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ABSTRACT

The problem of a cylindrical shell containing a circumferential through crack is considered by taking into account the effect of transverse shear deformations. The formulation is given for a specially orthotropic material within the confines of a linearized shallow shell theory. The particular theory used permits the consideration of all five boundary conditions regarding moment and stress resultants on the crack surface. Consequently, aside from multiplicative constants representing the stress intensity factors, the membrane and bending components of the asymptotic stress fields near the crack tip are found to be identical. Since these two fields are coupled in the shell problems, the difficulty in justifying the superposition of membrane and bending stresses and, particularly of the related stress intensity factors when applying the results to fracture problems has thus been removed. The stress intensity factors are calculated separately for a cylinder under a uniform membrane load, and that under a uniform bending moment. The stress intensity factors are calculated for values of $0 \leq \lambda_2 = [12(1-\nu^2)]^{\frac{1}{4}} a/\sqrt{Rh} \leq 10$ and $0<a/h\leq 10$, where 2a, h, and R are, respectively, the crack length, the thickness, and the (mean) radius of the cylinder. Sample results showing the nature of the out-of-plane crack surface displacement and the effect of the Poisson's ratio are also presented.

1. INTRODUCTION

Depending on the mechanical behavior of the material and the nature of the loading and environmental conditions, in designing the line pipes and cylindrical containers it is often necessary to consider fatigue crack propagation and fracture among the possible modes of failure. This requires, in addition to the application of standard techniques in the failure analysis of the component, the treatment of the problem from the viewpoint of fracture mechanics. In using this approach the initial flaws or certain types of imperfections which may exist in the material are treated as "cracks". These cracks may propagate under the fluctuating external loads as fatigue cracks, or under the sustained loads and sufficiently adverse environmental conditions as stress corrosion cracks, or a combination of both, namely as fatigue corrosion cracks. Aside from containing flaws and local damages in the form of notches, other stress raisers, or surface scratches which may be due to design, manufacture, or accidental causes, since pipes and containers are generally welded components they would almost always contain some imperfections in the form of lack of total fusion, elongated slag inclusions, voids, arc burns, and a variety of other surface irregularities which may act as fracture nuclei. Thus, it appears that in actual structural components, the existence of defects which may lead to fracture initiation and propagation is unavoidable and therefore, one must design around them, primarily through the appropriate selection of material and dimensions of the component and by setting up rational defect inspection and acceptance standards.

Fracture mechanics approach has been highly successful in dealing with this so-called fracture control problem which involves subcritical propagation of fatigue and stress corrosion cracks and unstable fracture. The most widely used correlation parameter in this type of problems is the stress intensity factor which has been used quite effectively to predict the crack propagation rate in the component and for a given crack size, its residual strength. Using, then, this information, the "working life" of the component may easily be estimated.

Up to now, fracture mechanics research as applied to pipeline and container safety, has dealt almost exclusively with the longitudinal flaws in the component. There is, of course, a good reason for this, namely that in pressurized cylinders under normal operating conditions, the hoop stress is by far the most dominant stress component. On the other hand, any secondary loads caused by known and unforseen factors would be primarily in the axial direction. These are mostly bending and axial constraint stresses. Some of the sources of these stresses are support misalignment, variety of thermal fluctuations, ground settlement and earthquake, hydroelastic or aeroelastic loading, axial constraint in buried pipes, gross bending in offshore piping caused by variety of oceanic disturbances and that generated during installation, stresses arising from vehicular vibrations and impact, and the residual stresses in and around the girth welds. One may also point out that the axial stresses would be much more time-varying than basically pressure-induced hoop stresses and hence would tend to facilitate the formation and propagation of fatigue cracks in the circumferential direction. If the thickness-to-radius ratio in the cylinder is sufficiently small, then a bending theory of shells may be used to formulate the circumferential crack problem and, by approximating the internal stresses (such as residual and thermal stresses) by a linear function in the thickness coordinate, the general solution may be obtained by appropriately superimposing the results of membrane and bending solutions. The existing solutions of this problem take into account the membrane loading only and are based on an 8th order shallow shell theory in which the effect of the transverse shear stress is ignored and the crack surface boundary conditions are approximated by using the Kirchhoff assumption(*) [1-3]. The consequence of this assumption in shells is the same as in the bending of cracked flat plates [4-7]. It has been shown that if a plate bending theory compatible with the number of independent boundary conditions is not used to solve the crack problem, the computed stress intensity factors may contain errors and the angular distribution of membrane and bending stress resultants would be different [4-7]. This would then make it difficult to justify the superposition of membrane and bending solutions in

(*)That is, on the traction-free crack surface instead of satisfying the conditions $M_{ns} = 0$, $V = 0$ separately, the condition $V + \partial M_{ns}/\partial s = 0$ is satisfied, thereby making the number of boundary conditions (four) and the order of the reduced differential equation (eighth) compatible. Here n and s are the coordinates normal and parallel to the crack, M_{ns} is the twisting moment, and V is the transverse shear.

fatigue crack propagation and fracture analysis.

The aim of this paper is to solve the problem of circumferentially cracked cylindrical shell by taking into account the effect of transverse shear and to obtain the stress intensity factors for bending moment as well as the membrane force as the external load. The problem for longitudinal crack was recently solved [8] by using a higher order shallow shell thoery described in [9]. In the present paper the effect of Poisson's ratio on the stress intensity factors and the nature of the out-of-plane displacement along the edges of the crack (i.e., bulging) will also be studied.

2. FORMULATION OF THE PROBLEM

In this problem the general shallow shell equations developed in [9] for an isotropic medium will be used. However, as shown in [10] and [11] for the classical shell theory, in this case too it can be shown that the related differential equations of the orthotropic shells car be factorized and reduced to the equations of isotropic shells if one assumes the material to be "specially orthotropic".

Referring to Figure 1 and following [9] the equilibrium equations for a shallow shell may be expressed as

$$N_{ij,j} = 0 \quad , \tag{2.1}$$

$$V_{i,i} + (Z_{,i}N_{ij})_{,j} + q(X_1,X_2) = 0 \quad , \tag{2.2}$$

$$M_{ij,j} - V_i = 0 \quad , \quad (i=1,2, \quad j=1,2) \tag{2.3}$$

where N_{ij}, M_{ij}, and V_i, $(i,j=1,2)$ are, respectively, membrane, moment and transverse shear resultants, and the indicial notation and the summation convention are used. The components of strains are given by

$$\varepsilon_{ij} = \frac{1}{2}[U_{i,j} + U_{j,i} + Z_{,i}W_{,j} + Z_{,j}W_{,i}] \quad , \quad (i,j=1,2) \tag{2.4}$$

where U_1, U_2 and W are, respectively, X_1, X_2, and Z-components of the displacement vector, and the function $Z(X_1,X_2)$ giving the equation of the middle surface is known. Let β_1 and β_2 be the angles of rotation of the normal to the shell surface. The (transverse) shear strains may then be expressed as

$$\theta_i = W_{,i} + \beta_i \quad , \quad i = 1,2 \quad . \tag{2.5}$$

With the Hooke's law

$$\varepsilon_{ij} = a_{ijk\ell}N_{k\ell}/h \tag{2.6}$$

and the relations giving M_{ij} and V_i in terms of β_i and θ_i, respectively, the formulation of the problem would be complete, technically the ten equations (2.1) - (2.5) accounting for the ten variables ε_{ij}, β_i, θ_i, U_i, and W.

First, eliminating U_1 and U_2 from (2.4) one obtains the following compatibility equation:

$$e_{ik}e_{j\ell}(\varepsilon_{ij,k\ell} + Z_{,ij}W_{,k\ell}) = 0 \quad . \tag{2.7}$$

where e_{ik} is the permutation symbol(*). Next, defining the stress function $F(X_1,X_2)$ by

$$N_{ij} = e_{ik}e_{j\ell}F_{,k\ell} \tag{2.8}$$

it is seen that (2.1) is satisfied and (2.2) and (2.7) reduce to

$$M_{ij,ij} + Z_{,ij}e_{ik}e_{j\ell}F_{,k\ell} + q = 0 \tag{2.9}$$

$$e_{im}e_{jn}e_{kp}e_{\ell q}a_{ijk\ell}F_{,mnpq} + hZ_{,ij}e_{ik}e_{j\ell}W_{,k\ell} = 0 \tag{2.10}$$

Even for simple shell geometries for anisotropic materials the

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(*) $e_{11} = 0 = e_{22}$, $e_{12} = 1 = -e_{21}$.

differential equations are not tractable. However, as in [10] if one assumes a special orthotropy, the related differential operators in these equations can be factorized and the problem can be made analytically tractable. Let the material be orthotropic and the stress strain relations be

$$\varepsilon_{11} = \frac{1}{hE_1}(N_{11} - \nu_1 N_{22}) \quad , \quad \varepsilon_{12} = \frac{N_{12}}{2hG_{12}} \quad ,$$
$$\varepsilon_{22} = \frac{1}{hE_2}(N_{22} - \nu_2 N_{11}) \quad , \quad \frac{\nu_1}{E_1} = \frac{\nu_2}{E_2} \tag{2.11}$$

In this case too the following is the condition for the factorization of the operators:

$$2G_{12} = \frac{\sqrt{E_1 E_2}}{1 + \sqrt{\nu_1 \nu_2}} \tag{2.12}$$

The material satisfying the condition (2.12) is said to be "specially orthotropic". It has been shown that for certain orthotropic materials the value of the shear modulus calculated from (2.12) and that measured experimentally are very nearly the same and consequently with the assumption of special orthotropy, the analysis can be simplified quite considerably [11]. If we now define

$$E = \sqrt{E_1 E_2} \quad , \quad \nu = \sqrt{\nu_1 \nu_2} \quad , \quad c = (E_1/E_2)^{1/4} \tag{2.13}$$

the stress-strain relations (2.11) become

$$\varepsilon_{11} = \frac{1}{hE}\left(\frac{N_{11}}{c^2} - \nu N_{22}\right) \quad , \quad \varepsilon_{12} = \frac{1+\nu}{hE} N_{12} \quad ,$$
$$\varepsilon_{22} = \frac{1}{hE}(c^2 N_{22} - \nu N_{11}) \quad . \tag{2.14}$$

From the linear thickness variation of the stress components σ_{ij} one obtains

$$M_{11} = D(c^2 \beta_{1,1} + \nu \beta_{2,2}) \quad , \quad M_{12} = \frac{D(1-\nu)}{2}(\beta_{1,2} + \beta_{2,1}) \quad ,$$
$$M_{22} = D(\nu \beta_{1,1} + \beta_{2,2}/c^2) \quad , \quad D = \frac{Eh^3}{12(1-\nu^2)} \quad . \tag{2.15}$$

Also, assuming a linear transverse shear stress-strain relationship, one finds

$$\theta_1 = \frac{1}{chB} V_1 \quad , \quad \theta_2 = \frac{c}{hB} V_2 \quad , \tag{2.16}$$

where B is the effective shear modulus. Referring to [12], it will be assumed that

$$B = \frac{5}{6} \frac{E}{2(1+\nu)} \tag{2.17}$$

Defining now the operator

$$\nabla_c^2 = c \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{c \partial X_2^2} \quad , \tag{2.18}$$

equations (2.9), (2.10), and (2.3) may be reduced to

$$\nabla_c^2 \nabla_c^2 F + hE\left(\frac{\partial^2 Z}{\partial X_1^2} \frac{\partial^2}{\partial X_2^2} - 2 \frac{\partial^2 Z}{\partial X_1 \partial X_2} \frac{\partial^2}{\partial X_1 \partial X_2} + \frac{\partial^2 Z}{\partial X_2^2} \frac{\partial^2}{\partial X_1^2}\right) W = 0 \quad , \tag{2.19}$$

$$D\nabla_c^2 \nabla_c^2 W - \left(1 - \frac{D}{Bh} \nabla_c^2\right)\left(\frac{\partial^2 Z}{\partial X_1^2} \frac{\partial^2}{\partial X_2^2} - 2 \frac{\partial^2 Z}{\partial X_1 \partial X_2} \frac{\partial^2}{\partial X_1 \partial X_2}\right.$$

$$\left. + \frac{\partial^2 Z}{\partial X_2^2} \frac{\partial^2}{\partial X_1^2}\right) F = \left(1 - \frac{D}{Bh} \nabla_c^2\right) q \quad , \tag{2.20}$$

$$\beta_1 + \frac{\partial W}{\partial X_1} = \frac{D}{hB} \left[\nabla_c^2 \beta_1 + \frac{1+\nu}{2c} \frac{\partial}{\partial X_2} \left(\frac{\partial \beta_2}{\partial X_1} - \frac{\partial \beta_1}{\partial X_2}\right)\right] \quad , \tag{2.21}$$

$$\beta_2 + \frac{\partial W}{\partial X_2} = \frac{D}{hB} \left[\nabla_c^2 \beta_2 + c \frac{1+\nu}{2} \frac{\partial}{\partial X_1} \left(\frac{\partial \beta_1}{\partial X_2} - \frac{\partial \beta_2}{\partial X_1}\right)\right] \quad . \tag{2.22}$$

Equations (2.19) - (2.22) provide the formulation for an arbitrary shallow shell in terms of the unknown functions F, W, β_1, and β_2.

Now let us assume that in the domain of interest the curvatures of the shell are constant. Then in (2.19) and (2.20) the terms involving Z may be replaced by

$$\frac{\partial^2 Z}{\partial X_1^2} = -\frac{1}{R_1} \quad , \quad \frac{\partial^2 Z}{\partial X_2^2} = -\frac{1}{R_2} \quad , \quad \frac{\partial^2 Z}{\partial X_1 \partial X_2} = -\frac{1}{R_{12}} \quad . \tag{2.23}$$

Also, following [8], if one introduces the dimensionless quantities given in Appendix A, equations (2.19) - (2.22) may further be simplified as

$$\nabla^4\phi - \frac{1}{\lambda^2}(\lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2})w = 0 \qquad (2.24)$$

$$\nabla^4 w + \lambda^2(1-\kappa\nabla^2)(\lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2})\phi$$

$$= \lambda^4(1-\kappa\nabla^2)\frac{a}{h} q \ , \qquad (2.25)$$

$$(1-\kappa\nabla^2)\beta_x + \frac{\partial w}{\partial x} = \kappa \frac{1+\nu}{2} \frac{\partial}{\partial y}(\frac{\partial \beta_y}{\partial x} - \frac{\partial \beta_x}{\partial y}) \ , \qquad (2.26)$$

$$(1-\kappa\nabla^2)\beta_y + \frac{\partial w}{\partial y} = \kappa \frac{1+\nu}{2} \frac{\partial}{\partial x}(\frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x}) \ . \qquad (2.27)$$

The constant a used in Appendix A to normalize various quantities is a characteristic length parameter in the shell. Usually in crack problems the shell is assumed to be "infinitely large" and a is taken to be the half crack length.

Defining now a new function Ω by

$$\Omega(x,y) = \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \qquad (2.28)$$

equations (2.26) and (2.27) may be expressed as

$$\left.\begin{aligned} \beta_x &= \kappa\nabla^2\beta_x - \frac{\partial w}{\partial x} - \kappa\frac{1+\nu}{2}\frac{\partial \Omega}{\partial y} \ , \\ \beta_y &= \kappa\nabla^2\beta_y - \frac{\partial w}{\partial y} + \kappa\frac{1+\nu}{2}\frac{\partial \Omega}{\partial x} \ . \end{aligned}\right\} \qquad (2.29)$$

and, it may easily be shown that

$$\left.\begin{aligned} \nabla^2\beta_x &= \frac{\partial \Omega}{\partial y} + \frac{\partial^2 \beta_x}{\partial x^2} + \frac{\partial^2 \beta_y}{\partial x \partial y} \ , \\ \nabla^2\beta_y &= -\frac{\partial \Omega}{\partial x} + \frac{\partial^2 \beta_x}{\partial x \partial y} + \frac{\partial^2 \beta_y}{\partial y^2} \ . \end{aligned}\right\} \qquad (2.30)$$

Also, if we define the function ψ by

$$\psi(x,y) = \kappa\left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}\right) - w \quad , \tag{2.31}$$

from (2.29) and (2.30) we obtain

$$\begin{aligned} \beta_x &= \frac{\partial \psi}{\partial x} + \kappa \, \frac{1-\nu}{2} \, \frac{\partial \Omega}{\partial y} \quad , \\ \beta_y &= \frac{\partial \psi}{\partial y} - \kappa \, \frac{1-\nu}{2} \, \frac{\partial \Omega}{\partial x} \quad . \end{aligned} \tag{2.32}$$

In (2.32) eliminating Ω and then using (2.31) we find

$$\kappa \nabla^2 \psi - \psi - w = 0 \quad . \tag{2.33}$$

Similarly, eliminating w, (2.29) yields

$$\kappa \, \frac{1-\nu}{2} \, \nabla^2 \Omega - \Omega = 0 \quad . \tag{2.34}$$

The solution of the shell problem must then satisfy the differential equations (2.24), (2.25), (2.33) and (2.34) and all the necessary boundary conditions.

3. CYLINDRICAL SHELL WITH A CIRCUMFERENTIAL CRACK

Consider now a cylindrical shell containing a circumferential through crack of length 2a shown in Figure 2. In this case $\lambda_1 = 0 = \lambda_{12}$, and if we further assume that $q = 0$, (2.24) and (2.25) become

$$\nabla^4 \phi - (\lambda_2/\lambda)^2 \, \frac{\partial^2 w}{\partial x^2} = 0 \quad , \tag{3.1}$$

$$\nabla^4 w + (\lambda\lambda_2)^2 (1-\kappa\nabla^2) \, \frac{\partial^2 \phi}{\partial x^2} = 0 \quad . \tag{3.2}$$

Eliminating ϕ, from (3.1) and (3.2) it follows that

$$\nabla^4 \nabla^4 w + \lambda_2^4 (1-\kappa\nabla^2) \, \frac{\partial^4 w}{\partial x^4} = 0 \quad . \tag{3.3}$$

The problem will be solved by using Fourier transforms. It will be assumed that through a proper superposition the original shell problem has been reduced to a perturbation problem in which self-equilibrating force and moment resultants acting on the crack surfaces are the only nonzero external loads (hence, the assumption $q = 0$). Thus, in some neighborhood of the crack in which the stresses are expected to be nonzero, the transform of w and its inversion may be expressed as

$$f(x,\alpha) = \int_{-\infty}^{\infty} w(x,y)e^{i\alpha y}dy \quad , \tag{3.4}$$

$$w(x,y) = \frac{1}{2\pi}\int_{\infty}^{\infty} f(x,\alpha)e^{-i\alpha y}d\alpha \quad . \tag{3.5}$$

Substituting from (3.5) into (3.3) one obtains an 8th order ordinary linear differential equation in f. Looking for a solution in the form $f = R(\alpha)\exp(mx)$, the characteristic equation of the problem is found to be

$$m^8 - (\kappa\lambda_2^4 + 4\alpha^2)m^6 + (6\alpha^4 + \lambda_2^4 \quad \kappa\lambda_2^4\alpha^2)m^4$$

$$- 4\alpha^6 m^2 + \alpha^8 = 0 \quad . \tag{3.6}$$

If we define

$$p = m^2 - \alpha^2 \tag{3.7}$$

equation (3.6) can be written in terms of p as follows:

$$p^4 - \kappa\lambda_2^4 p^3 + \lambda_2^4(1-2\kappa\alpha^2)p^2 + \lambda_2^4(2\alpha^2 - \kappa\alpha^4)p + \lambda_2^4\alpha^4 = 0 \tag{3.8}$$

It should be emphasized that the roots of (3.8) are in general complex and, of course, are not known as a function of α in closed form. Appendix B describes a convenient procedure for solving (3.8). After solving (3.8) let the roots of (3.6) be ordered such that

$$Re(m_j) < 0 \quad , \quad m_{j+4} = -m_j, \; j = 1,..,4 \quad . \tag{3.9}$$

The solution $f(x,\alpha)$ of the resulting, ordinary differential equation satisfying the regularity conditions at $x = \mp\infty$ may then be expressed as

$$f(x,\alpha) = \begin{cases} \sum_{j=1}^{4} R_j(\alpha)\exp(m_j x) \quad , \quad x>0 \quad , \\ \sum_{j=5}^{8} R_j(\alpha)\exp(m_j x) \quad , \quad x<0 \quad . \end{cases} \tag{3.10}$$

If we let

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x,\alpha)e^{-iy\alpha}d\alpha \quad , \tag{3.11}$$

from (3.1), (3.2) and (3.10) we find

$$g(x,\alpha) = \begin{cases} (\lambda_2/\lambda)^2 \sum_{1}^{4} (m_j/p_j)^2 R_j(\alpha)\exp(m_j x) \quad , \quad x>0 \quad , \\ (\lambda_2/\lambda)^2 \sum_{5}^{8} (m_j/p_j)^2 R_j(\alpha)\exp(m_j x) \quad , \quad x<0 \quad . \end{cases} \tag{3.12}$$

Similarly, assuming

$$\Omega(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x,\alpha)e^{-iy\alpha}d\alpha \tag{3.13}$$

from (2.34) we obtain

$$h(x,\alpha) = \begin{cases} A_1(\alpha)\exp(r_1 x) \quad , \quad x>0 \quad , \\ A_2(\alpha)\exp(r_2 x) \quad , \quad x<0 \quad , \end{cases} \tag{3.14}$$

where

$$r_1 = -[\alpha^2 + \frac{2}{\kappa(1-\nu)}]^{\frac{1}{2}} \quad , \quad r_2 = [\alpha^2 + \frac{2}{\kappa(1-\nu)}]^{\frac{1}{2}} \quad . \tag{3.15}$$

Also, let us assume that

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta(x,\alpha)e^{-iy\alpha}d\alpha \quad . \tag{3.16}$$

It can be shown that the remaining differential equation (2.33) is satisfied if θ is assumed to be

$$\theta(x,\alpha) = \begin{cases} \sum_{1}^{4} \frac{R_j(\alpha)}{\kappa p_j - 1} \exp(m_j x) \quad , \quad x>0 \\ \sum_{5}^{8} \frac{R_j(\alpha)}{\kappa p_j - 1} \exp(m_j x) \quad , \quad x<0 \quad . \end{cases} \tag{3.17}$$

The expressions given by (3.10), (3.12), (3.14), and (3.17) satisfy the differential equations of the problem(*). If one now determines the arbitrary functions $R_j(\alpha)$, $(j = 1,..,8)$ and $A_i(\alpha)$, $(i = 1,2)$ in such a way that the boundary conditions of the problem are also satisfied, one then has the solution.

4. BOUNDARY CONDITIONS

As mentioned earlier, the only external loads in the problem are the self-equilibrating force and moment resultants on the crack surfaces. These forces can be decomposed in such a way that in solving the problem one needs to consider (with respect to $x = 0$ plane) either symmetric or antisymmetric loading only. In these two cases the following conditions of symmetry will be satisfied:

$$\begin{aligned} &N_{xx}(x,y) = N_{xx}(-x,y) \quad , \quad N_{xy}(x,y) = -N_{xy}(-x,y) \quad , \\ &M_{xx}(x,y) = M_{xx}(-x,y) \quad , \quad M_{xy}(x,y) = -M_{xy}(-x,y) \quad , \\ &V_x(x,y) = -V_x(-x,y) \quad , \end{aligned} \tag{4.1}$$

(*)It should perhaps be pointed out that (3.17) is a solution satisfying the differential equation (2.33), and is not the most general solution. However, with (3.17), since the solution thus found satisfies all the differential equations and, as will be shown, all the boundary conditions, it must be the solution of the physical problem.

for the symmetric problem, and

$$N_{xx}(x,y) = -N_{xx}(-x,y) \quad , \quad N_{xy}(x,y) = N_{xy}(-x,y) \quad ,$$

$$M_{xx}(x,y) = -M_{xx}(-x,y) \quad , \quad M_{xy}(x,y) = M_{xy}(-x,y) \quad , \tag{4.2}$$

$$V_x(x,y) = V_x(-x,y) \quad ,$$

for the antisymmetric problem. One may note that (4.1) and (4.2) are valid for all values of x and y, and in the odd functions the discontinuity at $x = 0$ may be allowed only on the crack surface, outside the crack all these functions (indeed, all field quantities) must be continuous. It is therefore clear that in solving the problem one needs to consider only one half of the cylinder, say $x > 0$. Symmetry conditions similar to (4.1) and (4.2) are, of course valid for all the remaining field quantities. Thus, there are only five unknown functions $R_1,..,R_4$, and A_1 which may be determined from five conditions specified at $x = +0$.

Consider now the symmetric problem for a circumferentially cracked shell. Noting that before the superposition which led to the perturbation problem the crack surfaces were free from all external loads, and since outside the crack all quantities are continuous, from (4.1) it may be concluded that

$$N_{xy}(0,y) = 0 \quad , \quad M_{xy}(0,y) = 0 \quad , \quad V_x(0,y) = 0 \quad , \quad -\infty<y<\infty \quad , \tag{4.3}$$

and $N_{xx}(+0,y)$ and $M_{xx}(+0,y)$ are known functions in $-1<y<1$. The problem is then a mixed boundary value problem. Using (4.3) three of the five unknowns $(R_1,..,R_4,A_1)$ can be eliminated. The remaining two may be determined either from a system of dual integral equations or a system of singular integral equations arising from the mixed boundary conditions.

In terms of the normalized quantities the relevant force and moment resultants are given by

$$N_{xx} = \frac{\partial^2\phi}{\partial y^2} \quad , \quad N_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \quad ,$$

$$M_{xx} = \frac{a}{h\lambda^4} \left(\frac{\partial \beta_x}{\partial x} + \nu \frac{\partial \beta_y}{\partial y}\right) ,$$

$$M_{xy} = \frac{a}{h\lambda^4} \frac{1-\nu}{2} \left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) ,$$

$$\frac{\partial V_x}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial \beta_x}{\partial y} . \tag{4.4}$$

Using (4.4) and the results obtained in Sections 2 and 3, for x>0 these quantities may be expressed in terms of $R_1,\ldots,R_4$ and A_1 as follows:

$$N_{xx}(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} [(\frac{\lambda_2}{\lambda})^2 \alpha^2 \sum_1^4 (\frac{m_j}{p_j})^2 R_j(\alpha) e^{m_j x}] e^{-iy\alpha} d\alpha , \tag{4.5}$$

$$N_{xy}(x,y) = \frac{i}{2\pi} \int_{-\infty}^{\infty} [(\frac{\lambda_2}{\lambda})^2 \alpha \sum_1^4 \frac{m_j^3}{p_j^2} R_j(\alpha) e^{m_j x}] e^{-iy\alpha} d\alpha , \tag{4.6}$$

$$M_{xx}(x,y) = \frac{1}{2\pi} \frac{a}{h\lambda^4} \left[\int_{-\infty}^{\infty} \sum_1^4 \frac{m_j - \nu\alpha^2}{\kappa p_j - 1} R_j(\alpha) e^{m_j x} e^{-iy\alpha} d\alpha \right.$$
$$\left. - \frac{\kappa(1-\nu)^2}{2} \int_{-\infty}^{\infty} i\alpha r_1 A_1(\alpha) e^{r_1 x} e^{-iy\alpha} d\alpha \right] , \tag{4.7}$$

$$M_{xy}(x,y) = -\frac{1}{2\pi} \frac{a(1-\nu)}{h\lambda^4} \int_{-\infty}^{\infty} i\alpha \sum_1^4 \frac{m_j R_j(\alpha)}{\kappa p_j - 1} e^{m_j x} e^{-iy\alpha} d\alpha$$
$$- \frac{1}{2\pi} \frac{a\kappa}{h\lambda^4} \frac{(1-\nu)^2}{4} \int_{-\infty}^{\infty} (\alpha^2 + r_1^2) A_1(\alpha) e^{r_1 x} e^{-iy\alpha} d\alpha , \tag{4.8}$$

$$\frac{\partial}{\partial y} V_x(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} i\alpha \sum_1^4 \frac{\kappa p_j m_j}{\kappa p_j - 1} R_j(\alpha) e^{m_j x} e^{-iy\alpha} d\alpha$$
$$- \frac{1}{2\pi} \kappa \frac{1-\nu}{2} \int_{-\infty}^{\infty} \alpha^2 A_1(\alpha) e^{r_1 x} e^{-iy\alpha} d\alpha . \tag{4.9}$$

We will attempt to solve the problem by reducing it to a system of singular integral equations. The problem is "symmetric" and the "normal" membrane and moment resultants are specified on the crack surface. Therefore, the natural "dual" quantities which should be considered as the new unknown functions are

$$\frac{\partial}{\partial y} u(+0,y) = G_1(y) \quad , \quad \frac{\partial}{\partial y} \beta_x(+0,y) = G_2(y) \quad , \tag{4.10}$$

corresponding to the "normal" displacement and rotation on the crack surface, respectively. In (4.10) the derivatives of the dual quantities are used to make them dimensionally consistent with N_{xx} and M_{xx} and in order to insure that the resulting integral equations will have Cauchy type singularities [13]. The mixed boundary condition along $x = 0$, $-\infty<y<\infty$ may now be expressed as follows:

$$\lim_{x\to+0} N_{xx}(x,y) = F_1(y) \quad , \quad -1<y<1 \quad , \tag{4.11}$$

$$\lim_{x\to+0} M_{xx}(x,y) = F_2(y) \quad , \quad -1<y<1 \quad , \tag{4.12}$$

$$u(0,y) = 0 \quad , \quad 1<|y|<\infty \quad , \tag{4.13}$$

$$\beta_x(0,y) = 0 \quad , \quad 1<|y|<\infty \quad . \tag{4.14}$$

Referring to the definitions (4.10) and the physical conditions (4.13) and (4.14), it is clear that the functions G_1 and G_2 must satisfy the following single-valuedness conditions:

$$\int_{-1}^{1} G_1(y)dy = 0 \quad , \quad \int_{-1}^{1} G_2(y)dy = 0 \quad . \tag{4.15}$$

From (2.4), (2.14), (2.8), and (2.1) it may be shown that

$$\frac{\partial^2}{\partial y^2} u(+0,y) = -\frac{\partial^3}{\partial x^3} \phi(+0,y) + (\lambda_2/\lambda)^2 \frac{\partial}{\partial x} w(+0,y) \quad . \tag{4.16}$$

Then, after some manipulations, the quantities defined by (4.10) may be obtained as follows:

$$G_1(y) = \left(\frac{\lambda_2}{\lambda}\right)^2 \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha} \sum_1^4 \left(m_j - \frac{m_j^5}{p_j^2}\right) R_j(\alpha) e^{-iy\alpha} d\alpha \quad , \tag{4.17}$$

$$G_2(y) = -\frac{i}{2\pi} \int_{\infty}^{\infty} \alpha \sum_1^4 \frac{m_j R_j(\alpha)}{\kappa p_j - 1} e^{-iy\alpha} d\alpha$$

$$-\frac{1}{2\pi}\kappa\frac{1-\nu}{2}\int_{-\infty}^{\infty}\alpha^2 A_1(\alpha)e^{-iy\alpha}d\alpha \quad . \tag{4.18}$$

Now, using (4.3) and inverting (4.6), (4.8), (4.9), (4.17), and (4.18), we obtain five linear algebraic equations in $R_1,\ldots,R_4$, and A_1 which, after some modifications and by using (4.13) and (4.14), may be expressed as

$$A_1(\alpha) = 2\int_{-1}^{1} G_2(t)e^{i\alpha t}dt \quad , \tag{4.19}$$

$$\sum_1^4 \frac{m_j^3}{p_j^2} R_j(\alpha) = 0 \quad , \tag{4.20}$$

$$\sum_1^4 \frac{m_j^5}{p_j^3} R_j(\alpha) = \frac{1-\nu}{\lambda_2^4} i\alpha\int_{-1}^{1} G_2(t)e^{i\alpha t}dt \quad , \tag{4.21}$$

$$\sum_1^4 \frac{m_j^5}{p_j^4} R_j(\alpha) = i\frac{\kappa}{\lambda_2^4}\frac{1-\nu}{2\alpha}(\alpha^2 + r_1^2)\int_{-1}^{1} G_2(t)e^{i\alpha t}dt \quad , \tag{4.22}$$

$$\sum_1^4 \frac{m_j}{p_j^2} R_j(\alpha) = -i(\lambda/\lambda_2)^2\frac{1}{\alpha^3}\int_{-1}^{1} G_1(t)e^{i\alpha t}dt \quad . \tag{4.23}$$

The solution of the system of equations (4.20) - (4.23) is given in Appendix C. By substituting from (4.19), Appendix C, (4.5) and (4.7) into (4.11) and (4.12), we obtain two integral equations to determine the unknown functions G_1 and G_2 of the following form:

$$\lim_{x\to+0}\int_{-1}^{1}\sum_1^2 G_j(t)dt\int_{-\infty}^{\infty} H_{ij}(\alpha,x)e^{i(t-y)\alpha}d\alpha = F_i(y) \quad ,$$

$$i = 1,2, \quad -1<y<1 \tag{4.24}$$

By examining the functions H_{ij}, it can be seen that they are bounded everywhere in $-\infty<\alpha<\infty$. Therefore any possible singularity of the kernels in (4.24) at $y=t$ must be due to the behavior of $H_{ij}(\alpha,x)$ as $\alpha\to\mp\infty$. Note also that H_{ij} contains exponential damping terms of the form $\exp(m_j x)$ and $\exp(r_1 x)$, where $Re(m_j)<0$, $Re(r_1)<0$. However, since in limit x will go to zero, for $y=t$ this damping does not insure the convergence of the inner integrals in (4.24).

The major difficulty in this problem, of course, is that the functions $m_j(\alpha)$ are not known explicitly in terms of α. For the purpose of examining the singular behavior of the kernels in (4.24) and for extracting the singular parts, all one needs, however, is the asymptotic behavior of m_j and r_1 as $|\alpha|\to\infty$. Thus, from (3.6) - (3.8) and (3.15) it can be shown that for large values of $|\alpha|$ we have

$$m_j(\alpha) = -|\alpha| \ (1 + \frac{p_j}{2\alpha^2} - \frac{p_j^2}{8\alpha^4} + \cdots) \quad , \tag{4.25}$$

$$r_1(\alpha) = -|\alpha| \ (1 + \frac{1}{\kappa(1-\nu)\alpha^2} - \cdots \ ) \quad . \tag{4.26}$$

Using now the relations (4.25) and (4.26) and separating the asymptotic values of H_{ij} for large $|\alpha|$, the kernels in (4.24) may be expressed as

$$\int_{-\infty}^{\infty} H_{ij} e^{i(t-y)\alpha} d\alpha = \int_{-\infty}^{\infty} H_{ij}^{\infty}(\alpha,x) e^{i(t-y)\alpha} d\alpha + \int_{-\infty}^{\infty} [H_{ij}(\alpha,x) - H_{ij}^{\infty}(\alpha,x)] e^{i(t-y)\alpha} d\alpha \ . \tag{4.27}$$

where H_{ij}^{∞} is the asymptotic value of H_{ij} for $|\alpha|\to\infty$. On the right hand side of (4.27) the first term gives Cauchy type kernels $1/(t-y)$ on the main diagonal terms, and the second integrals are uniformly convergent for all t and y (in which, the limit $x=0$ can therefore be put under the integral sign). After the asymptotic analysis and some lengthy but straightforward manipulations the integral equations and the kernels may then be expressed as follows:

$$\int_{-1}^{1} \frac{G_1(t)}{t-y} dt + \sum_{1}^{2} \int_{-1}^{1} k_{1j}(y,t) G_j(t) dt = 2\pi F_1(y) \quad , \quad -1<y<1 \quad , \tag{4.28}$$

$$\frac{1-\nu^2}{\lambda^4} \int_{-1}^{1} \frac{G_2(t)}{t-y} dt + \sum_{1}^{2} \int_{-1}^{1} k_{2j}(y,t) G_j(t) dt = 2\pi \frac{h}{a} F_2(y) \quad , \quad -1<y<1 \quad , \tag{4.29}$$

$$k_{11}(y,t) = \int_0^\infty [(\frac{\lambda_2}{\lambda})^2 \alpha^2 \sum_1^4 \frac{2m_j^2}{p_j^2} Q_j(\alpha) - 1]\sin\alpha(t-y)d\alpha \quad , \tag{4.30}$$

$$k_{12}(y,t) = 2(\frac{\lambda_2}{\lambda})^2 \int_0^\infty \alpha^2 \sum_1^4 \frac{m_j^2}{p_j^2} N_j(\alpha)\sin\alpha(t-y)d\alpha \quad , \tag{4.31}$$

$$k_{21}(y,t) = -\frac{2\lambda_2^4}{\lambda^4} \int_0^\infty \sum_1^4 \frac{1}{p_j^4} (m_j^2 - \nu\alpha^2)m_j^4 Q_j(\alpha)\sin\alpha(t-y)d\alpha \quad , \tag{4.32}$$

$$k_{22}(y,t) = -2 \frac{1}{\lambda^4} \int_0^\infty [\sum_1^4 \frac{\lambda_2^4}{p_j^4} (m_j^2 - \nu\alpha^2)m_j^4 N_j(\alpha)$$

$$- \kappa(1-\nu)^2\alpha r_1 + \frac{1-\nu^2}{2}]\sin\alpha(t-y)d\alpha \quad , \tag{4.33}$$

where the functions $N_j(\alpha)$ and $Q_j(\alpha)$ are given in the Appendix C. Using the results of Appendix C it can be shown that $k_{12}(y,t) = k_{21}(y,t)$.

5. THE ASYMPTOTIC STRESS FIELD AROUND THE CRACK TIPS

The solutions of the singular integral equations (4.28) and (4.29) have integrable singularities at the end points $y = \mp 1$ and are of the following form:

$$G_i(y) = g_i(y)(1-y^2)^{-\frac{1}{2}} \quad , \quad i = 1,2 \quad , \tag{5.1}$$

where g_1 and g_2 are bounded in $-1 \le y \le 1$. Similar to the plane problems, it can be shown that the behavior of the stress distribution in the immediate neighborhood of the crack tips is dependent on $g_i(\mp 1)$ only. To show this, one needs to substitute (5.1) through the expressions of $R_1,\ldots,R_4$ given in Appendix C and A_1 given by (4.19) into the original expressions for N_{ij} and M_{ij}, $(i,j = x,y)$, such as those given by (4.5) - (4.8). Using now the relation [14]

$$\int_0^\infty \alpha^{\mu-1} e^{-b\alpha} \{{\sin \atop \cos}\}(c\alpha)d\alpha = \frac{\Gamma(\mu)}{(b^2+c^2)^{\mu/2}} \{{\sin \atop \cos}\}(\mu\tan^{-1}\frac{c}{b}) \quad ,$$

$$(b>0 \quad ,\mu>0) \quad , \tag{5.2}$$

and [8,10,11] for large values of $|\alpha|$

$$\int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}} e^{i\alpha t}dt = \left(\frac{\pi}{2|\alpha|}\right)^{\frac{1}{2}} \{g(1)\exp[i(\alpha - \frac{\pi}{4}\,\text{sign}(\alpha))]$$

$$+ g(-1)\exp[-i(\alpha - \frac{\pi}{4}\,\text{sign}(\alpha))] + 0\left(\frac{1}{|\alpha|}\right)\} \quad , \tag{5.3}$$

around the end point $y = 1$, $x = 0$ the leading terms of the asymptotic stress and moment resultants may be expressed as

$$N_{xx}(x,y) \cong \frac{g_1(1)}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1 + \alpha|x|)e^{-|x|\alpha}\sin[(1-y)\alpha - \frac{\pi}{4}]d\alpha \quad , \tag{5.4}$$

$$N_{yy}(x,y) \cong \frac{g_1(1)}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1 - \alpha|x|)e^{-|x|\alpha}\sin[(1-y)\alpha - \frac{\pi}{4}]d\alpha \quad , \tag{5.5}$$

$$N_{xy}(x,y) \cong \frac{g_1(1)}{2\sqrt{2\pi}} \int_0^\infty \sqrt{\alpha}\; xe^{-|x|\alpha}\cos[(1-y)\alpha - \frac{\pi}{4}]d\alpha \quad , \tag{5.6}$$

$$M_{xx}(x,y) \cong \frac{g_2(1)}{2\sqrt{2\pi}} \frac{h}{12a} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1 + \alpha|x|)e^{-|x|\alpha}\sin[(1-y)\alpha - \frac{\pi}{4}]d\alpha \quad , \tag{5.7}$$

$$M_{yy}(x,y) \cong \frac{g_2(1)}{2\sqrt{2\pi}} \frac{h}{12a} \int_0^\infty \frac{1}{\sqrt{\alpha}} (1 - \alpha|x|)e^{-|x|\alpha}\sin[(1-y)\alpha - \frac{\pi}{4}]d\alpha \quad , \tag{5.8}$$

$$M_{xy}(x,y) \cong \frac{g_2(1)}{2\sqrt{2\pi}} \frac{h}{12a} \int_0^\infty \sqrt{\alpha}\; xe^{-|x|\alpha}\cos[(1-y)\alpha - \frac{\pi}{4}]d\alpha \quad , \tag{5.9}$$

From (5.4) - (5.9) it is seen that, aside from the magnitudes as represented by $g_1(1)$ and $g_2(1)$, the asymptotic behavior of the membrane and bending stress distributions around the crack tip will be identical. This is, of course, in agreement with the uncoupled in-plane and bending results for flat plates [4-7]. Defining the polar coordinates r,θ by

$$x = r\sin\theta \quad , \quad y - 1 = r\cos\theta \quad , \tag{5.10}$$

evaluating the integrals, and observing that in dimensionless quantities (see Appendix A)

$$\frac{12az}{h} M_{ij} = \sigma_{ij} \quad , \quad (i,j = x,y) \quad , \tag{5.11}$$

from (5.4) - (5.9) the asymptotic stress distribution may be obtained as

$$\sigma_{xx}(r,\theta,z) \cong - \frac{g_1(1) + zg_2(1)}{2\sqrt{2r}} [\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{5\theta}{2}] \quad , \tag{5.12}$$

$$\sigma_{yy}(r,\theta,z) \cong - \frac{g_1(1) + zg_2(1)}{2\sqrt{2r}} [\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{5\theta}{2}] \quad , \tag{5.13}$$

$$\sigma_{xy}(r,\theta,z) \cong - \frac{g_1(1) + zg_2(1)}{2\sqrt{2r}} [- \frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{5\theta}{2}] \quad . \tag{5.14}$$

In this "symmetric" problem the stress intensity factor is defined by

$$k_1 = \lim_{X_2 \to a} \sqrt{2(X_2-a)} \, \sigma_{11}(0,X_2) \quad , \tag{5.15}$$

which, referring to Appendix A and (5.12) may be obtained in terms of $g_1(1)$ and $g_2(1)$ as follows:

$$k_1(Z) = - \frac{E\sqrt{a}}{2} [g_1(1) + \frac{Z}{a} g_2(1)] \quad . \tag{5.16}$$

It should be noted that even though the analysis given in the previous sections is valid for specially orthotropic as well as isotropic shells, (a) because of the dependence of x and y on the orthotropy constant $c = (E_1/E_2)^{1/4}$, r and θ defined by (5.10) are not the physical coordinates, and consequently the angular distribution of the stress state in orthotropic shells would be different from that of isotropic shells as given by (5.12) - (5.14); and (b) since the roots m_j and r_i are very heavily dependent on c, the numerical results obtained for the isotropic shells by assuming c = 1 cannot be readily adopted to the orthotropic shells.

6. CRACK SURFACE DISPLACEMENTS

After obtaining the functions G_1 and G_2 upon solving the integral equations (4.28) and (4.29), through the expressions for $R_1,\dots,R_4$, and

A_1, any desired field quantity in the shell can be expressed in terms of G_1 and G_2 and be easily evaluated. One such group of quantities of some practical interest is the components of crack surface displacement vector. In the symmetric problem under consideration, referring to the definition (4.10), the in-plane component of the crack surface displacement (i.e., the crack opening displacement) may easily be obtained as

$$u(+0,y) = -\int_y^1 G_1(t)dt \quad , \quad -1<y<1 \quad . \tag{6.1}$$

The component of the displacement vector which is perpendicular to the shell surface, i.e., $w(x,y)$, is given by (3.5) and (3.10). Expressing again $R_1,..,R_4$ in terms of G_1 and G_2 (Appendix C), w may be obtained as follows:

$$w(+0,y) = -\frac{1}{\pi}\int_{-1}^1 G_1(t)dt \int_0^\infty \sum_1^4 Q_j(\alpha)\sin\alpha(t-y)d\alpha$$

$$-\frac{1}{\pi}\int_{-1}^1 G_2(t)dt \int_0^\infty \sum_1^4 N_j(\alpha)\sin\alpha(t-y)d\alpha \quad ,$$

$$-1<y<1 \quad , \tag{6.2}$$

where Q_j and N_j are given in Appendix C.

7. NUMERICAL SOLUTION AND RESULTS

The singular integral equations (4.28) and (4.29) subject to single-valuedness conditions (4.15) are solved by using Gauss-Chebyshev integration formulas. Thus, equations (4.28), (4.29) and (4.15) are, respectively, replaced by

$$\sum_{j=1}^n W_j\left[\frac{g_1(t_j)}{t_j-y_i} + \sum_{m=1}^2 k_{1m}(y_i,t_j)g_m(t_j)\right] = 2\pi F_1(y_i) \quad ,$$

$$i = 1,..,n-1 \quad . \tag{7.1}$$

$$\sum_{j=1}^{n} W_j\left[\frac{1-\nu^2}{\lambda^4}\,\frac{g_2(t_j)}{t_j-y_i} + \sum_{m=1}^{2} k_{2m}(y_i,t_j)g_m(t_j)\right]$$

$$= 2\pi\,\frac{h}{a}\,F_2(y_i) \quad , \quad i=1,..,n-1 \quad , \tag{7.2}$$

$$\sum_{j=1}^{n} W_j g_1(t_j) = 0 \quad , \quad \sum_{j=1}^{n} W_j g_2(t_j) = 0 \quad , \tag{7.3}$$

where

$$t_j = \cos\left(\frac{j-1}{n-1}\,\pi\right) \quad , \quad j=1,..,n \quad , \tag{7.4}$$

$$y_i = \cos\left(\frac{2i-1}{2n-2}\,\pi\right) \quad , \quad i=1,..,n-1 \quad , \tag{7.5}$$

$$W_1 = W_n = \frac{\pi}{2(n-1)} \quad , \quad W_j = \frac{\pi}{n-1} \quad , \quad j=2,..,n-1 \quad . \tag{7.6}$$

As a numerical example a cracked cylindrical shell under uniform membrane and bending loads is considered. To make the practical applications of the results more convenient, these two loads are considered separately. The calculated results for the stress intensity factors are normalized with respect to the corresponding flat plate values. For example, if the applied loads are

$$N_{11}(0,X_2) = -N_{11} = -h\sigma_m \quad , \quad M_{11}(0,X_2) = 0 \quad , \quad -a<X_2<a \quad , \tag{7.7}$$

the input functions in the integral equations (4.28) and (4.29) become

$$F_1(y) = -N_{11}/hE \quad , \quad F_2(y) = 0 \quad , \tag{7.8}$$

and the corresponding flat plate stress intensity factor is $\sigma_m\sqrt{a}$. Then the membrane and bending stress intensity ratios k_{mm} and k_{bm} are obtained from (5.16) as follows:

$$k_{mm} = \frac{k_1(0)}{\sigma_m\sqrt{a}} = -\frac{E}{2\sigma_m}\,g_1(1) \quad , \tag{7.9}$$

$$k_{bm} = \frac{k_1(h/2) - k_1(0)}{\sigma_m\sqrt{a}} = -\frac{E}{2\sigma_m}\frac{h}{2a}\, g_2(1) \quad . \tag{7.10}$$

The "bending stress intensity factor" is thus based on the bending stresses at the outer layer of the shell, $Z = +h/2$.

Similarly, if the external loads are

$$N_{11}(0,X_2) = 0 \quad , \quad M_{11}(0,X_2) = -M_{11} = -\frac{h^2}{6}\sigma_b \quad , \quad -a<X_2<a \tag{7.11}$$

Then the input functions become

$$F_1(y) = 0 \quad , \quad F_2(y) = -\frac{M_{11}}{Eh^2} \quad , \quad -1<y<1 \quad . \tag{7.12}$$

In this case too the corresponding flat plate stress intensity factor is defined by $\sigma_b\sqrt{a}$, and the stress intensity factor ratios are obtained as follows:

$$k_{mb} = \frac{k_1(0)}{\sigma_b\sqrt{a}} = -\frac{E}{2\sigma_b}\, g_1(1) \quad , \tag{7.13}$$

$$k_{bb} = \frac{k_1(h/2) - k_1(0)}{\sigma_b\sqrt{a}} = -\frac{E}{2\sigma_b}\frac{h}{2a}\, g_2(1) \tag{7.14}$$

In the numerical calculations the effective transverse shear modulus for the shell is assumed to be $B = 5G/6$, G being the shear modulus of the material. Also, the Poisson's ratio is taken as $\nu = 1/3$ in all calculations except for one set of results where the effect of ν is investigated. One may note that λ_2 is used in this analysis is the standard shell parameter λ defined in the formulation of the problem by using the classical (i.e., the 8th order) shell theory. Also note that in the present analysis there is an additional parameter a/h, which, within the confines of the shallow shell theory, gives the thickness effect.

The numerical results are shown in Tables 1-5. To help visualizing the trends some of the results are also shown in Figures 3-6. Tables 1-4 show the stress intensity factor ratios k_{mm}, k_{bm}, k_{mb}, and k_{bb} defined by

(7.9), (7.10), (7.13) and (7.14), respectively. In these tables the values given for $\lambda_2 = [12(1-\nu^2)]^{1/4} \; a/\sqrt{Rh} = 0$ correspond to the flat plate. Both λ_2 and a/h have been varied from 0 to 10. For $\lambda_2 > 10$ and $a/h > 10$ the linearized shallow shell theory used in this study is probably not valid. For very thin shells (i.e., a/h = 10) under uniform membrane loading (Tables 1, 2 and Figures 3, 4) it is found that the membrane component of the stress intensity factor k_{mm} is indistinguishable from the results found by using the classical theory [1]. However the bending stress intensity factor k_{bm} is quite different. The results given in Tables 1-4 show that, particularly for large values of λ_2, the thickness parameter a/h may have considerable effect on the stress intensity factors. Table 3 and Figure 5 show that when $\lambda_2 \to 0$ the results are in good agreement with the flat plate bending results given in [5] and [6] and the axially cracked shell results given in [8].

For a specific geometry $\lambda_2 = 4$ and $a/h = 2$, Table 5 shows the effect of the Poisson's ratio ν on the stress intensity factors. For the two most important components k_{mm} and k_{bb} (i.e., for the primary stress intensity factors under membrane and bending loads) the effect of ν does not seem to be significant. Hence the results given in Tables 1-4 and calculated for $\nu = 1/3$ can be used for materials with a Poisson's ratio $0.2 < \nu < 0.4$, which may cover nearly all structural materials.

Figures 7 and 8 show some sample results for the out of plane displacement w(+0,y) calculated on the neutral surface along the line of the crack. The trend seems to be quite similar to the bulging results obtained for the axially cracked shell [11]. It should again be emphasized that in these numerical calculations the crack surface membrane stress is compressive and the bending moment is applied in such a way that the outer layer of the shell on the crack surface is again compressive. Under these loads, as seen from the figures, on the crack surface the displacement w is in the outward direction.

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APPENDIX A

The Dimensionless Quantities

$$x = \frac{1}{\sqrt{c}}\frac{X_1}{a} \quad , \quad y = \sqrt{c}\;\frac{X_2}{a} \quad , \quad z = \frac{Z}{a} \quad , \tag{A.1}$$

$$\left.\begin{aligned} &u = \sqrt{c}\;\frac{U_1}{a} \quad , \quad v = \frac{1}{\sqrt{c}}\frac{U_2}{a} \quad , \quad w = \frac{W}{a} \quad , \\ &\beta_x = \sqrt{c}\;\beta_1 \quad , \quad \beta_y = \frac{1}{\sqrt{c}}\,\beta_2 \quad , \quad \phi = \frac{F}{a^2hE} \end{aligned}\right\} \tag{A.2}$$

$$\sigma_{xx} = \frac{\sigma_{11}}{cE} \quad , \quad \sigma_{yy} = \frac{c\sigma_{22}}{E} \quad , \quad \sigma_{xy} = \frac{\sigma_{12}}{E} \tag{A.3}$$

$$N_{xx} = \frac{N_{11}}{chE} \quad , \quad N_{yy} = \frac{cN_{22}}{hE} \quad , \quad N_{xy} = \frac{N_{12}}{hE} \tag{A.4}$$

$$M_{xx} = \frac{M_{11}}{ch^2E} \quad , \quad {}_{yy} = \frac{cM_{22}}{h^2E} \quad , \quad M_{xy} = \frac{M_{12}}{h^2E} \tag{A.5}$$

$$V_x = \frac{V_1}{\sqrt{c}\;hB} \quad , \quad V_y = \frac{\sqrt{c}\;V_2}{hB} \quad , \tag{A.6}$$

$$\lambda_1^4 = 12(1-\nu^2)\,\frac{c^2a^4}{h^2R_1^2} \quad , \quad \lambda_2^4 = 12(1-\nu^2)\,\frac{a^4}{c^2h^2R_2^2} \quad ,$$

$$\lambda_{12}^4 = 12(1-\nu^2)\,\frac{a^4}{h^2R_{12}^2} \quad , \quad \lambda^4 = 12(1-\nu^2)\,\frac{a^2}{h^2} \quad , \quad \kappa = \frac{E}{B\lambda^4} \tag{A.7}$$

APPENDIX B

Procedure for Solving a Quartic Equation

Consider the following quartic equation:

$$p^4 + a_3p^3 + a_2p^2 + a_1p + a_o = 0 \tag{B.1}$$

Although in the shell problem the coefficients $a_o,..,a_3$ are real, for the sake of generality, here it will be assumed that they are complex. Therefore the roots of (B.1) are in general complex. Assume that (B.1) can be written as the difference of two squares:

$$(p^2 + Ap + B)^2 - (Cp + D)^2 = 0 \quad , \tag{B.2}$$

Then, from (B.2) it follows that

$$p^2 + (A + C)p + (B + D) = 0 \quad , \tag{B.3}$$

$$p^2 + (A - C)p + (B - D) = 0 \quad . \tag{B.4}$$

If the constants A, B, C, D can be determined in terms of a_3, a_2, a_1, a_o, then the four roots can be found in a straight-forward manner by solving equations (B.3) and (B.4).

Comparing (B.1) and (B.2), one can write:

$$2A = a_3 \quad , \tag{B.5}$$

$$2B + A^2 - C^2 = a_2 \quad , \tag{B.6}$$

$$2AB - 2CD = a_1 \quad , \tag{B.7}$$

$$B^2 - D^2 = a_o \quad . \tag{B.8}$$

Eliminating C and D in (B.6) - (B.8) and defining 2B = u, one finds:

$$u^3 + b_2 u^2 + b_1 u + b_o = 0 \tag{B.9}$$

where

$$b_2 = -a_2 \quad , \quad b_1 = a_1 a_3 - 4a_o \quad , \quad b_o = 4a_o a_2 - a_1^2 - a_o a_3^2 \quad . \tag{B.10}$$

Let u_1 be a root of the cubic equation (B.9), (see, for example [15] for determining the roots of a cubic). Then

$$B = \frac{u_1}{2} \tag{B.11}$$

and from (B.5)

$$A = \frac{a_3}{2} \quad . \tag{B.12}$$

Once A and B are known, one can determine C and D from (B.6) and (B.8) as follows:

$$C^2 = u_1 + \frac{a_3^2}{4} - a_2 \tag{B.13}$$

$$D^2 = \frac{u_1^2}{4} - a_o \tag{B.14}$$

It is seen that C and D are multiple-valued functions and one should choose the correct branch in order to obtain the correct solution.

C^2 and D^2 can be written in the complex form as follows:

$$C^2 = \rho_1 e^{i\theta_1} \quad , \quad D^2 = \rho_2 e^{i\theta_2} \tag{B.15}$$

where ρ_1, ρ_2, θ_1 and θ_2 can be determined by using (B.13) and (B.14). Thus

$$C_1 = \sqrt{\rho_1}\, e^{i\theta_1/2} \quad , \quad C_2 = \sqrt{\rho_1}\, e^{i(\theta_1/2 + \pi)}$$

$$D_1 = \sqrt{\rho_1}\, e^{i\theta_2/2} \quad , \quad D_2 = \sqrt{\rho_2}\, e^{i(\theta_2/2 + \pi)} \tag{B.16}$$

All pairs of (C,D) satisfy equations (B.6) and (B.8) but not (B.7). Therefore C and D must be selected such that equation (B.7) is satisfied, i.e.,

$$2CD = \frac{u_1 a_3}{2} - a_1 \quad . \tag{B.17}$$

APPENDIX C

Expressions for $R_j(\alpha)$, $j = 1,..,4$

$$R_j(\alpha) = i[Q_j(\alpha)\int_{-1}^{1} G_1(t)e^{i\alpha t}dt + N_j(\alpha)\int_{-1}^{1} G_2(t)e^{i\alpha t}dt] \quad , \tag{C.1}$$

$$\begin{aligned} N_1(\alpha) = \frac{1}{D(\alpha)} \frac{m_2m_3m_4}{(p_2p_3p_4)^2} \{ & -\alpha \frac{1-\nu}{\lambda_2^4} [m_2^2(\frac{m_3^4}{p_3^2} - \frac{m_4^4}{p_4^2}) \\ & - m_3^2 (\frac{m_2^4}{p_2^2} - \frac{m_4^4}{p_4^2}) + m_4^2 (\frac{m_2^4}{p_2^2} - \frac{m_3^4}{p_3^2})] \\ & + \frac{\kappa}{\lambda_2^4} \frac{1-\nu}{2\alpha} (\alpha^2 + r_1^2) [m_2^2(\frac{m_3^4}{p_3} - \frac{m_4^4}{p_4}) \\ & - m_3^2 (\frac{m_2^4}{p_2} - \frac{m_4^4}{p_4}) + m_4^2 (\frac{m_2^4}{p_2} - \frac{m_3^4}{p_3})]\} \quad , \end{aligned} \tag{C.2}$$

$$\begin{aligned} Q_1(\alpha) = \frac{1}{D(\alpha)} (\frac{\lambda}{\lambda_2})^2 \frac{1}{\alpha^3} \frac{(m_2m_3m_4)^3}{(p_2p_3p_4)} & [(\frac{m_3^2}{p_3} \frac{m_4^2}{p_4^2} - \frac{m_4^2}{p_4} \frac{m_3^2}{p_3^2}) \\ & - (\frac{m_2^2}{p_2} \frac{m_4^2}{p_4^2} - \frac{m_4^2}{p_4} \frac{m_2^2}{p_2^2}) + (\frac{m_2^2}{p_2} \frac{m_3^2}{p_3^2} - \frac{m_3^2}{p_3} \frac{m_2^2}{p_2^2})] \end{aligned} \tag{C.3}$$

$$\begin{aligned} D(\alpha) = \frac{m_1m_2m_3m_4}{(p_1p_2p_3p_4)^2} \{ & (m_1^2 - m_2^2)(m_3^2 - m_4^2)(\frac{m_1^4m_2^4}{(p_1p_2)^2} + \frac{m_3^4m_4^4}{(p_3p_4)^2}) \\ & + (m_1^2 - m_3^2)(m_4^2 - m_2^2)(\frac{m_1^4m_3^4}{(p_1p_3)^2} + \frac{m_2^4m_4^4}{(p_2p_4)^2}) \\ & + (m_1^2 - m_4^2)(m_2^2 - m_3^2)(\frac{m_1^4m_4^4}{(p_1p_4)^2} + \frac{m_2^4m_3^4}{(p_2p_3)^2})\} \end{aligned} \tag{C.4}$$

The expressions of N_2 and Q_2 are obtained from (C.2) and (C.3) by replacing the indices in m_j and p_j sequentially from 2, 3, 4 to 3, 4, 1. Similarly for N_3 and Q_3 the indices are replaced by 4, 1, 2, and for N_4 and Q_4 by 1, 2, 3.

Table 1 Stress intensity factor ratio k_{mm}

λ_2	a/h = 0.5	a/h = 1	a/h = 2	a/h = 5	a/h = 10
0.0	-	1.000	1.000	1.000	1.000
0.5	-	1.011	1.010	1.010	1.010
1.0	1.072	1.057	1.052	1.051	1.050
2.0	1.274	1.205	1.180	1.171	1.169
3.0	1.503	1.389	1.339	1.319	1.315
4.0	1.717	1.569	1.499	1.470	1.465
5.0	1.917	1.738	1.650	1.613	1.606
6.0	2.104	1.897	1.792	1.745	1.737
8.0	2.400	2.197	2.056	1.986	1.973
10.0	-	2.462	2.306	2.205	2.185

Table 2 Stress intensity factor ratio k_{bm}

λ_2	a/h = 0.5	a/h = 1	a/h = 2	a/h = 5	a/h = 10
0.0	-	0.000	0.000	0.000	0.000
0.5	-	0.050	0.047	0.046	0.046
1.0	0.101	0.100	0.098	0.098	0.099
2.0	0.103	0.110	0.119	0.130	0.139
3.0	0.066	0.052	0.052	0.073	0.089
4.0	0.036	-0.010	-0.037	-0.024	-0.004
5.0	0.017	-0.055	-0.117	-0.126	-0.106
6.0	0.002	-0.087	-0.179	-0.217	-0.202
8.0	-0.018	-0.130	-0.264	-0.359	-0.359
10.0	-	-0.161	-0.322	-0.463	-0.481

Table 3 Stress intensity factor ratio k_{bb}

λ_2	a/h = 0.5	a/h = 1	a/h = 2	a/h = 5	a/h = 10
0.5	-	0.737	0.693	0.660	0.653
1.0	0.738	0.672	0.636	0.612	0.608
2.0	0.648	0.549	0.508	0.493	0.493
3.0	0.615	0.484	0.422	0.401	0.401
4.0	0.602	0.455	0.376	0.343	0.342
5.0	0.595	0.441	0.350	0.308	0.303
6.0	0.592	0.434	0.335	0.284	0.277
8.0	0.589	0.426	0.319	0.254	0.242
10.0	-	0.423	0.311	0.236	0.219

Table 4 Stress intensity factor ratio k_{mb}

λ_2	a/h = 0.5	a/h = 1	a/h = 2	a/h = 5	a/h = 10
0.0	-	0.000	0.000	0.000	0.000
0.5	-	0.015	0.013	0.011	0.010
1.0	0.034	0.032	0.028	0.025	0.023
2.0	0.044	0.047	0.045	0.041	0.039
3.0	0.041	0.046	0.046	0.044	0.042
4.0	0.036	0.042	0.043	0.041	0.040
5.0	0.032	0.038	0.039	0.038	0.036
6.0	0.029	0.035	0.036	0.035	0.033
8.0	0.024	0.030	0.031	0.030	0.029
10.0	-	0.026	0.028	0.027	0.026

Table 5 The effect of Poisson's ratio on the stress intensity factors, $a/h = 2$, $\lambda_2 = 4$

ν	k_{mm}	k_{bm}	k_{bb}	k_{mb}
0.0	1.473	-0.013	0.387	0.031
0.1	1.479	-0.017	0.387	0.035
0.2	1.487	-0.023	0.385	0.038
0.3	1.495	-0.033	0.379	0.042
1/3	1.499	-0.037	0.376	0.043
0.4	1.506	-0.047	0.369	0.045
0.5	1.519	-0.068	0.354	0.048

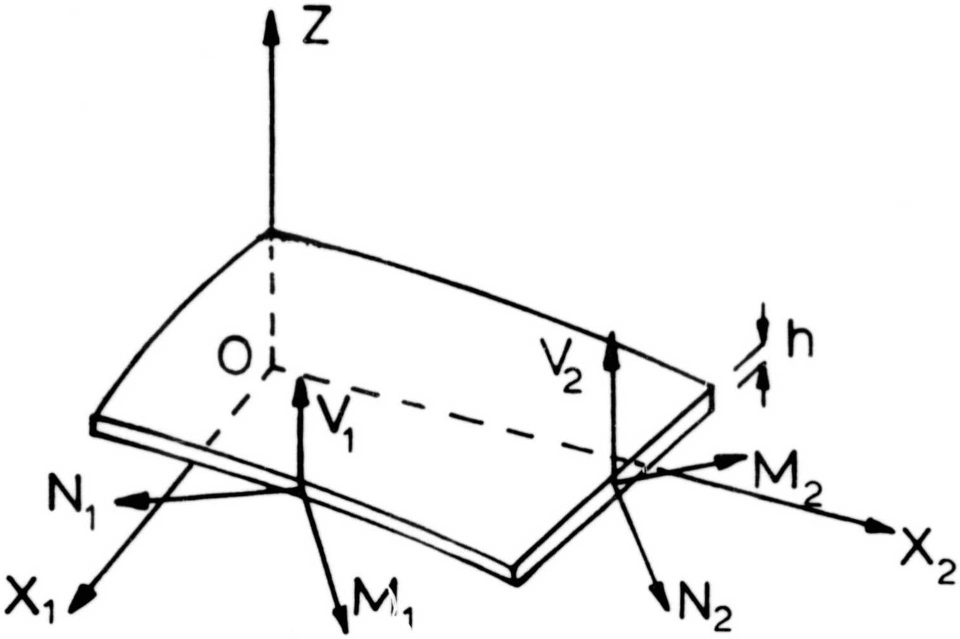

Figure 1 Stress and moment resultants in a shallow shell

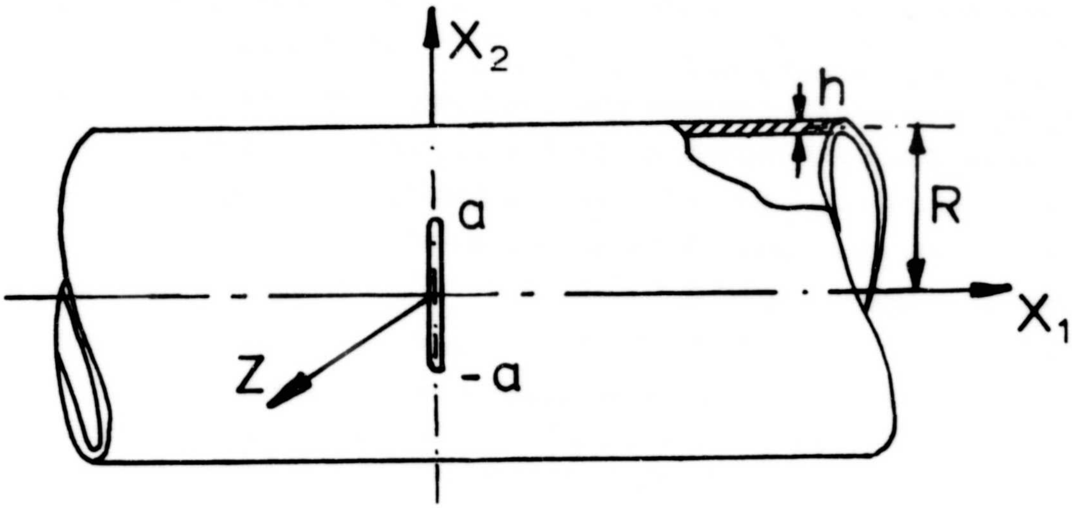

Figure 2 Cylindrical shell containing a circumfer ntial crack

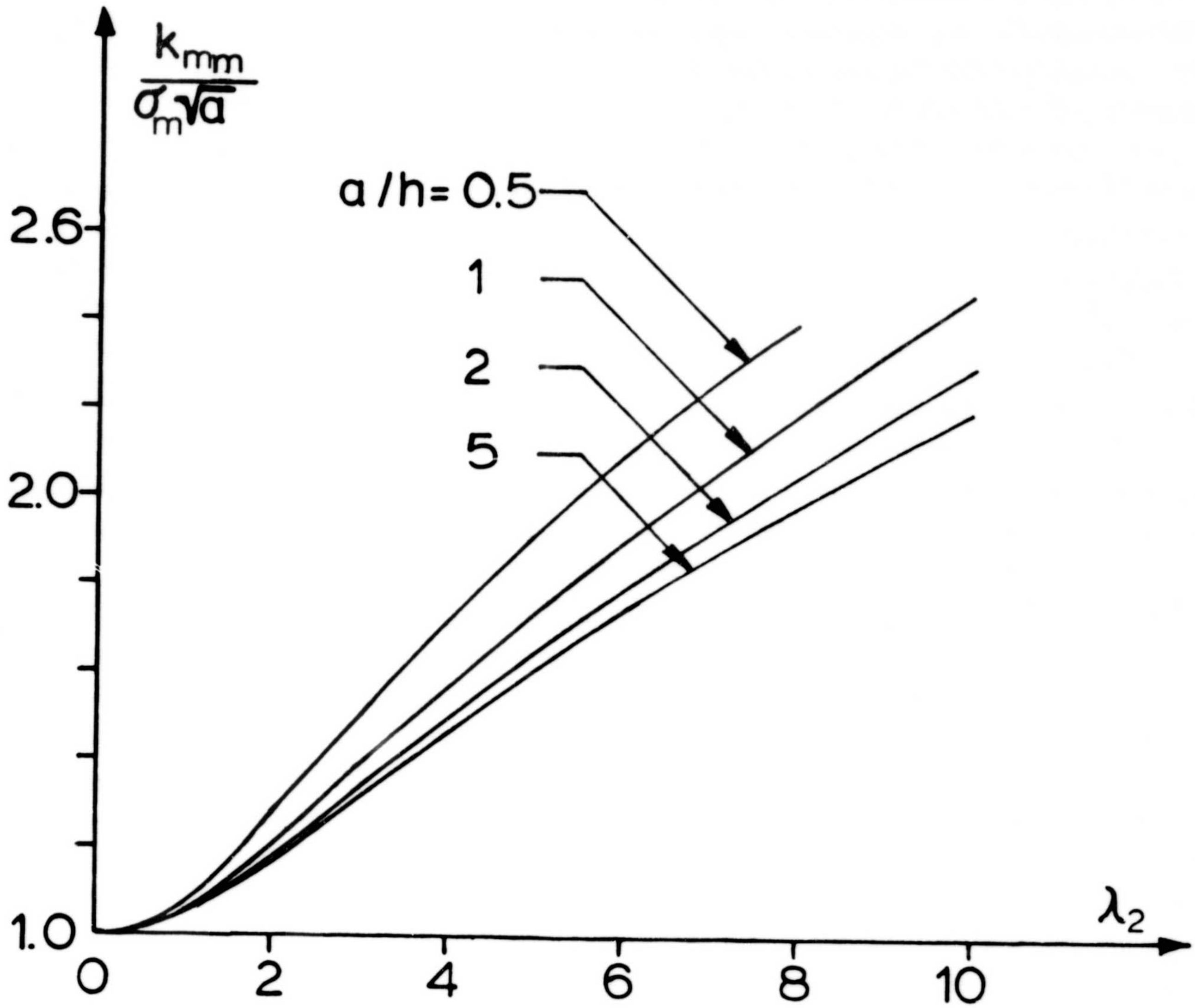

Figure 3 Membrane component of the stress intensity factor ratio
$k_{mm} = k_{shell}^{memb.}/k_{plate}$, $N_{xx} \neq 0$, $M_{xx} = 0$

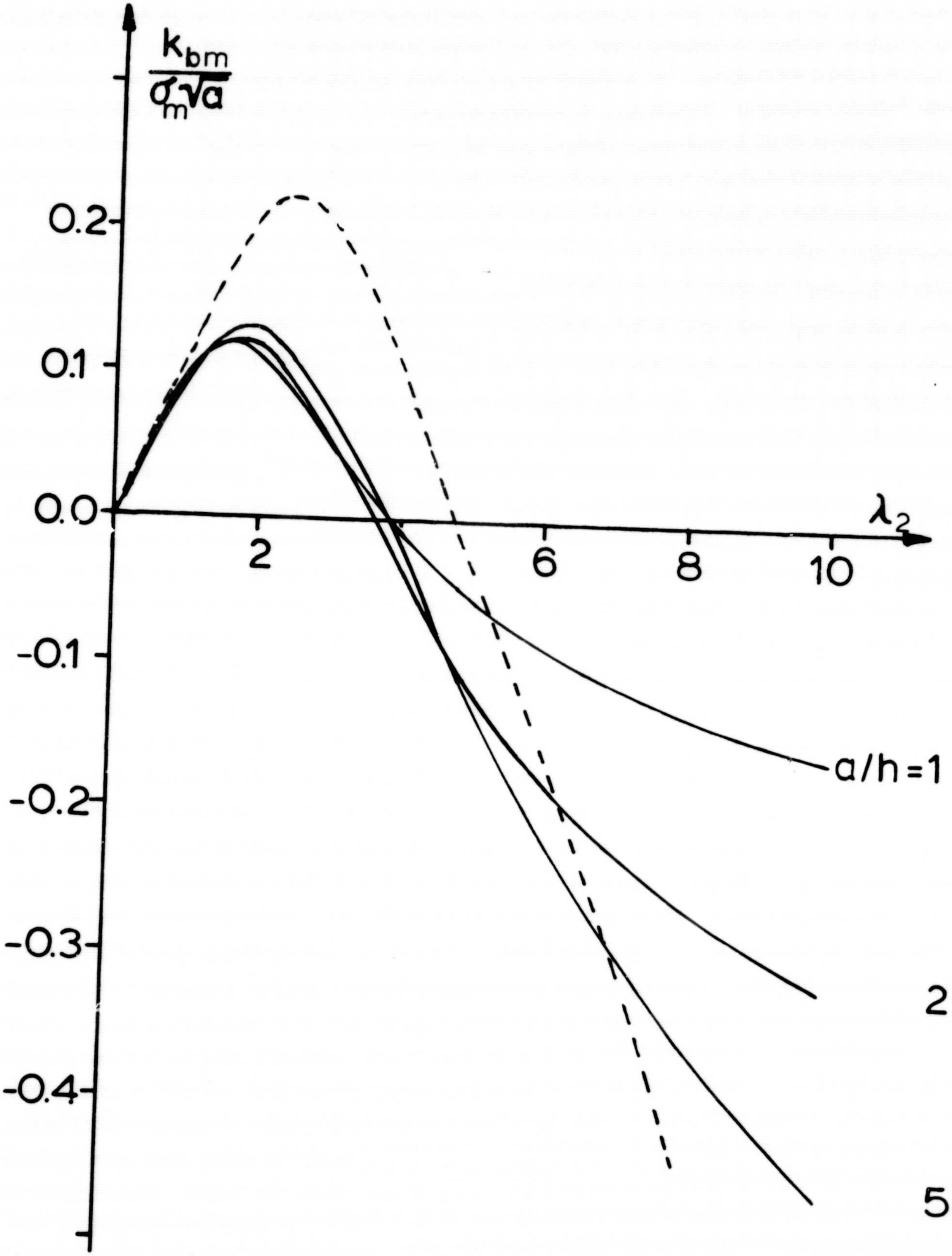

Figure 4 Bending component of the stress intensity factor ratio k_{bm}, $N_{xx} \neq 0$, $M_{xx} = 0$

----Result from the 8th order theory [1]

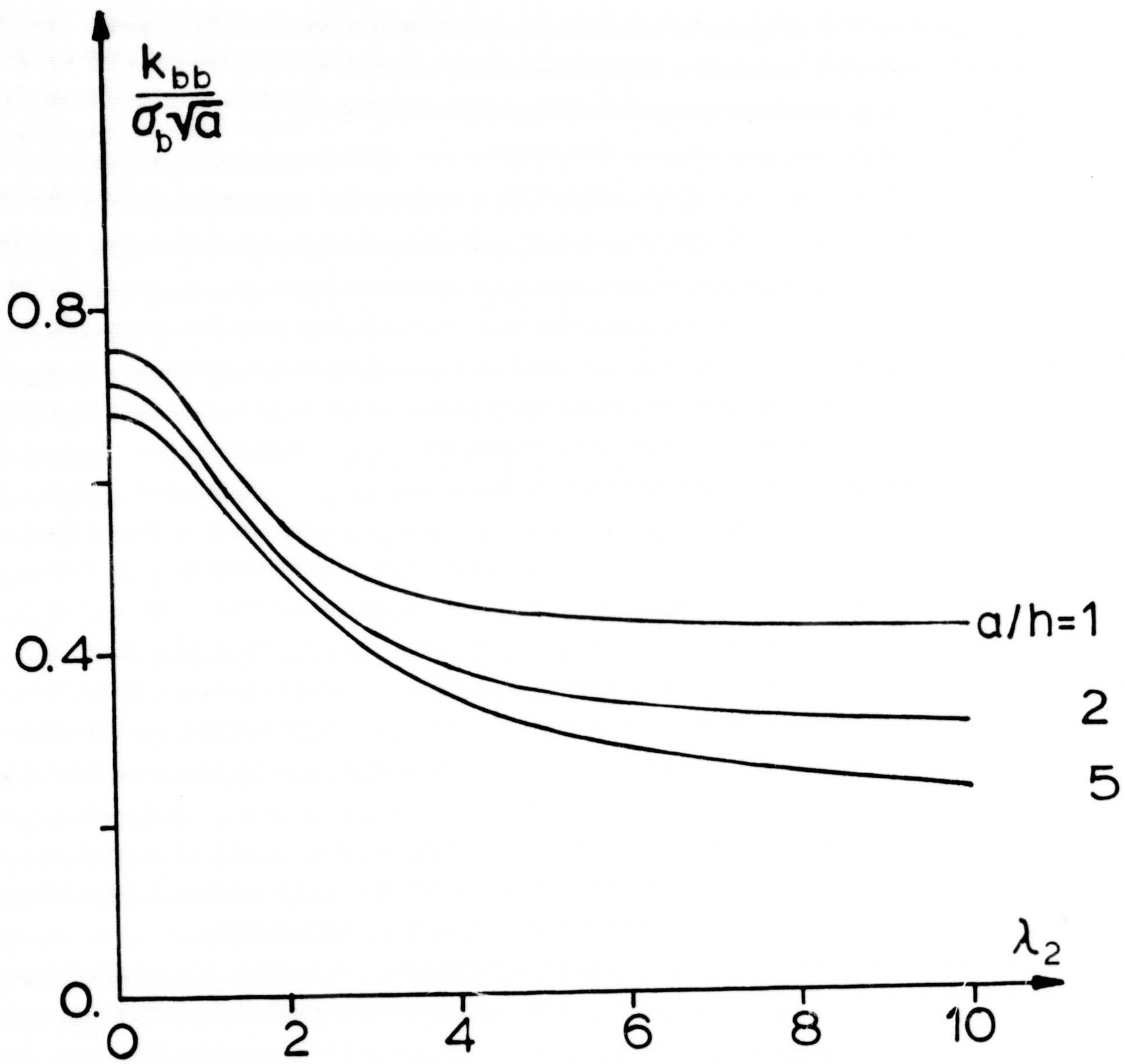

Figure 5 Bending component of the stress intensity factor ratio, k_{bb}, $N_{xx} = 0$, $M_{xx} \neq 0$

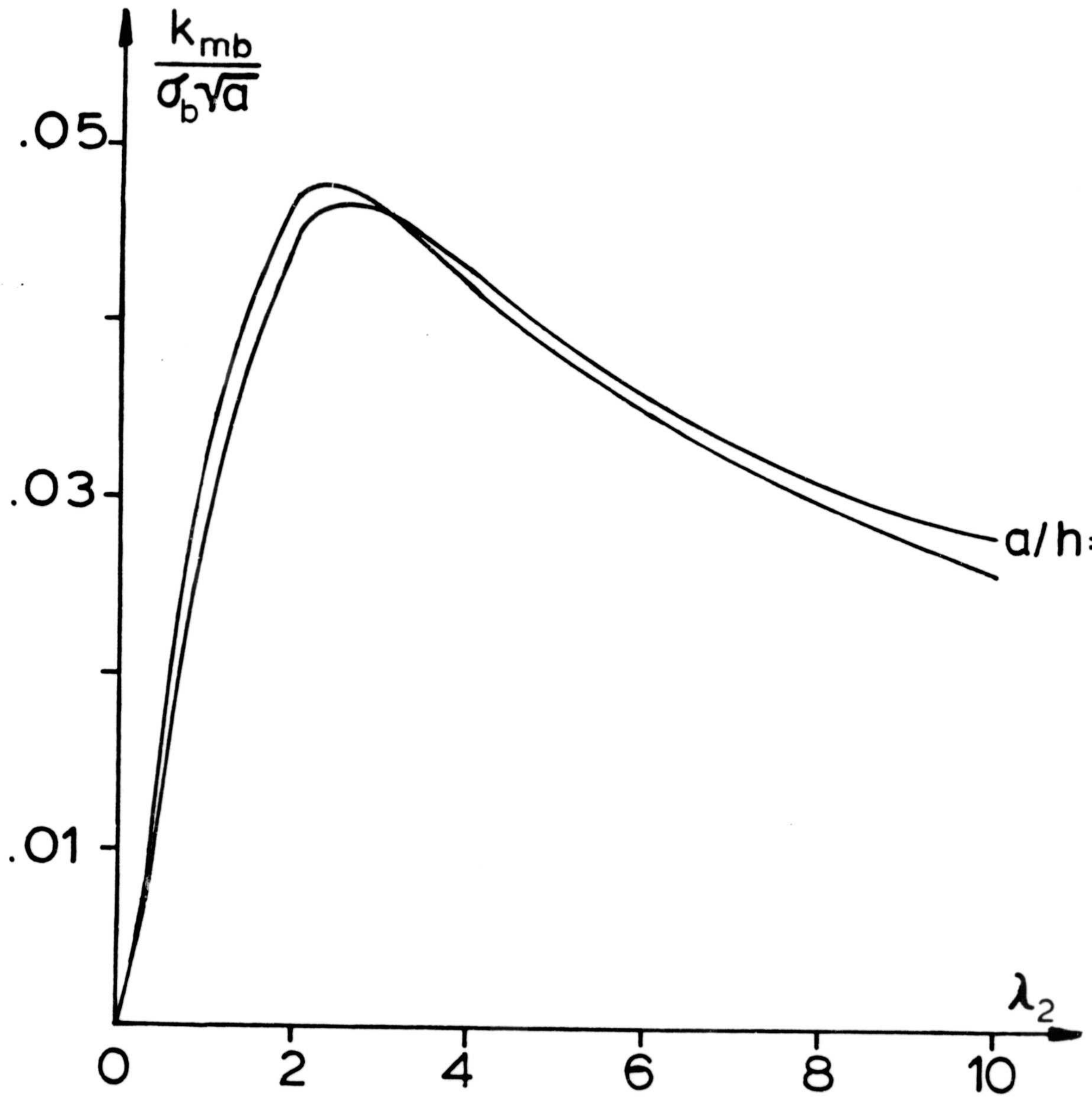

Figure 6 Membrane component of the stress intensity factor ratio k_{mb}, $N_{xx} = 0$, $M_{xx} \neq 0$

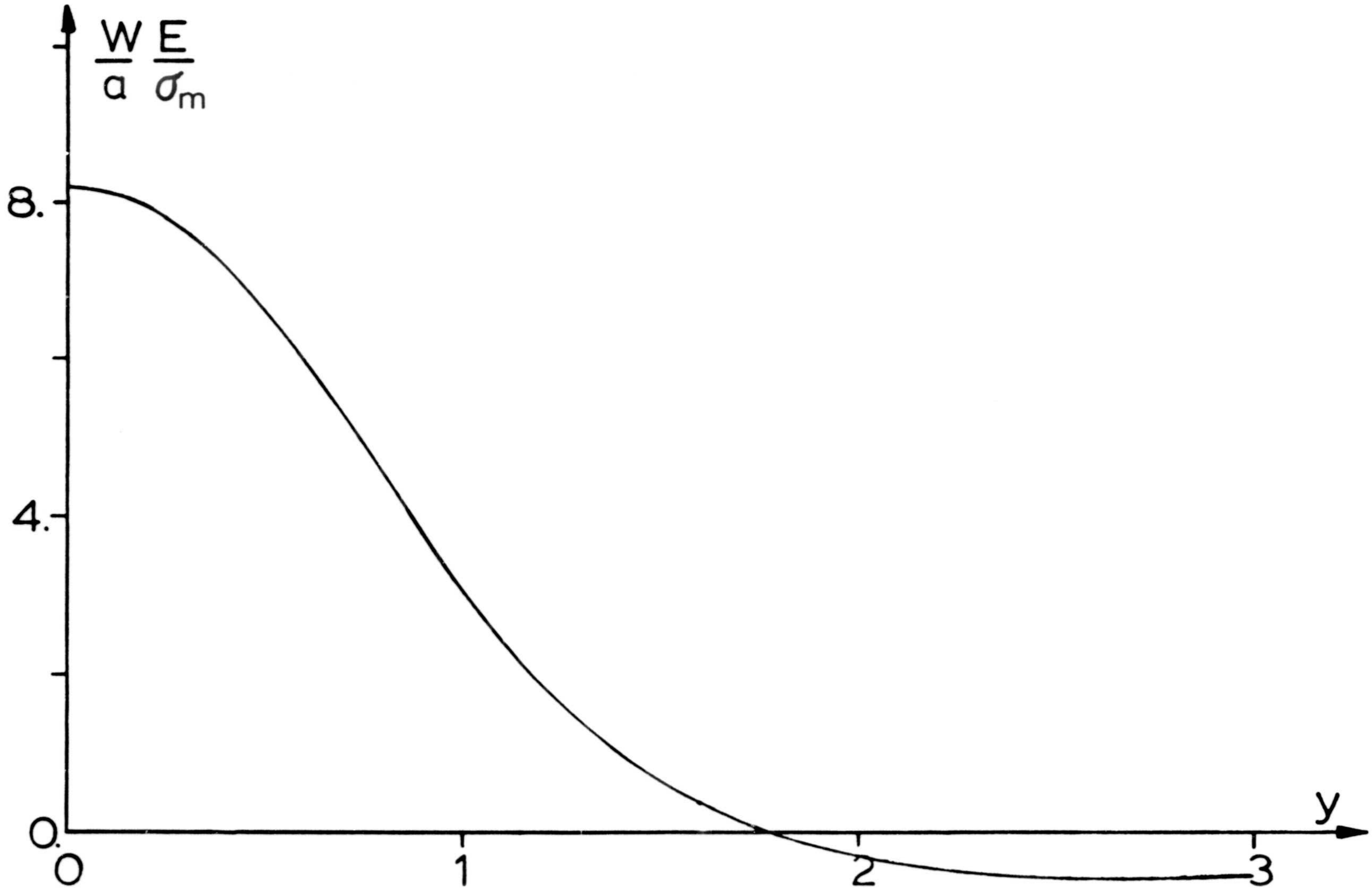

Figure 7 Displacement component perpendicular to the shell surface, $w(0,y)$. $N_{xx} \neq 0$, $M_{xx} = 0$, $a/h = 5$, $\lambda_2 = 4$

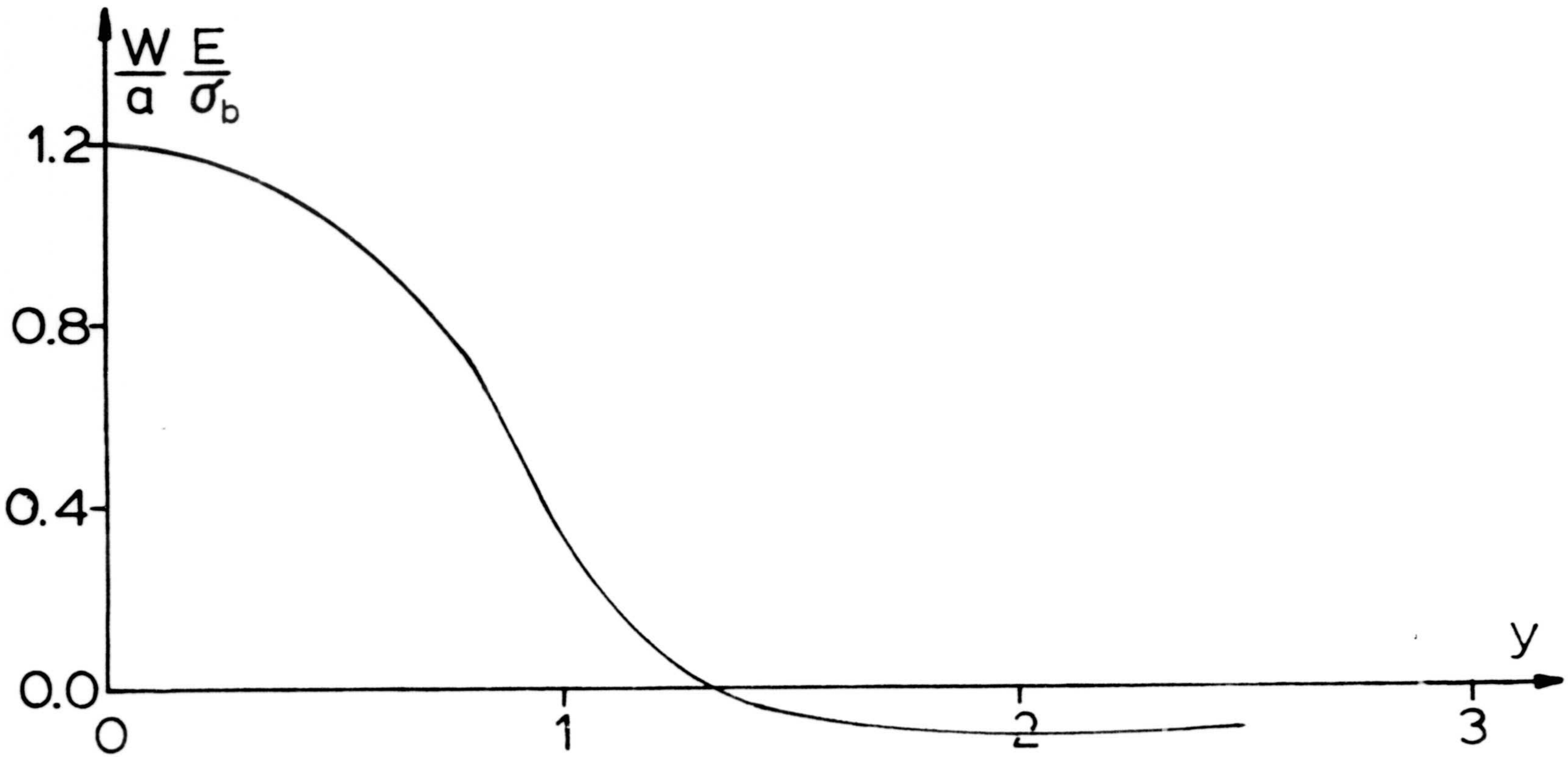

Figure 8 Displacement component perpendicular to the shell surface, $w(0,y)$, $N_{xx} = 0$, $M_{xx} \neq 0$, $a/h = 5$, $\lambda_2 = 4$