$ are then computed "on-line" as altimeter data in normal order are fed-through, and these are combined "on-line" with $[\hat{x}_b(t)]$ and $[p_b(t)]$ retrieved from storage to obtain the optimum smoothed estimate according to Equations (5.1) and (5.2). The major draw-back of this smoother is the complexity of data handling required; i.e.,

1. Reversing the ordering of altimeter data.

2. Reversing the order of output of backward filter

   $\underset{(3x1)}{[x_b^*(\tau_i^-)]}$ and $\underset{(3x3,\ symmetric)}{[p_b^*(\tau_i^-)]}$ and storing them for ready

   retrieval later.

---

*In these formulas, the forward filtered estimate is combined with the backward one-step prediction in order to obtain the smoothed estimate. One may, of course, reverse the roles and combine the forward prediction with the backward filtering.

If one can tolerate the appearance of smoothed geoid undulations in reverse order, an alternative recursive smoother (Reference 6) may be used which operates on the outputs of the forward filter in a backward recursion. This eliminates the need for step "1" above, although operations equivalent to step "2" above is still required.

5.2 A PRIORI ACCURACY ASSESSMENTS AND A SUB-OPTIMAL SMOOTHER

The covariance matrix gives us some information about the accuracy of the geoid undulation estimate. The equations governing the evolution of the covariance matrix in the preceding sections show that the covariance matrix depends on our model of the signal and noise process, on measurement frequencies, etc., but does not depend on the actual altimeter data. This means that if our model is reasonably accurate, the accuracy of the geoid undulation estimate may be assessed beforehand by solving the variance equations. The results form a standard by which one may measure the performance of simpler suboptimal smoothers. Also of interest are the convergence characteristics of the variance equation.

The optimum smoother, Equations (5.1) and (5.2), requires the input of the filtered state and covariance obtained during the first filter pass. In addition to the storage of the filtered outputs for retrieval in reverse order, inversions of the covariance matrices at each measurement time are required. Since the dimension of the geoid undulation state is 3, this portion of the smoother alone requires the reverse storage of (3 state variables + 6 independent elements of covariance matrix) * (N measurements) and the inversion of a 3x3 covariance matrix N times.† In order to reduce the computational and storage requirements, the following sub-optimal

† Equations (5.1) and (5.2) as they stand require 3N inversions. Other algorithms given in the Appendix show this may be reduced to N inversions.

smoother may be suggested. The forward and backward filtering per se do not involve particular difficulties and will be implemented. The optimal processing of the two filtered estimates to obtain the optimum smoothed estimate as described by Equations (5.1) and (5.2) will be replaced by their scalar versions involving the geoid undulation only; i.e., replacing $[\hat{x}]$ by $\hat{x}_3$ and [p] by P_{33} in these equations. It is expected that this sub-optimal estimate will be reasonably good for the following reason. Although there is a degradation in the estimate $(\hat{x}_3(t))_s$ resulting from not utilizing information about the other state variables, yet this degradation does not propagate in time because $(\hat{x}_3(\sigma))_s$ for $\sigma \neq t$ does not depend on $(\hat{x}_3(t))_s$. In any case, the following procedure may be suggested for assessing the quality of this sub-optimal smoother. The estimate covariance at a few selected instants during the data pass may be computed from Equation (5.2). Since it is only for a few points, the computational and storage requirements are minimal. These minimum geoid undulation variances may be compared with those obtained from the sub-optimal filter. In particular, one should expect the best accuracy exists near the middle of the data pass and the minimum variance obtained there provides a lower bound for the variance, or, an indication of the achievable accuracy.

To illustrate the ideas discussed above, let us consider a typical GEOS-III pass with altimeter measurements sampled at a nominal rate of 10 measurements per second (sampling interval = 0.102406 sec.). The geoid undulation is considered to have an a priori variance of 4 meters^2 and a correlation length of 50 Km. (β=0.3805/sec.). The altimeter noise is assumed uncorrelated from sample to sample and to have a variance of 0.36 meter^2, corresponding to an input signal to noise power ratio of $4/0.36 \approx 10$. Solutions to the variance equations give the following results:

1. Forward filtered error covariance matrix reaches a steady-state value of

$$\begin{bmatrix} 0.1528646 & & \\ 0.9721489\times10^{-1} & 0.1036214 & \\ 0.4034996\times10^{-1} & 0.6118031\times10^{-1} & 0.5292089\times10^{-1} \end{bmatrix}$$

2. Backward filtered one-step prediction error covariance matrix (untransformed) reaches a steady-state value of

$$\begin{bmatrix} 0.1581666 & & \\ 0.1052539 & 0.1158105 & \\ 0.4730372\times10^{-1} & 0.7172390\times10^{-1} & 0.6204108\times10^{-1} \end{bmatrix}$$

3. Forward-backward smoothed error covariance matrix reaches a steady-state value of

$$\begin{bmatrix} 0.3134349\times10^{-1} & & \\ 0.6365069\times10^{-2} & 0.1338882\times10^{-1} & \\ -0.6004677\times10^{-2} & 0.5315084\times10^{-2} & 0.1396868\times10^{-1} \end{bmatrix}$$

4. One-way forward filter gain reaches a steady-state value of

$$\begin{bmatrix} 0.1120832 \\ 0.1699453 \\ 0.1470024 \end{bmatrix}$$

5. In the steady-state, the contributions of the forward filtered and backward one-step predicted estimates to the smoothed estimates may be seen by the respective weighting matrices, i.e.,

$$\text{Forward Weighting Matrix} \quad [W]_F = \begin{bmatrix} 0.3486159 & -0.1313888 & -0.2273753 \\ -0.1989280 & 0.5260603 & -0.3560544 \\ 0.2653295\text{x}10^{-7} & -0.3293662 & 0.6447248 \end{bmatrix}$$

$$\text{Backward Weighting Matrix} \quad [W]_B = \begin{bmatrix} 0.6513841 & -0.1122795\text{x}10 & 0.7045924 \\ 0.1989280 & -0.7767080 & 0.8319259 \\ -0.2653295\text{x}10^{-7} & -0.3293661 & 0.6059229 \end{bmatrix}$$

Note $[W]_B$ is the weighting matrix for the untransformed backward estimate; i.e., $[\hat{x}_b]$ in Section 5.1. The weighting matrix for the transformed backward estimate $[x_b^*]$ is, of course, the difference between the identity matrix and $[W]_F$. It may be seen from these weighting matrices that the direct contributions of the estimates of x_1 to the smoothed value of the geoid undulation are insignificant.

6. The convergence characteristics of the one-way filter are described by the graphs in Figure 5.1. It is seen that the filter pretty much reaches the steady state after 30 measurements (3 seconds).

7. For the sub-optimal smoother described previously, the standard deviation of the geoid undulation reaches a steady-state value of $\sqrt{0.2855961\text{x}10^{-1}} = 0.169$ meter. This compares with the optimal value of $\sqrt{0.1396868\text{x}10^{-1}} = 0.118$ meter for the optimal smoother.

8. Sometimes, in addition to geoid undulation, the geoid undulation slope is also of interest. The slope = $x_2 - \beta\ x_3$, and simple calculation shows the smoothed undulation and slope covariance matrix reaches a steady-state value of

$$\begin{bmatrix} 0.1136643\text{x}10^{-1} & 0(10^{-9}) \\ 0(10^{-9}) & 0.1396869\text{x}10^{-1} \end{bmatrix}$$

It is of interest to note the smoothed geoid undulation and slope are almost uncorrelated. The slope estimation accuracy is specified by a standard deviation of $\sqrt{0.1136643\text{x}10^{-1}}$ = 0.1066135 meter/sec. ≈ 3.36 arc sec. at the approximate GEOS-III ground speed of 6.55 Km/sec.

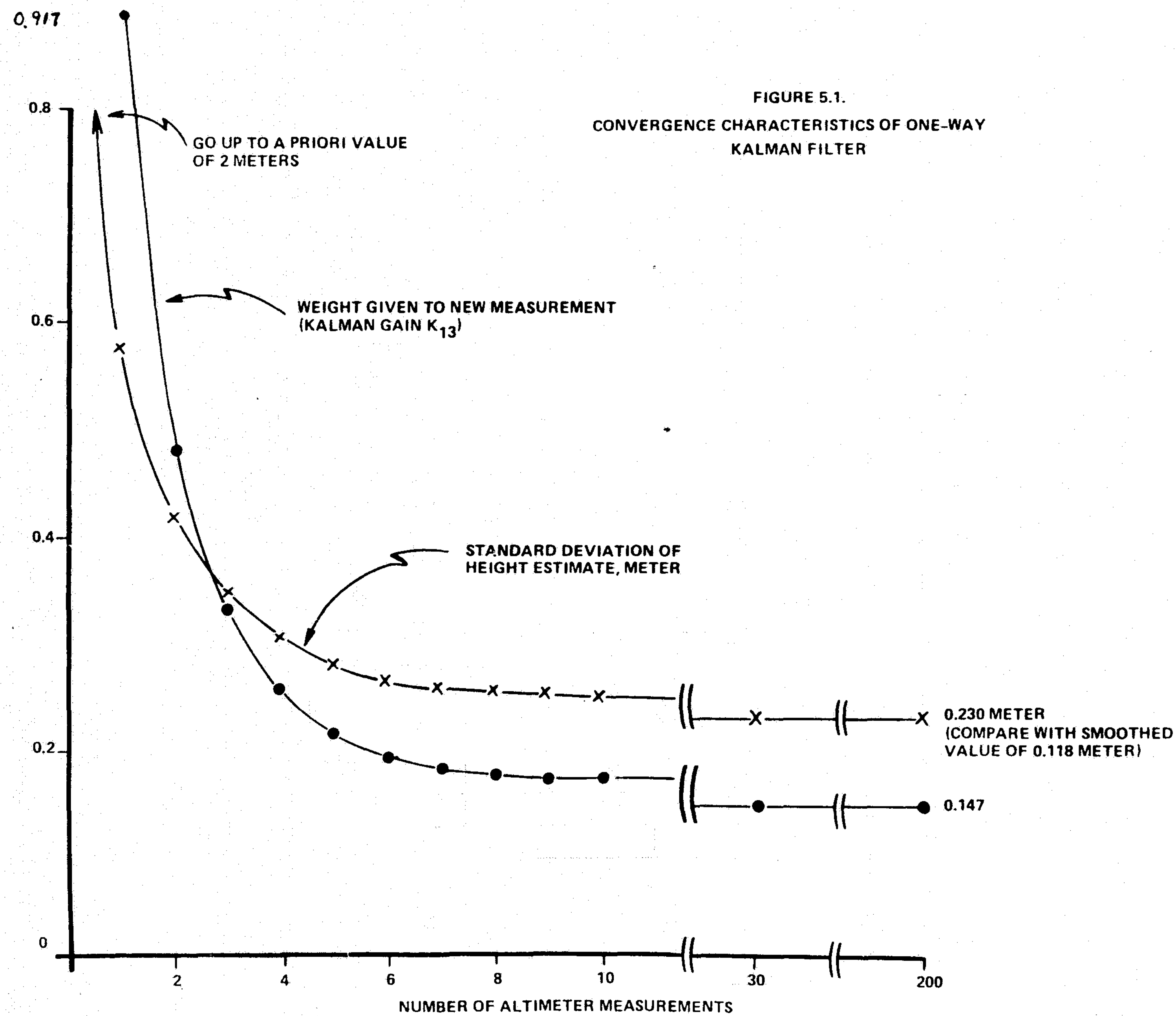


FIGURE 5.1.
CONVERGENCE CHARACTERISTICS OF ONE-WAY KALMAN FILTER

5.3 ASYMPTOTIC BEHAVIOR OF SMOOTHER-FREQUENCY RESPONSE AND DATA WEIGHTING SEQUENCE

Away from the beginning and the end of a long data sequence the smoother is governed essentially by its steady-state asymptotic behavior. The discrete transfer function describing the smoother asymptotic characteristics is derived in Appendix B. From the transfer function, quantities which give one intuitive understanding of the filter such as the frequency response and the data weighting sequence may be calculated immediately. To illustrate, the same numerical example as used in Section 5.2 will be considered. Substitution of the appropriate steady-state smoother parameters given in Section 5.2 into Equation (B.5), and after some algebra, one obtains the following transfer function relating $\hat{x}_3$, the geoid height estimate, to the altimeter measurement y.

$$H(z) = 0.0388018553 + f(z) + f(\tfrac{1}{z})$$

where

$$f(z) = \frac{0.0386814002z^2-0.0671610539z+0.0294466017}{z^2-2.726669377z^2+2.487576809z-0.7588967469}$$

The frequency response of the smoother is obtained from the transfer function by replacing z by $e^{i\omega T}$; i.e.,

$$H(e^{i\omega T}) = 0.0388018553 + f(e^{i\omega T}) + f(e^{-i\omega T})$$

$$= 0.0388018553 + 2 \text{ Re } \{f(e^{i\omega T})\}$$

That the above expression is real indicates there is no phase lag, a fact which could have been anticipated from the completely symmetrical forward and backward operation of the smoother. The amplitude response as a function of excitation frequency is computed, tabulated in Table 5.1 and plotted in Figure 5.2.

The data weighting sequence, or the variations of data weights with data "distance" from the estimate are coefficients of the polynomial expansion of the transfer function. As expected, because the dependence of H(z) on "z" and "1/z" is identical, there is no preference to "past" or "future" data. The data weighting sequence is computed, and is tabulated in Table 5.2, and plotted in Figure 5.3. Note that data weights exhibit a damped oscillatory behavior with a period of approximately 57 sampling intervals (5.8 seconds, or 38 Km). It is also of interest to note that as shown in the preceding section, the current measurement is given the weight 0.147 by the forward filter. With the smoother, the weight assigned is 0.0388, indicating the much improved smoothing effect.

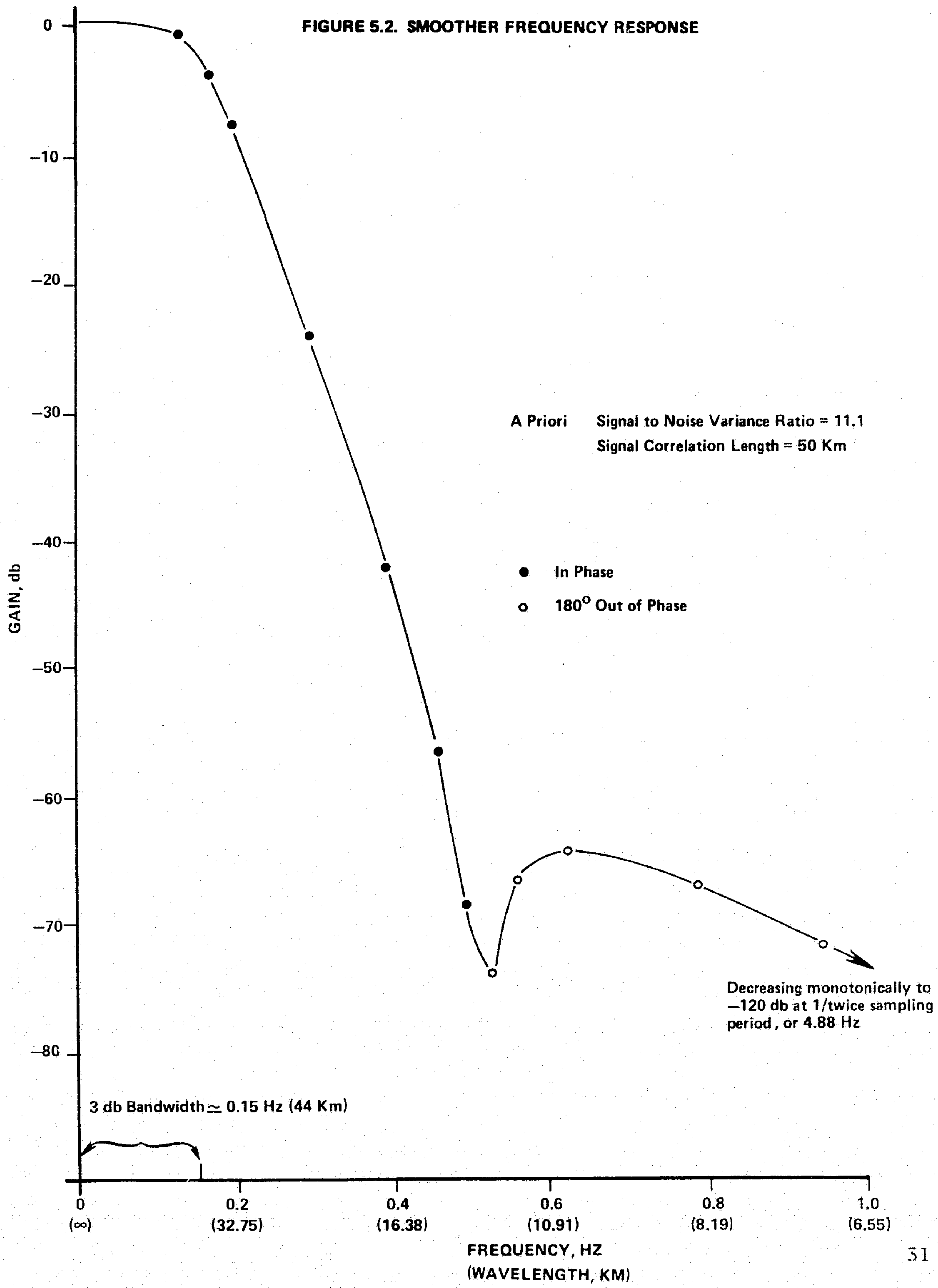


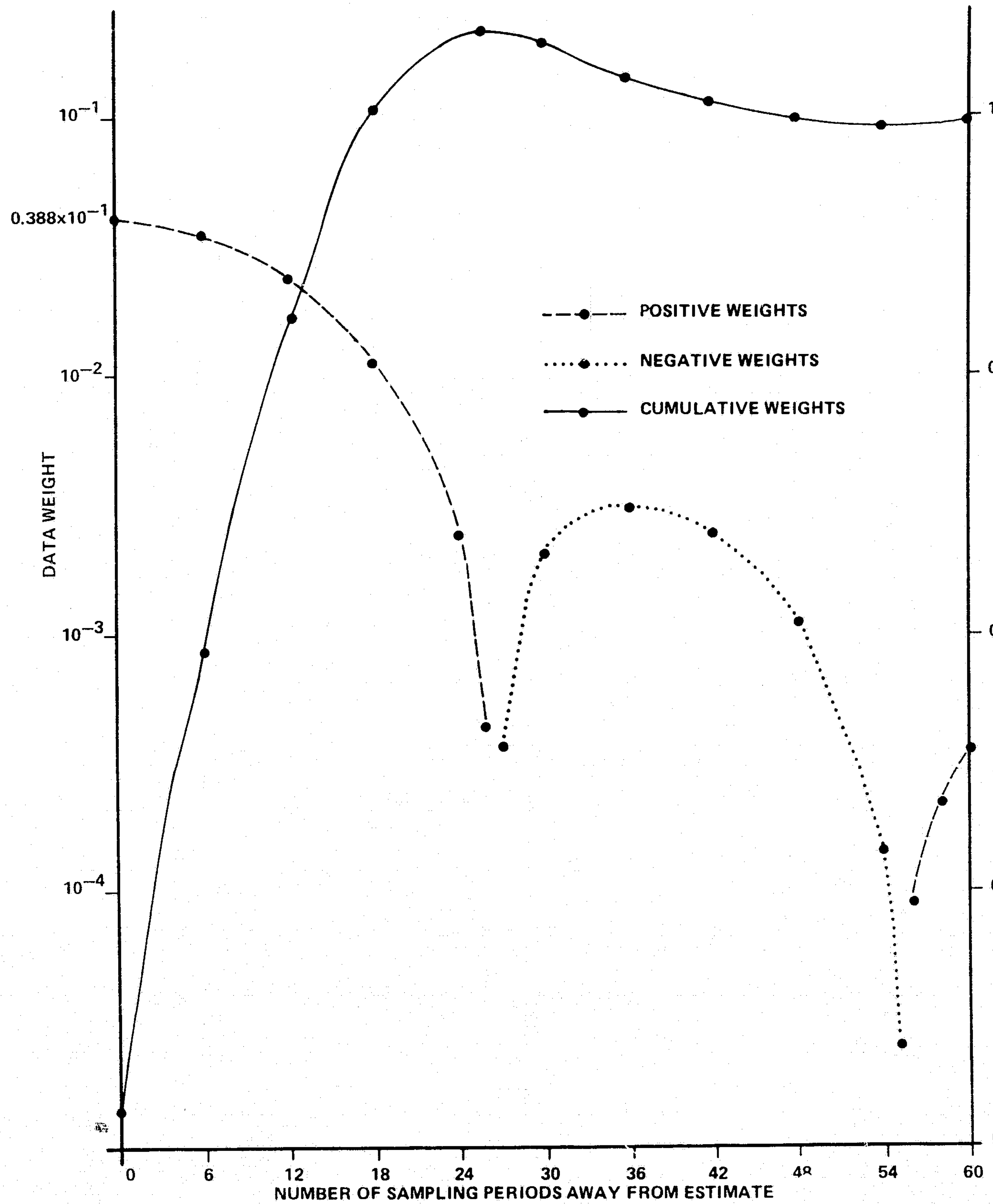
FIGURE 5.2. SMOOTHER FREQUENCY RESPONSE
0
-10
-20
-30
-40
-50
-60
-70
-80
GAIN, db
A Priori Signal to Noise Variance Ratio = 11.1
Signal Correlation Length = 50 Km
In Phase
180° Out of Phase
Decreasing monotonically to −120 db at 1/twice sampling period, or 4.88 Hz
3 db Bandwidth ≃ 0.15 Hz (44 Km)
0 (∞)
0.2 (32.75)
0.4 (16.38)
0.6 (10.91)
0.8 (8.19)
1.0 (6.55)
FREQUENCY, HZ
(WAVELENGTH, KM)

Table 5.1. Smoother Frequency Response

FREQUENCY, H_z	AMPLIFICATION, db
0	+0.005
0.0326	-0.037
0.0651	-0.190
0.0976	-0.580
0.130	-1.54
0.163	-3.64
0.195	-7.29
0.288	-12.3
0.260	-18.0
0.293	-24.0
0.326	-30.0
0.358	-36.0
0.391	-42.1
0.423	-48.7
0.455	-56.4
0.488	-68.4
0.520	-74.0
0.553	-66.3
0.586	-64.4
0.618	-63.9
0.651	-64.1
0.683	-64.6
1.01	-72.7
2.02	-95.6
3.03	-109.
4.88 (Nyquist Frequency)	-120

There is a 180° phase shift for responses to frequencies higher than ≈0.5 Hz

FIGURE 5.3. DATA WEIGHT VS DATA DISTANCE FROM ESTIMATE



(SAMPLING PERIOD = 0.1024 sec., equivalent to 0.67 Km. ground distance)

Table 5.2. Variation of Data Weights With "Distance" From Estimate

DATA "DISTANCE" FROM ESTIMATE, IN TERMS OF NUMBER OF SAMPLING PERIODS, N (Sampling Period=0.1024 sec. equivalent to 0.67 Km of ground distance)	DATA WEIGHTS	
	INSTANTANEOUS W(N)	CUMULATIVE $W(0)+2\sum_{n=1}^{N} W(n)$
0	0.388019D-01	0.038802
1	0.386814D-01	0.116165
2	0.383103D-01	0.192785
3	0.376833D-01	0.268151
4	0.368051D-01	0.341762
5	0.356889D-01	0.413140
6	0.343540D-01	0.481848
7	0.328245D-01	0.547497
8	0.311274D-01	0.609752
9	0.292918D-01	0.668335
10	0.273478D-01	0.723031
11	0.253252D-01	0.773681
12	0.232531D-01	0.820188
13	0.211594D-01	0.862506
14	0.190700D-01	0.900646
15	0.170086D-01	0.984663
16	0.149965D-01	0.964657
17	0.130526D-01	0.990762
18	0.111929D-01	1.013147
19	0.943070D-02	1.032009
20	0.777686D-02	1.047563
21	0.623958D-02	1.060042
22	0.482466D-02	1.069691
23	0.353564D-02	1.076762
24	0.237402D-02	1.081510
25	0.133940D-02	1.084189
26	0.429734D-03	1.085049
27	-0.358482D-03	1.084332
28	-0.102999D-02	1.082272
29	-0.159057D-02	1.079091
30	-0.204683D-02	1.074997
31	-0.240602D-02	1.070185
32	-0.267585D-02	1.064833
33	-0.286433D-02	1.059104
34	-0.297963D-02	1.053145
35	-0.302990D-02	1.047085
36	-0.302323D-02	1.041039
37	-0.296745D-02	1.035104
38	-0.287013D-02	1.029364

Table 5.2 (Cont.)

DATA "DISTANCE" FROM ESTIMATE, IN TERMS OF NUMBER OF SAMPLING PERIODS, N (Sampling Period=0.1024 sec. equivalent to 0.67 Km of ground distance)	DATA WEIGHTS	
	INSTANTANEOUS W(N)	CUMULATIVE $W(0)+2\sum_{n=1}^{N} W(n)$
39	-0.273846D-02	1.023887
40	-0.257919D-02	1.018729
41	-0.239859D-02	1.013931
42	-0.220246D-02	1.009526
43	-0.199602D-02	1.005534
44	-0.178399D-02	1.001966
45	-0.157054D-02	0.998825
46	-0.135930D-02	0.996107
47	-0.115339D-02	0.993800
48	-0.955425D-03	0.991889
49	-0.767552D-03	0.990354
50	-0.591470D-03	0.989171
51	-0.428469D-03	0.988314
52	-0.279458D-03	0.987755
53	-0.145005D-03	0.987465
54	-0.253708D-04	0.987414
55	0.794530D-04	0.987573
56	0.169710D-03	0.987913
57	0.245845D-03	0.988404
58	0.308466D-03	0.989021
59	0.358321D-03	0.989738
60	0.396259D-03	0.990531
61	0.423211D-03	0.991377
62	0.440161D-03	0.992257
63	0.448123D-03	0.993154
64	0.448121D-03	0.994050
65	0.441176D-03	0.994932
66	0.428284D-03	0.995789
67	0.410408D-03	0.996610
68	0.388463D-03	0.997386
69	0.363314D-03	0.998113
70	0.335762D-03	0.998785
71	0.306543D-03	0.999398
72	0.276327D-03	0.999950
73	0.245710D-03	1.000442
74	0.215222D-03	1.000872
75	0.185318D-03	1.001243
76	0.156390D-03	1.001556
77	0.128762D-03	1.001813
78	0.102696D-03	1.002019
79	0.783964D-04	1.002175
80	0 560145D-04	1.002287

Table 5.2 (Cont.)

DATA "DISTANCE" FROM ESTIMATE, IN TERMS OF NUMBER OF SAMPLING PERIODS, N (Sampling Period=0.1024 sec. equivalent to 0.67 Km of ground distance)	DATA WEIGHTS	
	INSTANTANEOUS W(N)	CUMULATIVE $W(0)+2\sum_{n=1}^{N} W(n)$
81	0.356515D-04	1.002359
82	0.173640D-04	1.002393
83	0.116954D-05	1.002396
84	-0.129497D-04	1.002370
85	-0.250413D-04	1.002320
86	-0.351785D-04	1.002249
87	-0.434554D-04	1.002162
88	-0.499830D-04	1.002063
89	-0.548854D-04	1.001953
90	-0.582960D-04	1.001836
91	-0.603541D-04	1.001715
92	-0.612023D-04	1.001593
93	-0.609836D-04	1.001471
94	-0.598392D-04	1.001351
95	-0.579066D-04	1.001236
96	-0.553178D-04	1.001125
97	-0.521980D-04	1.001021
98	-0.486645D-04	1.000923
99	-0.448261D-04	1.000834
100	-0.407820D-04	1.000752
101	-0.366221D-04	1.000679
102	-0.324263D-04	1.000614
103	-0.282649D-04	1.000557
104	-0.241985D-04	1.000509
105	-0.202784D-04	1.000468
106	-0.165471D-04	1.000435
107	-0.130384D-04	1.000409
108	-0.977857D-05	1.000390
109	-0.678637D-05	1.000376
110	-0.407406D-05	1.000368
111	-0.164793D-05	1.000365
112	0.491027D-06	1.000366
113	0.234643D-05	1.000370
114	0.392586D-05	1.000378
115	0.524024D-05	1.000389
116	0.630322D-05	1.000401
117	0.713061D-05	1.000416
118	0.773989D-05	1.000431
119	0.814967D-05	1.000447
120	0.837927D-05	1.000464
121	0.844836D-05	1.000481
122	0.837656D-05	1.000498
123	0.818316D-05	1.000514
124	0.788688D-05	1.000530
125	0.750561D-05	1.000545

Table 5.2 (Cont.)

DATA "DISTANCE" FROM ESTIMATE, IN TERMS OF NUMBER OF SAMPLING PERIODS, N (Sampling Period=0.1024 sec. equivalent to 0.67 Km of ground distance)	DATA WEIGHTS	
	INSTANTANEOUS W(N)	CUMULATIVE $W(0)+2\sum_{n=1}^{N} W(n)$
126	0.705627D-05	1.000559
127	0.655467D-05	1.000572
128	0.601538D-05	1.000584
129	0.545169D-05	1.000595
130	0.487555D-05	1.000605
131	0.429758D-05	1.000613
132	0.372703D-05	1.000621
133	0.317186D-05	1.000627
134	0.263877D-05	1.000632
135	0.213323D-05	1.000637
136	0.165959D-05	1.000640
137	0.122112D-05	1.000642
138	0.820151D-06	1.000644
139	0.458100D-06	1.000645
140	0.135603D-06	1.000645
141	-0.147403D-06	1.000645
142	-0.391592D-06	1.000644
143	-0.598158D-06	1.000643
144	-0.768726D-06	1.000642
145	-0.905276D-06	1.000640
146	-0.101006D-05	1.000638
147	-0.108555D-05	1.000636
148	-0.113434D-05	1.000633
149	-0.115910D-05	1.000631
150	-0.116257D-05	1.000629
151	-0.114742D-05	1.000626
152	-0.111629D-05	1.000624
153	-0.107175D-05	1.000622
154	-0.101620D-05	1.000620
155	-0.951940D-06	1.000618
156	-0.881099D-06	1.000616
157	-0.805631D-06	1.000615
158	-0.727312D-06	1.000613
159	-0.647733D-06	1.000612
160	-0.568301D-06	1.000611
161	-0.490238D-06	1.000610
162	-0.414586D-06	1.000609
163	-0.342217D-06	1.000608
164	-0.273838D-06	1.000608
165	-0.210002D-06	1.000607
166	-0.151121D-06	1.000607
167	-0.974754D-07	1.000607
168	-0.492279D-07	1.000607
169	-0.643593D-08	1.000607
170	0.309357D-07	1.000607
171	0.630025D-07	1.000607
172	0.899478D-07	1.000607
173	0.112011D-06	1.000607

Table 5.2 (Cont.)

DATA "DISTANCE" FROM ESTIMATE, IN TERMS OF NUMBER OF SAMPLING PERIODS, N (Sampling Period=0.1024 sec. equivalent to 0.67 Km of ground distance)	DATA WEIGHTS	
	INSTANTANEOUS W(N)	CUMULATIVE $W(0)+2\sum_{n=1}^{N} W(n)$
174	0.129478D-06	1.000608
175	0.142669D-06	1.000608
176	0.151928D-06	1.000608
177	0.157620D-06	1.000608
178	0.160114D-06	1.000609
179	0.159785D-06	1.000609
180	0.157001D-06	1.000609
181	0.152124D-06	1.000610
182	0.145499D-06	1.000610
183	0.137456D-06	1.000610
184	0.128303D-06	1.000610
185	0.118326D-06	1.000611
186	0.107787D-06	1.000611
187	0.969242D-07	1.000611
188	0.859483D-07	1.000611
189	0.750457D-07	1.000611
190	0.643771D-07	1.000612
191	0.540791D-07	1.000612
192	0.442648D-07	1.000612
193	0.350250D-07	1.000612
194	0.264301D-07	1.000612
195	0.185312D-07	1.000612
196	0.113618D-07	1.000612
197	0.493985D-08	1.000612
198	-0.730738D-09	1.000612
199	-0.565831D-08	1.000612
200	-0.986174D-08	1.000612

5.4 EFFECT OF INCORRECT GEOID UNDULATION MODEL

The geoid undulation has been considered as a stationary random process with a zero mean value and with a TASC correlation function, Equation (3.1). The TASC model is but one of many possible correlation functions which satisfy certain very general criteria for geoid undulations (Ref. 2). There is no a priori reason to believe the TASC correlation function is the model, much preferable to others. Instead one may adopt the point of view that the TASC model is one of many reasonable models based on which a filter may be designed rationally with the desirable features of being simple, symmetric and having fading memory. The two parameters which define the TASC correlation function are:

1. The variance σ^2.

2. The correlation length, or equivalently the "time constant" $\frac{1}{\beta}$.

The variance equation and the optimum gain given in Section 4 may be normalized with respect to the geoid undulation variance σ^2. It may be seen then that instead of σ^2, it is the signal-to-noise ratio σ^2/R which is of importance. The only direct effect of S/N ratio is on the gain. Naturally the desired gain increases and decreases with the increase and the decrease of the signal-to-noise ratio. The correlation length, or "time constant" $\frac{1}{\beta}$, determines the effective memory length of the filter, and if it is taken to be too large, sharp features will be smeared. On the other hand, if the correlation length is too short, the filter cannot differentiate the higher frequency noise from the lower frequency geoid undulation. Although the effects of the two parameters are somewhat coupled, broadly speaking, when examining the smoothed altimeter data, if one feels the amplitudes are too low, the S/N ratio assumed is probably too small; if the features are not sharp enough, the correlation length is likely too long.

Both the signal and noise are assumed to be zero mean random processes. The actual data may contain low frequency trends due to biases, orbital errors, and etc. Ideally, if geoid undulation is the output desired, these trends should be removed before the data are input to the filter. Sometimes, the operational objective is just to "smooth" the data, leaving the trends intact. Generally speaking, filters should have "low pass" features and we shall investigate the ability of our smoother in passing through the low frequency trends. The signal and noise models shall be the same as those considered in the preceding section. The trends considered are a constant bias and a constant slope. The following table summarizes the steady-state* smoother outputs resulting from these trends. The results, of course, agree with the frequency response given in the preceding section.

*Away from the beginning and the end of the data pass.

SIMULATION RESULTS

TREND	FORWARD ESTIMATE		BACKWARD PREDICTION		SMOOTHED VALUE	
	HEIGHT m.	SLOPE m/sec.	HEIGHT m.	SLOPE m/sec.	HEIGHT m.	SLOPE m/sec.
+ 1 Meter BIAS	0.976325	-0.038262	0.972245	0.039874	1.000611	0.000000
ZERO HEIGHT 0.097748 m/sec. SLOPE	-0.014554	0.073774	0.012844	0.072628	0.000000	0.091840

The simulation results show that a constant trend or bias in the data will have virtually no effect on the slope estimate but will cause a 0.06% height error. On the other hand, a linear trend, or constant drift in the data will have no effect on the height estimate but will cause a 6% slope error.

Note that the forward filter "underestimates" the height and slope of the bias. This is not unexpected since the signal is modeled as a stable system,* and will be damped between measurements. In particular, it will be damped to zero for a large data gap. This is not unreasonable as it says the geoid undulation estimate should be zero since the geoid undulation is just as likely to be positive as negative in the absence of any measurement information. That both the height and the slope are under-estimated is consistent with the positive correlation between these quantities as indicated by the forward filtered error covariance matrix given in Section 5.2. The backward filter exhibits a similar behavior to the bias. It underestimates the height and the "backward slope." But since the backward slope is the negative of the forward slope, the backward filter over-estimates the forward slope and also exhibits a negative correlation between the height and the "forward slope estimates." It is of interest to note that although both the forward and backward filters under-estimate the height, the differences in their slope estimates compensate each other and the resulting optimum smoothed results not only give an almost perfect slope estimate, but also bring along with it a slightly over-estimated height. A somewhat analogous situation exists for the linear trend.

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*Tracking unstable systems with accuracy is generally impractical. For a discussion of that subject see Ref. 8.

SECTION 6.0

OPTIMUM RECURSIVE FILTER AT HIGH ALTIMETER SAMPLING RATES-MODIFICATIONS FOR CORRELATED NOISE SEQUENCE

Some correlations may exist among the altimeter noises when measurements are made at rates greater than 10 samples/sec. Modifications of the results of Section 4.2 to take into account this correlation are presented in this section. For a general treatment of correlated noise see Reference 7.

Consider altimeter measurements at two instants, t_i and t_j; i.e.,

$$y_i = x_3(t_i) + n_i \qquad (6.1)$$

$$y_j = x_3(t_j) + n_j \qquad (6.2)$$

The existence of correlation between the noises n_i and n_j means knowledge of n_i conveys information about n_j, or, past measurements enable us to improve our a priori knowledge of future noises. As a matter of fact, an application of Equation (4.0) enables us to relate n_i and n_j in the following way,

$$n_j = \hat{n}_j + (\hat{n}_j - n_j) \triangleq \frac{E(n_i - n_j)}{E(n_i^2)} n_i + u_j \qquad (6.3)$$

$$E(n_i u_j) = 0$$

Notice that Equation (3.17), given before, is just a special case of Equation (6.3) for exponentially correlated noise, which is our model of the altimeter noise at high sampling rates.

By making use of Equations (3.17) and (6.1) and taking $j=i+1$, one may replace Equation (6.2) by

$$y_{i+1}-e^{b(t_{i+1}-t_i)}y_i = x_3(t_{i+1})-e^{b(t_{i+1}-t_i)}x_3(t_i)+e^{b(t_{i+1}-t_i)}u_i.$$

Furthermore, because geoid undulations at "adjacent locations" are related by Equation (3.8), one obtains finally

$$y_{i+1}-e^{bT_i}y_i = e^{-\beta T_i}\left[\frac{T_i^2}{2} \quad T_i \quad 1-e^{\beta T_i}\right] \underset{3\times 1}{[x(t_i)]}$$

$$+\int_{t_i}^{t_{i+1}} \frac{(t_{i+1}-\tau)^2}{2} e^{-\beta(t_{i+1}-\tau)} w(\tau)\, d\tau+e^{bT_i}u_i,$$

where $T_i = t_{i+1}-t_i$ is the sampling period. The above equation is in the standard form of the measurement equation in the Kalman filter; i.e.,

$$Z_i = [H(t_i)]\ [x(t_i)] + \nu_i\ , \qquad (6.4)$$

with

$$Z_i = y_{i+1}-e^{bT_i}\, y_i$$

$$[H(t_i)] = e^{-\beta T_i}\left[\frac{T_i^2}{2} \quad T_i \quad 1-e^{\beta T_i}\right]$$

$$\nu_i = \int_{t_i}^{t_{i+1}} \frac{(t_{i+1}-\tau)^2}{2} e^{-\beta(t_{i+1}-\tau)} w(\tau)\, d\tau + e^{bT_i}\, u_i$$

$$E(\nu_i) = 0\ , \quad E(\nu_i\nu_j) = G_i\ \delta_{ij}$$

$$G_i = R_i\ (e^{2bT_i}-1) + q_{33}$$

With Equation (6.4) replacing the original measurement equation, a Kalman Filter for the correlated altimeter data may be constructed immediately. Note the following complexities are introduced because of the altimeter noise correlations:

1. The original altimeter data have to be converted to an equivalent data set. This involves some computations and introduces a one-sample-period lag to the filter output.

2. The converted measurements are related to all three state variables instead of just the geoid undulation.

3. Correlations now exist between the measurement noise ν_i and the dynamic noise; i.e.,

$$E\left(\nu_i \int_{t_i}^{t_{i+1}} e^{-\beta(t_{i+1}-\tau)} \begin{bmatrix} 1 \\ t_{i+1}-\tau \\ (t_{i+1}-\tau)^2 \end{bmatrix} w(\tau) \quad d\tau\right)$$

$$= q^2 \; E \int_{t_i}^{t_{i+1}} e^{-2\beta(t_{i+1}-\tau)} \begin{bmatrix} 1 \\ t_{i+1}-\tau \\ \frac{(t_{i+1}-\tau)^2}{2} \end{bmatrix} \frac{(t_{i+1}-\tau)^2}{2} d\tau$$

$$= \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix}$$

Although "2" and "3" do not present any difficulties, they do make the filter structure more complex.

SECTION 7.0
PROGRAM IMPLEMENTATION

A computer program, ALTKAL, which implements the algoriths described in the preceding sections has been developed at the Wallops Flight Center, Wallops Island, Virginia. The program is designed specifically for the smoothing of sea surface heights derived from GEOS-3 altimeter data. The program computes, and outputs, filtered sea surface heights from both a forward and a backward pass through a data set. The forward and backward filter results are then combined to obtain the optimally smoothed heights. The computations include the smoothed sea surface slope, which is also output by the program.

Although ALTKAL was intended to be used primarily with cumulative altitude data (data period of 0.1024 seconds), any regular or irregular data rate can be used. However, the implemented algorithms assume that measurement noise is uncorrelated between samples, so correlated input (such as instantaneous altitudes from GEOS-3) would not be optimally treated.

7.1 PROGRAM INPUT

The program will accept as input either of two types of input tapes. These tapes are:

- GEODYN binary residual tape
- ARC tape

Detailed formats for these tapes are given in references 9 and 10, respectively.

7.2 PROGRAM OUTPUT

The complete output of the program includes:

- A list of the input parameters used in the TASC model.
- A complete listing of the processing results.
- The RMS of the residuals of the measurements about the final smoothed heights.
- An output tape containing relevant variables to be saved or for use in plotting (optional).
- An output tape suitable for input to the SEAHT program (optional).

Specific examples and discussion of the output can be found in Section 8.3.

7.3 OPTIONS

ALTKAL has the capability of allowing specification of the input processing parameters used. Through the use of keyword option cards (Section 8.2) certain options can be implemented. These include:

- Adjusting the editing threshold.
- Adjusting the correlation length.
- Adjusting the geoid undulation sigma.
- Adjusting the altimeter noise sigma.
- The manual editing (culling) of data points.

SECTION 8.0
PROGRAM OPERATION

The operational ALTKAL program resides as a card deck at the Wolf Research and Development Group office in Pocomoke City, Maryland. The deck is operational on the Honeywell 625/35 system at NASA/Wallops Flight Center. The run deck consists of the appropriate control cards, the program (binary cards), and the desired keyword cards. The standard set up for a GEOS-3 data run consists of the following:

- Specifying the FORTRAN logical units used for input and output procedure.

- Designating the appropriate options desired.

A specific example of a complete set up can be found in Section 8.3.1.

8.1 FORTRAN LOGICAL UNITS

FORTRAN LOGICAL UNIT	USE
1	GEODYN binary residual tape input+
2	ARC tape input+
11,12	Disk storage used during execution
*	Output tape used for plotting
**	Output tape used as input to SEAHT program

+The input tape number on which the individual pass is found must be specified on either 1 or 2 as indicated on the DATA option card.

*This can be any unit except 1, 2, 11, or 12 as specified on the PLOT option card.

**This can be any unit except 1, 2, 11, or 12 as specified on the SEAHT option card.

8.2 KEYWORD OPTION CARDS

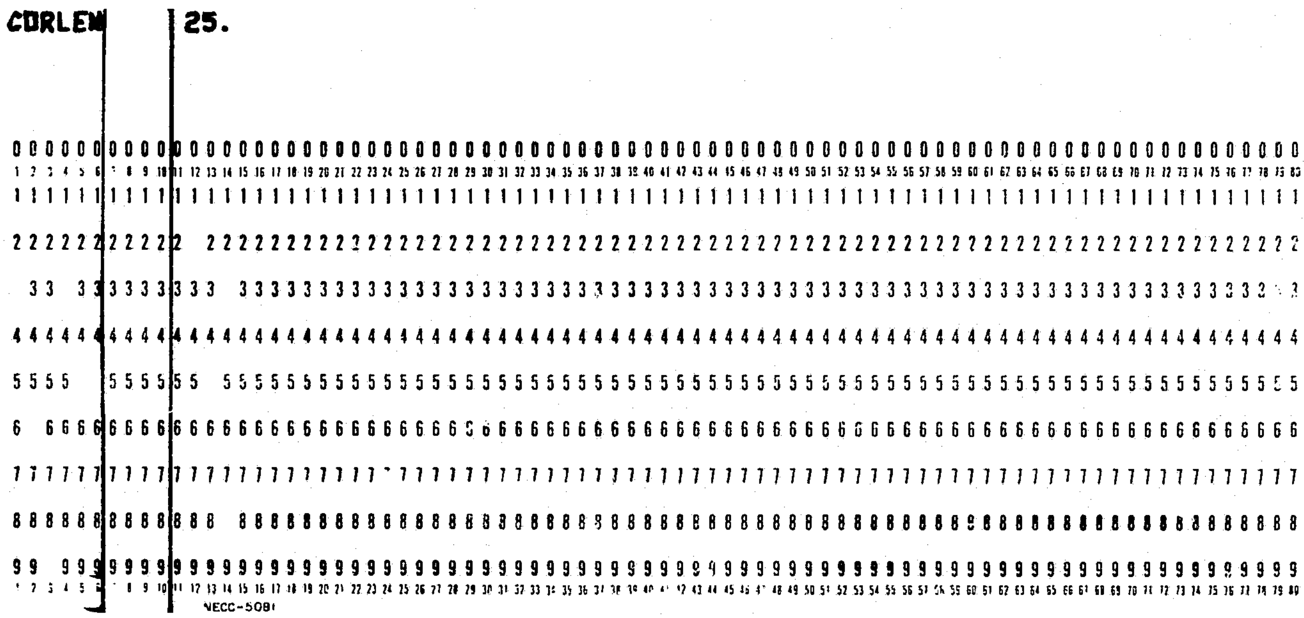
Each option card is identified by the name beginning in column 1 and is read under an A6 format. The name is followed by 5 fields:

Field Columns	Format
1-6	A6
7-10	I4
11-25	E15.6
26-40	E15.6
41-55	E15.6
56-70	E15.6

The option cards can be placed in any order and, if the card is omitted, all option variables have default values as indicated.

The option cards are:

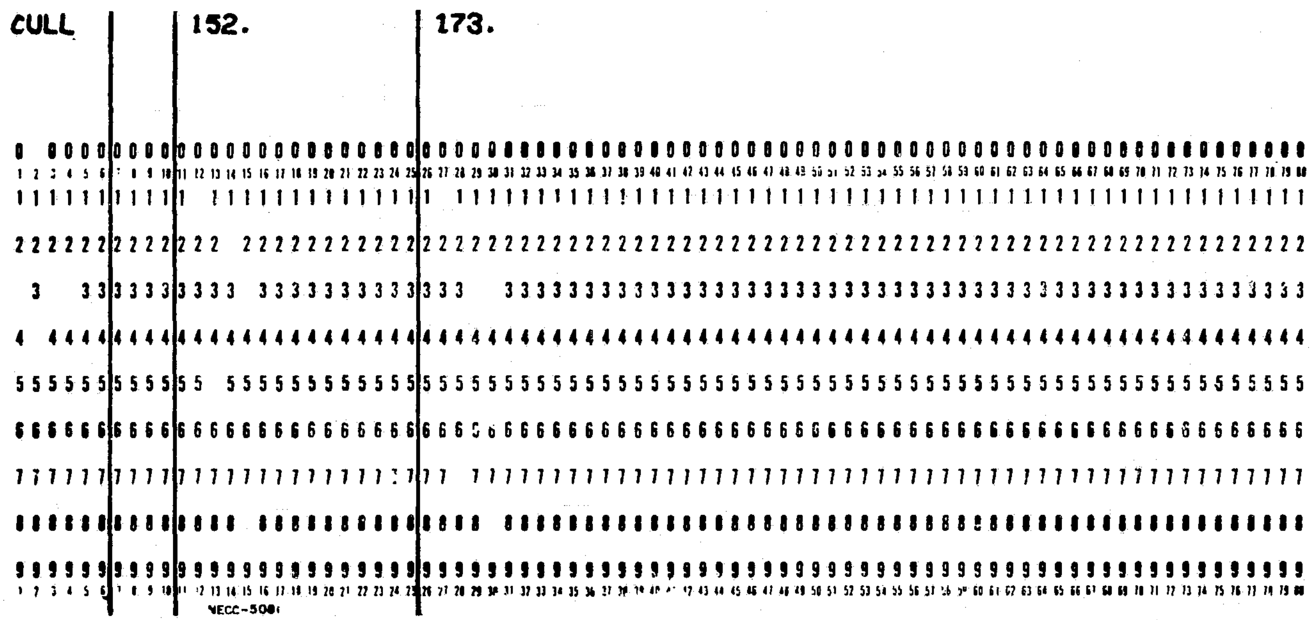
CULL
CORLEN
DATA
EDIT
PLOT
REVNUM
SEAHT
SIGMA



NAME	COLUMN	FORMAT	DESCRIPTION
CORLEN	1-6	A6	Modifies correlation length used in TASC model.
	11-25	E15.6	Correlation length in meters = CORLEN.

IF CARD OMITTED: Default value of CORLEN = 50.0 is used.

CULL



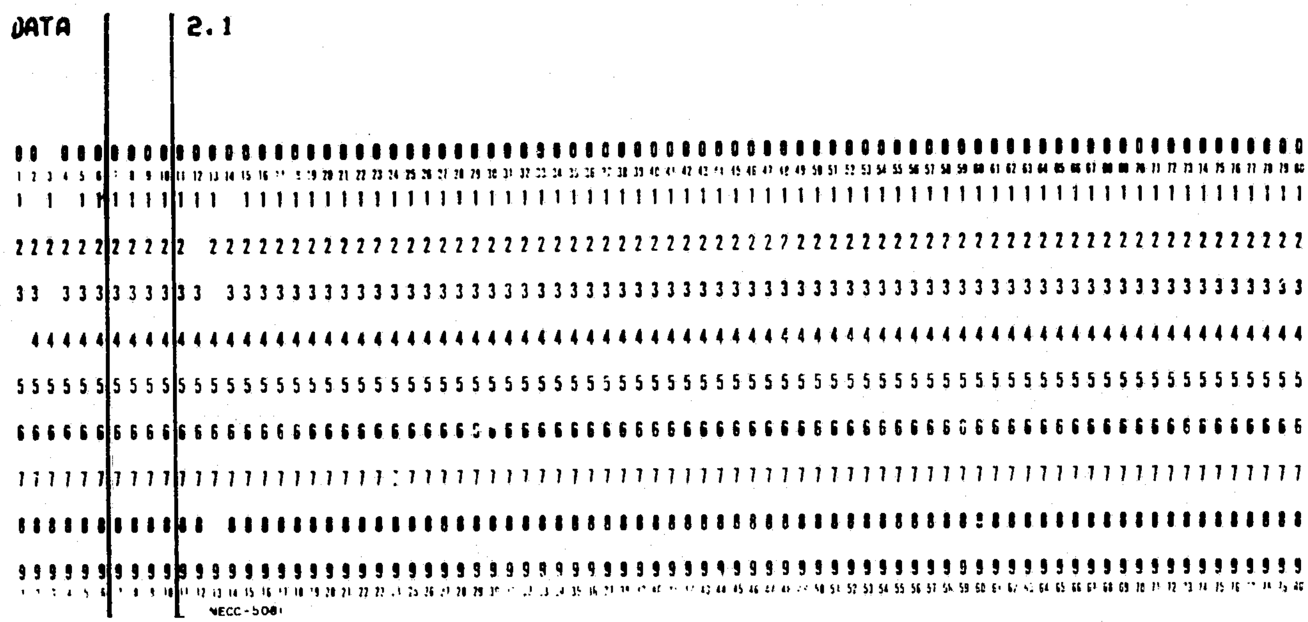
NAME	COLUMN	FORMAT	DESCRIPTION
CULL	1-4	A4	Used to delete individual measurements from the solution.
	11-25	E15.6	The number* of the first measurement in a series to be deleted.
	26-40	E15.6	The number* of the last measurement in a series to be deleted.

IF CARD OMITTED: No measurements are deleted.

NOTE: A maximum of 50 series of points per arc may be culled. The series may fall in any order. Only one series per card is permitted. If only one measurement is to be culled, then that number must appear both as the start and stop number of the series.

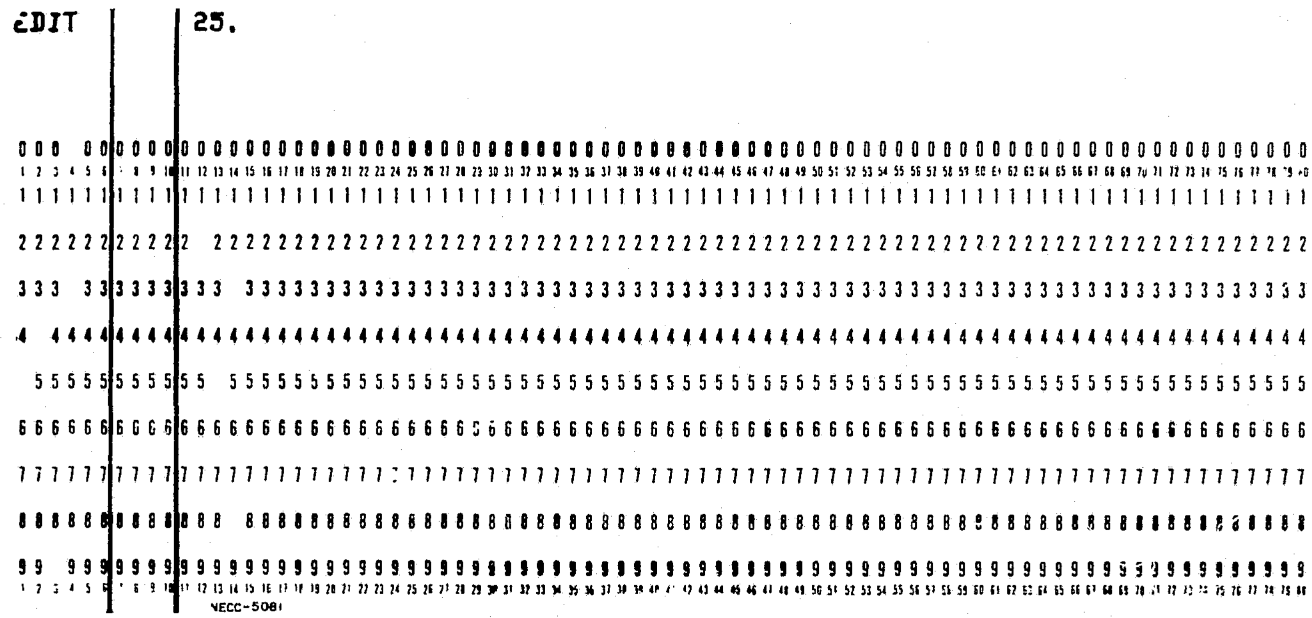
*The numbers which correspond to each measurement are printed to the far left on the printout.

DATA



NAME	COLUMN	FORMAT	DESCRIPTION
DATA	1-4	A4	Specifies type of input data tape.
	11-25	E15.6	Integer part of number indicates type of tape used = INDATA: 1 = residual tape 2 = ARC tape Decimal part of number indicates file of ARC tape on which arc is found = IFILE.

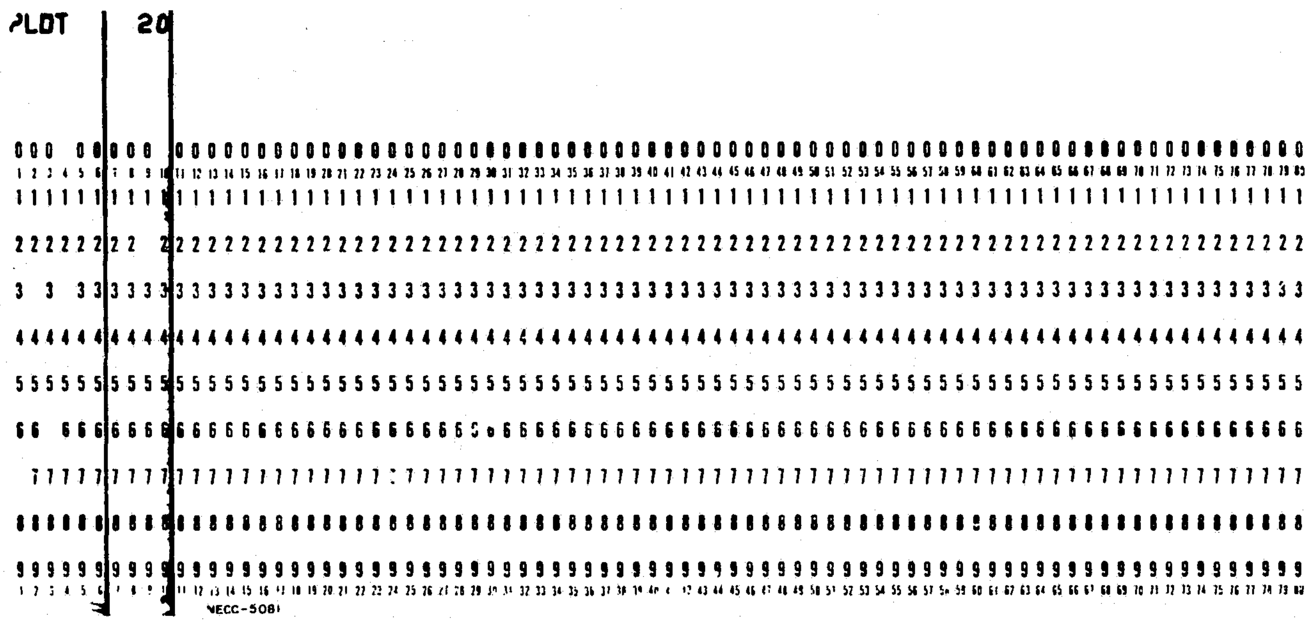
IF CARD OMITTED: Default value of INDATA = 1 is used. IFILE is not used.



NAME	COLUMN	FORMAT	DESCRIPTION
EDIT	1-4	A4	Determines editing threshold level.
	11-25	E15.6	Edit threshold value in meters.

IF CARD OMITTED: Default value of EDIT = 50.0 is used.

PLOT



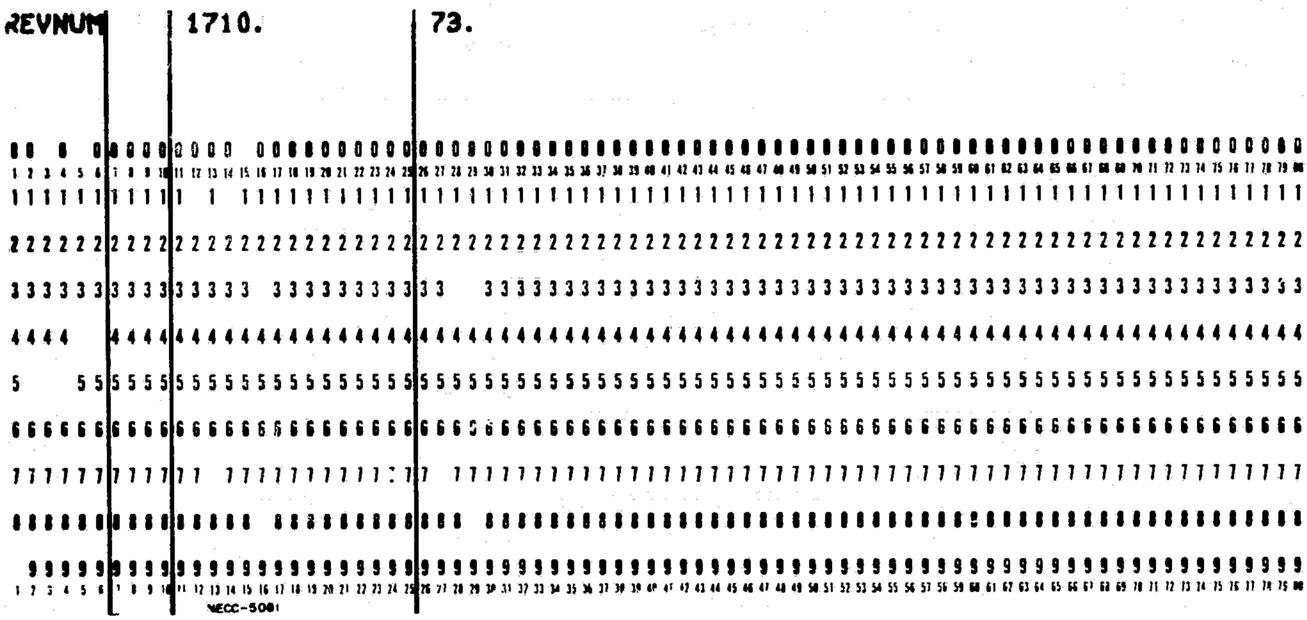
NAME	COLUMN	FORMAT	DESCRIPTION
PLOT	1-4	A4	Determines if output tape is to be generated containing variables: time, measurement, forward filter results, backward filter results, smoothed results, residuals of measurements about smoothed results.

NAME	COLUMN	FORMAT	DESCRIPTION
	7-10	I4	If greater than 0, indicates unit* to be used for writing tape = IPLOT.

IF CARD OMITTED: Default value of IPLOT = 0 is used and no output tape is generated.

*Logical units 1, 2, 11, and 12 cannot be used as output units.

REVNUM

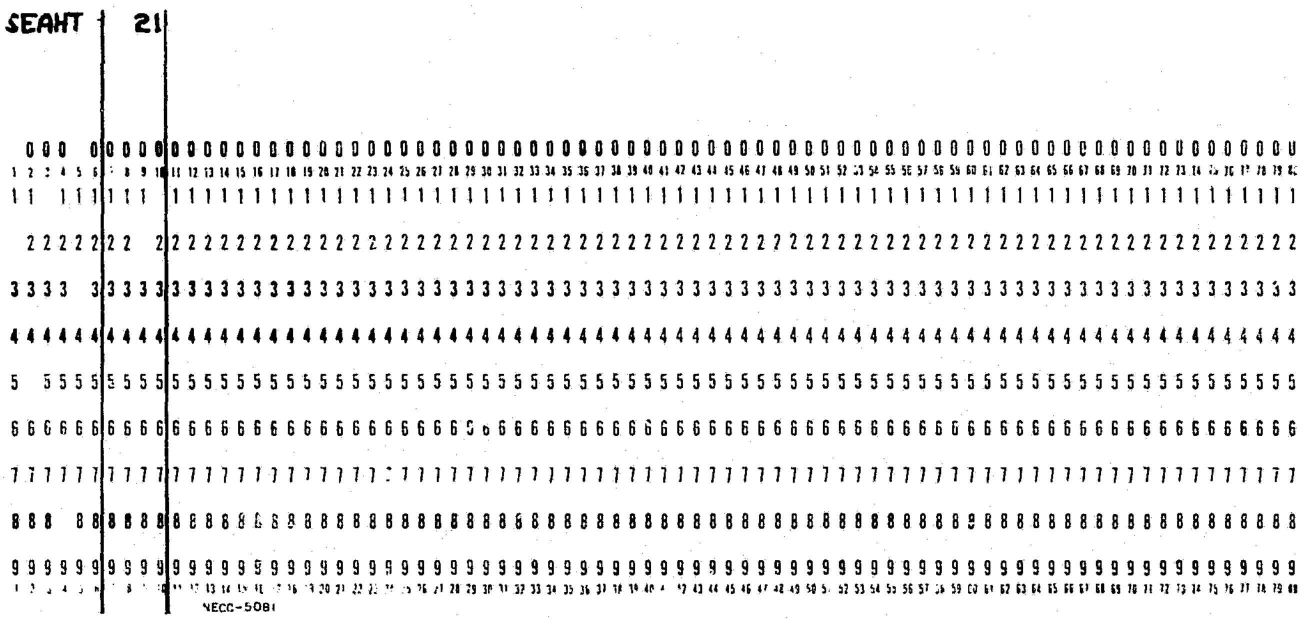


NAME	COLUMN	FORMAT	DESCRIPTION
REVNUM	1-6	A6	Inputs rev number and segment number to be used on SEAHT tape.
	11-25	E15.6	Rev number = REVNUM
	26-40	E15.6	Segment number = SEGNUM

IF CARD OMITTED: Default values of REVNUM = 0 and SEGNUM = 0 are used.

NOTE: If SEAHT card is present and ISEAHT is greater than 0 then this card must be present to give correct values to REVNUM and SEGNUM.

SEAHT

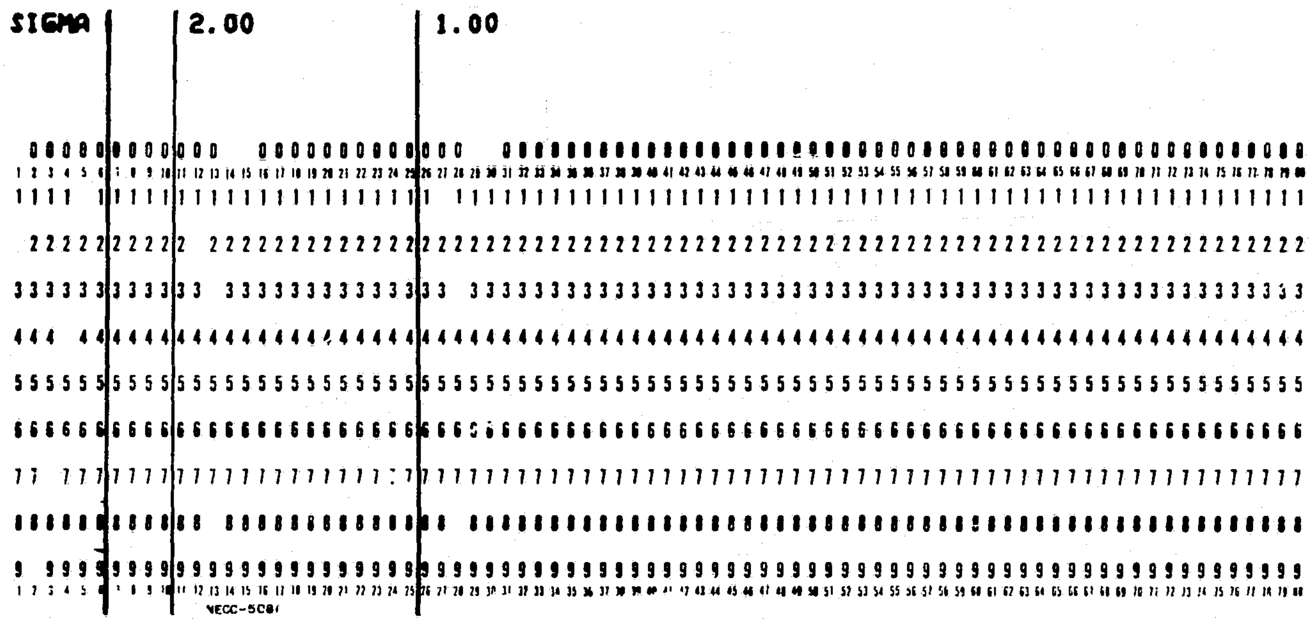


NAME	COLUMN	FORMAT	DESCRIPTION
SEAHT	1-5	A5	Determines if output tape is to be generated for use as input to SEAHT.
	7-10	I4	If greater than 0: indicates unit* to be used for writing tape = ISEAHT.

IF CARD OMITTED: Default value of ISEAHT = 0 is used and no output tape is generated.

*Logical units 1, 2, 11, and 12 cannot be used as output units.

SIGMA



NAME	COLUMN	FORMAT	DESCRIPTION
SIGMA	1-5	A5	Determines (i) Geoid undulation sigma and (ii) Altimeter noise sigma.
	11-25	E15.6	Geoid undulation sigma in meters = SIGEO.
	26-40	E15.6	Altimeter noise sigma in meters = SIGNO.

IF CARD OMITTED: Default values of SIGEO = 2.00 and SIGNO = 0.60 are used.

8.3 SPECIFIC EXAMPLES

8.3.1 Sample Deck Set-up and Optional Input

The following is a sample of a working deck set-up. This run would operate on sea surface heights obtained from file 1 of tape 5344 which contains the pass for REV 1710, segment number 73. This job will produce two output tapes. The output tape used for plotting is written on unit 20 and the tape used as input to SEAHT is written on unit 21. The default values for CORLEN = correlation length, EDIT, SIGEO = geoid undulation sigma, and SIGNO = altimeter noise sigma are used because of the absence of their respective keyword option cards. Also, no measurements are culled.

COL.

```
1      8  11          16          26
$      IDENT           300902,REV1710
$      OPTION          FORTRAN
         .
         .
         .
   PROGRAM BINARY DECK
         .
         .
         .
$      EXECUTE
$      LIMITS          20,30K,,10K
$      TAPE            02,X2D,,5344
$      TAPE            20,X3D,,,,SAVE
$      TAPE            21,X4D,,,,SSHT
$      FFILE           02,NSTDLB,MLTFIL
$      FILE            11,F6S,20R
$      FILE            12,F7S,20R
$      INCODE          IBMF
PLOT     20
SEAHT    21
REVNUM      1710.                73.
DATA        2.1
$      ENDJOB
```

8.3.2 Sample Output

Specific examples of the program output are reproduced on the following pages. A sample of the optional input parameters and a portion of the program output are reproduced in Tables 8.1 and 8.2. The RMS can be found printed out after the processed results. Table 8.3 is a list of the indices corresponding to the places where the state transition matrix is recomputed. These indices indicate where time gapes occur in the data. Notice that the indices of the state transition matrix occur in pairs. The actual time gap in the data occurs between the first of these data points and the previous data point (e.g., in table 8.3 a time gap occurs between data points 1717 and 1718). Table 8.4 is an example of the output produced when data points are culled. In addition to the list, points which are culled are tagged with an asterisk on the printout for each recognition, as shown in Table 8.5. Equivalent messages and tags are also output when data points are edited (i.e., deleted).

EDIT= 50.00

GEOID UNDULATION SIGMA= 2.00

ALTIMETER NOISE SIGMA= 0.60

CORRELATION LENGTH= 50.00

Table 8.1

INDX	TIME OF DATA YYMMDD HHMM SS,SSSS	ELAPSED TIME (SEC)	MEASURMENT (METERS)	FORWARD HEIGHT (METERS)	BACKWARD HEIGHT (METERS)	SMOOTHED HEIGHT (METERS)	RESIDUAL (METERS)	SLOPE (ARCSEC)	SIGMA SLOPE (ARCSEC)	SIGMA SMOOTHED (METERS)
91	750706 1638 36,8657	65,8465	-45,6250	-45,3593	-45,5060	-46,6794	1,0544	-1,3416	3,3574	0.1182
92	750706 1638 36,9684	65,9489	-46,2344	-45,3121	-45,6715	-46,6845	0,4501	-1,2151	3,3574	0.1182
93	750706 1638 37,0708	66,0513	-46,4453	-45,3001	-45,7529	-46,6890	0,2437	-1,1055	3,3574	0.1182
94	750706 1638 37,1732	66,1537	-46,9219	-45,3599	-45,8064	-46,6930	-0,2289	-1,0086	3,3574	0.1182
95	750706 1638 37,2756	66,2561	-47,6094	-45,5169	-45,7873	-46,6967	-0,9127	-0,9222	3,3574	0.1182
96	750706 1638 37,3780	66,3585	-47,4375	-45,6368	-45,6564	-46,7001	-0,7374	-0,8469	3,3574	0.1182
97	750706 1638 37,4804	66,4609	-46,9141	-45,6700	-45,5420	-46,7034	-0,2107	-0,7841	3,3574	0.1182
98	750706 1638 37,5828	66,5633	-45,8516	-45,5436	-45,5020	-46,7065	0,8549	-0,7358	3,3574	0.1182
99	750706 1638 37,6852	66,6657	-46,7578	-45,5594	-45,6282	-46,7095	-0,0483	-0,7033	3,3574	0.1182
100	750706 1638 37,7876	66,7681	-46,6484	-45,5578	-45,6205	-46,7125	0,0640	-0,6877	3,3574	0.1182
101	750706 1638 37,8900	66,8705	-46,6797	-45,5606	-45,6297	-46,7155	0,0358	-0,6903	3,3574	0.1182
102	750706 1638 37,9924	66,9729	-47,0000	-45,6101	-45,6348	-46,7186	-0,2814	-0,7132	3,3574	0.1182
103	750706 1638 38,0948	67,0753	-47,4922	-45,7278	-45,5896	-46,7219	-0,7703	-0,7584	3,3574	0.1182
104	750706 1638 38,1972	67,1777	-47,4922	-45,8358	-45,4633	-46,7255	-0,7667	-0,8283	3,3574	0.1182
105	750706 1638 38,2996	67,2802	-46,4609	-45,7826	-45,3283	-46,7294	0,2684	-0,9246	3,3574	0.1182
106	750706 1638 38,4020	67,3826	-45,9375	-45,6550	-45,3487	-46,7336	0,7961	-1,0471	3,3574	0.1182
107	750706 1638 38,5044	67,4850	-47,1406	-45,7132	-45,4560	-46,7382	-0,4025	-1,1936	3,3574	0.1182
108	750706 1638 38,6068	67,5874	-46,7031	-45,7021	-45,3828	-46,7432	0,0400	-1,3599	3,3574	0.1182
109	750706 1638 38,7092	67,6898	-46,1094	-45,6038	-45,3754	-46,7485	0,6392	-1,5408	3,3574	0.1182
110	750706 1638 38,8117	67,7922	-46,0156	-45,4989	-45,4640	-46,7543	0,7387	-1,7293	3,3574	0.1182
111	750706 1638 38,9141	67,8946	-46,7891	-45,5160	-45,5765	-46,7605	-0,0286	-1,9172	3,3574	0.1182
112	750706 1638 39,0165	67,9970	-46,1406	-45,4370	-45,5764	-46,7670	0,6264	-2,0960	3,3574	0.1182
113	750706 1638 39,1189	68,0994	-46,6094	-45,4337	-45,6804	-46,7738	0,1644	-2,2575	3,3574	0.1182
114	750706 1638 39,2213	68,2018	-46,5859	-45,4282	-45,7189	-46,7809	0,1950	-2,3938	3,3574	0.1182
115	750706 1638 39,3237	68,3042	-47,2813	-45,5264	-45,7645	-46,7883	-0,4930	-2,4984	3,3574	0.1182
116	750706 1638 39,4261	68,4066	-46,0625	-45,4386	-45,7033	-46,7958	0,7333	-2,5655	3,3574	0.1182
117	750706 1638 39,5285	68,5090	-46,5938	-45,4363	-45,8310	-46,8033	0,2095	-2,5909	3,3574	0.1182
118	750706 1638 39,6309	68,6114	-46,8047	-45,4661	-45,8840	-46,8107	0,0060	-2,5710	3,3574	0.1182
119	750706 1638 39,7333	68,7138	-46,8906	-45,5070	-45,9068	-46,8180	-0,0727	-2,5042	3,3574	0.1182
120	750706 1638 39,8357	68,8162	-47,2578	-45,5993	-45,9166	-46,8249	-0,4329	-2,3907	3,3574	0.1182

Table 8.2

SNUMB = 49136, ACTIVITY # = 02, REPORT CODE = 92, RECORD COUNT = 000048

STATE TRANSITION MATRIX # 1
STATE TRANSITION MATRIX # 2
STATE TRANSITION MATRIX # 364
STATE TRANSITION MATRIX # 365
STATE TRANSITION MATRIX # 781
STATE TRANSITION MATRIX # 782
STATE TRANSITION MATRIX # 1322
STATE TRANSITION MATRIX # 1323
STATE TRANSITION MATRIX # 1718
STATE TRANSITION MATRIX # 1719
STATE TRANSITION MATRIX # 1994
STATE TRANSITION MATRIX # 1995
STATE TRANSITION MATRIX # 2264
STATE TRANSITION MATRIX # 2265
STATE TRANSITION MATRIX # 2326
STATE TRANSITION MATRIX # 2327
STATE TRANSITION MATRIX # 2409
STATE TRANSITION MATRIX # 2410
STATE TRANSITION MATRIX # 2436
STATE TRANSITION MATRIX # 2437
STATE TRANSITION MATRIX # 2881
STATE TRANSITION MATRIX # 2882
STATE TRANSITION MATRIX # 3098
STATE TRANSITION MATRIX # 3099
STATE TRANSITION MATRIX # 4243
STATE TRANSITION MATRIX # 4242
STATE TRANSITION MATRIX # 3097
STATE TRANSITION MATRIX # 3096
STATE TRANSITION MATRIX # 2880
STATE TRANSITION MATRIX # 2879
STATE TRANSITION MATRIX # 2435
STATE TRANSITION MATRIX # 2434
STATE TRANSITION MATRIX # 2408
STATE TRANSITION MATRIX # 2407
STATE TRANSITION MATRIX # 2325
STATE TRANSITION MATRIX # 2324
STATE TRANSITION MATRIX # 2263
STATE TRANSITION MATRIX # 2262
STATE TRANSITION MATRIX # 1993
STATE TRANSITION MATRIX # 1992
STATE TRANSITION MATRIX # 1717
STATE TRANSITION MATRIX # 1716
STATE TRANSITION MATRIX # 1321

Table 8.3

SNUMB = 49592, ACTIVITY # = 02, REPORT CODE = 06, RECORD COUNT = 005459

MEASUREMENT NO. 2611CULLED

MEASUREMENT NO. 2612CULLED

MEASUREMENT NO. 2613CULLED

MEASUREMENT NO. 2614CULLED

MEASUREMENT NO. 2615CULLED

MEASUREMENT NO. 2840CULLED

MEASUREMENT NO. 2841CULLED

MEASUREMENT NO. 2842CULLED

MEASUREMENT NO. 2843CULLED

MEASUREMENT NO. 2844CULLED

MEASUREMENT NO. 2845CULLED

MEASUREMENT NO. 3180CULLED

MEASUREMENT NO. 3181CULLED

MEASUREMENT NO. 3182CULLED

MEASUREMENT NO. 3183CULLED

MEASUREMENT NO. 3184CULLED

MEASUREMENT NO. 3504CULLED

MEASUREMENT NO. 3505CULLED

MEASUREMENT NO. 3506CULLED

MEASUREMENT NO. 3507CULLED

MEASUREMENT NO. 3508CULLED

Table 8.4

INDX	TIME OF DATA YYMMDD	HHMM	SS.SSSS	ELAPSED TIME (SEC)	MEASURMENT (METERS)	FORWARD HEIGHT (METERS)	BACKWARD HEIGHT (METERS)	SMOOTHED HEIGHT (METERS)	RESIDUAL (METERS)	SLOPE (ARCSEC)	SIGMA SLOPE (ARCSEC)	SIGMA SMOOTHED (METERS)
2821	750831	58	24.9293	422.9346	-15.2500	-14.4103	-16.3058	-15.6341	0.3841	-22.6946	3.4301	0.1186
2822	750831	58	25.0318	423.0370	-15.1563	-14.4857	-16.5171	-15.7017	0.5454	-21.3716	3.4446	0.1189
2823	750831	58	25.1342	423.1394	-14.3594	-14.4359	-16.7538	-15.7646	1.4052	-19.7844	3.4599	0.1192
2824	750831	58	25.2366	423.2418	-15.0391	-14.4878	-17.1276	-15.8221	0.7830	-17.9451	3.4756	0.1196
2825	750831	58	25.3390	423.3442	-15.8672	-14.6554	-17.4119	-15.8738	0.0066	-15.8756	3.4911	0.1201
2826	750831	58	25.4414	423.4466	-16.7578	-14.9388	-17.5745	-15.9193	-0.8385	-13.6096	3.5058	0.1207
2827	750831	58	25.5438	423.5490	-19.4375	-15.5910	-17.5943	-15.9584	-3.4791	-11.1933	3.5190	0.1214
2828	750831	58	25.6462	423.6514	-20.5703	-16.3540	-17.1715	-15.9909	-4.5794	-8.6859	3.5300	0.1222
2829	750831	58	25.7486	423.7538	-21.8281	-17.2339	-16.5135	-16.0171	-5.8110	-6.1568	3.5380	0.1231
2830	750831	58	25.8510	423.8563	-20.9219	-17.8996	-15.5728	-16.0369	-4.8850	-3.6803	3.5424	0.1241
2831	750831	58	25.9534	423.9587	-18.1016	-18.0821	-14.6741	-16.0505	-2.0511	-1.3272	3.5427	0.1252
2832	750831	58	26.0558	424.0611	-14.3359	-17.6777	-14.1484	-16.0580	1.7221	0.8429	3.5383	0.1263
2833	750831	58	26.1582	424.1635	-13.1641	-17.1140	-14.2235	-16.0598	2.8957	2.7874	3.5292	0.1275
2834	750831	58	26.2606	424.2659	-13.4922	-16.6271	-14.5308	-16.0562	2.5640	4.4815	3.5154	0.1287
2835	750831	58	26.3630	424.3683	-14.8438	-16.3656	-14.8281	-16.0478	1.2040	5.9145	3.4975	0.1298
2836	750831	58	26.4654	424.4707	-16.8359	-16.4092	-14.8947	-16.0353	-0.8006	7.0855	3.4763	0.1309
2837	750831	58	26.5678	424.5731	-15.9688	-16.3158	-14.4972	-16.0194	0.0507	7.9999	3.4530	0.1319
2838	750831	58	26.6702	424.6755	-14.1641	-15.9589	-14.1946	-16.0008	1.8367	8.6692	3.4294	0.1327
2839	750831	58	26.7726	424.7779	-14.3047	-15.6466	-14.3233	-15.9799	1.6752	9.1097	3.4072	0.1335
2840	750831	58	26.8751	424.8803	-14.9688*	-15.5386	-14.4452	-15.9571	0.9884	9.3405	3.3884	0.1340
2841	750831	58	26.9775	424.9827	-15.4141*	-15.4224	-14.5550	-15.9330	0.5190	9.3803	3.3747	0.1344
2842	750831	58	27.0799	425.0851	-15.0078*	-15.2987	-14.6508	-15.9080	0.9002	9.2456	3.3675	0.1345
2843	750831	58	27.1823	425.1875	-14.7813*	-15.1680	-14.7522	-15.8825	1.1012	8.9513	3.3675	0.1345
2844	750831	58	27.2847	425.2899	-15.3828*	-15.0307	-14.8593	-15.8569	0.4741	8.5113	3.3747	0.1344
2845	750831	58	27.3871	425.3923	-16.3828*	-14.8873	-14.9519	-15.8317	-0.5511	7.9395	3.3884	0.1340
2846	750831	58	27.4895	425.4947	-15.7813	-15.0479	-15.0399	-15.8075	0.0262	7.2512	3.4073	0.1335
2847	750831	58	27.5919	425.5971	-14.6719	-14.8531	-15.0058	-15.7846	1.1127	6.4640	3.4294	0.1327
2848	750831	58	27.6943	425.6996	-14.9688	-14.7686	-15.1478	-15.7635	0.7948	5.5984	3.4531	0.1319
2849	750831	58	27.7967	425.8020	-15.4922	-14.8018	-15.2558	-15.7448	0.2526	4.6769	3.4763	0.1309
2850	750831	58	27.8991	425.9044	-15.2969	-14.7846	-15.2908	-15.7287	0.4318	3.7220	3.4975	0.1298

Table 8.5

Note the asterisks after measurements 2840 - 2845.

8.3.3 PLOT Tape Format

This type of output tape contains information on the variables produced as output from the filter. The tapes are 800 BPI, standard label tapes. The main use of these tapes is in plotting certain output variables versus time, as exemplified in Figures 8.1 and 8.2. The data record format for this tape is as follows:

WORD	FORMAT	DESCRIPTION
1	R	Time in seconds from the first data point of the arc.
2	R	The input measurement as read from the input tape.
3	R	The value* returned from filtering in the forward direction.
4	R	The value* returned from filtering in the backward direction.
5	R	The final combined smoothed value.
6	R	The residual of the measurement about the smoothed value (WORD(2) - WORD(5)).

*These values correspond to the a posteriori values returned from the filter after improvement based on the individual measurement.

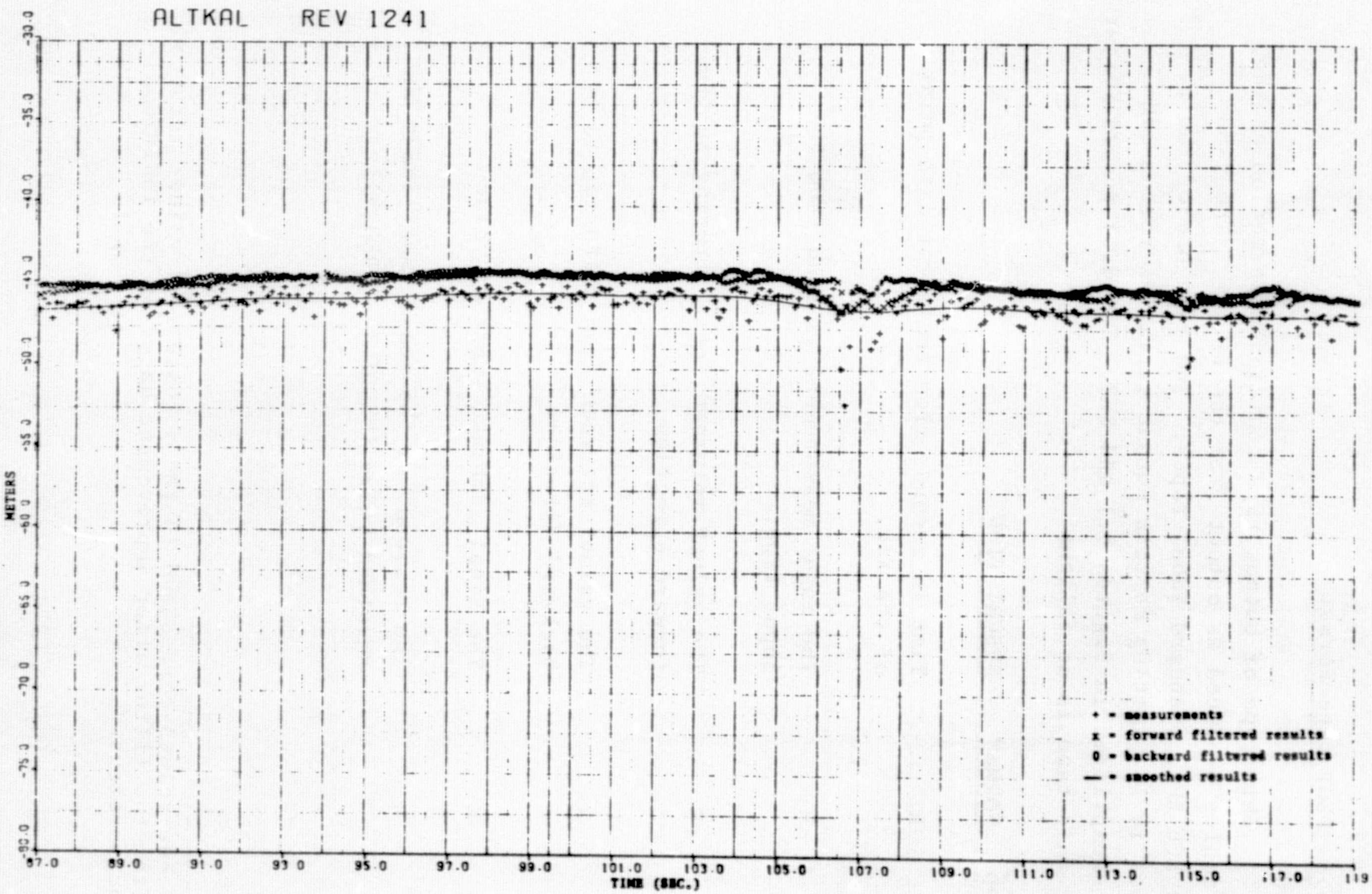

ALTKAL REV 1241
METERS
TIME (SEC.)
+ = measurements
x = forward filtered results
0 = backward filtered results
— = smoothed results

Figure 8.1

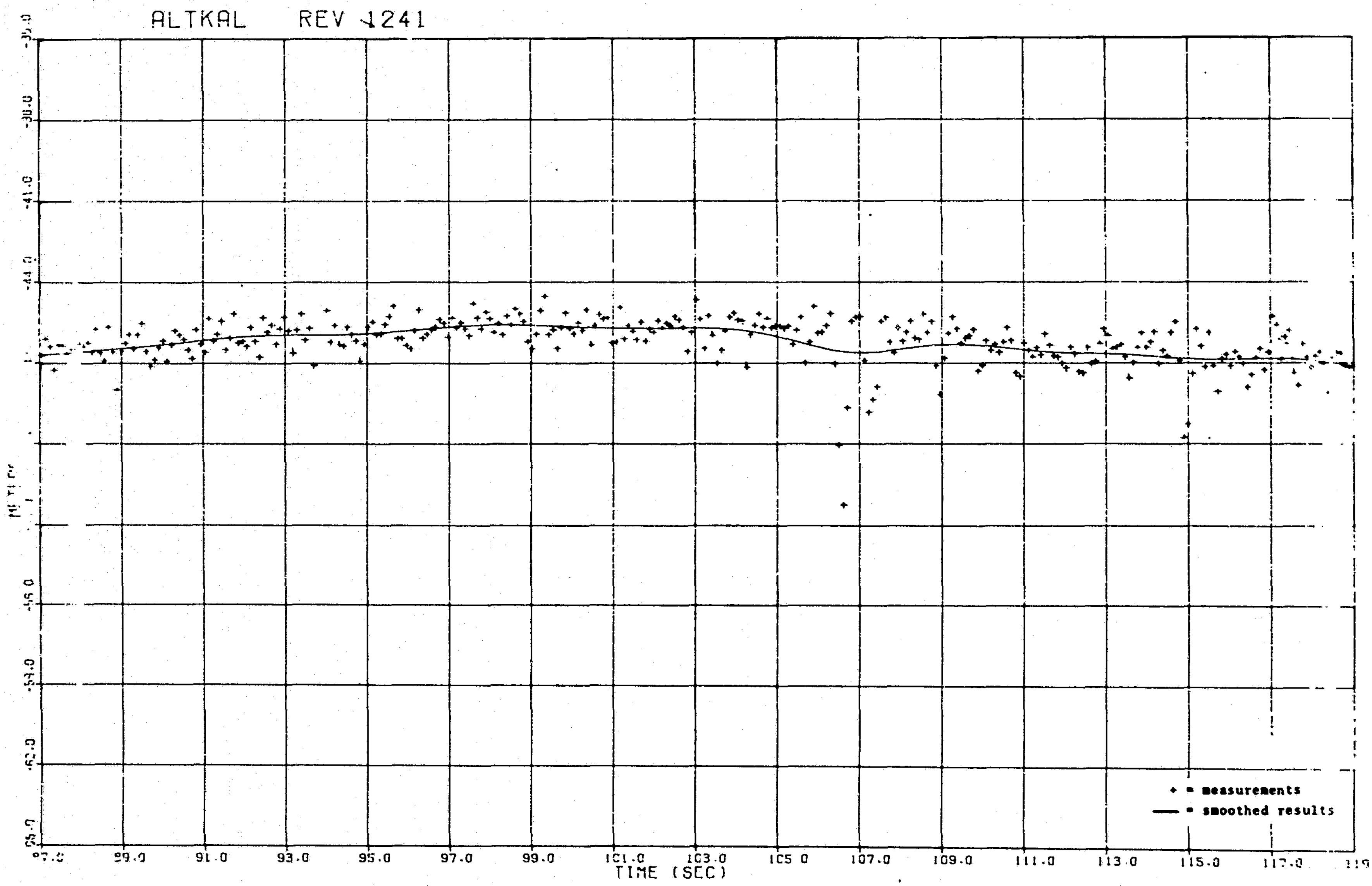


Figure 8.2

8.3.4 SEAHT Tape Format

This type of output tape is a tape for use as input to the SEAHT program. The tape is an 800 BPI, standard label tape. It contains one header record and then subsequent data records. The record formats are as follows:

SEAHT INPUT FORMAT
HEADER RECORD

WORD NO.	TYPE	DESCRIPTION
1	I	Year and day of year
2-3	DP(FRAMTI)	Frame time of day (seconds)
4	I	Segment number - must be input to ALTKAL
5	I	Rev number - must be input to ALTKAL
6	I(MTYPE)	Measurement type (40 or 41)
7-19	I	Zeroes (used for making all records same length)

SEAHT INPUT FORMAT
DATA RECORD

WORD NO.	TYPE	DESCRIPTION
1	R	Time from first data point (minutes)
2	R(LAT)	Latitude of first smoothed sea surface height in record
3	R(LON)	Longitude of first smoothed sea surface height in record
4-13	R	Smoothed sea surface heights
14	R(THITE)	Tide height
15	R(TREF)	Tropospheric refraction correction
16	R(RAGCAV)	Average AGC
17-19	R	Zeroes for future additions

APPENDIX A
PROCESSING TWO SETS OF UNCORRELATED DIRECT MEASUREMENTS

A number of alternative algorithms for obtaining an optimum estimate from two sets of uncorrelated estimates, or direct measurements, are presented in this Appendix.

Given: Two sets of estimates and estimate error covariances

$$[x_a] \quad , \quad [p_a] = E([x_a - x][x_a - x]^T)$$

$$[x_b] \quad , \quad [p_b] = E([x_b - x][x_b - x]^T)$$

$$E([x_a - x][x_b - x]^T) = [0] \Longleftrightarrow$$ the estimates are uncorrelated.

$[x_a]$ and $[x_b]$ may either be estimates, or direct measurements.

Determine:

New estimate and covariance

Solution:

It is well known and easily proven that the optimum estimate and its covariance are given by

$$[\hat{x}] = [p]\ ([p_a]^{-1}[x_a] + [p_b]^{-1}[x_b])$$

$$[p] = ([p_a]^{-1} + [p_b]^{-1})^{-1}$$

These expressions are not in the most convenient computational forms. We shall give several alternative forms and show the only matrix inversion required is $([p_a] + [p_b])^{-1}$.

<u>Alternative Formulations:</u>

$$[p] = ([p_a]^{-1}+[p_b]^{-1})^{-1} = ([p_a]^{-1}([p_b] + [p_a])\ [p_b]^{-1})^{-1}$$

$$= [p_b]([p_b]+[p_a])^{-1}[p_a] = [p_a]([p_a]+[p_b])^{-1}[p_b]$$

$$= ([p_b]+[p_a]-[p_a])([p_b]+[p_a])^{-1}[p_a]$$

$$= [p_a]-\underbrace{[p_a]([p_a]+[p_b])^{-1}}[p_a] \longleftrightarrow \text{Variance Equation}$$

$$\triangleq [p_a]-[K][p_a]$$

$$[K] \triangleq [p_a]([p_a]+[p_b])^{-1} \longleftrightarrow \text{Optimum Gain}$$

$$= [p]\ [p_b]^{-1}$$

$$[\hat{x}] = [p]([p_a]^{-1}[x_a]+[p_b]^{-1}[x_b])$$

$$= [p]\ \{([p_a]^{-1}+[p_b]^{-1}-[p_b]^{-1})\ [x_a]+[p_b]^{-1}[x_b]\}$$

$$= [x_a]+[p][p_b]^{-1}([x_b]-[x_a])$$

$$= [x_a]+[K]([x_b]-[x_a])$$

Applications:

1. $[x_a]$ and $[x_b]$ may be considered as the forward and backward filtered estimates.

2. Application to altimeter measurements.

In this case one may take $[x_b] = \begin{bmatrix} x_{a_1} \\ x_{a_2} \\ y \end{bmatrix}$

and $[p_b] = \begin{bmatrix} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & R \end{bmatrix}_{\ell \to \infty}$

Therefore,

$$([p_a]+[p_b])^{-1} = \begin{bmatrix} 1/\ell & 0 & 0 \\ 0 & 1/\ell & 0 \\ 0 & 0 & (R+p_{a_{33}})^{-1} \end{bmatrix}_{\ell \to \infty}$$

$$K = [p_a]([p_a]+[p_b])^{-1}$$

$$= \frac{1}{R+p_{a_{33}}} \begin{bmatrix} 0 & 0 & p_{a_{13}} \\ 0 & 0 & p_{a_{23}} \\ 0 & 0 & p_{a_{33}} \end{bmatrix}$$

$$[K]([x_b]-[x_a]) = \frac{1}{R+p_{a_{33}}} \begin{bmatrix} p_{a_{13}} \\ p_{a_{23}} \\ p_{a_{33}} \end{bmatrix} (y-x_{a_3})$$

$$[K][p_a] = \frac{1}{R+p_{a_{33}}} \begin{bmatrix} p_{a_{13}} \\ p_{a_{23}} \\ p_{a_{33}} \end{bmatrix} [p_{a_{13}} \quad p_{a_{23}} \quad p_{a_{33}}]$$

One sees these are exactly the same equations relating the a priori and a posteriori estimates given previously in Section 4.2.

APPENDIX B

DERIVATION OF SMOOTHER DISCRETE TRANSFER FUNCTION

In processing the mid-portion of a long data sequence,* the forward and backward filter gains approach their steady-state value.* The smoother becomes stationary and may be represented by a transfer function. Since the transfer function gives additional insight to the smoother behavior, it is derived in this Appendix.

From Equations (4.4) and (4.10) one obtains the following equation governing the "a posteriori" forward state estimate,

$$[x^f_{n+1}] = [\phi^f]\ [x^f_n] + [K]\ y_{n+1} \tag{B.1}$$

where

$$[\phi^f] = ([E] - [K\,\vdots\,0\,\vdots\,0])\ [\phi].$$

$$[E] = \text{identity matrix}$$

In the steady-state $[\phi^f]$ and [K] are constants and one may take the z-transform of Equation (B.1) to obtain

$$z[X^f_n] = [\phi^f]\ [X^f_n] + z[K]\ \ Y_n \quad ,$$

or

$$[X^f_n] = z(z[E] - [\phi^f])^{-1}\ [K]\ \ Y_n \tag{B.2}$$

*For constant data sampling rates and no missing or edited data.

where

$$[X_n] = Z\{[x_n]\}$$

$$Y_n = Z\{y_n\}$$

Similarly, the "a priori" backward state estimate is given by,

$$[x^{b-}_{n+1}] = [\phi]\ [x^b_{n+2}]$$

$$= [\phi]\ \{[\phi^f]\ [x^b_{n+3}] + [K]\ y_{n+2}\} \qquad (B.3)$$

For brevity in the above equations we have suppressed the superscript "+" for "a posteriori" estimate, but have retained the superscript "-" for "a priori" estimate.

By taking the z-transform of Equation (B.3), one obtains

$$Z\{[x^{b-}_{n+1}]\} = [\phi]\ [X^b_{n+2}]$$

$$= [\phi]\ \{z[\phi^f]\ [X^b_{n+2}] + z^2[K]\ Y_n\}$$

or

$$Z\{[x^{b-}_{n+1}]\} = z^2[\phi]\ ([E] - z[\phi^f])^{-1}\ [K]\ Y_n \qquad (B.4)$$

The smoothed estimate is taken as the optimally-weighted combination of the "a posteriori" forward estimate and the "a priori" backward estimate; i.e.,

$$[x_{n+1}] = [W]_F \; [x^f_{n+1}] + [W]_B \; [x^{b-}_{n+1}]$$

Taking the z-transform of this equation, and making use of Equation (B.1) and (B.4), one obtains

$$z[X_n] = Y_n \; z^2\{([W]_F(z[E]-[\phi^f])^{-1} + [W]_B[\phi] \; ([E]-z[\phi^f])^{-1})\}[K] \, ,$$

or, the transfer matrix,

$$[H(z)] \triangleq \frac{1}{Y_n} \; [X_n] = (z[W]_F[F(z)] + [W]_B[F(\tfrac{1}{z})])[K] \qquad \text{(B.5)}$$

where

$$F(z) = (z[E] - [\phi^f])^{-1} \; .$$

The third component of [H(z)] is of particular interest as it is the z transfer function relating the geoid undulation estimate to the altimeter measurements. Notice that the transfer matrix contains negative as well as positive powers of z, indicating the state estimate depends on "past" as well as "future" data, which is, of course, the essence of smoothing. The gain [K] and weighting matrices $[W]_F$ and $[W]_B$ appearing in the expression (B.5) for the transfer matrix are obtainable from the steady-state solution of the filter variance equation.

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NOMENCLATURE

[ ]	= matrix
E( )	= expectation of ( )
b	= altimeter noise correlation function decay constant, 1/b is the (1/e) noise correlation time
h	= geoid undulation
G	= converted measurement noise variance
[K]	= optimum gain matrix
n	= altimeter noise
[p]	= estimate error covariance matrix
[Q]	= increment in covariance matrix resulting from dynamic noise excitation
R	= altimeter noise variance
t	= time
T	= sampling period
u	= white noise generating exponentially correlated altimeter noise
x_1, x_2, x_3	= geoid undulation state variables as defined in Equation (3.5). x_3 is the geoid undulation.

y = altimeter measurement with orbit height and other known corrections subtracted

w = white noise exciting the geoid undulation state

α = $1/2\beta$

β = geoid undulation correlation decay constant, $2.903/\beta$ is the (1/e) geoid undulation correlation time

γ = $\frac{8}{3}\sigma^2\beta^4$

ν = white measurement noise in converted measurement defined in Equation (6.4).

$[\phi]$ = geoid undulation state transition matrix defined in Equation (3.7)

σ^2 = a priori geoid undulation variance

τ = time

Subscripts

i,j,k = indices for time
b = output from backward filtering
s = output from smoother

Superscripts

+ = immediately after measurement
- = immediately before measurement
$\wedge$ = estimated quantity