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NASA Contractor Report 145246

DIGITAL FLIGHT CONTROL SYSTEMS

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CONTENTS

I. INTRODUCTION

Under NASA Grant NGR-47-004-116, two major problems have been studied. The first problem studied is the design of stable feedback control laws for sampled-data systems with variable rate sampling. These types of sampled-data systems arise naturally in digital flight control systems which use digital actuators. In these control systems, it is desirable to decrease the number of control computer output commands in order to save wear and tear of the associated equipment. Variable sampling also provides the designer with the capability of a more efficient utilization of the flight control computer than the standard fixed sampling-rate approach. Therefore, a variable sampling approach can also be of value to digital flight control systems using analog actuators. For instance, more time can be devoted to the identification of aircraft parameters or to some other task by reducing the control calculations using variable sampling.

The second major problem studied under NASA Grant NGR-47-004-116 is the design of aircraft control systems which are optimally tolerant of sensor and actuator failures. The first problem to be resolved is the detection of the failed sensor or actuator. If the estimate of the state is used in the control law, then it is also desirable to have an estimator which will give the optimal state estimate even under the failed conditions. Both the detection of sensor and actuator failures and the optimal state estimation with sensor and actuator failures are important control system problems which, if not resolved, can seriously (even fatally) degrade the control system performance of an aircraft.

II. OPTIMAL CONTROL OF SAMPLED-DATA SYSTEMS WITH VARIABLE SAMPLING

At each flight condition, the airplane dynamics can be modeled by a continuous, linear, time-variant, dynamic system [1]. Comparative simulations have indicated that a model-follower scheme in which the error between the model states and the plant states is penalized continuously in time was a more suitable approach in the design of control laws for sampled-data systems with variable sampling than designs based on a minimization of error at only the sampling instants. The aircraft dynamics are continuous and the gust inputs affecting the airplane are continuous random processes, but the measurements are made only at the sampling instants and the control is constrained to be constant between the sampling instants. Thus, the problem can be cast into the format of the stochastic sampled-data regulator problem of linear stochastic optimal control theory. The first problem to be resolved was to find out whether the separation theorem of linear optimal control continues to hold for the stochastic sampled-data regulator problem. This problem has been resolved and the results are reported in reference [2] along with a discrete-time stochastic problem which is equivalent to the stochastic sampled-data regulator problem. A summary of the results follows.

The stochastic sampled-data regulator problem is to find the stochastic optimal control for the dynamical system represented by

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \qquad t\varepsilon[t_o, t_N] \qquad (1)$$

where x is the n-dimensional state vector, u is the r-dimensional control vector, and w is the white Gaussian plant noise vector of dimension n with $Ew(t) = 0$ and $Ew(t)w'(s) = \Psi_w \delta_D(t-s)$ for some positive semidefinite

matrix Ψ_w. E denotes the expectation operator and δ_D is the Dirac delta function. A and B are matrices of appropriate order.

The plant noise is a continuous random process; however, the measurements are available only at the sampling instants:

$$y(t_k) = Cx(t_k) + v(t_k) \qquad k = 0, 1, 2, \ldots, N-1 \qquad (2)$$

where y is the m-dimensional measurement vector, and v is a Gaussian sequence of uncorrelated zero-mean random vectors with $Ev(t_k)v'(t_j) = \Psi_v \delta_{kj}$.

The cost functional penalizes the state and the control continuously in time

$$J = \frac{1}{2} E\int_{t_0}^{t_N} [x'(t)Q_c x(t) + u'(t)R_c u(t)] \, dt \qquad (3)$$

where Q_c is positive semidefinite and R_c is positive definite. The stochastic sampled-data regulator problem is to find the control sequence $\{u^*(t_k)\}_{k=0}^{N-1}$ with the constraint

$$u^*(t) = u(t_k) \qquad t\epsilon(t_k, t_{k+1}) \qquad k = 0, 1, 2, \ldots, N-1 \qquad (4)$$

and also the additional constraint that $u^*(t_k)$ will depend only on the past measurement sequence $y(t_i)$, $i = 1, 2, \ldots, k$ which will minimize the cost functional (3).

In reference [2], it is shown that this stochastic optimal control problem is equivalent to finding the optimal control for the discrete system

$$x_{k+1} = \Phi(t_{k+1}, t_k)x_k + \Gamma(t_{k+1}, t_k)u_k + w_k$$

$$y_k = C_k x_k + v_k \qquad k = 0, 1, \ldots, N-1 \qquad (5)$$

where $x_k = x(t_k)$, $u_k = u(t_k)$, $\Phi(t_{k+1}, t_k) = \exp A(t_{k+1} - t_k)$, $\Gamma(t_{k+1}, t_k) =$

$$\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},s) \, dsB, \text{ and } w_k \text{ is a zero-mean Gaussian sequence of random}$$

vectors with

$$Ew_k w_j' = [\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},s) \, \Psi_w \Phi'(t_{k+1},s)ds]\delta_{kj} \tag{6}$$

with the cost functional

$$J = \frac{1}{2} E \sum_{k=0}^{N-1} (x_k'Q_k x_k + 2x_k'M_k u_k + u_k'R_k u_k) \tag{7}$$

where

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi'(t,t_k)Q_c \, \Phi(t,t_k)dt \tag{8}$$

$$M_k = \int_{t_k}^{t_{k+1}} \Phi'(t,t_k)Q_c \, \Gamma(t,t_k)dt \tag{9}$$

$$R_k = \int_{t_k}^{t_{k+1}} [R_c + \Gamma'(t,t_k)Q_c \, \Gamma(t,t_k)]dt \tag{10}$$

The stochastic optimal control is shown to be given by

$$u_i^* = -[R_i^{-1}M_i' + (R_i + \Gamma_i'K_{i+1}\Gamma_i)^{-1}\Gamma_i'K_{i+1} \overline{\Phi}_i]\hat{x}_i$$

$$\tag{11}$$

$$i = 0, 1, 2, \ldots, N-1$$

where K_i is the solution to the Riccati difference equation

$$K_i = \overline{\Phi}_i'[K_{i+1} - K_{i+1}\Gamma_i(R_i + \Gamma_i'K_{i+1}\Gamma_i)^{-1}\Gamma_i'K_{i+1}] \overline{\Phi}_i + \overline{Q}_i \tag{12}$$

with $K_N = 0$ and where $\Gamma_i = \Gamma(t_{i+1},t_i)$, $\overline{\Phi}_i = \Phi(t_{i+1},t_i) - \Gamma_i'R_i^{-1}M_i'$,

$\overline{Q}_i = Q_i - M_i R_i^{-1}M_i'$, and $\hat{x}_i$ is the conditional expectation of x_i given the

observation sequence $\{y_0, y_1, \ldots, y_i\}$.

Looking at equations (11) and (12), it is seen that the optimal con-

trol law for the stochastic sampled-data regulator problem is the same as

the deterministic sampled-data regulator problem [3] with x_k replaced by

its estimate $\hat{x}_k$. Therefore, the separation between estimation and con-

trol continues to hold for the stochastic sampled-data regulator problem.

The only modification needed for stochastic control is the use of the derived equivalent discrete plant covariance (6) in the Kalman filter equations for the state estimates.

As a byproduct of this investigation, new results were also obtained concerning the geometric relationship between the optimal solutions to the sampled-data and the continuous regulator problems. The findings are reported in [4]. A brief summary of findings is given below.

Since the dynamical system (1) and the integral cost function (3) used are the same for both continuous and the sampled-data regulators, the optimal sampled-data control must be intuitively an approximation to the continuous one in some sense. The relationship between the two optimal solutions, that is, the sense in which the sampled-data solution is an approximation to the continuous solution, has been obscured due to the separate formulations of these two problems in the control literature. The continuous problem has been solved by using the Pontryagin's minimum principle, by using the Hamilton-Jacobi-Bellman partial differential equation for the optimal cost function, and by some other methods. The sampled-data problem has been solved by converting it into an unconstrained discrete minimization problem through the integration of the cost functional and the system differential equations over each sampling interval and then applying dynamic programming or the discrete minimum principle. In our study, the two problems have been formulated in the same Hilbert space as minimum norm problems. In this geometric formulation, it is shown in reference [4] that the optimal sampled-data control is a "projection" of the optimal continuous control. Specifically, it is shown, that if $[x^*(t), u^*(t)]$ is the optimal continuous regulator solution and $[x^{**}(t), u^{**}(t)]$ is the optimal sampled-data solution to the corres-

ponding deterministic regulator problems, then

$$[Hx^{**}(t), u^{**}(t)] = [H\Phi(t,t_o)x(t_o), 0] +$$

$$P_N[Hx^*(t) - H\Phi(t,t_o)x(t_o), u^*(t)]$$

(13)

where H is defined by $H^T H = Q_c$, N is the sub-Hilbert space of all output-input pairs satisfying the sampled-data constraint with zero initial condition, and P denotes the projection operator with respect to the Hilbert space norm

$$||(x,u)|| = \frac{1}{2} \int_{t_o}^{t_f} [x'(t)Q_c x(t) + u'(t)R_c u(t)]dt$$

(14)

One of the implications of the result is that if an optimal sampled-data tracking problem is to be solved where the desired trajectory to be tracked is the optimal continuous regulator solution, then the solution obtained would be the same as the solution to the original sampled-data regulator problem. To phrase it precisely, the optimal sampled-data regulator solution is the sampled-data control that minimizes.

$$\frac{1}{2} \int_{t_o}^{t_f} \{[x(t) - x^*(t)]'Q_c[x(t) - x^*(t)] +$$

$$[u(t) - u^*(t)]'R_c[u(t) - u^*(t)]\}dt$$

(15)

The above equation clearly shows the sense of the approximation of the sampled-data control. Equation (14) shows that the optimal sampled-data solution is the projection of the optimal continuous solution onto the set of all solutions that satisfy the sampled-data constraint. Furthermore, the specific projections have been converted into recursive algorithms to compute the optimal sampled-data control. These algorithms are new; however, they are not necessarily less complicated than the known Riccati equations in the literature.

Stochastic Modeling Approach to Variable Sampling

During the course of the project, a number of sampling interval adaptation control laws have been developed based on minimization of local objective functions. Although these algorithms proved to be successful in various simulations, they suffered from being only locally optimal. A control law that would take into account the changing of the sampling intervals was needed. This was accomplished by modeling the sampling interval sequence as a finite-state Markov chain with known transition probabilities. The finite-state assumption, that is, the constraint that the sampling interval can assume values only from a finite number of sampling intervals, was necessary to avoid an infinite set of equations. Specifically, the system equations are given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{16}$$

where x, u, A, B are as in equation (1). The sampling interval sequence $\{T_k, k = 0, 1, 2, \ldots\}$ is a finite-state Markov chain that assumes values $\{S_1, S_2, \ldots, S_n\}$ with the transition probability

$$P(T_{k+1} = S_j \mid T_k = S_i) = P_{ij} \tag{17}$$

and the initial probability distribution

$$P(T_0 = S_i) = P_i, \quad i = 1, 2, \ldots, n \tag{18}$$

Then, the discrete-time stochastic system is given by

$$x_{k+1} = \Phi(t_k + T_k, t_k)x_k + \Gamma(t_k + T_k, t_k)u_k,$$

$$k = 0, 1, \ldots, N-1 \tag{19}$$

where Φ, Γ are as in equation (5). The observations are modeled as

$$y_k = (x_k, T_k) \qquad (20)$$

That is, at each sampling instant t_k, the state vector at t_k is known and the sampling interval T_k to be applied at the instant t_k is known. Note that the sampling intervals to be used at t_{k+1} and further on are uncertain at time t_k. Only a statistical knowledge of their uncertainty is known through the transition probabilities. Measurement noise and plant noise are not included in the model to keep the equations simple. The problem is to find the stochastic optimal control $\{u_k, k = 0, 1, \ldots, N-1\}$ that depends only on the past and present measurements $\{y_i, i = 0, 1, 2, \ldots, k\}$ that minimizes

$$J = \frac{1}{2} E[\sum_{k=0}^{N-1} (x_k'Qx_k + u_k'Ru_k) + x_N'Hx_N] \qquad (21)$$

The stochastic optimal control has been obtained by dynamic programming and is given by

$$u_k = -F(T_k)x_k \qquad (22)$$

where $F(T_k) = F_{k,i}$ for $T_k = S_i$, given by the recursive equations

$$F_{k,i} = [R + \Gamma_i'(\sum_{j=1}^{n} p_{ij}K_{k+1,j})\Gamma_i]^{-1}\Gamma_i(\sum_{j=1}^{n} p_{ij}K_{k+1,j})\Phi_i \qquad (23)$$

$$K_{k,i} = Q + (\Phi_i - \Gamma_i F_{k,i})'(\sum_{j=1}^{n} p_{ij}K_{k+1,j})(\Phi_i - \Gamma_i F_{k,i}) + F'_{k,i}RF_{k,i} \qquad (24)$$

with $K_{N,i} = H$; $i = 1, 2, \ldots, n$; $k = 0, 1, \ldots, N-1$; and $\Phi_i = \Phi(S_i, 0)$, $\Gamma_i = \Gamma(S_i, 0)$.

That is, at each sampling instant t_k the control is given by a linear feedback law and the feedback gain at the instant t_k is $F_{k,i}$ where the "i" corresponds to the sampling interval S_i at the instant t_k. The recursive equations are similar to the standard Riccati equations of

linear optimal control. As it can be seen from equation (24), n coupled Riccati equations are to be solved recursively. The coupling comes from the terms P_{ij}, corresponding to the transition probabilities for the sampling interval sequence. In the case when n = 1 and P_{11} = 1, we get the standard Riccati equation of optimal control.

Steady-state gains can be obtained as k → -∞, then lim. k → -∞

$$F_{k,i} = F_i, \quad i = 1, 2, \ldots, n.$$

The stochastic optimal control for the infinite time problem becomes

$$u_k = -F_i x_k \text{ for all k such that } T_k = S_i.$$

The infinite time optimization solution is very easy to implement. Steady-state gains, $\{F_i, i = 1, 2, \ldots, n\}$, where n is the total number of possible sampling intervals, are calculated off-line and stored. At each sampling instant t_k the feedback gain F_i is used where "i" is determined by the sampling interval S_i at time t_k. The algorithms for the stochastic optimal control, equations (23) and (24), are not yet available in the literature.

The difficulty of applying these results can be the selection of a sampling interval adaptation law, since the problem formulation does not address that question. However, for any sampling interval adaptation law whose sampling interval sequence can be modeled by a Markov chain with appropriate transition probabilities, the stochastic optimal control derived can be used. Moreover, several examples have been simulated on the digital computer where the sampling interval sequence was chosen arbitrarily and the results have been very successful. A computer program that calculates the stochastic optimal control gains has been written. Given next is an example showing how the results can be applied to a specific problem.

Example

Let the system matrices be given by

$$A = \begin{bmatrix} 0 & 6.28 \\ -6.28 & -3.14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{25}$$

The control weighting matrices have been chosen to be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = [1] \tag{26}$$

Sampling intervals are chosen as

$$S_1 = .1 \qquad S_2 = .01 \tag{27}$$

The transition probabilities have been chosen to be the simplest case

$$P_{ij} = .5 \qquad i,j = 1,2 \tag{28}$$

That is, if the sampling interval at t_k is S_1, the chance of the next interval being the same and the chance of the sampling interval jumping to S_2 are equally likely.

The stochastic optimal control gains have been solved by using equations (23) and (24) to obtain the steady-state values

$$F_1 = [-.0181 \qquad .4455]$$

$$F_2 = [.01246 \qquad .0612]$$

The stochastic optimal closed-loop system has been simulated by using arbitrary sampling interval sequences. Figure 1 shows one of these runs. For this fixed sampling interval sequence, the deterministic optimal control feedback gains have been found also, and the closed-loop simulation of this system is in Figure 2. An overlay of the two figures would show that the two trajectories are virtually the same. For a
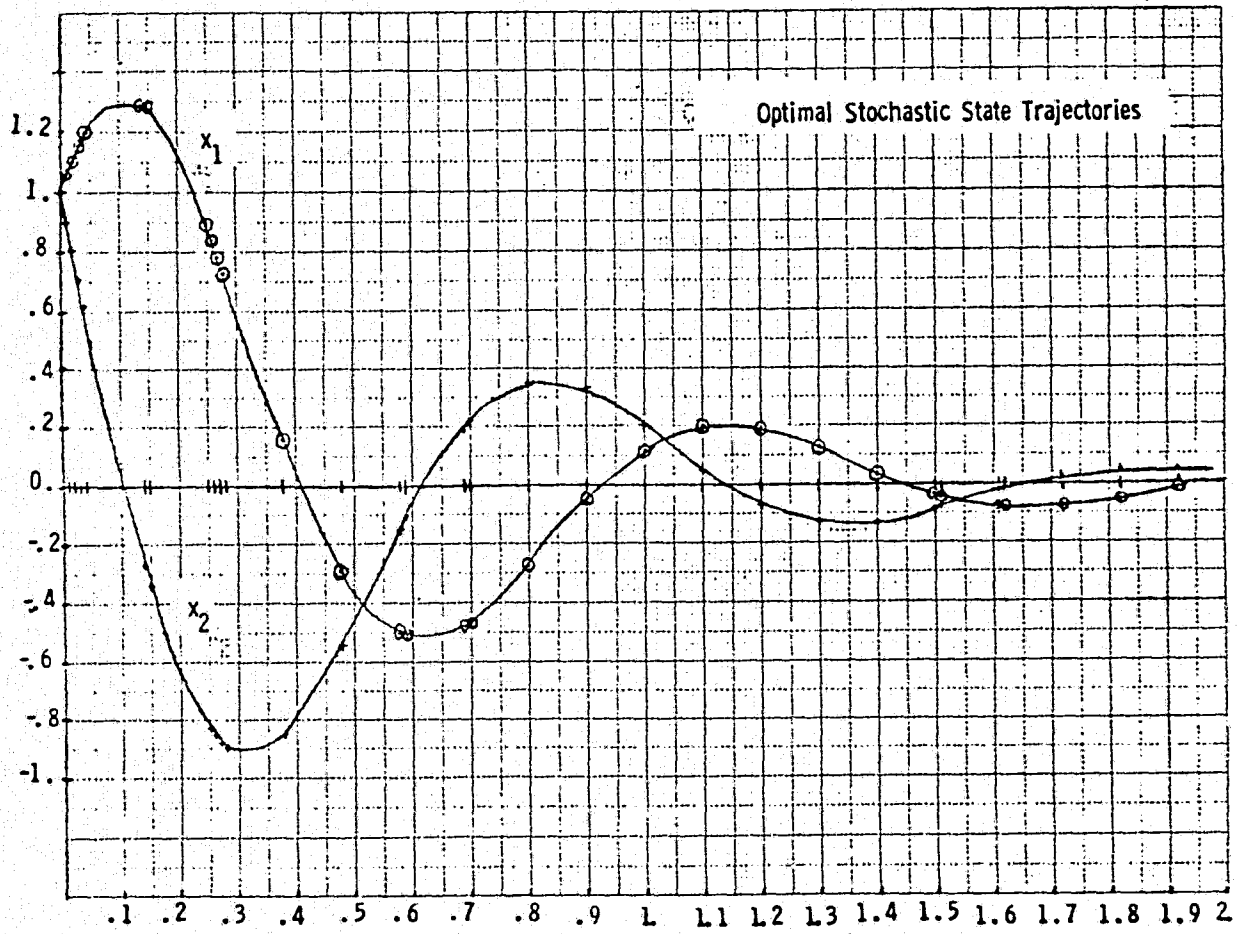
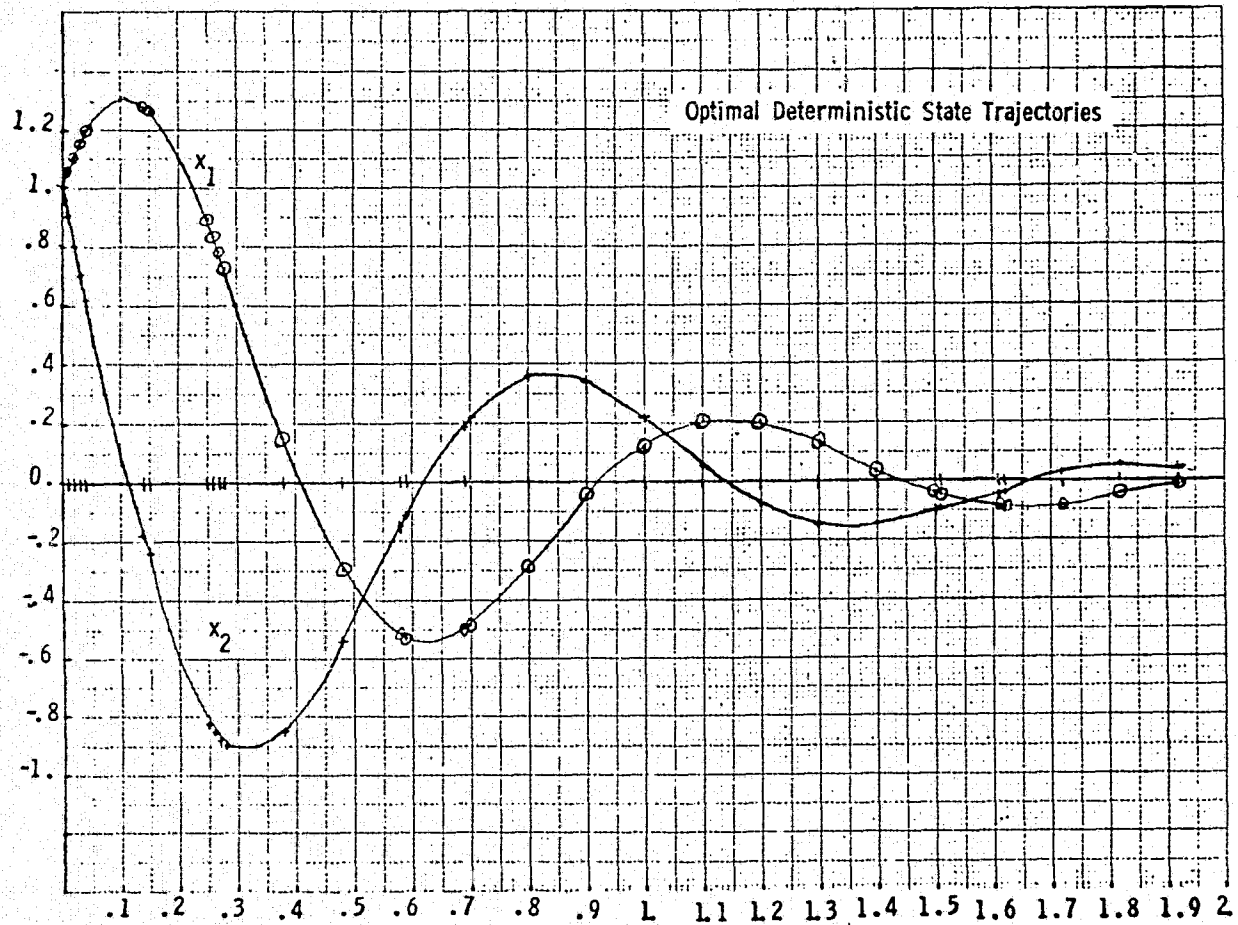Figure 1. Optimal Stochastic State Trajectories

Figure 2. Optimal Deterministic State Trajectories

fixed sampling interval sequence and the cost function of (21), expecta-
tion can be dropped when the sampling interval sequence is fixed because
the system becomes deterministic in this case. Since this resulting
deterministic time-varying linear quadratic regulator solution is optimal,
the closeness of the stochastic and the deterministic trajectories are
quite noteworthy. The remarkable aspect of the stochastic optimal control
is that it not only matches the ultimate optimal deterministic performance
for this fixed sampling interval sequence, but it also matches the per-
formance of several other sampling interval sequences in the simulations.
Of course, the deterministic optimal control is <u>not</u> realizable for the
problem we are considering since it requires the absolute knowledge of
the whole sampling interval sequence.

Stability of Sampled-Data Systems with Variable Sampling

The deterministic stability conditions of the stochastic model of
the previous section has been investigated jointly with Dr. D. P. Stanford
of the College William and Mary. The results have been reported in [5].
From the formulation in the previous section, it is seen that "n" closed-
loop discrete system matrices will be obtained

$$\Phi_1 - \Gamma_1 F_1, \ \Phi_2 - \Gamma_2 F_2, \ \cdot \ \cdot \ \cdot \ , \ \Phi_n - \Gamma_n F_n, \tag{29}$$

where "n" is the total number of possible sampling intervals and the "F_i"
are the stochastic optimal steady-state gains obtained through equations
(23) and (24). The question of whether any initial condition can be
brought to zero by a repeated application of the matrices in (29) in some
order has been investigated in the stability analysis [5]. If a sequence
of matrices can be found whose terms are selected from the set of matrices
in (23) for each initial condition such that the state goes to zero in the

limit when this sequence of matrices are applied, then the set of matrices will be called convergent. In [5], it is shown that "precontractiveness" is a necessary and sufficient condition for a set of matrices to be convergent. It is also shown that "contractiveness" is a sufficient condition for a set of matrices to be convergent. The stochastic optimal feedback gains have been found for a number of examples. It has been found that the closed-loop system matrices resulting from the stochastic optimal control turn out to be contractive in each case. However, an analytical justification has not been found yet.

III. FAILURE ACCOMMODATION IN CONTROL SYSTEMS

The second major problem which has been investigated in NASA Grant NGR-47-004-116 is the design of aircraft control systems which are optimally tolerant of sensor and actuator failures. A design method has been developed, and the results have been reported in Ref. [6]. The method developed is based on Bayesian decision theory.

Each sensor and actuator failure mode (including the normal operation mode) is formulated as one hypothesis. Using M-ary hypothesis testing, the corresponding likelihood ratios are computed for each hypothesis. The computations of likelihood ratios require M different Kalman filters corresponding to M different failure modes of the system where M is the total number of different hypotheses. By comparing the likelihood ratios, the most likely failure mode of the system is selected in the Bayesian sense.

The unique feature of this method is the flexibility of modeling the sensor failures as noise with unknown mean and variance. The mean of each sensor in a failed mode is computed on line by employing maximum likelihood estimation. This estimate for the mean is used in the calcu-

lation of the likelihood ratios which make them generalized likelihood
ratios. The advantage of the method is that it is not required for the
designer to know how the sensor will fail. This point has been demon-
strated in the simulation [7] by the superior performance of the system
in the detection of both increased noise type and hard-over type failures.
·Although the variances of the sensors in failed modes have been fixed
in Ref. [6], the same approach for the estimation of the mean can be used
for the variance as well. The maximum likelihood estimation of the para-
meters requires the storage of a moving window of innovations of each
Kalman filter.

The applicability of the fault-tolerant system design has been demon-
strated by using a real-time hybrid simulation for a space shuttle orbiter
developed jointly with Dr. R. C. Montgomery of NASA/Langley Research Center.
The failures were identified in two or three sampling periods. The simu-
lations indicate that the use of steady-state Kalman filters were adequate.

Parameter Adaptive Estimation

From the preceding failure detection problem in aircraft control
systems research, the additional benefit of obtaining the optimal state
estimate under failed conditions resulted. This problem is exceedingly
important if state variable feedback is used and if a filter is used to
get the state estimates. This estimation problem has been resolved in a
Ph.D. thesis [7] and extensions of the work have been reported in [8].
This problem is known as parameter adaptive estimation in the literature.
The following is a summary of the work reported in [7] and [8]. Consider
the observation model

$$y(t) = z(\Theta, t) + v(t) \tag{30}$$

where Θ is a random variable and, for each fixed value of Θ, z is a random process and v is a white Gaussian random process. The problem is to find the minimum mean-square estimate of the signal z. Detection problems can be easily modeled by this formulation with a suitable choice of the random variable Θ [7]. The general case when Θ has an arbitrary distribution has been worked in [8]. When the parameter Θ has a discrete distribution

$$P(\Theta = \Theta_i) = P_i, \ i = 1, 2, \ldots ; \tag{31}$$

it is shown in [8] and [9] that the minimum mean-square estimate of the signal z is given by

$$\hat{z} = \sum_{i=1}^{\infty} \pi_i(t) \hat{z}_i(t) \tag{32}$$

where $\hat{z}_i(t)$ is the estimate of the signal $z(\Theta_i, t)$ given the observation $y(s) = z(\Theta_i, s) + v(s)$, $0 \le s \le t$ and where $\pi_i(t)$ is given by

$$\pi_i(t) = P(\Theta = \Theta_i \mid y(s), \ 0 \le s \le t) \tag{33}$$

That is, $\{\hat{z}_i(t)\}_{i=1}^{\infty}$ are the parameter conditioned estimates and $\{\pi_i(t)\}_{i=1}^{\infty}$ are the <u>posterior</u> probabilities of the parameter Θ.

The <u>posterior</u> probabilities satisfy the stochastic differential equations

$$d\pi_i(t) = \pi_i(t)[\hat{z}_i(t) - \sum_{j=1}^{\infty} \pi_j(t)\hat{z}_j(t)]R^{-1}(t)[dy(t) - \sum_{j=1}^{\infty} \pi_j(t)\hat{z}_j(t)] \tag{34}$$

with the initial conditions

$$\pi_i(0) = P_i, \ i = 1, 2, \ldots \ . \tag{35}$$

The parameter conditioned estimator has two parts: A non-adaptive part in which the parameter conditioned estimates are found and an adaptive part in which the <u>posterior</u> probabilities are found. The form of the solution given by (32) and (34) implies that if recursive equations

are known for the parameter conditioned estimates, then a completely recursive solution is found to the problem. The recursive form of the solution is, of course, very advantageous in terms of implementation of the filter. The following example demonstrates how these results can be applied to a nonlinear filtering problem.

Example: Consider the random telegraph signal, $Z(t)$, with values ± 1 and transition density α where α is a random variable with prior distribution $P(\alpha = \alpha_i) = P_i$, $i = 1, 2, \ldots, M$. The observation model is given by

$$y(t) = Z(t) + v(t)$$

where v is a unit-variance Gaussian process, from (32) and (34), it is seen that the minimum mean-square estimate of the signal is given by

$$\hat{Z}(t) = \sum_{i=1}^{M} \pi_i(t) \hat{Z}_i(t)$$

where the parameter conditioned estimates $\hat{Z}_i(t)$ are known to be given by [8]

$$d\hat{Z}_i(t) = 2\alpha_i \hat{Z}_i(t) + [1 - \hat{Z}_i^2(t)][dy(t) - \hat{Z}_i(t)dt]$$

which, in turn, derives the stochastic differential equation (34)

$$d\pi_i(t) = \pi_i(t)[\hat{Z}_i(t) - \sum_{j=1}^{M} \pi_j(t)\hat{Z}_j(t)][dy(t) - \sum_{j=1}^{M} \pi_j(t)\hat{Z}_j(t)dt]$$

with $\pi_i(0) = P_i$, $i = 1, 2, \ldots, M$.

Directions for Further Research

In the work on optimal sampled-data regulators, a promising and interesting problem is to find early computable algorithms to perform the projection in equation (13). In this way, optimal sampled-data regulator gains can be obtained as a function of the optimal continuous regulator gains. So far the equations obtained are not any simpler than the known equations available in the literature. However, this study can lead to an understanding of how the sampled-data optimal gains vary with the sampling interval even though the algorithms may turn out to be complicated.

In the work on the stochastic modeling approach to variable sampling, more analytical and experimental study is needed. Analytical conditions that guarantee the convergence of gains for the steady-state gains [equations (23) and (24)] would be very useful. Simulation of a wide range of applications to different problems is needed to see how far the stochastic optimal law can be "stretched." So far, the simulations indicate that the stochastic optimal control law gives virtually the ultimate deterministic performance.

In the work on parameter adaptive estimation, more work is necessary on the following questions. What happens when the parameter does not have the prescribed distribution? In this case, for instance, does the set of _posterior_ probabilities converge to a value that is nearest to the actual value?

Also, additional effort is necessary to understand the stability characteristics of the stochastic differential equations for the _posterior_ probabilities (34).

REFERENCES

1. Hartman, G. L., Hauge, J. A., and Hendrick, R. C., "F8C Digital CCV Flight Control Laws," NASA CR-2629, Feb. 1976.

2. Caglayan, A. K. and Halyo, N., "A Separation Theorem for the Stochastic Sampled-Data Regulator Problem," International Journal of Control, 1976, Vol. 23, No. 2. (Appendix A)

3. Levis, A. H., Schlueter, R. A., and Athans, M., "On the Behaviour of Optimal Sampled-Data Regulators," 1971, Vol. 13, No. 2, pp. 343-361.

4. Caglayan, A. K. and Halyo, N., "On the Relation Between the Sampled-Data and the Continuous Optimal Linear Regulator Problems," Proceedings of the IEEE Conference on Decision and Control, Paper No. FP3-3, Dec. 1975, Houston, Texas. (Appendix B)

5. Caglayan, A. K. and Stanford, D. P., "Stability of Multirate Sampled-Data Systems," NASA Contractor Report CR-2589, Washington, D.C., Oct. 1975. (Appendix E)

6.[†]Montgomery, R. C. and Caglayan, A. K., "Failure Accommodation in Digital Flight Control Systems by Bayesian Decision Theory," AIAA Journal of Aircraft, Vol. 13, No. 2, Feb. 1976, pp. 69-75. (Appendix C)

7. Caglayan, A. K., "Optimal Parameter Adaptive Estimation of Stochastic Processes," Ph.D. Dissertation, Electrical Engineering Department, Virginia Polytechnic Institute and State University, Oct. 1974.(Ap. F)

8. Caglayan, A. K. and VanLandingham, H. F., "Parameter Adaptive Estimation of Random Processes," Proceedings of the IEEE Conference on Decision and Control, Dec. 1975, Houston, Texas. (Appendix D)

[†] Presented as "A Self-Reorganizing Digital Flight Control System for Aircraft", AIAA 12th Aerospace Sciences Meeting, Washington, D.C., January 1974.

A separation theorem for the stochastic sampled-data LQG problem†

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This paper considers the control of a continuous linear plant disturbed by white plant noise when the control is constrained to be a piecewise constant function of time; i.e. a stochastic sampled-data system. The cost function is the integral of quadratic error terms in the state and control, thus penalizing errors at every instant of time while the plant noise disturbs the system continuously. The problem is solved by reducing the constrained continuous problem to an unconstrained discrete one. It is shown that the separation principle for estimation and control still holds for this problem when the plant disturbance and measurement noise are Gaussian.

1. Introduction

Various studies of the deterministic discrete-time regulator problem have been made since the original work of Kalman and Koepcke (1958). A tutorial review of these basic results can be found in the paper by Dorato and Levis (1971). The deterministic sampled-data regulator problem has also been studied by Levis *et al.* (1971). The stochastic counterparts of the continuous and discrete linear quadratic Gaussian problems result in the separation principle for estimation and control (Joseph and Tou 1961, Gunckel and Franklin 1963, Potter 1964). Various extensions of the separation theorem to include a larger class of continuous and discrete cost functions have also been made (e.g. Streibel 1965, Wonham 1968, Halyo and Foulkes 1974, and the references therein).

In this paper the control of a stochastic linear sampled-data system (i.e. the stochastic counterpart of the work by Levis *et al.* (1971)) is considered. A continuous linear plant is disturbed by a continuous Gaussian white plant noise, while the control is constrained to be constant in between sampling instants, say $\{t_k\}$, but can change at every sampling instant. Measurements, which can be expressed as linear combinations of the state variables corrupted by Gaussian white noise, are made at the sampling instants. t_k. Such systems are often encountered, particularly if the control law is to be implemented on a digital computer. On the other hand, the performance of the system depends on the deviations of the state vector from a steady-state condition (represented as the zero vector) at every instant of time, not only at the sampling instants. Hence, a cost function which penalizes deviations in the state continuously is

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more appropriate as a performance criterion† than a discrete cost function.
The problem is solved by reducing it to a discrete stochastic regulator where the
states are measured with no error.

2. Statement of the stochastic sampled-data problem

Consider the following continuous stochastic dynamical system represented
by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)w(t), \quad t \in [t_0, t_f] \tag{1}$$

where x is the n-dimensional state vector, u is the r-dimensional control vector,
and w is the white Gaussian plant noise vector of dimension p with $Ew(t) = 0$
and $Ew(t)w'(s) = F(t)\delta_{ij}(t-s)$ for some positive semidefinite matrix F. A, B
and D are time-varying matrices of compatible order. Equation (1) is inter-
preted as the following stochastic integral equation (Wong 1971) :

$$x(t) = x(t_0) + \int_{t_0}^{t} A(s)x(s)\,ds + \int_{t_0}^{t} B(s)u(s)\,ds + \int_{t_0}^{t} D(s)\,dW(s) \tag{2}$$

where $W(t)$ is a Wiener process with $EW(t) = 0$ and

$$E(W(t) - W(t_0))(W(s) - W(t_0))' = \int_{t_0}^{\min(t,s)} F(\tau)\,d\tau$$

The plant noise can be considered as the formal time derivative of the Wiener
process W.

In the stochastic sampled-data regulator the plant noise is a continuous
random process, whereas the measurement noise is a discrete random process.
Measurements of some of the linear combinations of states with additive noise
are available at the sampling instants

$$y(t_k) = C_k x(t_k) + v(t_k), \quad k = 0, 1, \dots, N, \quad t_0 < t_1 < \dots < t_N = t_f \tag{3}$$

where y is the m-dimensional measurement vector, $\{v(t_k)\}$ is the measurement
noise vector which is a Gaussian sequence of uncorrelated zero mean random
vectors with $Ev(t_i)v'(t_j) = \theta_i \delta_{ij}$ for some positive definite matrix θ_i, and
$EW(t)v'(t_i) = Ex(t_0)v'(t_i) = Ex(t_0)W'(t) = 0$ for all t in $[t_0, t_f]$ and $i = 0, 1, \dots, N$.

The controls are constrained by

$$u(t) = u(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \dots, N-1 \tag{4}$$

In order for a solution to exist to eqn. (2), the following further assumptions
will be made : the elements of $A(t)$ are bounded and measurable real functions
of time, $B(t)$ is integrable, and $\int_{t_0}^{t_f} D(t)F(t)D'(t)\,dt$ is finite.

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† If the sampling rate can be chosen high enough a discrete cost function may be
adequate ; however, in many cases this increases the cost of the computer by placing
stringent requirements on its speed of operation. This trade-off makes the design
of control laws, which do not degrade at low sampling rates, important.

The following cost functional is used to achieve the desired system perfor-
mance :

$$J = \tfrac{1}{2} E \left[\int_{t_0}^{t_f} (x'(t)Q_c(t)x(t) + u'(t)R_c(t)u(t))\, dt + x'(t_f)Hx(t_f) \right] \qquad (5)$$

where Q_c is positive semidefinite and R_c is positive definite on $[t_0, t_f]$.

Now the stochastic optimal control problem can be stated as follows :
given the linear stochastic dynamical system (1), and a partition $t_0, t_1, ..., t_N$
of the interval $[t_0, t_f]$, find a control sequence $(u^*(t_k), k = 0, 1, ..., N-1)$ with
the constraint (4) and also the additional constraint that $u^*(t_k)$ will depend
only on the past measurement sequence $(y(t_i), i = 0, 1, ..., k-1)$ which will
minimize the cost functional (5).

3. Equivalent stochastic discrete-time problem

It will be shown that the constrained continuous stochastic optimization
problem can be transformed into an unconstrained discrete stochastic optimi-
zation problem by integrating the system differential equations and the cost
functional over each sampling interval. The problem will be embedded into
the known format of the standard discrete linear quadratic Gaussian regulator
problem. Thus, it will be proved that the separation between estimation and
control is still valid for this constrained continuous stochastic optimization
problem.

Under the assumptions made in § 2, the solution to eqn. (1), for any bounded
control, is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)\, ds + \int_{t_0}^{t} \Phi(t, s)D(s)\, dW(s) \qquad (6)$$

where $\Phi(t, s)$ is the state-transition matrix of $A(t)$ defined as the solution of
matrix differential equation

$$\frac{d}{dt}\Phi(t, s) = A(t)\Phi(t, s), \quad (t > s)$$

with $\Phi(s, s) = I$.

Using (6), the state for $t_k \leqslant t < t_{k+1}$ can be computed from the state at time t_k :

$$x(t) = \Phi(t, t_k)x_k + \Gamma(t, t_k)u_k + \xi(t, t_k) \qquad (7\,a)$$

where $x_k = x(t_k)$, $u_k = u(t_k)$, and

$$\Gamma(t, t_k) = \int_{t_k}^{t} \Phi(t, s)B(s)\, ds \qquad (7\,b)$$

$$\xi(t, t_k) = \int_{t_k}^{t} \Phi(t, s)D(s)\, dW(s) \quad \text{for} \quad t_k < t \leqslant t_{k+1} \qquad (7\,c)$$

The term given by eqn. (7 c) shows that the plant noise corrupts the state
in a continuous fashion, but the resulting discrete-time system will be given by

$$x_{k+1} = \Phi(t_{k+1}, t_k)x_k + \Gamma(t_{k+1}, t_k)u_k + \xi_k \qquad (8\,a)$$

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where $\xi_k = \xi(t_{k+1}, t_k)$ is a zero mean white Gaussian sequence of random vectors
with (Jazwinski 1970)

$$E\xi_k\xi_j' = \left[\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)D(s)F(s)D'(s)\Phi'(t_{k+1}, s)\, ds \right] \delta_{kj} = F_k \delta_{kj} \qquad (8\,b)$$

The assumptions made about the independence of plant noise and $v(t_k)$, $x(t_0)$
in § 2 will still be valid for the new discrete plant noise (ξ_k).

The cost integral can also be written as the sum of N integrals as in the
deterministic case. The control at time t_k must only depend on the measure-
ments made until t_{k-1} and the initial estimate $\hat{x}(t_0)$. This can be expressed
by restricting $u(t_k)$ to be measurable with respect to Y_k, where Y_k is the
minimal σ-algebra generated by the measurement sequence $(y(t_i),\ i = 0, 1, ...,$
$k-1)$ and the initial estimate $\hat{x}(t_0)$ while the initial estimate satisfies the
equation $E(x(t_0)|\hat{x}(t_0)) = \hat{x}(t_0)$. The usual choice of $\hat{x}(t_0)$ as the mean of $x(t_0)$
satisfies this condition.

Using (7 a), the cost functional (5) can be put into the following form :

$$J = E\left[\tfrac{1}{2}x_N'Hx_N + \tfrac{1}{2}\sum_{k=0}^{N-1} (x_k'Q_kx_k + 2x_k'M_ku_k + u_k'R_ku_k \right.$$

$$+ 2x_k'\int_{t_k}^{t_{k+1}} \Phi'(t, t_k)Q_c(t)\xi(t, t_k)\, dt$$

$$\left. + \int_{t_k}^{t_{k+1}} \xi'(t, t_k)Q_c(t)\xi(t, t_k)\, dt) \right] \qquad (9\,a)$$

where Q_k, M_k, R_k are given by

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi'(t, t_k)Q_c(t)\Phi(t, t_k)\, dt \qquad (9\,b)$$

$$M_k = \int_{t_k}^{t_{k+1}} \Phi'(t, t_k)Q_c(t)\Gamma(t, t_k)\, dt \qquad (9\,c)$$

$$R_k = \int_{t_k}^{t_{k+1}} (R_c(t) + \Gamma'(t, t_k)Q_c(t)\Gamma(t, t_k))\, dt \qquad (9\,d)$$

Note that R_k will be positive definite since R_c is positive definite. Similarly,
Q_k will be positive semidefinite.

We shall now show that the last two terms in eqn. (9 a) can be ignored as
far as the minimization of the cost functional is concerned :

Lemma 1

The control sequence $(u_k, k = 0, 1, ..., N-1)$ minimizes the cost functional
J given by eqn. (9 a) if, and only if, it minimizes the following cost functional
J_1 :

$$J_1 = E\left(\tfrac{1}{2}x_N'Hx_N + \tfrac{1}{2}\sum_{k=0}^{N-1} x_k'Q_kx_k + 2x_k'M_ku_k + u_k'R_ku_k \right) \qquad (10)$$

Proof

From (7 c), note that $\xi(t, t_k)$ depends only on the increments of $W(s)$ in the interval $[t_k, t]$, which are independent of $x(t_0)$ and $(W(t), t_0 \leqslant t \leqslant t_k)$; hence, so is $\xi(t, t_k)$ of $(W(s), t_0 \leqslant s \leqslant t_k)$ and $x(t_0)$. On the other hand, $x(t_k)$ depends only on $(W(s), t_0 \leqslant s \leqslant t_k)$ and $x(t_0)$, so that $\xi(t, t_k)$ and $x(t_k)$ are independent whenever $t \geqslant t_k$. Thus,

$$E x_k' \int_{t_k}^{t_{k+1}} \Phi'(t, t_k) Q_c(t) \xi(t, t_k)\, dt = E(x_k') \int_{t_k}^{t_{k+1}} \Phi'(t, t_k) Q_c(t) E(\xi(t, t_k))\, dt = 0$$

since $\xi(t, t_k)$ has zero mean. Furthermore, the last term in (9 a) is a constant which does not depend on the control sequence; hence, it can be excluded from the minimization. So the lemma follows.

At this stage, in the case of the deterministic regulator, a preliminary feedback of the form $u_k = -R_k^{-1} M_k' x_k + \bar{u}_k$ would reduce the problem into the standard regulator problem. However, as x_k is not available, the same cannot be done in this problem. Instead, we now show that minimization of the cost functional J_1 is equivalent to the minimization of the same expression with x_k's replaced by their conditional expectations $\hat{x}_k$.

Lemma 2

The control sequence $(u_k^*,\ k = 0, 1, \ldots, N-1)$ minimizes the cost functional J_1 given by (10), if, and only if, it minimizes the following cost functional J_2:

$$J_2 = E\left[\tfrac{1}{2} \hat{x}_N' H \hat{x}_N + \tfrac{1}{2} \sum_{k=0}^{N-1} (\hat{x}_k' Q_k \hat{x}_k + 2\hat{x}_k' M_k u_k + u_k' R_k u_k) \right] \qquad (11)$$

where $\hat{x}_k = E(x_k \mid Y_k)$.

Proof

Since u_k is measurable with respect to Y_k, using well-known theorems on conditional expectations,

$$E(x_k' M_k u_k) = E(E(x_k' M_k u_k \mid Y_k)) = E(E(x_k' \mid Y_k) M_k u_k) = E\hat{x}_k' M_k u_k \qquad (12)$$

Similarly, by letting $\tilde{x}_k = x_k - \hat{x}_k$:

$$E(x_k' Q_k x_k) = E(\hat{x}_k' Q_k \hat{x}_k + 2\hat{x}_k' Q_k \tilde{x}_k + \tilde{x}_k' Q_k \tilde{x}_k)$$

Using the fact that $E(\hat{x}_k' Q_k \tilde{x}_k) = 0$, we get

$$E(x_k' Q_k x_k) = E(\hat{x}_k' Q_k \hat{x}_k + E(\tilde{x}_k' Q_k \tilde{x}_k \mid Y_k)) \qquad (13)$$

Since the σ-algebras generated by $\tilde{x}_k$ and Y_k are independent, it follows that

$$E(\tilde{x}' Q_k \tilde{x}_k \mid Y_k) = E(\tilde{x}' Q_k \tilde{x}_k) \quad \text{a.e.} \qquad (14)$$

Note that the right-hand side of (14) does not vary with u_k. Substituting (12), (13) and (14) into (11), we get

$$J_1 = E\left[\tfrac{1}{2}\hat{x}_N'H\hat{x}_N + \tfrac{1}{2}\sum_{k=0}^{N-1}(\hat{x}_k'Q_k\hat{x}_k + 2\hat{x}_k'M_k u_k + u_k'R_k u_k) \right]$$
$$+ \tfrac{1}{2}E(\tilde{x}_N'H\tilde{x}_N) + \tfrac{1}{2}\sum_{k=0}^{N-1}E(\tilde{x}_k'Q_k\tilde{x}_k) \qquad (15)$$

Since the second term in (15) does not depend on ($u_k, k = 0, 1, ..., N-1$), it may be dropped as far as the minimization with respect to ($u_k, k = 0, 1, ..., N-1$) is concerned. So J_1 reduces to J_2 given by (11); this completes the proof of the lemma.

Thus, to find the optimal control for the original cost functional J, it is sufficient to find the control sequence which minimizes the cost functional J_2 of (11). Note that the cost J_2 depends only on the estimates of the state at the sampling instants. Looking back at eqns. (8 a), (8 b) and (3), it is clear that these estimates will be given by the well-known Kalman filter equations (Jazwinski 1970, Kalman 1960). We can now introduce a preliminary feedback of the form $u_k = -R^{-1}M_k'\hat{x}_k + \bar{u}_k$ and embed the problem into the standard discrete linear quadratic Gaussian problem. We sum up these results in the next theorem.

Theorem

Consider the stochastic optimal control problem described in § 2. A unique control sequence ($u_k^*, k = 0, 1, ..., N-1$) which minimizes the cost functional J of (4) exists and is given by

$$u_i^* = -[R_i^{-1}M_i' + (R_i + \Gamma_i'K_{i+1}\Gamma_i)^{-1}\Gamma_i'K_{i+1}\Phi_i]\hat{x}_i \qquad (16)$$

where K_i is the solution to the Riccati difference equation

$$K_i = \Phi_i'(K_{i+1} - K_{i+1}\Gamma_i(R_i + \Gamma_i'K_{i+1}\Gamma_i)^{-1}\Gamma_i'K_{i+1})\Phi_i + Q_i \qquad (17)$$

with the boundary condition $K_N = H$

where $\Gamma_i = \Gamma(t_{i+1}, t_i)$, $\Phi_i = \Phi(t_{i+1}, t_i) - \Gamma_i R_i^{-1}M_i'$, $Q_i = Q_i - M_i R_i^{-1}M_i'$, and $i = 0, 1, ..., N-1$.

Proof

Introducing the preliminary feedback (Payne and Silverman 1973, Thomasson and Cook 1973)

$$u_k = -R_k^{-1}M_k'\hat{x}_k + \bar{u}_k$$

the cost functional becomes

$$J_2 = E\left[\frac{1}{2}\hat{x}_N' H \hat{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}(\hat{x}_k'(Q_k - M_k R_k^{-1} M_k')\hat{x}_k + \bar{u}_k' R_k \bar{u}_k) \right] \quad (18)$$

The estimates $\hat{x}_k$ can be obtained by the Kalman filter equations (Jazwinski 1970, Kalman 1960)

$$\hat{x}_{k+1} = \Phi_k \hat{x}_k + \Gamma_k \bar{u}_k + G_k \nu_k \quad (19)$$

where ν_k is the white innovation sequence.

Hence, the original constrained continuous problem is reduced to minimizing J_2 as given in (18) with the dynamics of (19) where $\bar{u}_k$ may depend on $\hat{x}_k$. However, this is a discrete LQG problem and its solution is given by (16) and (17); e.g. see Kushner (1971).

Looking at eqns. (16) and (17), it is seen that the optimal control law for this stochastic optimization problem is the same as the deterministic sampled-data regulator problem (Levis *et al.* 1971) with x_k replaced by its estimate $\hat{x}_k$. Therefore, the separation between estimation and control continues to hold for this constrained continuous stochastic optimization problem.

4. Conclusions

In this paper it is shown that the separation principle between estimation and control continues to hold for the stochastic sampled-data regulator. The problem is solved by reducing the constrained continuous stochastic optimization problem into an unconstrained discrete stochastic one. The results are obtained by embedding the problem into the standard discrete stochastic regulator problem.

Using similar techniques to the one described here, it can also be shown that the separation of estimation and control continues to hold for linear systems using nth-order holds and having computational delays in the control loop.

REFERENCES

DORATO, P., and LEVIS, A. H., 1971, *I.E.E.E. Trans. autom. Control*, 16, 613.
GUNCKEL, T. L., and FRANKLIN, G. F., 1963, *J. bas. Engng*, D, 85, 197.
HALYO, N., and FOULKES, R. H., 1974, *Proc. I.E.E.E. International Conference on Systems Man and Cybernetics*, p. 99.
JAZWINSKI, A. H., 1970, *Stochastic Processes and Filtering Theory* (New York: Academic Press), p. 195.
JOSEPH, P. D., and TOU, J. T., 1961, *A.I.E.E. Trans. appl. Ind.*, 80, 193.
KALMAN, R. E., 1960, *J. bas. Engng*, 82, 34.
KALMAN, R. E., and KOEPCKE, R. W., 1958, *Trans. A.S.M.E.*, 80, 1800.
KUSHNER, H., 1971, *Introduction to Stochastic Control* (New York: Holt, Rinehart & Winston), p. 229.
LEVIS, A. H., SCHLUETER, R. A., and ATHANS, M., 1971, *Int. J. Control*, 13, 343.
MEDITCH, J. S., 1969, *Stochastic Optimal Linear Estimation and Control* (New York: McGraw-Hill), p. 248.
PAYNE, H. J., and SILVERMAN, L. M., 1973, *I.E.E.E. Trans. autom. Control*, 18, 223.

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POTTER. J. E., 1964. Report RE-11. Experimental Astronomical Laboratory,
        Massachusetts Institute of Technology. Cambridge. U.S.A.
SAGE. A. P., and MELSA. J. L., 1971, *Estimation Theory with Applications to Communi-
        cations and Control* (New York : McGraw-Hill). pp. 64, 82, 282.
STREIBEL. C., 1965, *J. math. Analysis Applic.*, **12**, 576.
THOMASSON. F. Y.. and COOK. G.. 1973. *Int. J. Control*. **17**, 1117.
WONG, E.. 1971, *Stochastic Processes in Information and Dynamical Systems* (New
        York : McGraw-Hill).
WONHAM, W. M., 1968, *J. SIAM Control*. **6**, 312.

ON THE RELATION BETWEEN THE SAMPLED-DATA AND THE CONTINUOUS OPTIMAL LINEAR REGULATOR PROBLEMS

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Abstract

In this paper, the geometric relationship between the optimal solutions to the sampled-data and continuous linear quadratic regulator problems is investigated in a Hilbert space framework. It is shown that the optimal sampled-data solution, excluding the response due to the initial condition, is the projection of the optimal continuous solution onto the set of all solutions that satisfy the sampled-data constraint. That is, the optimal sampled-data solution is an optimal approximation to the continuous solution. In fact, it is shown that the sampled-data solution can be obtained by solving a sampled-data tracking problem with the continuous solution as the desired trajectory.

1. Introduction

This paper is concerned with the relationship between the optimal continuous and the optimal sampled-data linear quadratic regulator problem [1], [2]. Since the dynamical system and the integral cost function used are the same for both the continuous and the sampled-data regulator, intuitively, the optimal sampled-data control must be an approximation to the optimal continuous one in some sense. In this paper the precise relationship between these two optimal solutions will be investigated in a vector space setting.

In the control literature, the continuous and the sampled-data regulator problems have usually been treated using different methods: The continuous regulator problem has been solved by using Pontryagin's minimum principle, by using the Hamilton-Jacobi-Bellman partial differential equation for the optimal cost function, and by a few other methods. The sampled data problem has been solved by converting it into an unconstrained discrete optimization problem through the integration of the cost functional and the system differential equations over each sampling interval and then applying dynamic programming or the discrete minimum principle. The relationship between the two optimal solutions is, thus, obscured due to these separate formulations. In this paper, the two problems are formulated in the same framework by using a Hilbert space approach. This geometric formulation reduces the problem to one of finding the element of minimum norm in two linear varieties in a Hilbert space. (This approach has been suggested for a simple control problem in [3]). It is shown that, excluding the unforced response of the system due to the initial condition, the sampled-data control and the resulting state trajectory is the projection of the optimal continuous control and its corresponding state trajectory; that is, the optimal sampled-data solution is, in fact an optimal approximation of the continuous solution with respect to an appropriate Hilbert space norm.

2. A Minimum Norm Theorem in a Hilbert Space

Before proceeding to formulate the problem in a Hilbert space setting, we shall prove a general theorem concerning the elements of minimum norm of two linear varieties in a Hilbert space which will be required in the following derivation. We now state the projection theorem and its extension to linear varieties for ease of reference; the proofs can be found in [3].

Lemma 2.1 (Projection Theorem) Let H be a Hilbert space and M be a closed subspace of H. Corresponding to any vector x in H, there exists a unique vector m* in M such that m* is the closest element in M to x in the sense of the inner product norm. Furthermore, a necessary and sufficient condition that m* be this unique vector is that x − m* be orthogonal to M. (We will denote the projection operator onto M by P_M; i.e., $m^* = P_M x$).

Corollary 2.2 Let H and M be as in Lemma 2.1. Let x_o be a fixed element in H and let V be the linear variety $x_o + M$. Then there exists a unique vector v* in V of minimum norm. Furthermore, a necessary and sufficient condition that v* be this unique vector is that v* be orthogonal to the subspace M.

The following theorem describes the relationship between the elements of minimum norm of two linear varieties in a Hilbert space where the generator subspace of one variety is a subset of the other corresponding subspace.

Theorem 2.3 Let H be a Hilbert space and let M and N be closed subspaces of H such that N is a subset of M. Let x_o be a fixed element in H and let V and W be the linear varieties defined by

$x_o + M$ and $x_o + N$ respectively. Then there exists a unique vector v^* in V and a unique vector w^* in W of minimum norm. Furthermore, w^* can be obtained from v^* by

$$w^* = x_o + P_N m^* \qquad (1)$$

where m^* is the unique element of M defined by the relation

$$v^* = x_o + m^* \qquad (2)$$

Proof: By Corollary 2.2, there exists a unique element v^* in V and a unique element w^* in W of minimum norm. Since v^* is in V, it must have a representation of the form $v^* = x_o + m^*$ where m^* is an element of M. Clearly, m^* must be unique. Similarly, there exists a unique element n^* in N such that $w^* = x_o + n^*$. Since N is a sub-Hilbert space of M, there exists a direct sum decomposition of M [4] given by

$$M = N + N^\perp \qquad (3)$$

where N is the orthogonal complement of $N^\perp$ with respect to M, and $+$ denotes the direct sum. In order to prove that $w^* = x_o + P_N m^*$, we have to show

$$< x_o + P_N m^*, n > = 0 \text{ for all } n \text{ in } N \qquad (4)$$

where $< . , . >$ denotes the inner product. Since $< P_{N^\perp} m^*, n >$ is identically zero, we have

$$< x_o + P_N m^*, n > = < x_o + P_N m^* + P_{N^\perp} m^*, n > \qquad (5)$$

Since m^* has the direct sum decomposition $P_N m^* + P_{N^\perp} m^*$, we get

$$< x_o + P_N m^*, n > = < x_o + m^*, n > = $$
$$< v^*, n > = 0 \qquad (6)$$

The last equality follows from Corollary 2.2 and the proof is complete.

3. Formulation of the Problem

In this section, we shall formulate the two regulator problems in a Hilbert space framework so that we can apply the theorem of the last section to find the precise mathematical sense in which the optimal sampled-data solution is an approximation of the continuous solution. Consider the linear dynamical system represented by the differential equation

$$\dot{x}(t) = A(t) x(t) + B(t) u(t) \qquad (3.1)$$
$$a \le t \le b$$
$$y(t) = C(t) x(t)$$

where $x(t)$ is the n-dimensional state vector, $u(t)$ is the m-dimensional control vector, $y(t)$ is the r-dimensional measurement vector, and $x(a)$ is the initial condition. $A(t)$, $B(t)$, and $C(t)$ are matrices of appropriate order with continuous elements. The following cost functional will be used to achieve the desired system performance:

$$J(x(t), u(t)) = (\int_a^b (y'(t)y(t) + u'(t)R(t)u(t))dt)^{1/2} \qquad (3.2)$$

where $R(t)$ is a positive definite matrix with continuous elements. We will denote the R^m-valued

functions on $[a, b]$ that are square integrable in the Lebesque sense by $L_2^m(a, b)$, i.e.,

$$L_2^m(a,b) = \{u(t): \int_a^b u'(t)u(t)dt < \infty \} \qquad (3.3)$$

Similarly, let $L_2^r(a, b)$ be the space of R^r-valued square integrable functions on $[a, b]$. We will formulate the two problems in the Hilbert space $H = L_2^r(a, b) \times L_2^m(a,b)$, the Cartesian product of $L_2^r(a, b)$ and $L_2^m(a,b)$, with the inner product defined by

$$<(y_1(t), u_1(t)), (y_2(t), u_2(t))> = \int_a^b (y_1'(t)y_2(t) + u_1'(t)R(t)u_2(t))dt \qquad (3.4)$$

so that the norm induced by the inner product 3.4 gives the desired cost functional of 3.2. Since the solution of the differential equation 3.1 is given by

$$x(t) = \Phi(t,a) x(a) + \int_a^t \Phi(t,s)B(s)u(s)ds \qquad (3.5)$$
$$y(t) = C(t)x(t)$$

where $\Phi(t,s)$ is the state transition matrix of $A(t)$, we will be concerned with the ordered pairs $(y(t), u(t))$ in H that satisfy the integral constraint 3.5 for the continuous regulator. It is easy to show that the subset M of H defined by

$$M = \{(y(t), u(t)) \subset L_2^r \times L_2^m: y(t) = \int_a^t T(t,s)u(s)ds$$
$$T(t,s) = C(t)\Phi(t,s)B(s) \qquad (3.6)$$

is a subspace, so that we can define the linear variety in H by

$$V = (C(t)\Phi(t,a)x(a), 0) + M \qquad (3.7)$$

Thus, the continuous regulator problem is one of finding the element of minimum norm in V. To apply the Hilbert space theory of the previous section, we have to show that M is a closed subspace. To this end, let us define the linear transformation L from $L_2^m(a,b)$ into $L_2^r(a,b)$ by

$$L(u(t)) = \int_a^t T(t,s)u(s)ds \qquad (3.8)$$

Using Fubini's theorem, it can easily be shown that for all y in $L_2^r(a,b)$

$$<Lu, y> = <u, \int_t^b T'(s,t) y(s)ds> \qquad (3.9)$$

which proves that L is a bounded and, therefore, continuous linear transformation. (See Theorem 5 in [5], p. 48). We also note the linear transformation L^* from L_2^m defined by

$$L^* y = \int_t^b T'(s,t)y(s)ds \qquad (3.10)$$

is the adjoint of L. So if (y_n, u_n) is a sequence of elements in M converging to an element (y,u) in H, we have

$$\lim_n (y_n, u_n) = \lim_n (\int_a^t T(t,s) u_n(s)ds, u_n(s)) = $$
$$(\int_a^t T(t,s)\lim_n u_n(s)ds, \lim_n u_n(s)) \in M$$

This follows since L is a continuous operator and, therefore, M is closed.

In the sampled-data problem, we will be given a partition $(a = t_0, t_1, \ldots, t_N = b)$ of $[a, b]$ such that over each sampling interval the control remains constant, i.e., $u(t) = u(t_k)$ for $t_k \leq t < t_{k+1}$.

In this case, the subspace of interest will be

$$N = \{(y,u) \ L_2^r \times L_2^m : y(t) = \int_a^t T(t,s)u(s)ds, \ u \epsilon L_2^m,$$

$$u(t) = u(t_k) \text{ for } t_k \leq t < t_{k+1}\} \qquad (3.11)$$

Clearly, N is a subset of M. To show that N is closed, we note that the same linear operator L defined 3.8 is used except now its domain is a subset of L_2^m. So we only have to show that this domain is closed in L_2^m. Clearly, the set of all functions that are constant over each sampling interval is isometrically isomorphic to $R^m \times R^m \times \ldots \times R^m$ (N times) so that N is closed.

So the sampled-data regulator problem is one of finding the element of minimum norm in the linear variety W defined by

$$W = (C(t)\Phi(t,a)x(a), \ 0) + N \qquad (3.12)$$

Thus, we have cast the two regulator problems into the vector space format of the previous section.

4. Optimal Sampled-Data Solution as an Approximation to the Optimal Continuous Solution

In this section, we will first show that the Hilbert space formulations of the sampled-data and the continuous regulator problems in the previous section do indeed give the standard results. Then, using the theorem concerning two linear varieties in Section 2., we will state the precise sense in which the optimal sampled-data solution is an approximation of the optimal continuous solution. We will also show that the optimal sampled-data solution can be obtained by taking the appropriate projection of the optimal continuous one.

Consider the continuous regulator problem formulation described in the previous section by equations 3.1-3.7. A necessary and sufficient condition for (y^*, u^*) to be the element of minimum norm in V by Corollary 2.2, is that (y^*, u^*) be orthogonal to M, which implies

$$\langle (y^*,u^*), \ (y,u) \rangle = 0 \text{ for all } (y, u) \text{ in } M \qquad (4.1)$$

Since the inner product on $L_2^r \times L_2^m$ is the sum of inner products on L_2^r and L_2^m

$$\langle y^*(t), \ \int_a^t T(t,s)u(s)ds \rangle + \langle u^*(t), \ R(t)u(t) \rangle = 0$$

$$\text{for all } u \text{ in } L_2^m(a,b) \qquad (4.2)$$

By using 3.9, we get

$$\langle \int_t^b T'(s,t)y^*(s)ds, \ u(t) \rangle + \langle u^*(t), \ R(t)u(t) \rangle = 0$$

$$\text{for all } u \text{ in } L_2^m \qquad (4.3)$$

It follows that

$$R(t)u^*(t) + \int_t^b T'(s,t) \ y^*(s)ds = 0 \qquad (4.4)$$

Therefore, the optimal continuous solution is given by

$$u^*(t) = -R^{-1}(t) \int_t^b T'(s,t) \ y^*(s)ds \qquad (4.5)$$

or, equivalently

$$u^*(t) = -R^{-1}(t)B'(t) \int_t^b \Phi'(s,t)C'(s)C(s) \ x^*(s)ds \qquad (4.6)$$

It is easy to show that there exists a positive semidefinite matrix $K(t)$ such that

$$\int_t^b \Phi'(s,t)C'(s)C(s)x^*(s)ds = K(t)x^*(t) \qquad (4.7)$$

$$\text{for } t \text{ in } [a, b]$$

where $K(t)$ is the unique positive semidefinite solution of

$$\dot{K}(t) = -A'(t)K(t) - K(t)A(t) + K(t)B(t)R^{-1}(t)B'(t)$$

$$K(t) - C'(t)C(t) \text{ with } K(b) = 0 \qquad (4.8)$$

which is the Riccati equation of the linear continuous optimal control.

Consider now the formulation of the sampled-data regulator problem described by equations 3.1 - 3.5 and 3.11 - 3.12. A necessary and sufficient condition that (y^{**}, u^{**}) be this optimal solution is by Corollary 2.2 that

$$\langle (y^{**},u^{**}), \ (y,U) \rangle = 0 \text{ for all } (y,u) \text{ in } N \qquad (4.9)$$

which implies (after some manipulation)

$$\{\int_{t_0}^{t_1} (x'^{**}(t) \ C'(t)C(t)\Gamma(t, t_0) + u'^{**}(t)R(t))dt$$

$$+ \int_{t_1}^{t_N} x'^{**}(t) \ C'(t)C(t)dt\Gamma(t_1,t_0)\} u_0 +$$

$$\{\int_{t_1}^{t_2} (x'^{**}(t) \ C'(t)C(t)\Gamma(t, t_1) + u'^{**}(t)R(t))dt$$

$$+ \int_{t_2}^{t_N} x'^{**}(t) \ C'(t)C(t)dt\Gamma(t_2, t_1)\}u_1$$

$$+ \ldots \ldots$$

$$+ \{\int_{t_{N-1}}^{t_N} (x'^{**}(t)C'(t)C(t)\Gamma(t,t_{N-1}) + u'^{**}(t)$$

$$R(t))dt\}u_{N-1} = 0 \qquad (4.10)$$

$$\text{for all vectors } (u_0, u_1 \ldots, u_{N-1}) \text{ in } R^m$$

where $\Gamma(t, t_i) = \int_{t_i}^t \Phi(t,s)B(s)ds$. This implies that the terms in the brackets in (4.10) must be zero. Starting from the last term in 4.10 and using the fact that $x^{**}(t) = \Phi(t, t_{N-1})x^{**}_{N-1} + \Gamma(t, t_{N-1})u^{**}_{N-1}$, it can be shown that the optimal sampled-data control law is given by

$$u_i^{**} = -(R_i + \Gamma_i' K_{i+1}\Gamma_i)^{-1} (\Gamma_i' K_{i+1}\Phi_i + M_i')x_i^{**} \qquad (4.11)$$

where $\Gamma_i = \Gamma(t_{i+1}, t_i), \Phi_i = \Phi(t_{i+1}, t_i)$, and

$$Q_i = \int_{t_i}^{t_{i+1}} \Phi'(t,t_i) \ Q(t)\Phi(t,t_i)dt \qquad (4.12)$$

$$M_i = \int_{t_i}^{t_{i+1}} \Phi'(t,t_i)Q(t)\Phi(t,t_i)dt \qquad (4.13)$$

$$R_i = \int_{t_i}^{t_{i+1}} (R(t) + \Gamma'(t,t_i)Q(t)\Gamma(t,t_i))dt \qquad (4.14)$$

where K_i satisfies the discrete Riccati equation [6]

$$K_i = (\Phi_i'K_{i+1}\Phi_i+Q_i) - (\Gamma_i'K_{i+1}\Phi_i + M_i')'(R_i +$$
$$\Gamma_i'K_{i+1}\Gamma_i)^{-1} \; (\Gamma_i'K_{i+1}\Phi_i + M_i') \qquad (4.15)$$

$$i = N - 1, \ldots, 0 \text{ with } K_N = 0$$

We will state the exact relationship between the two regulator solutions in the next theorem.

Theorem 4.1 Consider the continuous regulator problem described 3.1 - 3.5, 3.11, 3.12. Let (y^*, u^*) be the optimal solution to the continuous problem given by 4.5 - 4.8 and let (y^{**}, u^{**}) be the optimal solution to the sampled-data problem given by 4.11 - 4.15. Then, excluding the response due to the initial condition, the optimal sampled-data solution is the projection of the optimal continuous solution onto the set of ordered pairs $(y(t), u(t))$ that satisfy the sampled-data integral constraint of 3.11 or more precisely:

$$(y^{**}(t), u^{**}(t)) = (C(t)\Phi(t,a)x(a), 0) +$$
$$+ P_N(y^*(t) - C(t)\Phi(t,a)x(a), u^*(t)) \qquad (4.16)$$

or, equivalently, (y^{**}, u^{**}) is the element of W which minimizes

$$\int_a^b [(y(t)-y^*(t))'(y(t)-y^*(t))+(u(t)-u^*(t))'R(t)(u(t)-u^*(t))]dt \qquad (4.17)$$

among all (y,u) in W. A necessary and sufficient condition that (y^{**}, u^{**}) be the optimal sampled-data solution is that

$$\int_a^b [(y^*(t)-y^{**}(t))'y(t)+(u^*(t)-u^{**}(t))'R(t)u(t)]dt=0$$

for all (y,u) in N \qquad (4.18)

Proof: Equation 4.16 follows from Theorem 2.3 with $x_0 = (C(t)\Phi(t,a)x(a),0)$, $m^* = (y^*,u^*)$, and N defined by 3.11.

To see 4.18, we note that be Lemma 2.1

$$||(y^{**}(t) - C(t)\Phi(t, a)x(a), u^{**}(t))|| =$$

$$\text{minimum} \; ||(y(t) - (y^*(t) - C(t)\Phi(t,a)x(a)),$$

$$u(t) - u^*(t))|| \quad (y, u)\epsilon N \qquad (4.19)$$

which implies

$$||(y^{**}(t), u^{**}(t))||^2 = \text{minimum} \; ||(y(t)-y^*(t),u(t)-u^*(t))||^2 \quad {(y,u)\epsilon W} \qquad (4.20)$$

which verifies 4.17. Equation 4.18 follows from the last part of Lemma 2.1 and the proof is complete.

Theorem 4.1 implies (see equation 4.17) that if an optimal sampled-data tracking problem is solved by any method(such as dynamic programming, discrete minimum principle, etc.), where the desired trajectory to be tracked is the optimal continuous regulator solution, then the solution obtained would be the same as the solution to the original sampled-data regulator problem.

5. Conclusions

The sampled-data and continuous linear regulator problems are formulated in the same Hilbert space as minimum norm problems. The geometric relationship between the two optimal solutions is investigated. It is shown that the optimal sampled-data solution is an optimal approximation to the continuous regulator solution in an appropriate Hilbert space norm. Specifically, it is shown that, excluding the response due to the initial condition, the optimal sampled-data solution is the projection of the optimal continuous solution onto the set of all solutions that satisfy the sampled-data constraint. It should be also noted that extensions to the case where [a,b] is an infinite interval can be easily obtained with this method with slight modifications.

References

[1] Kalman, R. E., "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mex., Vol. 5, pp. 102-119, 1960.

[2] Levis, A. H.; and Schlueter, R. A.; and Athans, B., "On the Behavior of Optimal Linear Sampled-Data Regulators," Int. J. Control, Vol. 13, pp. 343-361, 1971.

[3] Luenberger, D. G., Optimization by Vector Space Methods, John Wiley and Sons, Inc., New York, 1969.

[4] Bachman, G.; and Narici, L.; Functional Analysis, Academic Press, New York, 1966.

[5] Akhiezer, N. I., and Glazman, I. M.; Theory of Linear Operators in a Hilbert Space, Frederick Ungar Publishing Co., New York, 1961.

[6] Dorato, P.; and Levis, A. H.; "Optimal Linear Regulators: The Discrete Time Case," IEEE Trans. Automat. Contr., Vol. AC-16, pp. 613-620, Dec. 1971.

Failure Accommodation in Digital Flight Control Systems by Bayesian Decision Theory

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A design method for digital control systems which is optimally tolerant of failures in aircraft sensors is presented. The functions of this system are accomplished with software instead of the popular and costly technique of hardware duplication. The approach taken, based on *M*-ary hypothesis testing, results in a bank of Kalman filters operating in parallel. A moving window of the innovations of each Kalman filter drives a detector that decides the failure state of the system. The detector calculates the likelihood ratio for each hypothesis corresponding to a specific failure state of the system. It also selects the most likely state estimate in the Bayesian sense from the bank of Kalman filters. The system can compensate for hardover as well as increased noise-type failures by computing the likelihood ratios as generalized likelihood ratios. The design method is applied to the design of a fault tolerant control system for a current configuration of the space shuttle orbiter at Mach 5 and 120,000 ft. The failure detection capabilities of the system are demonstrated using a real-time simulation of the system with noisy sensors.

Introduction

THE most striking impact of new technology in aircraft flight control stems from the advent of the modern, high-speed, digital computer. Control concepts previously considered untractable can now be considered because of the flexibility and speed of information processing made available by this new technology. One important new potential that exists is the ability of digital system to reorganize itself to accommodate for failures in sensors and actuators. This reorganization is possible, provided there is enough duplication of function between the actuators or the sensors in a given control system. This paper presents a design method for digital flight control systems that will be optimally tolerant of sensor failures.

Modern control methods allow one to determine the part of the state space of an aircraft that can be dynamically influenced by a given actuator (the controllability subspace) and the part of the state space that a given sensor can produce information about using state estimator theory (observability subspace). Reference 1 provides a good treatment of theoretical considerations involved in determination of these subspaces. Redundancy is provided in either sensors or actuators when there is overlapping of the subspaces of the various sensors or actuators in a given system. For example, consider the longitudinal dynamics of an airplane. If there are three sensors on the aircraft, say an accelerometer, to measure normal acceleration, a pitch-rate gyro, and an elevator position transducer, and if the aircraft state is completely observable from outputs of either sensor, then it is possible, using say a minimum order observer,[2] to estimate the behavior of one sensor based on the output of another one. Redundancy, in that situation, does exist and can be used by cross-checking state estimates obtained by one sensor with those obtained from another one.

Theoretical considerations for determining the absolute level of redundancy that exists in a given system were developed in Ref. 3. Reference 3 also presented a failure detection filter designed to make use of the system redundancy. One limitation of that work was that no consideration

of the practical noise environment of the sensors was made and failure detection depended on observing a steady-state bias in an error plane in a state space. For aircraft applications, however, a design process is desired that enables rapid detection of failures during maneuvering transients and accounts for the normal operational noise environment of the aircraft and the control system actuators and sensors.

A design method is presented for resolving both problems in that it accounts for noise in sensors and is capable of determining hardover as well as increased noise-type failures during maneuvering transients. Incorporation of failure detection and recovery into an aircraft control system design is a joint detection, estimation, and control problem. The design method presented here produces a decision for detecting system failures which is optimal in the Bayesian sense. In addition, because of the theoretical development, one is able to account for uncertainty in the aircraft's stability derivatives, mass, inertia, and geometric characteristics. Although the method developed can be applied to both sensor and actuator failures, only sensor failure detection and recovery are considered.

The approach taken here uses *M*-ary hypothesis testing with generalized likelihood ratios. The elements of this theory were originally developed at the close of World War II for a binary hypothesis testing problem of determining whether a radar return signal respresented a target or not. In that case there are clearly two hypotheses—either there is a target or there is not. Theoretically, one can assign a cost to either failing to detect a real target or creating a false alarm. A performance index can be constructed which expresses the cost of making a decision based on a given radar return. This index can be minimized by selection of threshold points for decision whether or not the return represents a target. Elements of this problem are outlined in Ref. 4, which also contains a brief description of the *M*-ary hypothesis testing and generalized likelihood ratios. In this paper the set of hypotheses used is, first, that all sensors are functioning properly and, then, $M-1$ further hypotheses stating that the ith sensor group has failed $i=1,2,...,M-1$. In the next section the theory for applying *M*-ary hypothesis testing to self-reorganizing systems is presented. Then, it is applied to an example aircraft problem.

Sensor Failure Accommodation Using *M*-ary Hypothesis Testing

Consider the equations of motion of an aircraft to be represented by

$$\dot{x} = Ax + Bu + w \tag{1}$$

where x is an n-dimensional state vector, u is an m-dimensional control vector, w is a zero mean Gaussian white noise process with a covariance matrix $W \delta_D (t - \tau)$. In Eq. (1) matrices A and B are determined from the aircraft's stability and control derivatives, its mass and inertia characteristics, and its geometric characteristics. The variable w may, but need not, represent turbulence. It may represent uncertainty in the designer's knowledge of the characteristics of the aircraft. Basically, it can be thought of as representing the error in calculation of $\dot{x}$, given x and u. We will be concerned with the digital control of the plant where the control is constrained to be constant with sampling interval T, that is $u(t) = u(kT)$ for $kT \le t < (k+1)T$. By integrating the system differential Eq. (1) over each sampling interval,[4] we get the discrete equations of motion for the aircraft

$$x(k+1) = \Phi x(k) + \Gamma u(k) + w(k) \tag{2}$$

where $x(k) = x(kT)$, $u(k) = u(kT)$, $\Phi = \Phi(T)$, $\Phi(s) = e^{As}$

$$\Gamma = \int_0^T \Phi(s) \, ds \, B$$

and $w(k)$ is a zero mean, white Gaussian sequence with covariance $E w(k) w'(j) = Q \delta_{kj}$ where

$$Q = \int_0^T \Phi'(s) W \Phi(s) \, ds \tag{3}$$

Let us assume that the control system has $M - 1$ sensor failure modes for each mode

$$y(k) = C_i x(k) + v_i(k) \quad i = 1, 2, \dots, M - 1$$

where $v_i(k)$ is a Gaussian white noise sequence where

$$E[v_i(k)] = (0, \dot{m}_i, 0)' \equiv m_i'$$

and

$$E[v_i(k) v_i'(j)] = R_i \delta_{kj}$$

The quantity $\dot{m}_i$ is an unknown (nonrandom) parameter vector.

We shall solve the problem as if m_i were known and then use the maximum likelihood estimate of m_i under the ith hypothesis. This procedure is known as generalized likelihood ratio approach in the communication literature.[5] This approach to failure modeling enables the designer to compensate for hardover failures of arbitary magnitude. Increased sensor noise-type failures can be modeled by appropriate selection of the noise variances R_i.

For the normal unfailed condition we will assume

$$y(k) = C_0 x(k) + v_0(k)$$

where $E[v_0(k)] = 0$ and $E[v_0(k) v_0'(j)] = R_0 \delta_{kj}$. Hence, for a system with three failure modes, as considered in the next section, we have four hypotheses to consider

$$H_0: y(k) = C_0 x(k) + v_0(k)$$

$$H_1: y(k) = C_1 x(k) + v_1(k)$$

$$H_2: y(k) = C_2 x(k) + v_2(k)$$

$$H_3: y(k) = C_3 x(k) + v_3(k)$$

where C_i ($i = 1, 2, 3$) is C_0 matrix with the rows corresponding to the ith group of sensors replaced by zeros.

We will be concerned with the selection of the most probable hypothesis, based on a finite set of measurements, $Y(K) = \{y(1), y(2), y(2) \dots y(K)\}$. To do this we construct a Bayesian cost function for the M-ary problem

$$\mathfrak{R} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} P_{H_j} C_{ij} \int_{Z_i} p_{Y|H}(\alpha | H_j) \, d\alpha \tag{4}$$

subject to

$$\sum_{i=0}^{M-1} P_{H_i} = 1$$

and where the sets Z_i, $i = 0, 1 \dots M - 1$, are disjoint and their union represents the entire observation space. P_{H_i} is the a priori probability of hypothesis H_i being true, C_{ij} is the cost of selecting H_i when H_j is true, and $P_{Y|H}(\alpha | H_j)$ is the conditional probability density of the measurement sequence Y given that H_j is true. The symbol $\int_{Z_i}$ implies that the integral is carried over the decision region Z_i in the observation space. Decision regions Z_i are subsets of observation space such that if Y is in Z_i then the hypothesis H_i is to be selected. Note that the integral in Eq. (4) represents nothing more than the probability of making the incorrect decision of selecting hypothesis H_i when H_j is true for $i \ne j$. So the Bayes risk $\mathfrak{R}$, represents the sum of probabilities corresponding to different decisions weighted by the a priori probabilities P_{H_i} and the design weights C_{ij}. The problem is to choose the boundaries of decision regions Z_i that will result in minimum Bayes risk. These boundaries are, in effect, switching hypersurfaces for the decision logic in the measurement space.

The minimization of Bayes risk can be performed easily by rewriting the cost function (4) in the form

$$\mathfrak{R} = \sum_{i=0}^{M-1} \int_{Z_i} \psi_i(\alpha) \, d\alpha \tag{5}$$

where

$$\psi_i(\alpha) = \sum_{j=0}^{M-1} P_{H_j} C_{ij} P_{Y|H}(\alpha | H_j) \tag{6}$$

The Bayes risk is minimized by selecting H_i at each point α in the observation space such that $\psi_i(\alpha)$ is the smallest of M possible values of $\psi_i(\alpha)$ ($k = 0, 1 \dots M - 1$). Hence, the optimal decision regions are

$$Z_i \{\alpha | \psi_i(\alpha) = \min \psi_k(\alpha), \quad 0 \le k \le M - 1\} \tag{7}$$

From a computational point of view, it is convenient to introduce a dummy hypothesis H_M with a priori probability $P_{H_M} = 0$ with $H_M: y(k) = v_0(k)$. Then, an equivalent decision criterion can be given in terms of likelihood ratios, $\Lambda_i(\alpha)$

$$\Lambda_i(\alpha) = P_{Y|H}(\alpha | H_i) / P_{Y|H}(\alpha | H_M) \quad i = 0, 1 \dots M - 1 \tag{8}$$

Dividing each ψ_i in Eq. (6) by the probability density of $Y(K)$ under H_M, we get an equivalent decision criterion in terms of the likelihood ratios

$$\lambda_i(\alpha) = \sum_{j=0}^{M-1} P_{H_j} C_{ij} \Lambda_j(\alpha) \tag{9}$$

Then

$$Z_i = \{\alpha | \lambda_i(\alpha) = \min \lambda_k(\alpha), 0 \le k \le M - 1\} \tag{10}$$

The advantage of using likelihood ratios is that the boundaries of the decision regions are linear hyperplanes and not general hypersurfaces in the likelihood ratio space $\Lambda_1, \Lambda_2, \dots$,

Λ_{M-1}. From the chain rule of probability densities and the Gaussian density of the observations, it can be shown[5] that the likelihood ratio for the problem considered is given by

$$\Lambda_i[Y(K)] = \left\{ \prod_{k=1}^{\Lambda} \frac{(\det R_0)^{\frac{1}{2}}}{[\det Q_i(k)]^{\frac{1}{2}}} \right\}$$

$$\exp\left\{ -\frac{1}{2} \sum_{k=1}^{\Lambda} [r_i'(k)Q_i^{-1}(k)r_i(k) \right.$$

$$\left. - y'(k)R_0^{-1}y(k)] \right\} \quad i=0,1...M-1 \quad (11)$$

where $r_i(k)$ is the innovation[6] of the measurements under the ith hypothesis given by

$$r_i(k) = y(k) - C_i\hat{x}_i(k|k-1) - m_i(k) \quad (12)$$

with $\hat{x}_i(k|k-1) = E[x(k)|Y(k-1), H_i]$. The matrix $Q_i(k)$ in Eq. (11) is given by

$$Q_i(k) = C_i V_{\hat{x}_i}(k|k-1)C_i' + R_i \quad (13)$$

where $V_{\hat{x}_i}(k|k-1)$ is the prediction error variance of the estimate of the state $x(k)$ under the ith hypothesis defined by

$$V_{\hat{x}_i}(k|k-1) = E\{[x(k)-\hat{x}_i(k|k-1)][x(k)$$

$$-\hat{x}_i(k|k-1)]'|Y(k-1),H_i\}$$

In Eq. (12), the true value of $m_i(k)$ should be used to get the exact likelihood ratio. Since this is not available, we will use the sample mean of $[0, \hat{m}_i(j), 0]'$, $j=1, 2,...k$, that is the maximum likelihood estimate of m_i at the kth instant under the ith hypothesis. That makes Λ_i a generalized likelihood ratio.[5]

To compute $\hat{x}_i(k|k-1)$ and $V_{\hat{x}_i}(k|k-1)$, M Kalman[7] filters are required. A bank of Kalman filters operting in parallel has been used for parameter adaptive control in Ref. 8. The filter equations are listed as follows for completeness

$$\hat{x}_i(k) = \hat{x}_i(k|k-1) + K_i(k)r_i(k) \quad (14)$$

$$\hat{x}_i(k|k-1) = \Phi\hat{x}_i(k) + \Gamma u(k) \quad (15)$$

where $\hat{x}_i(k)$ is the estimate of the aircraft state under the ith hypothesis defined by

$$\hat{x}_i(k) = E(x(k)|Y(k), H_i)$$

The filter gain $K_i(k)$ in Eq. (13) can be calculated recursively from the algorithm

$$K_i(k) = V_{\hat{x}_i}(k|k-1)C_i'Q_i^{-1}(k) \quad (16)$$

where $Q_i(k)$ is given by Eq. (13) and the prediction error variance $V_{\hat{x}_i}(k|k-1)$ is given by

$$V_{\hat{x}_i}(k|k-1) = \Phi V_{\hat{x}_i}(k-1)\Phi' + Q \quad (17)$$

where $V_{\hat{x}_i}(k)$ is the filter error variance given by

$$V_{\hat{x}_i}(k) = [I-K_i(k)C_i]V_{\hat{x}_i}(k|k-1) \quad (18)$$

Note that, because of the special structure of the matrices C_i, the unknown mean m_i does not enter the filter equations. That is, the estimates of the hypothesis conditioned filters will be exact. Thus, for each hypothesis we have a Kalman filter, as previously indicated, that can be used to determine the likelihood ratios, which can, in turn, be used to make the decision as to which hypothesis is most likely. The structure of the system is schematically indicated in Fig. 1.
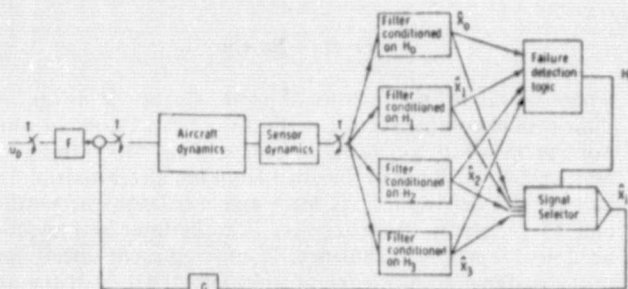
Fig. 1   Fault tolerant control system structure.

Considerable simplification occurs if one considers $C_{ij} = C_{ji} = 1$, $(j \neq 1)$ and $C_{ii} = 0$. Ramifications of this assumption are discussed in the example to follow. Under those conditions the equations for Λ_i may be modified without loss of generality to select the maximum of

$$\{\ln P_{H_i} - \ln \sum_{j=1}^{K} |Q_i(j)| - \frac{1}{2} \sum_{j=1}^{K} r_i'(j)Q_i^{-1}(j)r_i(j),$$

$$i=0,1,...M-1\}$$

where K is the total number of measurements used to make the decision. If the steady-state Kalman filter is used, we can select the largest of

$$\{\ln P_{H_i} - \frac{K}{2} \ln|Q_i| - \frac{1}{2} \sum_{j=1}^{K} r_i'(j)Q_i^{-1}r_i(j),$$

$$i=0,1,...M-1\}$$

Also, if the a priori probabilities of H_i are equal, without loss of generality, we may take

$$\tau_i[Y(K)] = \frac{K}{2} \ln|Q_i| + \frac{1}{2} \sum_{j=1}^{K} r_i'(j)Q_i^{-1}r_i(j) \quad (19)$$

and select the hypothesis H_i corresponding to the smallest τ_i, $i=0,1...M-1$. The next section demonstrates the application of this method to a practical problem.

Application to Aircraft Flight Control

The theory developed in the previous section has been applied to the design of a control system for one space shuttle orbiter configuration at a Mach number of 5 and an altitude of 120,000 ft. Taking the state to be defined as $x = (p, \vartheta, r, \beta)'$ and the only effective control $u = \delta_a$ the aircraft equations of motion can be written as

$$\dot{x} = \begin{bmatrix} -0.0580 & 0 & 0.0170 & -5.791 \\ 1.0 & 0 & 0.5773 & 0 \\ -.0029 & 0 & -0.0085 & -0.7438 \\ 0.5 & 0.0055 & -0.8660 & -0.0009 \end{bmatrix} x$$

$$+ \begin{bmatrix} 2.256 \\ 0 \\ 0.0553 \\ 0 \end{bmatrix} u + w$$

Table 1 Evaluation of W

W component	Level of certainty[a]	Error scale
W_1 (p component)	2	0.05 rad/sec
W_2 (ϕ component)	0	1 rad
W_3 (r component)	2	0.01 rad/sec
W_4 (β component)	3	0.001 rad/sec

[a] 0 implies absolute certainty, 1 implies a high level of certainty, 2 implies only moderate certainty, and 3 implies not too sure.

The selection of the variance W for the last equation involves consideration of 1) the uncertainty that we, as designers, feel related to our knowledge of the equations of motion, 2) the relative scales of the variables, and 3) the environment, with regard to turbulence, under which the vehicle must operate. We will only consider the first two items here. Table 1 shows the authors' interpretation of the level of certainty and scale considerations. Concerning the level of certainty, it was felt that the $\dot\phi$ equation was well understood since it represents a well-known kinematic relationship. A high level of certainty was assigned to the $\dot\phi$ equation. On the other hand, the $\dot p$ and $\dot r$ equations were felt to be better defined than the $\dot\beta$ equation. Turning to scale considerations we have, in effect, equated an error of 1/0.05 in the computation of $\dot p$ to one of 1/0.001 in the computation of $\dot\beta$. The W matrix selected is constructed from the elements of Table 1 as follows

$$W = \text{diag}\ [2(0.05)^2, 0(1)^2, 2((1.01)^2, 3(0.001)^2]$$

The discretized equations of motion using a zero-order-hold with a sampling interval of 0.1 sec is

$$x(k+1) = \begin{bmatrix} 0.9798 & -0.0002 & 0.0267 & -0.5752 \\ 0.0992 & 1 & 0.0587 & 0.0310 \\ 0.0021 & 0 & 1.002 & -.0740 \\ 0.0497 & 0.0006 & -0.0862 & 0.9887 \end{bmatrix} x(k) + \begin{bmatrix} 0.2240 \\ 0.0139 \\ 0.00536 \\ 0.00538 \end{bmatrix} u(k) + w(k)$$

and the discrete variance matrix for the process $w(k)$ is

$$Q = \begin{bmatrix} 0.4757 & 0.04757 & -0.0066 & 0.0236 \\ 0.04757 & 0.00654 & 0.00100 & 0.00309 \\ -0.006 & 0.00100 & 0.02015 & -0.00185 \\ 0.0236 & 0.00309 & -0.00185 & 0.00211 \end{bmatrix} \times (10)^{-3}$$

which was evaluated using Eq. (3). Note that, because of the sampling, even though the ϕ equation was considered absolutely certain, uncertainty does result in the $\dot\phi$ equation of the discrete model. Also, the components of the plant noise vector are correlated in the discrete model.

For illustration, consider that the vehicle has three sensors: a roll-rate gyro, a yaw-rate gyro, and a sideslip indicator. There will, therefore, be four hypotheses to consider, as follows

$$H_0:\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + v_0$$

$$H_1:\ y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + v_1$$

$$H_2:\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + v_2$$

$$H_3:\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + v_3$$

Variances of the measurement error are taken, consistent with current technological capability, to be

$$R_0 = \text{diag}\ [\ (0.05)^2,\ (0.01)^2,\ (0.01)^2\ ]$$

Failure covariances are assumed to be larger than the unfailed ones. (The behavior of the resulting system will, however, be illustrated for both statistical failures—increased variance—and for hardover failures. This capability is a direct result of not assuming a zero-mean measurement error in the failure states.) The values of R_1, R_2, and R_3 used are

$$R_1 = \text{diag}\ (0.025, 0.0001, 0.0001)$$

$$R_2 = \text{diag}\ (0.0025, 0.001, 0.0001)$$

$$R_3 = \text{diag}\ (0.0025, 0.0001, .01)$$

For each hypothesis the state is observable but, given the measurement errors and uncertainties in the vehicle equations of motion, each hypothesis has a different capability of estimating the state of the aircraft. Hence, embedded in the theory is the consideration of the capability of any given sensor group, corresponding to each hypothesis, to estimate the state of the aircraft. This is reflected in the error covariance matrix elements of each hypothesis. As an example, $E[\ (\dot p -p)^2]$ under each hypothesis is indicated as the (1, 1) element of the error covariance matrix and is 0.00075, 0.0015, 0.00082, 0.00087 for H_0, H_1, H_2, and H_3, respectively. As expected, H_0 has the smallest value of $E[\ (\dot p -p)^2]$, indicating that this hypothesis, if true, can produce the best estimate of p. Also indicated, however, is the fact that H_1 produces the worst estimate. Again, this is expected since H_1 corresponds to deletion of roll-rate gyro information.

In this example, The Bayesian risk weights C_{ij} are taken as $C_{ij} = 1$ for $i \neq j$ and $C_{ii} = 1$. Also, steady-state Kalman filters are used so that Eq. (18) is applicable. For the example here, Eq. (18) becomes (using a memory size of five samples)

$$\tau_0 = -\frac{5}{2}\ (22.508) + \frac{1}{2} \sum_{k=1}^{5} r_0(k)$$

$$\begin{bmatrix} 300 & 0.6 & 340 \\ 0.6 & 6350 & 550 \\ 340 & 550 & 3552 \end{bmatrix} r_0(k)$$

$$\tau_1 = -\frac{5}{2}(20.020) + \frac{1}{2}\sum_{k=1}^{5} r_1'(k)$$

$$\begin{bmatrix} 40 & 0 & 0 \\ 0 & 6335 & 542 \\ 0 & 542 & 2000 \end{bmatrix} r_1(k)$$

$$\tau_2 = -\frac{5}{2}(20.438) + \frac{1}{2}\sum_{k=1}^{5} r_2'(k)$$

$$\begin{bmatrix} 300 & 0 & 340 \\ 0 & 1000 & 0 \\ 340 & 0 & 2893 \end{bmatrix} r_2(k)$$

$$\tau_3 = -\frac{5}{2}(18.897) + \frac{1}{2}\sum_{k=1}^{5} r_3'(k)$$

$$\begin{bmatrix} 262 & -52.5 & 0 \\ -52.5 & 6106 & 0 \\ 0 & 0 & 100 \end{bmatrix} r_3(k)$$

During control system operation the scalars τ_i should be using the innovations r_i of the Kalman filter bank stored over the past five samples. Then, the hypothesis corresponding to the minimum τ_i should be selected.

The behavior of the system has been studied using a hybrid computer facility in which the equations of motion of the vehicle were programmed on an analog computer and the control system was mechanized in a digital computer. Figure 2 illustrates the unaugmented step response of the vehicle to an aileron input. This aircraft is a nonminimum phase system indicated by roll reversal. Also, the aircraft possesses a large coupling of the Dutch roll into the aileron response. Digital feedback was employed at a cycle time of 0.1 sec using feedback gains $(-4.9, 0.4, 14.5, -6)$ for $(p, \emptyset, r, \beta)$, respectively, to the aileron. The gains were selected to be constrained to a control system operating with only roll control. Figure 3 shows the response of the closed-loop aircraft to the same pilot step input when state variable feedback (perfect measurement of each state) is employed. Considerable improvement in flying qualities could be obtained if yaw control were available. Figure 4 illustrates the same step response using noisy measurements and accepting H_0. No actuators and sensors have been failed in Fig. 4. In Fig. 5 the responses of the system are indicated for the case where H_2 is true but for each hypothesis being accepted at different times. The failure mode considered in Fig. 5 is an increase in measurement noise. Note that at the start of the record H_0 is selected and produces poor characteristics, as can be seen by comparing the H_0 true portion of the roll-rate trace of Fig. 5 with that of Fig. 4. Had there been no failure, those traces would be almost identical. When H_1 is selected at approximately 5 sec, poor characteristics are still produced. However, when H_2 is selected at approximately 10 sec the system moves to a normal operation, only to return to its poor characteristics when H_3 is selected at approximately 15 sec. This figure illustrates the effect of accepting hypothesis H_i when H_j is true. It indicates the effect of cost selection of the C_{ij} terms in the Bayesian risk function. Figure 6 shows the fault tolerant system in operation when failures of increased noise type are introduced. By looking at the (p, r, β) measurements, it can be seen that the following sensor failure modes have been simulated: $\{H_0, H_1, H_2, H_3\}$. The plot showing the hypothesis accepted indicates the performance of the detector logic. Figure 7 deals with the detection of hardover failures. A hardover failure in the beta sensor has been simulated. Note that, although detection logic is able to detect sensor failures in all cases quite rapidly, the detector logic takes a longer time to reject a failure hypothesis when the

Fig. 2 Response of unaugmented aircraft to aileron step.
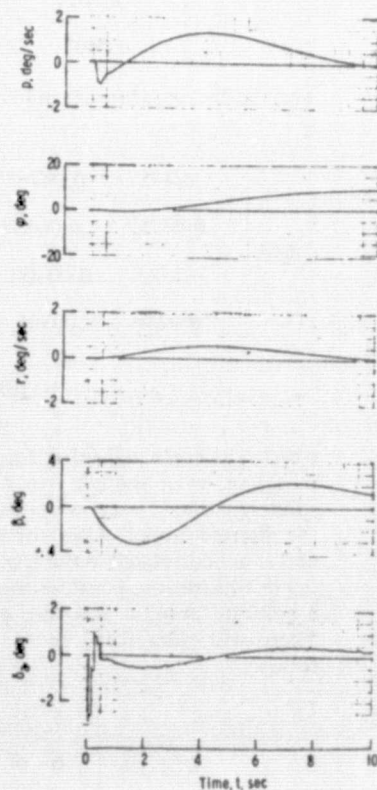
Fig. 3 Response of aircraft to step pilot input when digital state variable feedback (with no measurement noise) is used.
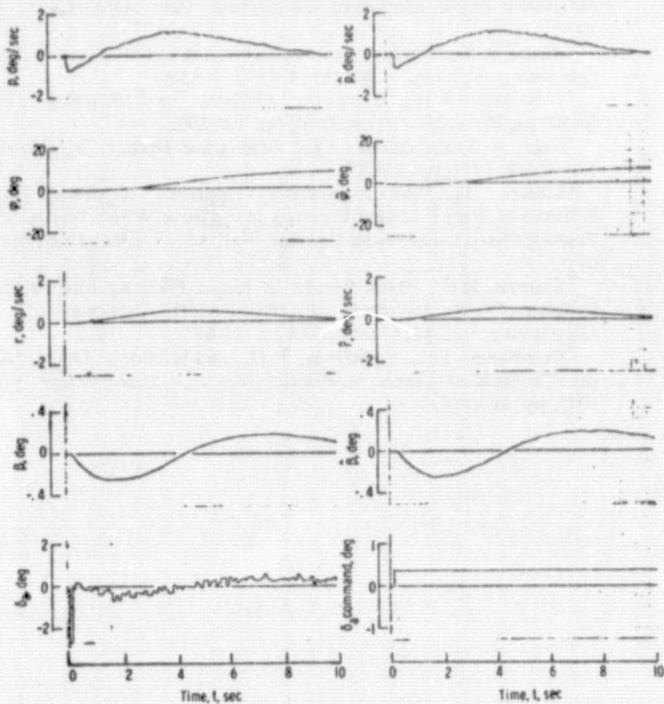
Fig. 4   Response of closed-loop system with noisy measurements under normal operation.
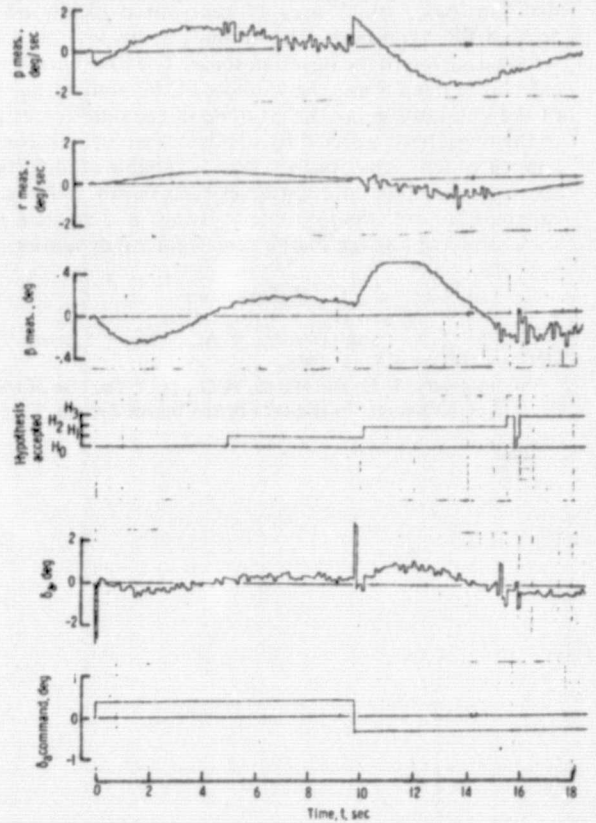

Fig. 6   Operation of fault tolerant system during failures resulting in increased noise.
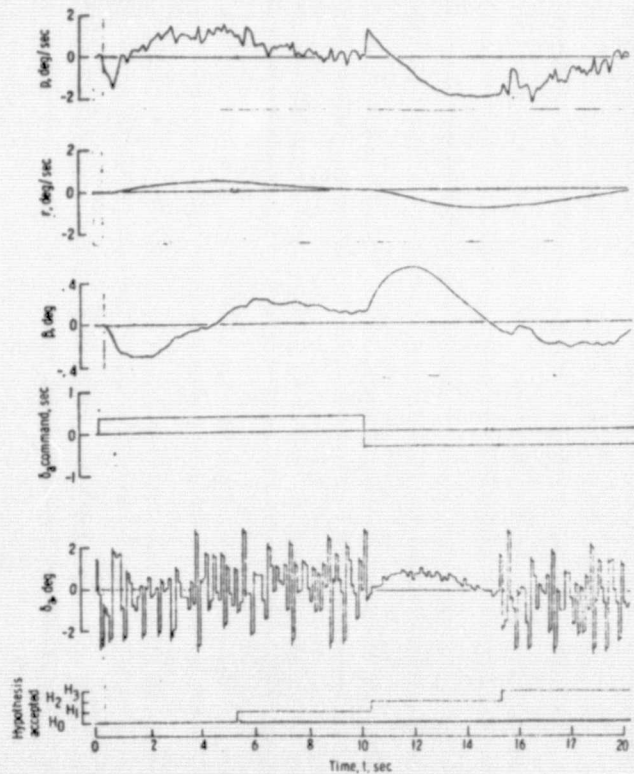

Fig. 5   Response of closed-loop system demonstrating effects of accepting hypothesis H_0, H_2, H_2, when H_2 is true.

Fig. 7   Operation under saturation hardover failure.

system is already in one. Further, note also that only the steady-state Kalman filters are used and overall performance may be improved using time-varying Kalman filters.

Conclusions

A digital fault tolerant control system design that accommodates for aircraft sensor failures has been presented.

Each sensor failure mode and the normal operation of sensors are modeled as M different hypotheses. Then, using the Bayesian M-ary hypothesis testing approach, a detection logic is developed that results in a bank of M Kalman filters. The

decision logic, which uses M generalized likelihood ratios, selectes the hypothesis that minimizes the cost of making a wrong decision in the Beyesian sense. The likelihood ratios are calculated from a moving window of the innovations in each of the Kalman filters. The estimate of the state corresponding to the hypothesis selected by the detection logic is used in the control system. The design system is capable of identifying increased noise type and hardover-type sensor failures. These capabilities are demonstrated using a real-time hybrid simulation for a space shuttle vehicle lateral dynamics.

References

[1] Zadeh, L.A., and Desoer, C.A., *Linear Systems Theory*, McGraw-Hill, New York, 1963.

[2] Montgomery, R.C. and Hatch, H.G., Jr., "The Use of Minimum Order State Observers in Digital Flight-Control System," *AIAA Atmospheric Flight Mechanics Conference*, Palo Alto, Calif., Sept. 1972.

[3] Beard, R.V., "Failure Accommodation in Linear Systems through Self-Reorganization," CR-118314, 1971, NASA.

[4] Jazwinski, A.H., *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1970, pp. 195-200.

[5] Sage, A.P. and Melsa, J.L., *Estimation Theory*, McGraw-Hill, New York, 1971, pp. 116-171.

[6] Kailath, T., "An Innovations Approach to Least Equares Estimation, Part I: Linear Filtering in Additive White Noise," *IEEE Transactions on Automatic Control*, Vol. AC-13, Dec. 1968, pp. 646-655.

[7] Kalman, R.E., "New Results in Linear Filtering and Prediction Problems," *Transactions of the ASME, Journal of Basic Engineering*, Vol. 82, March 1960, pp. 35-45.

[8] Desphande, J.G., Upadhyay, T.N., and Lainiotis, D.G., "Adaptive Controls of Linear Stochastic Systems," *Automatica*, Vol. 9, 1973, pp. 107-115.

PARAMETER ADAPTIVE ESTIMATION OF RANDOM PROCESSES

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Abstract

This paper is concerned with the parameter
adaptive least squares estimation of random proces-
ses. The main result is a general representation
theorem for the conditional expectation of a random
variable on a product probability space. Using this
theorem along with the general likelihood ratio
expression, the least squares estimate of the pro-
cess is found in terms of the parameter conditioned
estimates. The stochastic differential for the a
posteriori probability and the stochastic differen-
tial equation for the a posteriori density are
found by using simple stochastic calculus on the
representations obtained. The results are special-
ized to the case when the parameter has a discrete
distribution. The results can be used to construct
an implementable recursive estimator for certain
types of nonlinear filtering problems. This is
illustrated by some simple examples.

1. Introduction

This paper is concerned with the parameter
adaptive estimation of random processes corrupted
by additive noise. The problem may be briefly
stated as follows. From a given (possibly uncount-
able) collection of random processes with known
distributions, one process is observed with additive
noise. The a priori probability, that a specific
random process in this collection is observed, is
specified for each one in the collection. The
problem is to find the least squares estimate of the
observed signal process.

There is a large class of physical problems
that can be considered in this context. For
instance, joint detection and estimation [9], [18],
estimation under uncertainty [14], joint estimation
and identification [10], parameter adaptive self-
organizing control [15], [16], [17].

Parameter adaptive estimation of discrete
Gaussian processes with linear dynamic models has
been treated by Magill [12]. Lainiotis [10] has
investigated the parameter adaptive estimation
problem for continuous Gaussian processes with
linear dynamic models. Other related work along

these lines can be found in [18] - [21]. The first
systematic treatment of joint detection-estimation
in a general setting was done by Middleton and
Esposito [9].

In the mathematical theory of probability,
related works can be outlined as follows. Parameter
adaptive estimation for a specific jump Markov
process with a finite number states has been stud-
ied by Wonham [22]. The same problem for a two
dimensional Markov process with the nonobservable
component a jump Markov process and the observable
component, a diffusion process, has been resolved
by Shiryaev [23]. Related work on the estimation
of arbitrary stochastic systems can be found in the
work by Kallianpur and Striebel [5] and for Markov
processes in [24].

This paper generalizes the results of [10],
[9]. The approach taken is along the lines of
[13], [5]. The problem is formulated in the usual
Bayesian decision theoretic framework. In the
second section, a mathematical statement of the
problem is given. In section 3, a general repre-
sentation theorem is proved for the conditional
expectation of a random variable on a product prob-
ability space using the properties of Radon-
Nikodym derivatives. This theorem also yields a
Radon-Nikodym representation for the a posteriori
probability which can be thought of as a general-
ized Bayes theorem. By using the representation
theorems along with the general likelihood ratio
expression of Kailath-Duncan [6], [7], the para-
meter adaptive estimation problem of random pro-
cesses is solved in section 4. The main theorem
of this section can be thought of as a generalized
partition theorem. In section 5, by using simple
stochastic calculus, the stochastic differential
corresponding to the a posteriori probability and
the stochastic differential equation for the a
posteriori density are found. In section 6, the
results are specialized to the case when the para-
meter space is countably infinite, i.e., when the
parameter has a discrete distribution. The results
of this section include the case of finite para-
meter space such as M-ary hypotheses. Section 7
includes some illustration and applications.

2. Statement of the Problem

We will be concerned with two underlying prob-
ability spaces [2]: (Ω_1, A_1, P_1) and (Ω_2, A_2, P_2).
In the Bayesian sense, Ω_1 will correspond to the
parameter space and Ω_2 will designate the sample
space. Elements of Ω_1 and Ω_2 will be denoted

where $\mu(\cdot) = P(Y^{-1}(\cdot))$, $\dfrac{d\mu_0}{d\mu_o}$ is the Radon-Nikodym

derivative of μ_θ w.r.t. μ_o, E denotes expectation w.r.t. the measure P (i.e. $P_1 \times P_2$), and $[\mu]$ denotes almost everywhere with respect to the measure μ.

Proof: See Appendix.

Corollary 3.3 Under the assumptions of Theorem 3.2, for P_1 - almost all $\theta \in \Omega_1$ and $\mu_\theta \ll \mu$, the R-N derivative is given by

$$\frac{d\mu_\theta}{d\mu}(y) = \frac{\dfrac{d\mu_\theta}{d\mu}(y)}{\int_{\Omega_1} \dfrac{d\mu_\xi}{d\mu_o}(y)\ dP(\xi)} \quad [\mu]. \tag{3.3}$$

So that

$$E(g(\theta, \eta) | Y(\theta, \eta) = y) = \int_{\Omega_1} E_2(g_\theta(\eta) | Y_\theta = y)$$

$$\frac{d\mu_\theta}{d\mu}(y)\ dP_1(\theta). \tag{3.4}$$

Also

$$P(\hat{A}_1 \times \Omega_2 | Y = y) = \int_{\hat{A}_1} \frac{d\mu_\theta}{d\mu}(y)\ dP_1(\theta). \tag{3.5}$$

Proof: See Appendix.

Note that $p(\hat{A}_1 \times \Omega_2 | Y = y)$ is the *a posteriori* probability of the event $\hat{A}_1$ in the parameter space conditioned on Y. $P(A_1 \times \Omega_2 | Y = y)$ defines a measure π on A_1-sets for each $y \in Y$ by:

$$\pi(\hat{A}_1, y) = (\hat{A}_1 \times \Omega_2 | Y = y) \tag{3.6}$$

We also note that the representation for the *a posteriori* probability is a generalized Bayes theorem. To see this, let Ω_1 be countable, $\Omega_1 = \{\theta_1, \theta_2, \dots, \theta_n, \dots\}$, and $\Omega_2 = R$ and let μ_o be Lebesgue measure on R, then by 3.3 and 3.8

$$\pi(\theta_j, y) = \frac{P_1(\theta_j)\dfrac{d\mu_{\theta_j}}{d\mu_o}(y)}{\sum_{i=1}^{\infty} P_1(\theta_i)\dfrac{d\mu_{\theta_i}}{d\mu_o}(y)}. \tag{3.7}$$

Note that $\dfrac{d\mu_{\theta_j}}{d\mu_o}$ is by definition, the conditional probability density of Y given θ_j so that (3.7) is one form of the celebrated theorem of Bayes.

We shall now apply the results of the Theorem 3.2 and Corollary 3.3, to the parameter adaptive estimation of random processes.

4. Parameter Conditioned Estimation of Random Processes

In this section we will apply the results of the previous section to parameter adaptive estimation of random processes. The representation theorem for the conditional expectation of the last section will be used along with the general likelihood ratio expression of Kailath and Duncan [6], [7], [26], [27]. We now state the general likelihood ratio results in the next lemma which is a slightly different restatement of Theorem 2 in [6].

Lemma 4.1 Let (Ω_1, A_1, P_1) and (Ω_2, A_2, P_2) be two probability spaces. Assume that for each $\theta \in \Omega_1$, we are given three measurable vector random processes (taking their values in R^n) $z_\theta(t, \eta)$, $y_\theta(t, \eta)$,

and $v(t, \eta)$ with $\eta \in \Omega_2$ and $t \in [0, T]$ such that $v(t, \eta)$ is a zero mean Wiener process with incremental covariance

$$E_2\ dv(t)\ dv'(s) = R(t)\ dt$$

and the three random processes are related by the equation

$$y_\theta(t, \eta) = \int_0^t z_\theta(s, \eta)\ ds + v(t, \eta). \tag{4.1}$$

Let Y_t be the space of all R^n-valued continuous functions on $[0, t]$ and B_t be the Borel sets of Y_t. Define the measurable transformations $Y_{t\theta}$ and Y_{tv} from (Ω_2, A_2, P_2) into (Y_t, B_t) by:

$$Y_{t\theta} : (\Omega_2, A_2, P_2) \to (Y_t, B_t)$$

$$Y_{t\theta}(\eta) = (y_\theta(s, \eta), 0 \le s \le t) \tag{4.2}$$

$$Y_{tv} : (\Omega_2, A_2, P_2) \to (Y_t, B_t)$$

$$Y_{tv}(\eta) = (v(s, \eta), 0 \le s \le t). \tag{4.3}$$

Let $P_2 Y_{t\theta}^{-1}$ and $P_2 Y_{tv}^{-1}$ be the measures induced on (Y_t, B_t) by:

$$P_2 Y_{t\theta}^{-1}(B) = P_2(Y_{t\theta}^{-1}(B)) \qquad B \in B_t \tag{4.4}$$

$$P_2 Y_{tv}^{-1}(B) = P_2(Y_{tv}^{-1}(B)) \qquad B \in B_t. \tag{4.5}$$

If, for some $\theta \in \Omega_1$,

$$E_2 \int_0^t || z_\theta(t, \eta) ||_1\ dt < \infty \tag{4.6}$$

and

$$\int_0^t z_\theta'(t, \eta)\ z_\theta(t, \eta)\ dt < \infty \tag{4.7}$$

and the future increments $\{v(t, \eta) - v(s, \eta)\}$ are independent of the σ-algebra generated by $\{v(\tau, \eta), z_\theta(\tau, \eta), 0 \le \tau \le s\}$, then $P_2 Y_{t\theta}^{-1} \ll P_2 Y_{tv}^{-1}$ and the R-N derivative is given by

$$\frac{dP_2 Y_{t\theta}^{-1}}{dP_2 Y_{tv}^{-1}}(y) = \exp\{\int_0^t \hat{z}_\theta'(s,y)\ R^{-1}(s)\ dy(s) -$$

$$\frac{1}{2}\int_0^t \hat{z}_\theta'(s,y)\ R(s)^{-1}\ \hat{Z}_\theta(s,y)\ ds\}\ [P_2 Y_{tv}^{-1}]$$

$$\tag{4.8}$$

where

$$\hat{Z}_\theta(s,y) = E_2(z_\theta(s, \eta) \mid Y_{s\theta} = y(\xi), 0 \le \xi \le s). \tag{4.9}$$

Proof: See Theorem 2 in [6] and also Theorem 1 in [7].

Theorem 4.2 Let the assumptions be as in Lemma 4.1. Also assume that $z_\theta(t, \eta)$ is a measurable function in (t, η, θ) so that we can now define the measurable random processes $z(t, \eta, \theta)$ and $y(t, \eta, \theta)$ on the product probability space $(\Omega, A, P) = (\Omega_1 \times \Omega_2, A_1 \times A_2, P_1 \times P_2)$ by

$$z(t, \theta, \eta) = z_\theta(t, \eta) \left.\begin{array}{l}\end{array}\right\} \quad t \in [0, T], \theta \in \Omega_1, \eta \in \Omega_2. \tag{4.10}$$

$$y(t, \theta, \eta) = y_\theta(t, \eta) \qquad\qquad\qquad\qquad\qquad \tag{4.11}$$

The three random processes are related by

$$y(t, \theta, \eta) = \int_0^t z(s, \theta, \eta)\ ds + v(t, \eta)$$

$$t \in [0, T]. \tag{4.12}$$

Let Y_t be the measurable transformation

$$Y_t : (\Omega, A, P) \to (\mathcal{Y}_t, \mathcal{B}_t)$$

$$Y_t(\theta, \eta) \to (y(s, \theta, \eta), 0 \le s \le t). \qquad (4.13)$$

Note that $Y_{t\theta}$ is the section Y_t at θ. If there exists a set N_1 in A_1 of P_1 measure zero such that for all $\theta \in \Omega_1 - N_1$, the assumptions in Lemma 4 are satisfied, and $E(z(t, \eta, \theta))$ exists, then

$$E(z(t, \theta, \eta) \mid Y_t(\theta, \eta) = y) =$$

$$\frac{\int_{\Omega_1} \hat{z}_\theta(t, y) \Lambda_\theta(t, y) dP_1(\theta)}{\int_{\Omega_1} \Lambda_\theta(t, y) dP_1(\theta)} \quad [PY_t^{-1}] \qquad (4.14)$$

where $\hat{z}_\theta(t, y)$ is given by

$$\hat{z}_\theta(t, y) = E_2(z_\theta(t, \eta) \mid Y_{t\theta} = y(\xi), 0 \le \xi \le t) \qquad (4.15)$$

and the R-N derivative $\Lambda_\theta(t, y)$ is given by

$$\Lambda_\theta(t, y) = \frac{dP_2 Y_{t\theta}^{-1}}{dP_2 Y_{tv}^{-1}} (y) = \exp \{ \int_o^t \hat{z}_\theta'(s, y) R^{-1}(s)$$

$$dy(s) - \frac{1}{2} \int_o^t \hat{z}_\theta'(s, y) R^{-1}(s) \hat{z}_\theta(s, y) ds\}. \qquad (4.16)$$

Proof: For all $\theta \in \Omega_1 - N_1$ by Lemma 4.1, we have $P_2 Y_{t\theta}^{-1} \ll P_2 Y_{tv}^{-1}$, and the R-N derivative is given by $\Lambda_\theta(t, y)$. We can now apply Theorem 3.2 with $\mathcal{Y} = \mathcal{Y}_t$, $\mathcal{B} = \mathcal{B}_t$, $\mu_\theta(\cdot) = P_2 Y_{t\theta}^{-1}(\cdot)$, $\mu(\cdot) = P Y_t^{-1}(\cdot)$, $\mu_o(\cdot) = P_2 Y_{tv}^{-1}(\cdot)$.

Corollary 4.3 Under the assumptions of Theorem 4.2, for P_1-almost all $\theta \in \Omega_1$, $P_2 Y_{t\theta}^{-1} \ll P Y_t^{-1}$, and the R-N derivative is given by

$$\frac{dP_2 Y_{t\theta}^{-1}}{dP Y_t^{-1}} (y) = \frac{\Lambda_\theta(t, y)}{\int_{\Omega_1} \Lambda_\xi(t, y) dP_1(\xi)} \quad [PY_t^{-1}]. \qquad (4.17)$$

So that

$$E(z \mid Y_t = y) = \int_{\Omega_1} \hat{z}_\theta(t, y) \frac{dP_2 Y_{t\theta}^{-1}}{dPY_t^{-1}} (y)$$

$$dP_1(\theta) [PY_t^{-1}]. \qquad (4.18)$$

Also

$$P(\hat{A}_1 \times \Omega_2 \mid Y_t = y) = \int_{\hat{A}_1} \frac{dP_2 Y_{t\theta}^{-1}}{dPY_t^{-1}} (y)$$

$$dP_1(\theta) [PY_t^{-1}]. \qquad (4.19)$$

Proof: Follows from Corollary 3.3.

Corollary 4.4 Under the assumptions of Theorem 4.2, we have the following representation for the R-N derivative $\dfrac{dP_2 Y_{t\theta}^{-1}}{dPY_t^{-1}} (y)$

$$\frac{dP_2 Y_{t\theta}^{-1}}{dPY_t^{-1}} (y) = \frac{\Lambda_0(t, y)}{\Lambda(t, y)} \qquad (4.20)$$

where

$$\Lambda(t, y) = \frac{dP Y_t^{-1}}{dP_2 Y_{tv}^{-1}} (y) = \exp \{ \int_o^t \hat{z}(s, y) R^{-1}(s) dy(s) -$$

$$\frac{1}{2} \int_o^t \hat{z}'(s, y) R^{-1}(s) \hat{z}(s, y) ds\}. \qquad (4.21)$$

Proof: By Corollary 4.3, we have

$$\frac{dP_2 Y_{t\theta}^{-1}}{dP Y_t^{-1}} (y) = \frac{\Lambda_\theta(t, y)}{\int_{\Omega_1} \Lambda_\xi(t, y) dP_1(\xi)}$$

By Remark 4 in the proof of Theorem 3.2

$$\int_{\Omega_1} \Lambda_\theta(t, y) dP_1(\theta) = \frac{dP Y_t^{-1}}{dP_2 Y_{tv}^{-1}} (y).$$

The result now follows from Lemma 4.1.

Comments: The R-N derivative $\Lambda_\theta(t, y)$ is the likelihood ratio for the detection problem (with θ fixed)

$$h_\theta: \quad y_\theta(t, \eta) = \int_o^t z_\theta(s, \eta) ds + v(t, \eta)$$

$$h_o: \quad y_\theta(t, \eta) = v(t, \eta).$$

$\Lambda(t, y)$ in Corollary 4.4 is the L.R. for the composite hypotheses testing problem

$$h : \quad y(t, \eta, \theta) = \int_o^t z(s, \eta, \theta) ds + v(t, \eta)$$

$$h_o: \quad y(t, \eta, \theta) = v(t, \eta).$$

Note that

$$\lambda_\theta(t, y) = \frac{\Lambda_\theta(t, y)}{\int_{\Omega_1} \Lambda_\xi(t, y) dP_1(\xi)}$$

represents a normalized likelihood ratio. This normalized LR is the ratio of the conditional probability density of the parameter conditioned on the observation and the probability density of the parameter; that is, $\lambda_\theta(t, y)$ is the normalized conditional density conditioned on the observation.

Note also that $P(\hat{A}_1 \times \Omega_2 \mid Y_t = y)$ is the a posteriori probability of the event $\hat{A}_1$ in the parameter space conditioned on the observation. Since

$$P(\hat{A}_1 \times \Omega_2 \mid Y_t = y) = \int_{\hat{A}_1} \frac{dP_2 Y_{t\theta}^{-1}}{dP Y_t^{-1}} (y) dP_1(\theta) ,$$

we have proved the existence of the conditional probability density of the parameter.

Theorems 3.2 and 4.2 unify a number of known results in the literature. For instance, in the case, where the parameter space contains two points, each one corresponding to the presence or the absence of a signal, we obtained the relationship between the optimal estimate of the signal under uncertainty and the optimal estimate of sure signal in the least squares sense derived in reference [9]. In the case, where the signal and the measurement noise are independent and the signal is a Gaussian signal with a linear dynamic model we get the results in [10].

The expression derived for the conditional expectation in this section is useful in the case of fixed observation time interval. If the data is coming continously, a stochastic differential equation implementation is more practical. In the next section, by using simple Ito calculus on the representations obtained in this section, we shall find the stochastic differential equations that the a posteriori probability and probability density satisfy.

5. Stochastic Differential Equations for the A Posteriori Probability

The results of the previous section are useful when the observation interval is fixed. Since in most technological applications the data is obtained continuously, the differential form of the results is more practical to implement. In this section, we shall find the stochastic differential equations for the a posteriori probability density and a stochastic differential representation for the a posteriori probability by a simple application of Ito's differentiation rule on the normalized conditional probability density representation in Corollary 4.4.

Theorem 5.1 If the assumptions of Corollary 4.4 hold, then normalized conditional probability density

$$\frac{dP_2 Y_{t\theta}^{-1}}{dP Y_t^{-1}} (y) = \lambda_\theta (t, y)$$

exists P_1-almost everywhere and is the unique solution of the following stochastic differential equation

$$d\lambda_\theta (t,y) = \lambda_\theta (t,y)(\hat{z}_\theta (t,y) - \hat{z}(t,y))' R^{-1}(t)(dy(t) - \hat{z}(t,y)dt) \tag{5.1}$$

with the initial condition unity.

Proof: From Corollary 4.4.

$$\lambda_\theta (t, y) =$$

$$\frac{\exp\{ \int_o^t \hat{z}_\theta '(s, y) R^{-1}(s) dy(s) -}{\exp\{ \int_o^t \hat{z}' (s, y) R^{-1}(s) dy(s) -}$$

$$\frac{\frac{1}{2} \int_o^t \hat{z}_\theta '(s, y) R^{-1}(s) \hat{z}_\theta (s, y) ds\}}{\frac{1}{2} \int_o^t \hat{z}' (s, y) R^{-1}(s) \hat{z} (s, y) ds\}} . \tag{5.2}$$

Let R be the identity matrix with no loss of generality.

Since the required partials exist and are continuous, by Ito's differentiation rule [25] we get

$$d\lambda_\theta (t, y) = \lambda_\theta (t, y)\{\hat{z}_\theta '(t, y) dy - \hat{z}(t, y) dy$$

$$- \frac{1}{2} \hat{z}_\theta '(t,y)\hat{z}_\theta (t,y)dt + \frac{1}{2}\hat{z}'(t,y)\hat{z}(t,y)dt\}$$

$$+ \frac{1}{2}\lambda_\theta (t,y)(\hat{z}_\theta (t,y) - \hat{z}(t,y))'(\hat{z}_\theta (t,y)$$

$$- \hat{z}(t,y)) dt$$

$$d\lambda_\theta (t, y) = \lambda_\theta (t,y)(\hat{z}_\theta (t,y) - \hat{z}(t,y))'$$

$$(dy(t) - \hat{z}(t,y)dt). \tag{5.3}$$

To show uniqueness, let λ_1 and λ_2 be two solutions of 5.3, then

$$d(\lambda_1/\lambda_2) = \frac{\lambda_2 d\lambda_1 - \lambda_1 d\lambda_2}{\lambda_2^2} + \frac{\lambda_1 d\lambda_2 d\lambda_2 - \lambda_2 d\lambda_1 d\lambda_2}{\lambda_2^3} .$$

By 5.3 it is clear that $\lambda_2 d\lambda_1 - \lambda_1 d\lambda_2 = 0$. By using 5.3 and stochastic calculus

$$\lambda_1 d\lambda_2 d\lambda_2 = \lambda_1 \lambda_2^2 (\hat{z}_\theta (t,y) - \hat{z}(t,y))'(\hat{z}_\theta (t,y) - \hat{z}(t,y))dt$$

$$\lambda_2 d\lambda_1 d\lambda_2 = \lambda_2^2 \lambda_1 (\hat{z}_\theta (t,y) - \hat{z}(t,y))'(\hat{z}_\theta (t,y) - \hat{z}(t,y))dt.$$

We, therefore, have

$$d(\lambda_1/\lambda_2) = 0 \Rightarrow \frac{\lambda_1(t)}{\lambda_2(t)} - \frac{\lambda_1(0)}{\lambda_2(0)} = 0 \quad \text{for all } t\epsilon[0,T],$$

or $\tag{5.4}$

$$\lambda_2(t) = \frac{\lambda_2(0)}{\lambda_1(0)} \lambda_1(t) \quad \text{for all } t \text{ in } [0,T]. \tag{5.5}$$

So if $\lambda_1(0) = \lambda_2(0)$, we must have $\lambda_1(t) = \lambda_2(t)$ for all t in [0, T]. Detailed arguments can be found in [13].

From the stochastic differential equation that the normalized conditional probability density satisfies in Theorem 5.1, it is now easy to find a stochastic differential representation for the a posteriori probability in $P(\hat{A}_1 \times \Omega_2 \mid Y = y)$. From Equation 3.6, we know that $\pi(\hat{A}_1, y) = P(\hat{A}_1 \times \Omega_2 \mid Y = y)$ defines a measure on A_1 sets for PY^{-1} almost everywhere, and $\pi \ll P_1$. The R-N derivative is given by

$$\frac{d\pi}{dP_1} = \frac{dP_2 Y_{t\theta}^{-1}}{dP Y_t^{-1}} (y). \tag{5.6}$$

By using Theorem 5.1 and equation 5.6, we shall find the stochastic differential for the a posteriori probability.

Theorem 5.2 Let the assumptions of Corollary 4.3 hold. For a fixed parameter set $\hat{A}_1$, the a posteriori probability

$$\pi_t(\hat{A}_1 \times \Omega_2 \mid Y = y),$$

admits the following differential

$$d\pi_t(\hat{A}_1, y) = \int_{\hat{A}_1} (\hat{z}_\theta (t,y) - \hat{z}(t,y))' d\pi_t(\theta,y)$$

$$R^{-1}(t)(dy(t) - \hat{z}(t,y)dt) \tag{5.7}$$

with $\pi_0(\hat{A}_1, y) = P_1(\hat{A}_1)$ or what is the same

$$d\pi_t(\hat{A}_1, y) = [\int_{\hat{A}_1} \hat{z}_\theta (t, y) d\pi_t(\theta, y) - \pi_t(\hat{A}_1, y)$$

$$\hat{z}(t, y)] R^{-1}(t)(dy(t) - \hat{z}(t, y) dt). \tag{5.8}$$

Proof: See appendix.

In the next section, we shall specialize the results to the case where the parameter space is countably infinite.

6. Discrete Parameter Case

The real advantage of parameter conditioned approach to estimation occurs when the parameter space is finite or countably infinite since the solution may then be readily implemented on a digital computer. This will be the case when the parameter has a discrete distribution or it has been suitably quantized to be put on a digital computer. The parameter adaptive approach will of course be more rewarding when the parameter conditioned estimates can be easily obtained. We now give the results for the discrete parameter case.

Theorem 6.1

Let the assumption of Theorem 4.2 hold. If the parameter space is countably infinite (or finite), let

$$\Omega_1 = \{\theta_1, \theta_2 ..., \theta_n, ...\} ,$$

then

$$E(z|Y_t=y) = \frac{\sum_{i=1}^{\infty} P_1(\theta_i) \Lambda_{\theta_i}(t,y) \hat{z}_{\theta_i}(t,y)}{\sum_{i=1}^{\infty} P_1(\theta_i) \Lambda_{\theta_i}(t,y)} [PY_t^{-1}] \quad (6.1)$$

where

$$\hat{z}_{\theta_i}(t,y) = E_2(z_{,i}|Y_t\theta_i = y(s), \ 0 \leq s \leq t) \quad (6.2)$$

and the R-N derivative is given by

$$\Lambda_{\theta_i}(t,y) = \frac{dP_2 Y_{t\theta_i}^{-1}}{dP_2 Y_{tv}^{-1}} (y) = \exp \{ \int_0^t \hat{z}_{\theta_i}(s,y) R^{-1}(s) dy(s) $$

$$- \frac{1}{2} \int_0^t \hat{z}_{\theta_i}'(s,y) R^{-1}(s) \hat{z}_{\theta_i}(s,y) ds \}. \quad (6.3)$$

Proof: Follows from Theorem 4.2 by integrating with respect of P_1 over Ω_1.

Corollary 6.2 Under the assumption of Theorem 6.1, for all $\theta_j \epsilon \Omega_1$, $P_2 Y_{t\theta_j}^{-1} \ll P Y_t^{-1}$, and the R-N derivative is given by

$$\frac{dP_2 Y_{t\theta_j}^{-1}}{dPY_t^{-1}} (y) = \frac{\Lambda_{\theta_j}(t,y)}{\sum_{i=1}^{\infty} \Lambda_{\theta_i}(t,y) P_1(\theta_i)} [PY_t^{-1}]. \quad (6.4)$$

So that

$$E(z|Y_t=y) = \sum_{i=1}^{\infty} \hat{z}_{\theta_i}(t,y) \frac{dP_2 Y_{t\theta_i}^{-1}}{dPY_t^{-1}} (y) P_1(\theta_i) [PY_t^{-1}] \quad (6.5)$$

$$P(\theta_j \times \Omega_2 | Y_t=y) = P_1(\theta_j) \frac{dP_2 Y_{t\theta_j}^{-1}}{dPY_t^{-1}} (y) \quad [PY_t^{-1}]. \quad (6.6)$$

Theorem 6.3

Under the assumptions of Theorem 5.1 and Theorem 6.1 the a posteriori probability for a fixed value θ_i of the parameter admits the following stochastic differential for

$$\pi_t(\theta_i,y) = P(\theta_i \times \Omega_2 | Y_t=y). \quad (6.7)$$

$$d\pi_t(\theta_i,y) = \pi_t(\theta_i,y)(\hat{z}_{\theta_i}(t,y) - \hat{z}(t,y))' R^{-1}(t)$$

$$(dy(t) - \hat{z}(t,y)dt) \quad (6.8)$$

with $\pi_0(\theta_i,y) = P_1(\theta_i)$.

Proof: Follows form Theorem 5.2 by carrying out the integration with respect to P_1.

In practice, since $\hat{z}(t,y)$ will be computed by

$$\hat{z}(t,y) = \sum_{i=1}^{\infty} \pi_t(\theta_i,y) \hat{z}_{\theta_i}(t,y),$$

it is therefore of interest to look at the existence and uniqueness of the solution of the infinite set of simultaneous equations that Equation 6.8 implies.

Theorem 6.4

Under the assumption of Theorem 6.1, the set of a posteriori probabilities

$$\{\pi_t(\theta_i,y), \ i = 1, 2, 3, \ldots\}$$

is the unique solution of the infinite order system of stochastic differential equations

$$d\pi_t(\theta_i,y) = \pi_t(\theta_i,y)(\hat{z}_{\theta_i}(t,y) - \sum_{j=1}^{\infty} \hat{z}_{\theta_j}(t,y)\pi_t(\theta_j,$$

$$y)) R^{-1}(t) (dy(t) - \sum_{j=1}^{\infty} \hat{z}_{\theta_j}(t,y) \pi_t(\theta_j,y)dt)$$

$$i = 1,2,\ldots$$

with $\pi_0(\theta_i,y) = P_1(\theta_i)$.

Proof: See appendix.

Theorems 6.1 and 6.4 imply an implementable estimator structure which consists of two parts: (1) a nonadaptive part in which parameter conditioned estimates are found, (2) an adaptive part in which the a posteriori probabilities are found. If the differential equations for the parameter conditioned estimated are available, then these estimates along with the differential equations for the a posteriori probabilities (Theorem 6.4) provide a recursive solution to the problem. In the next section, we shall give some examples to illustrate this estimator structure.

7. Examples

The usefulness of parameter adaptive approach to estimation problems has been illustrated in several papers [10], [12], [18], [20], [21], [22]. In this section, we shall outline some further possible applications of parameter conditioned approach to estimation.

Example 1 Consider the nonlinear stochastic dynamic system described by the following stochastic differential equations

$$dx = x \theta^2 dt + dw$$

$$d\theta = 0$$

with the observation model

$$dy = x \theta^3 dt + dv$$

where $x(0)$ is a Gaussian random variable with mean μ and variance σ^2 and $\theta(0)$ is an arbitrary random variable with a discrete distribution $P_1(\theta_i)$. The Wiener processes w and v have zero mean and unit variance. Also, $x(0)$, $\theta(0)$, $v(t)$, and $w(t)$ are independent. By applying Theorems 6.1 and 6.4, with $\theta(0)$ as the parameter we get the least squares estimate for $x(t)$

$$\hat{x}(t) = \sum_{i=1}^{\infty} \pi_i(t) \hat{x}_i(t)$$

where $\hat{x}_i(t)$ is the solution of the Kalman-Bucy filter

$$\frac{d}{dt} \hat{x}_i(t) = \theta_i^2 \hat{x}_i(t) + K_i(t) (\dot{y}(t) - \theta_i^3 \hat{x}_i(t))$$

where $V_i(t)$ is the solution of Riccati equation with $x_i(0) = \mu$ and $K_i(t) = \theta_i^3 V_i$

$$\frac{d}{dt} V_i(t) = 2 \theta_i^2 V_i(t) - \theta_i^6 V_i^2(t) + 1$$

with $V_i(0) = \sigma^2$ and the set of a posteriori probabilities

$$\{\pi_i(t), \ i = 1, 2, \ldots\}$$

is the unique solution of the following system of stochastic differential equations

$$d\pi_i(t) = \pi_i(t) \left(\alpha_i^3 x_i(t) - \sum_{j=1}^{\infty} \alpha_j^3 \pi_j(t) \hat{x}_j(t) \right)$$

$$\left(dy(t) - \sum_{j=1}^{\infty} \alpha_j^3 \pi_j(t) \hat{x}_j(t) dt \right)$$

with $\pi_i(0) = P_0(\theta_i)$.

Example 2 In this example, we shall consider the random telegraph signal $z(t)$, with values ± 1 and transition density, α jump/second where α is a discrete random variable with a range of values $\{\alpha_1, \alpha_2, \ldots \alpha_M\}$ and a priori probability distribution $\{P_1(\alpha_j), j = 1, 2, \ldots, M\}$. The observation model is taken to be

$$dy(t) = z(t)dt + dv$$

where v is a unit Wiener process. By applying Theorem 6.1 and 6.4 we get

$$\hat{z}(t) = \sum_{i=1}^{M} \pi_i(t) \hat{z}_i(t)$$

where the parameter conditioned estimated $\hat{z}_i(t)$ satisfied the stochastic differential equation derived by Wonham in [22]:

$$d\hat{z}_i(t) = 2 \alpha_i \hat{z}_i(t) + (1 - \hat{z}_i^2(t))(dy(t) - \hat{z}_i(t)dt)$$

which in turn derives the stochastic differential equations for a posteriori probabilities given by Theorem 6.4

$$d\pi_i(t) = \pi_i(t) \left(\hat{z}_i(t) - \sum_{j=1}^{M} \pi_j(t) \hat{z}_j(t) \right) \left(dy(t) - \sum_{j=1}^{M} \pi_j(t) \hat{z}_j(t)dt \right)$$

with

$$\pi_i(0) = P_1(\alpha_i).$$

8. Conclusions

The problem of optimal parameter adaptive estimation for random processes is formulated in the Bayesian framework. A general R-N derivative representation for the least squares estimate of a random variable on a product space is derived. The representation theorem is applied to the optimal parameter adaptive estimation problem for random processes to find the least squares estimate of the observed signal and the a posteriori probability of parameter conditioned estimates. The stochastic differential equation for the a posteriori probability is derived. The results are specialized to the case where the parameter has a discrete distribution. The approach is illustrated by simple examples.

Appendix

The appendix contains the proofs that are not given in the text. To prove Theorem 3.2, we shall need the following lemma.

Lemma 3.1 Let (Ω_1, A_1, P_1) be a probability space and (Y, B) be a measurable space where B is generated by a countable class of sets. Suppose that for each θ in Ω_1 we are given a measure $\mu_\theta(y)$ on (Y, B) such that, for fixed $B \in B$, $\mu_\theta(B)$ is measurable in θ. If there exists a set N_1 in A_1 of P_1 measure zero such that, for all θ in $\Omega_1 - N_1$, $\mu_\theta \ll \mu_0$ where μ_0 is a measure on (Y, B), then there is an $A_1 \times B$ measurable function f such that

$$f(\theta, y) = \frac{d\mu_\theta}{d\mu_0}(y) \qquad \theta \in \Omega_1 - N_1.$$

Proof: See [4] pp. 616-617.

Proof of Theorem 3.2: Let us define the signed measures ν and $\{\nu_\theta, \theta \in \Omega_1\}$ on (Y, B) by

$$\nu(B) = \int_{Y^{-1}(B)} g(\theta, \eta) \, dP(\theta, \eta) \qquad B \in B \qquad (A-1)$$

and

$$\nu_\theta(B) = \int_{Y_\theta^{-1}(B)} g_\theta(\eta) \, dP_2(\eta) \qquad B \in B \qquad (A-2)$$

where $Y_\theta(\eta)$ are the sections of Y and g at $\theta \in \Omega_1$. We have from the definitions of conditional expectations

$$E(g(\theta, \eta)|Y(\theta, \eta) = y) = \frac{d\nu}{d\mu}(y) \qquad (A-3)$$

$$E_2(g_\theta(\eta)|Y_\theta(\eta) = y) = \frac{d\nu_\theta}{d\mu_\theta}(y). \qquad (A-4)$$

So we are to prove that

$$\frac{d\nu}{d\mu}(y) = \frac{\int_{\Omega_1} \frac{d\nu_\theta}{d\mu_\theta}(y) \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta)}{\int_{\Omega_1} \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta)} \qquad (A-5)$$

Remark 1

$$\nu(B) = \int_{\Omega_1} \nu_\theta(B) \, dP_1(\theta). \qquad (A-6)$$

This follows from Fubini's theorem and the fact that $Y_\theta^{-1}(B) = (Y^{-1}(B))_\theta$ so that

$$\nu(B) = \int_{\Omega_1} \left[\int_{Y_\theta^{-1}(B)} g_\theta(\eta) \, dP_2(\eta) \right] dP_1(\theta). \qquad (A-7)$$

Remark 2 $E_2(g_\theta(\eta)|Y_\theta(\eta) = y)$ can be chosen to be $A_1 \times B$ - measurable.

Since $\nu_\theta \ll \mu_\theta$ and $\mu_\theta \ll \mu_0$, then by the chain rule of R-N derivatives (Theorem 32.A in [1])

$$\frac{d\nu_\theta}{d\mu_0} = \frac{d\nu_\theta}{d\mu_\theta} \frac{d\mu_\theta}{d\mu_0} \quad [\mu_0]. \qquad (A-8)$$

Since μ_θ and μ_0 are measures, $\frac{d\mu_\theta}{d\mu_0}$ is nonnegative. Since $\{y \in Y: \frac{d\mu_\theta}{d\mu_0}(y) = 0\}$ has μ_θ measure zero and $\{y \in Y: \frac{d\mu_\theta}{d\mu_0}(y) = \infty\}$ has μ_0 and consequently, μ_θ-measure zero, it follows that

$$0 < \frac{d\mu_\theta}{d\mu_0} < \infty \quad [\mu_\theta]. \qquad (A-9)$$

So

$$\frac{d\nu_\theta}{d\mu_\theta} = \frac{d\nu_\theta}{d\mu_0} \Big/ \frac{d\mu_\theta}{d\mu_0} \quad [\mu_\theta]. \qquad (A-10)$$

By Lemma 1.1 $\frac{d\nu_\theta}{d\mu_0}$ and $\frac{d\mu_\theta}{d\mu_0}$ can be chosen to be

$A_1 \times B$ - measurable, so that $\frac{d\nu_0}{d\mu_0}$ can be chosen to be a $A_1 \times B$ measurable function.

Remark 3

$$\frac{d\nu}{d\mu_0}(y) = \int_{\Omega_1} E_2(g_\theta(\eta)|Y_\theta(\eta) = y) \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta). \tag{A-11}$$

From Remark 1 we have

$$\nu(B) = \int_{\Omega_1} [\int_{Y_\theta^{-1}(B)} g_\theta(\eta) \, dP_2(\eta)] \, dP_1(\theta). \tag{A-12}$$

Let $g_\theta(\eta) = g_\theta^+(\eta) - g_\theta^-(\eta)$ where g_θ^+ and g_θ^- are the positive and negative parts of $g_\theta[1]$, then

$$\nu(B) = \int_{\Omega_1} [\int_{Y_\theta^{-1}(B)} g_\theta^+(\eta) \, dP_2(\eta)] \, dP_1(\theta)$$
$$-\int_{\Omega_1} [\int_{Y_\theta^{-1}(B)} g_\theta^-(\eta) \, dP_2(\eta)] \, dP_1(\theta). \tag{A-13}$$

From the definition of conditional expectations

$$\nu(B) = \int_{\Omega_1} [\int_B E_2(g_\theta^+|Y_\theta = y) \, d\mu_\theta(y)] \, dP_1(\theta)$$
$$-\int_{\Omega_1} [\int_B E_2(g_\theta^-|Y_\theta = y) \, d\mu_\theta(y)] \, dP_1(\theta). \tag{A-14}$$

From the chain rule of R-N derivatives (Theorem 32.B in [1]),

$$\nu(B) = \int_{\Omega_1} [\int_B E_2(g_\theta^+|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) \, d\mu_0(y)] \, dP_1(\theta)$$
$$-\int_{\Omega_1} [\int_B E_2(g_\theta^-|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) \, d\mu_0(y)] \, dP_1(\theta). \tag{A-15}$$

By Lemma 1.1 and Remark 2, the terms $E_2(g_\theta^+|Y_\theta = y)$, $E_2(g_\theta^-|Y_\theta = y)$, and $\frac{d\mu_\theta}{d\mu_0}$ can be chosen to be $A_1 \times B$ - measurable. By applying Fubini's theorem for non-negative functions, we can write that $\nu(B)$

$$= \int_B [\int_{\Omega_1} E_2(g_\theta^+|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta)] \, d\mu_0(y)$$
$$- \int_B [\int_{\Omega_1} E_2(g_\theta^-|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta)] \, d\mu_0(y). \tag{A-16}$$

Since the difference is well-defined, at least one of the integrals should be finite and, consequently, the integrand corresponding to that integral must be finite valued almost everywhere $P_1 \times \mu_0$. So the following term is well-defined.

$$E_2(g_\theta^+|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) - E_2(g_\theta^-|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y)$$
$$= [E_2(g_\theta^+|Y_\theta = y) - E_2(g_\theta^-|Y_\theta = y)] \frac{d\mu_\theta}{d\mu_0}(y). \tag{A-17}$$

From Theorem 6.5.2 in [3] this expression

$$= E_2(g_\theta|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y). \tag{A-18}$$

So that

$$\nu(B) = \int_B [\int_{\Omega_1} E_2(g_\theta|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta)] \, d\mu_0(y). \tag{A-19}$$

Since the term in brackets is B-measurable by Fubini's theorem and since the R-N derivative is unique a.e.,

$$\frac{d\nu}{d\mu_0} = \int_{\Omega_1} E_2(g_\theta|Y_\theta = y) \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta) \; [\mu_0]. \tag{A-20}$$

Remark 4

$$\frac{d\mu}{d\mu_0}(y) = \int_{\Omega_1} \frac{d\mu_\theta}{d\mu_0}(y) \, dP_1(\theta). \tag{A-21}$$

This follows from Remark 3 by setting $g(\theta,\eta) = I_{\Omega_1 \times \Omega_2}(\theta,\eta)$ where $I_{\Omega_1 \times \Omega_2}$ is the characteristic function of the set $\Omega_1 \times \Omega_2$. This can be seen as follows

$$\nu(B) = \int_{Y^{-1}(B)} I_{\Omega_1 \times \Omega_2}(\theta,\eta) \, dP(\theta,\eta) = P(Y^{-1}(B))$$
$$= \mu(B). \tag{A-22}$$

Now since $\nu \ll \mu$ and $\nu \ll \mu_0$ (Remark 4), we have

$$\frac{d\nu}{d\mu} \frac{d\mu}{d\mu_0} = \frac{d\nu}{d\mu_0} \; [\mu_0]. \tag{A-23}$$

Since

$$0 < \frac{d\mu}{d\mu_0} < \infty \; [\mu], \tag{A-24}$$

we have

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\mu_0} / \frac{d\mu}{d\mu_0} \; [\mu]. \tag{A-25}$$

The result now follows from Remark 3 and Remark 4.

Proof of Corollary 3.3: Let $h_\theta(y)$ denote the right hand side of 3.3. From Theorem 3.2 we have

$$E(g|Y=y) = \int_{\Omega_1} E_2(g_\theta|Y_\theta = y) h_\theta(y) \, dP_1(\theta). \tag{A-26}$$

With $g = I_{\hat{A}_1 \times \Omega_2}(\theta,\eta)$ where $\hat{A}_1 \subset A_1$

$$E(I_{\hat{A}_1 \times \Omega_2}|Y=y) = \int_{\hat{A}_1} h_\theta(y) \, dP_1(\theta). \tag{A-27}$$

This proves the last assertion in the statement of the corollary.

Integrating both sides w.r.t. PY^{-1}

$$\int_B E(I_{\hat{A}_1 \times \Omega_2}|Y=y) \, d\mu(y) = \int_B [\int_{\hat{A}_1} h_\theta(y) \, dP_1(\theta)] \, d\mu(y). \tag{A-28}$$

Applying the definition of the conditional expectation to the L.H.S. and Fubini's theorem to the right hand side ($h_\theta(y)$ is $A_1 \times B$ - measurable and non-negative), we get

$$\int_{Y^{-1}(B)} I_{\hat{A} \times \Omega_2} \, dP = \int_{\hat{A}_1} [\int_B h_\theta(y) \, d\mu(y)] \, dP_1(\theta). \tag{A-29}$$

Using Fubini's theorem on the left hand side gives

$$\int_{\hat{A}_1} P_2(Y_\theta^{-1}B)) \, dP_1(\theta) = \int_{\hat{A}_1} [\int_B h_\theta(y) \, d\mu(y)] \, dP_1(\theta). \tag{A-30}$$

So for P_1-almost everywhere on Ω_1 we have

$$\mu_\theta(B) = P_2(Y_\theta^{-1}(B)) = \int_B h_\theta(y) \, d\mu(y) \; [\mu]. \tag{A-31}$$

Since $h_\theta(y)$ is B - measurable for P_1-almost all θ, it must be that

$$\frac{d\mu_\theta}{d\mu}(y) = h_\theta(y). \tag{A-32}$$

Proof of Theorem 5.2: There is no loss of generality in assuming that R is the identity matrix. We shall first show that the following equality holds almost surely.

$$\int_{\Omega_1} [\int_o^t \Lambda_\theta (s,y) \ \hat{z}_\theta'(s,y) \ dy(s)] \ dP_1(\theta)$$

(A-33)

$$= \int_o^t [\int_{\Omega_1} \Lambda_\theta(s,y) \ \hat{z}_\theta'(s,y) \ dP_1(\theta)] \ dy(s).$$

By applying the differential rule on the representation of Λ_θ in Theorem 4.2, we have

$$\Lambda_\theta(t,y) - \Lambda_\theta (0,y) = \int_o^t \Lambda_\theta(s,y) \ \hat{z}_\theta'(s,y) \ dy(s). \quad \text{(A-34)}$$

Integrating with respect to P_1 over Ω_1 we get

$$\Lambda(t,y) - \Lambda(0,y) = \int_{\Omega_1} [\int_o^t \Lambda_\theta(s,y) \ \hat{z}_\theta'(s,y) \ dy(s)]$$

$dP_1(\theta).$

(A-35)

On the other hand, by an application of the differential rule on the representation of $\Lambda(t,y)$ in Corollary 4.4 we get

$$\Lambda(t,y) - \Lambda(0,y) = \int_o^t \Lambda(s,y) \ \hat{z}'(s,y) \ dy(s). \quad \text{(A.36)}$$

Also, by Theorem 4.2, we have

$$\hat{z}(s,y) = \frac{\int_{\Omega_1} \hat{z}_\theta'(s,y) \ \Lambda_\theta(s,y) \ dP_1(\theta)}{\Lambda(s,y)} . \quad \text{(A-37)}$$

So that the right hand sides of A-36 and A-35 must be equal almost surely which proves A-33.

Now by integrating Equation 5.1 in Theorem 5.1 over $\hat{A}_1$ with respect to P_1 we get

$$\int_{\hat{A}_1} \lambda_\theta(t,y) \ dP_1(\theta) = \int_{\hat{A}_1} \lambda_\theta (0,y) \ dP_1(\theta) \quad \text{(A-38)}$$

$$+ \int_{\hat{A}_1} \{\int_o^t \lambda_\theta(s,y)[\hat{z}_\theta(s,y) - \hat{z}(s,y)]'[dy(s) - \hat{z}(s,y)$$

$ds]\} \ dP_1(\theta).$

With $\dfrac{I_{\hat{A}_1 \times \Omega_2} \ z(s,\theta,\eta)}{\Lambda(s,y)}$ in place of $z(s,\theta,\eta)$ in A.33, we get the following equality.

$$\int_{\hat{A}_1} [\int_o^t \lambda_\theta(s,y) \ \hat{z}_\theta'(s,y) \ dy(s)] \ dP_1 \ \theta \quad \text{(A-39)}$$

$$= \int_o^t [\int_{\hat{A}_1} \lambda_\theta(s,y) \ \hat{z}_\theta'(s,y) dP_1(\theta)] \ dy(s).$$

So that

$$\pi_t(\hat{A}_1,y) = \int_{\hat{A}_1} \lambda_\theta(t,y) \ dP_1(\theta) = \pi_o(\hat{A}_1,y) +$$

$$\int_o^t [\int_{\hat{A}_1} \lambda_\theta(s,y)[z_\theta(s,y) - \hat{z}(s,y)] \ dP_1(\theta)]$$

$[dy(s) - \hat{z}(s,y)ds].$

Since, by Remark 3.2, $d\pi_t(\theta, y) = \lambda_\theta(t, y) \ dP_1(\theta)$,

$$\pi_t(\hat{A}_1,y) = \pi_o(\hat{A}_1,y) +$$

$$\int_o^t [\int_{\hat{A}_1} [z_\theta(s,y) - \hat{z}(s,y)]' \ d\pi_t(\theta, y)]$$

$[dy(s) - \hat{z}(s,y) \ ds]$

which proves Theorem 5.2.

Proof of Theorem 6.4: The existence and uniqueness of this type of infinite order stochastic differential equation has been studied by Rozovskii and Shiryaev [28]. To prove Theorem 6.4, we shall apply Theorem 1 in [28]. To this end, we have to show that $\pi_t(\theta_1, y)$ is in class M. It is clear that, for each i, $\pi_t(\theta_1,y)$ is continuous with probability one. This follows from the continuity property of the Ito integral which states that the points of continuity of the random process defined by the Ito integral are the points of continuity of the Wiener process with respect to which the integral is defined [4]. Measurability conditions are satisfied due to the definition of $\pi_t(\theta_1, y)$.

$$\sum_{i=1}^{\infty} \pi_t(\theta_1,y) = 1$$

Also

$$P \{ \int_o^T [\sum_{i=1}^{\infty} \hat{z}_{\theta_1}(s,y) \ \pi_{\theta_1}(s,y)]^2 \ ds < \infty \}$$

$$= P \{ \int_o^T \hat{z}^2(s,y) \ ds < \infty \} = 1 .$$

Therefore, $\pi_t(\theta_1, y)$ is in class M. By Theorem 1 in [28], the system of equations 6.9 has a unique solution in the sense if π and $\tilde{\pi}$ are two solutions with $\pi_o(\theta_1,y) = \tilde{\pi}_o(\theta_1,y)$, i = 1, 2,..., then

$$P \{ \sup_i \ \sup_t \ | \ \pi_t(\theta_1,y) - \tilde{\pi}_t(\theta_1,y) \ | > 0 \} = 0.$$

References

1. Halmos, P. R., Measure Theory, D. Van Nostrand Company, Inc., Princeton, NJ, 1950.

2. Neveu, J., Mathematical Foundations of the Calculus of Probability, Holden-Day, Inc., San Francisco, CA, 1965.

3. Ash, R. B., Real Analysis and Probability, Academic Press, New York, 1972.

4. Doob, J. L., Stochastic Processes, John Wiley and Sons, Inc., New York, 1953.

5. Kallianpur, G., and Striebel, C., "Estimation of Stochastic Systems," Ann. Math. Statist., Vol. 39, No. 3, pp. 785-801, 1968.

6. Kailath, T., "The Structure of Radon-Nikodym Derivatives with Respect to Wiener and Related Measures," Ann. Math. Statist., Vol. 42, No. 3, pp. 1054-1067, 1971.

7. Duncan, T. E.,"Likelihood Functions for Stochastic Signals in White Noise," Ann. Math. Statist, Vol. 16, pp. 303-310, 1970.

8. Van Trees, H. L., Detection Estimation and Modulation Theory, Part I, John Wiley and Sons, Inc., New York, 1968.

9. Middleton, D. and Esposito, R., "Simultaneous

Optimum Detection and Estimation of Signals in Noise," _IEEE Trans. Inf. Theory_, Vol. IT-14, pp. 434-444, 1968.

10. Lainiotis, D. G., "Optimal Adaptive Estimation Structure and Parameter Adaptation," _IEEE Trans. on Automat. Control_, Vol. AC-16, pp. 160-170, April 1972.

11. Loeve, M., _Probability Theory_, D. Van Nostrand Company, Inc., Princeton, NJ, 1963.

12. Magill, D. T., "Optimal Adaptive Estimation of Sampled Stochastic Processes," _IEEE Trans. Automat. Control_, Vol. AC-10, pp. 434-439, October 1965.

13. Caglayan, A. K., "Optimal Parameter Adaptive Estimation of Stochastic Processes," _Ph.D. Dissertation_, Dept. of Elect. Engrg., VPI and SU, Blacksburg, VA, October 1974.

14. Nahi, N. E., "Optimal Recursive Estimation with Uncertain Observation," _IEEE Trans. Inf. Theory_, Vol. IT-15, pp. 457-462, 1969.

15. Stein, G., and Saridis, G. N., "A Parameter-Adaptive Control Technique," _Automatica_, Vol. 5, pp. 731-739, 1969.

16. Saridis, G. N., and Dao, T. K., "A Learning Approach to the Parameter Adaptive Self-Organizing Control Problem," _Automatica_, Vol. 8, pp. 589-597, 1972.

17. Deshpande, J. G., Upadhyay, T. N., and Lainiotis, D. G., "Adaptive Control of Linear Stochastic Systems," _Automatica_, Vol. 9, pp. 107-115, 1973.

18. Park, S. K., and Lainiotis, D. G., "Joint Detection-Estimation of Gaussian Signals in White Gaussian Noise," _Inf. Sciences_, Vol. 4, pp. 315-325, 1972.

19. Lo, J. T. H., "On Optimal Nonlinear Estimation," Part I, _Information Sciences_, Vol. 6, pp. 19-32, 1973.

20. Lainiotis, D. G., "Optimal Nonlinear Estimation," _International J. of Control_, Vol. 14, No. 6, pp. 1137-1148, 1971.

21. Haddad, A. H., and Cruz, J. B., "Nonlinear Filtering for Systems With Random Models," _Proc. of Second Nonlinear Estimation Conference_, Sept. 1971, San Diego, CA.

22. Wonham, W. M., "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering," _J. SIAM Control_, Vol. 12, pp. 347-369, 1965.

23. Shiryaev, A. N., "Stochastic Equations of Nonlinear Filtering of Markovian Jump Processes," _Problemy Peredachi Informatsii_, Vol. 2, No. 3, pp. 3-22, 1966.

24. Kallianpur, G., and Striebel, C., "Stochastic Differential Equations Occurring in the Estimation of Continuous Parameter Stochastic Processes," _Theory of Probability and Its Applications_, Vol. XIV, No. 4, pp. 567-594, 1969.

25. Gihman, I. I., and Skorohod, A. V., _Stochastic Differential Equations_, Springer-Verlag, New York, 1972.

26. Kailath, T., "A General Likelihood Ratio Formula for Random Signals in Gaussian Noise," _IEEE Trans. Inform. Theory_, Vol. IT-15, pp. 350-361, May 1969.

27. Kailath, T., "A Further Note on a General Likelihood Formula for Random Signals in Gaussian Noise," _IEEE Trans. Inf. Theory_, Vol. IT-16, July 1970.

28. Rozovskii, B. L., and Shiryaev, A. N., "On Infinite Order Systems of Stochastic Differential Equations Arising in the Theory of Optimal Nonlinear Filtering," _Theory of Probability and Its Applications_, Vol. 17, No. 2, pp. 218-226, 1972.

ORIGINAL PAGE IS
OF POOR QUALITY

1. Report No. NASA CR-2589	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Stability of Multirate Sampled-Data Systems		5. Report Date OCTOBER 1975
		6. Performing Organization Code 14.140
7. Author(s) David F. Stanford and Alper K. Caglayan		8. Performing Organization Report No.
9. Performing Organization Name and Address College of William and Mary Williamsburg, VA 23185		10. Work Unit No. 766-75-02-01
		11. Contract or Grant No. NAS1-11958-11
12. Sponsoring Agency Name and Address National Aeronautics & Space Administration Washington, DC 20546		13. Type of Report and Period Covered Contractor Report 6/10/74 - 8/15/74
		14. Sponsoring Agency Code
15. Supplementary Notes TOPICAL REPORT		

16. Abstract

Current adaptive sampling schemes that can be used in sampled-data systems with variable rate sampling does not guarantee the stability of the resulting closed-loop system. Therefore, this study has been undertaken with the objective of finding a stable control law for multirate sampled-data systems. The problem is formulated such that the sampling interval is selected from a set of fixed number of sampling intervals. A necessary and sufficient condition under which these types of systems can be stabilized is given. For a certain subclass of these types of systems, a sampling selection algorithm is given which results in a stable closed-loop system.

17. Key Words (Suggested by Author(s)) Adaptive sampling, stability; Discrete control systems; Variable or multirate sampling		18. Distribution Statement Unclassified - Unlimited Subject Category 33
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 31 22. Price* $3.75

OPTIMAL PARAMETER ADAPTIVE ESTIMATION OF STOCHASTIC PROCESSES

by

Alper K. Caglayan

(ABSTRACT)

This study is concerned with the simultaneous detection and least squares estimation of vector random processes. The problem is formulated in the following context: A random process, out of a countably infinite collection of (not necessarily Gaussian) vector random processes with known distributions, is observed with additive white Gaussian noise. The a priori probability, that a specific random process will be observed, is specified for each one in the collection. The least squares estimate of the random process that is being observed is to be found in terms of the hypothesis conditioned estimates.

It is shown that the best estimate is the linear combination of the hypothesis conditioned estimates weighted by the a posteriori probabilities of the hypotheses conditioned on the observations. A Radon-Nikodym derivative representation is derived for the a posteriori probability by using the specific structure of the product probability measure for this problem. It is shown that this Radon-Nikodym derivative can be expressed in terms of the Radon-Nikodym derivatives of measures induced by the random processes in the collection with respect to Wiener measure. By using the recent results on likelihood functions, an expression for the a posteriori probability is found in terms of the conditioned estimates. In this connection, an extended version of the partition theorem of parameter adaptive estimation

is proved. The unique stochastic differential equation, that each
a posteriori probability satisfies with its associated a priori
probability as the initial condition, is derived for the case of
finitely many hypotheses along with an expression for the conditional
error covariance in terms of the hypothesis conditioned error
covariances.

The results are applied to the parameter adaptive estimation
problem in linear continuous and discrete stochastic dynamic systems.
In the continuous case, the solution is also obtained through an
alternate approach using nonlinear filtering theory. An application of
the theory to the design of a digital flight control system which is
tolerant of sensor failures is presented with real-time hybrid computer
simulation results. A review of random processes and statistical
decision theory is also included.