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Avoidance Control

G. Leitmann[†] and J. Skowronski[††]

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[†] University of California, Berkeley, California, USA
[††] University of Queensland, Brisbane, Australia

Abstract

We consider dynamical systems subject to control by two agents, one of whom desires that no trajectory of the system, emanating from outside a given set, intersects the set no matter what the admissible actions of the other agent. Conditions are given whose satisfaction assures that a given control results in avoidance. Furthermore, these conditions are constructive in that they yield an avoidance feedback control. Some examples are presented.

1. Introduction

We consider dynamical systems subject to control by two agents, one of whom desires that no trajectory of the system, emanating from outside of a given set, intersects that set no matter what the actions of the other agent; that is, the first agent desires _avoidance_ of a prescribed set. A number of problems fall into this category. Among them is that of evasion by one agent (evader) from one or from more than one pursuer, e.g., Refs. 1,2,3. The pursuer(s) may act with fixed (predetermined) strategies or as active pursuers capable of choosing any one out of a given set of strategies. Also of interest are problems in which avoidance is sought in the presence of uncertainty. This latter problem encompasses evasion by one agent from another one who is not a purposeful pursuer but whose unplanned actions may result in collision, e.g., Refs. 4,5.

Heretofore problems of the type mentioned above were treated in one of two ways (Refs. 6,7): as games of degree (quantitative games) with time or distance of approach as cost, e.g., Refs. 1,2,3,8,9, or as games of kind (qualitative games) involving the construction of barriers, e.g., Refs. 4,5. Even for systems of low dimensionality (≤ 3), these techniques usually require numerical integration so that only particular cases can be discussed. Furthermore, results are obtained by use of _necessary_ conditions. Here we propose an alternative approach, namely, the constructive utilization of conditions _sufficient_ to assure avoidance. This method is simple and elementary; its main drawback lies in the requirement for a Lyapunov-type function for whose construction no general recipe is given.

2. Problem Statement

We consider dynamical systems in the sense of Filippov (Ref. 10). In particular, let

$$p^i(\cdot) : R^n \times R \to \text{nonempty subsets of } R^{d_i} \quad, \quad i = 1, 2.$$

be (feedback) strategies belonging to given classes of possibly set-valued functions, u_i, with control values, u^i, ranging in given[3] sets, U_i; that is, given $(x, t) \in R^n \times R$

$$u^i \in p^i(x, t) \subseteq U_i \subseteq R^{d_i} \quad, \quad i = 1, 2 .$$

Now consider a given function

$$f(\cdot) : R^n \times R \times R^{d_1} \times R^{d_2} \to R^n$$

and for given $p^i(\cdot) \in u_i$, $i = 1, 2$, the set-valued function $F(\cdot)$ defined by

$$F(x, t) \overset{\triangle}{=} \{z \mid z = f(x, t, u^1, u^2), \ u^i \in p^i(x, t)\}$$

$$= f(x, t, p^1(x, t), p^2(x, t)) .$$

Then a dynamical system is defined by the relation

$$\dot{x} \in F(x, t) . \tag{1}$$

Given $(x_0, t_0) \in R^n \times R$, the solutions of (1) are absolutely continuous functions on intervals of R

$$x(\cdot) : [t_0, t_1] \to R^n , \ x(t_0) = x_0 \tag{2}$$

satisfying (1) a.e.: namely,

$$\dot{x}(t) \in f(x, t, p^1(x(t), t), p^2(x(t), t)) \quad \text{a.e. } [t_0, t_1] \tag{3}$$

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[3] We allow state and time-dependent constraints $U_i = U_i(x, t)$.

Sometimes it may be convenient to restrict x by $x \in \Delta$, where Δ is open or the closure of an open set in R^n. In that event, $(x_o, t_o) \in \Delta \times R$ and R^n is replaced by Δ in (2)[4].

Now let there be given an <u>anti-target</u>, T, in Δ, that is a given set into which no solution of (1) must enter for some $p^1(\cdot) \in u^1$ no matter what $u^2(\cdot) \in u_2$. Furthermore, consider a closed subset, A, of Δ such that

$$A \supset T$$

and the closure, Δ_ϵ, of an open subset of Δ such that

$$\Delta_\epsilon \supset A$$

and

$$\partial \Delta_\epsilon \cap \partial A \cap \text{int } \Delta = \phi$$

Figure 1, Anti-target, Avoidance Set
and Safety Zone

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[4] We shall employ Δ henceforth with the understanding that Δ may be R^n itself.

We term A the <u>avoidance set</u> and

$$\Delta_A \triangleq \Delta_\epsilon - A$$

the <u>safety zone</u>. This nomenclature is employed for the following reasons. If a solution avoids A then it cannot enter T, and if a policy is implemented in Δ_A that guarantees avoidance of A then a solution originating outside of A cannot reach A.

It is our purpose to determine an <u>avoidance strategy</u> $p^1(\cdot) \in U_1$ such that, given $(x_0, t_0) \in \Delta$, no solution of (1) intersects A no matter what $p^2(\cdot) \in U_2$. To put this more precisely we introduce

$$K(x_0, t_0, t) \triangleq \{x(t) \mid \text{given } (x_0, t_0) \in \Delta \text{ and } p^1(\cdot) \in U_1, \text{ all } p^2(\cdot) \in U_2\}$$

$$= \text{attainable set of motions from } (x_0, t_0) \text{ at } t \geq t_0,$$
$$\text{given } p^1(\cdot) \in U_1, \text{ for all } p^2(\cdot) \in U_2$$

$$K(x_0, t_0, [t_0, \infty)) \triangleq \bigcup_{t \in [t_0, \infty)} K(x_0, t_0, t)$$

$$= U_2 \text{ funnel of motions from } (x_0, t_0)$$

$$K(\Omega_A, R) \triangleq \bigcup_{(x_0, t_0) \in \Omega_A} K(x_0, t_0, [t_0, \infty))$$

where

$$\Omega_A \triangleq \Delta_A \times R .$$

Now we have

<u>Definition 2.1</u> Given relation (1), and U_i, $i = 1, 2$, a prescribed set A is <u>avoidable</u> iff there is a $p^1(\cdot) \in U_1$ and a $\Omega_A \neq \phi$ such that

$$K(\Omega_A, R) \cap A = \phi \tag{4}$$

Remark 2.1 Note that (4) implies global avoidance; tnat is, satisfaction of (4) implies

$$K[(\Delta - A) \times R, R] \cap A = \phi \ .$$

Remark 2.2 Avoidance set A can be any set containing anti-target T . In specific cases it may be convenient, or indeed necessary, to select A different from T ; e.g., see Sec. 6.

3. Avoidance Strategies

The following theorem embodies __sufficient__ conditions for avoidance of a given set A .

Theorem 3.1 A given set A is avoidable if there exist an Ω_A and two functions, a strategy $p^1(\cdot) \in u_1$ and a C^1 function $V(\cdot) : \bar{\Omega}_A \to R$, such that for all $(x, t) \in \Omega_A$

(i) $V(x, t) > V(x^1, t_1) \quad \forall x^1 \in \partial A \ , \quad \forall t_1 \geq t$

and $\forall u^1 \in p^1(x, t)$

(ii) $\dfrac{\partial V(x, t)}{\partial t} + \nabla_x V(x, t) f(x, t, u^1, u^2) \geq 0 \quad \forall u^2 \in U_2$

where $\tilde{p}^1(\cdot) = p^1(\cdot) \big|_{\Omega_A}$.

Proof Suppose that for some $(x_0, t_0) \in \Omega_A$ there is a $t_2 > t_0$ such that $K(x_0, t_0, t_2) \cap A \neq \phi$. Then there is a $t_1 \in (t_0, t_2]$ and an $x^1 \in K(x_0, t_0, t_1) \cap \partial A$ such that

$$V(x_0, t_0) > V(x^1, t_1)$$

by (i). By (ii), however, $V(x, t)$ is nondecreasing along every solution of (1); e.g., see Ref. 11.

Theorem 3.1 has an immediate corollary. Let

$$H(x, t, u^1, u^2) \triangleq \frac{\partial V(x, t)}{\partial t} + \nabla_x V(x, t) f(x, t, u^1, u^2)$$

Then we have at once

Corollary 3.1 Given $(x, t) \in \Omega_A$, if there is $(\tilde{u}^1, \tilde{u}^2) \in U_1 \times U_2$ such that

$$H(x, t, \tilde{u}^1, \tilde{u}^2) = \max_{u^1 \in U_1} \min_{u^2 \in U_2} H(x, t, u^1, u^2) \tag{5}$$

and

$$H(x, t, \tilde{u}^1, \tilde{u}^2) \geqslant 0 \tag{6}$$

then condition (ii) of Theorem 3.1 is met, with $\tilde{u}^1 \in \tilde{p}^1(x, t)$ provided the resulting $p^1(\cdot) \in U^1$; that is, it may be possible to deduce an avoidance strategy from (6).

Proof The corollary follows directly from

$$H(x, t, \tilde{u}^1, \tilde{u}^2) \leqslant H(x, t, \tilde{u}^1, u^2) \quad \forall u^2 \in U_2$$

Remark 3.1 Note that an avoidance strategy need be known only on Ω_A ; that is, only $\tilde{p}^1(\cdot)$, not $p^1(\cdot)$, is required.

Remark 3.2 As is usually the case with conditions involving Lyapunov-type functions, the most difficult part of applying these conditions is the determination of V-functions. In the next section we address this problem for a special case.

Remark 3.3 Theorem 3.1 is related to Theorem 6.2 of Ref. 7.

4. Linear Systems

Consider a linear system for which eq. (3) is

$$\dot{x}(t) = A x(t) + B u^1 + C u^2 \tag{7}$$

where $x \in \Delta \subset R^n$, $u^i \in U_i \subset R^{d_i}$, $i = 1, 2$, and A, B, C are constant matrices of appropriate dimensions.

Now suppose that matrix $-A$ is stable. Let Q be a negative definite $n \times n$ matrix, and consider the Lyapunov equation

$$P A + A' P + Q = 0 \tag{8}$$

Then matrix P is positive definite (Ref. 12). We choose $V(\cdot)$ such that

$$V(x, t) = x' P x \tag{9}$$

If the avoidance set

$$A = \{x \mid x' P x \leq constant\} \tag{10}$$

that is, if it is a ball in R^n, then condition (i) of Theorem 3.1 is satisfied.[5]

Furthermore, if there is a matrix D such that $C = BD$, and if

$$U_i \overset{\Delta}{=} \{u^i \mid \| u^i \| \leq \rho_i = constant > 0\} \quad i = 1, 2 \tag{11}[6]$$

with $\rho_1 \geq \| D \| \rho_2$, then condition (ii) of Theorem 3.1 is met and the corresponding avoidance strategy is given by

$$\tilde{p}^1(x, t) = \frac{B'Px}{\|B'Px\|} \rho_1 \quad \forall (x, t) \notin N \times R \tag{12}$$

where

$$N \overset{\Delta}{=} \{x \mid B'Px = 0\} .$$

For $(x, t) \in N \times R$, $\tilde{p}^1(x, t)$ may take on any admissible value.

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[5] That is, anti-target, T, must belong to a ball containing $\{0\}$.

[6] Here, $\| \cdot \|$ denotes Euclidean norm.

The satisfaction of (i) follows at once, whereas that of (ii) is readily seen by considering

$$H(x, t, u^1, u^2) = 2x'P [Ax + Bu^1 + Cu^2]$$

$$= x'(PA + A'P) x + 2x'PBu^1 + 2x'PBDu^2$$

$$= -x'Qx + 2[x'PBu^1 + x'PBDu^2]$$

$$\geqslant -x'Qx + 2[x'PBu^1 - \|B'Px\| \|D\| \rho_2] .$$

Remark 4.1 If $-A$ is not stable but $(-A, -B)$ is stabilizable so that there exists a matrix E such that $-A -BE$ is stable, then (12) is replaced by

$$\tilde{p}^1(x, t) = E x + \frac{B'Px}{\|B'Px\|} \|D\| \rho_2 \tag{13}$$

and for this strategy to be admissible, $U_1 = U_1(x,t)$ must be such that $\tilde{p}^1(x, t) \in U_1$ for all $(x, t) \in \Omega_A$.

5. Example - Linear System

Consider the linear system with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad B = C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and with

$$U_2 = \{u^2 \mid |u^2| \leqslant 1\}$$

Avoidance set A and constraint set U_1 will be specified subsequently.

Since $-A$ is not stable but $(-A, -B)$ is stabilizable, we determine first the linear part of $\tilde{p}^1(\cdot)$. It is readily deduced to be Ex where, for instance,

$$E = [-1 \quad 1]$$

Then, if

$$Q = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the solution of (8), with A replaced by $A + BE$, is

$$P = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

so that the nonlinear part of $\tilde{p}^1(\cdot)$ is $\text{sgn}(-\frac{1}{2}x_1 + x_2)$. Thus, provided A is a ball

$$A = \{x \mid x'Px \leq a = \text{constant} > 0\}$$

an avoidance control is given by

$$\tilde{p}^1(x, t) = -x_1 + x_2 + \text{sgn}(-\frac{1}{2}x_1 + x_2)$$

for all

$$(x, t) \notin N \times R = \{(x, t) \mid x_1 = 2x_2\} \ .$$

Constraint set U_1 depends on the choice of safety zone Δ_A. For instance, we may choose

$$\Delta_\epsilon = \{x \mid x'Px \leq a + \epsilon, \ \epsilon > 0\}$$

so that U_1 must be such that $\tilde{p}^1(x, t) \in U_1$ for all $x \in \Delta_A = \Delta_\epsilon - A$, $t \in R$.

6. Evasion of Pure Pursuit

Here we consider the problem of evasion from a slower pursuer whose strategy is one of pure pursuit, that is, such that the pursuer's velocity is directed along the line-of-sight (see Figure 2).

Figure 2, Pure Pursuit

For constant pursuer and evader speeds, v_p and v_E, the kinematic equations are

$$\dot{r} = v_E \cos \phi - v_p$$

$$r\dot{\theta} = v_E \sin \phi$$

while the evader's normal acceleration is

$$a_n = v_E (\dot{\theta} + \dot{\phi})$$

Let

$$k \triangleq \frac{v_p}{v_E} \in (0, 1)$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleq \begin{pmatrix} r \\ \phi \end{pmatrix} , \quad u^1 = u \triangleq a_n$$

Then

$$\dot{x}_1 = v_E (\cos x_2 - k)$$

$$\dot{x}_2 = \frac{u}{v_E} - \frac{v_E}{x_1} \sin x_2$$

With control constraint

$$U_1 = U = \{ u \mid |u| \leq \bar{u} = \text{constant} > 0 \}$$

Note that here the pursuer's strategy -- that is, his normal acceleration -- is fully specified. In other words, U_2 is a singlet.

Now suppose that evader E wishes to assure that the distance r from pursuer P remain greater than a specified length a; that is, anti-target

$$T = \{ x \mid x_1 \leq a = \text{constant} > 0 \}$$

In choosing avoidance set $A \supset T$, we are guided by the following considerations. When

$$\dot{x}_1 = \dot{x}_{1\,max} = v_E - v_P > 0 \qquad (\cos x_2 = 1)$$

P can be allowed to approach to within the minimum distance of approach, a . However, when

$$\dot{x}_1 = \dot{x}_{1\,min} = -v_E - v_P < 0 \qquad (\cos x_2 = -1)$$

P must be kept farther away to give E sufficient time to evade. Thus if[7]

$$\Delta = \{ x \mid x_1 \in R_+ \,, \quad |x_2| \leq \pi \}$$

we are led to an avoidance set of the form[8] (see Figure 3)

$$A = \{ x \mid x_1 \leq a + \pi + \delta - \sqrt{(\pi + \delta)^2 - x_2^2} \,, \quad \delta > 0 \}$$

Figure 3, Avoidance Set

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[7] We shall see subsequently that $|x_2| \leq \pi$ is assured.

[8] Of course, there are other possibilities; for example, instead of a circular arc one can choose a parabolic one for ∂A .

Upon applying Corrollarry 3.1, one obtains the expected avoidance control

$$u = \bar{u} \quad \text{if} \quad x_2 < 0$$

$$u = -\bar{u} \quad \text{if} \quad x_2 > 0$$

For $x_2 = 0$, any admissible control may be used; clearly, $u = 0$ is the reasonable one.

Condition (6) of Corollary 3.1 is met for all $x \in \Delta_A$ if

$$\bar{u} \geqslant \frac{v_E^2(k - \cos x_2)\ \sqrt{(\pi \div \delta)^2 - x_2^2}}{|x_2|} \qquad \forall\ x \in \Delta_A$$

Since $x_2 \in [-\pi,\ \pi]$, a conservative bound is given by

$$\bar{u} \geqslant \frac{v_E^2(k + 1)\ \sqrt{(\pi + \delta)^2 - (\cos^{-1} \cdot)^2}}{\cos^{-1} k}$$

Note that the larger δ the closer P may approach E when $\dot{x}_1 = \dot{x}_{1\ min}$, and hence the larger the required control $\bar{u}$. Also, as k increases so does the lower bound of $\bar{u}$.

The treatment utilized above may be applied to the problem of planar evasion from an _active_ pursuer. This problem is treated by the method of barrier trajectory construction in Ref. 5 , and by the method of this paper in Ref. 13 .

List of Captions

Figure 1, Anti-Target, Avoidance Set and Safety Zone
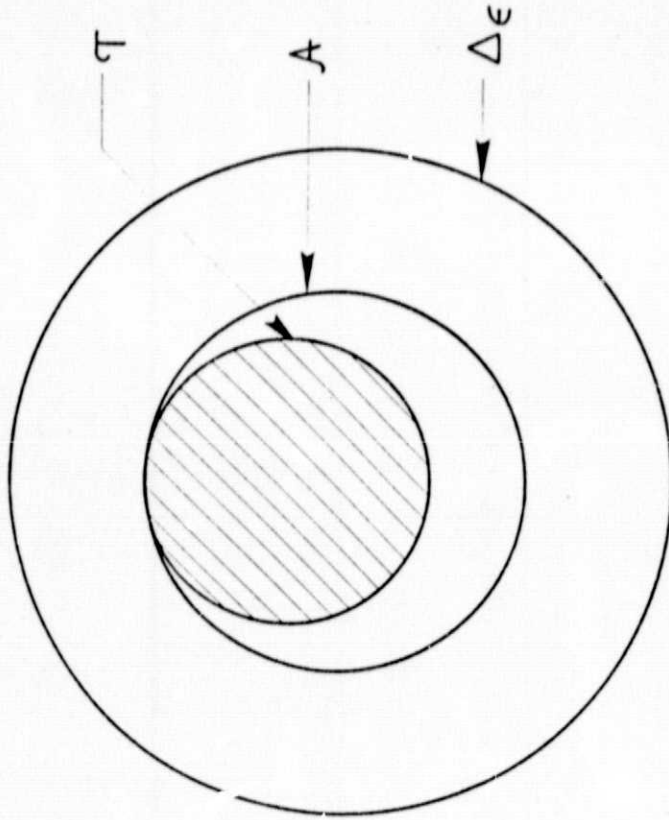
Figure 2, Pure Pursuit

Figure 3, Avoidance Set

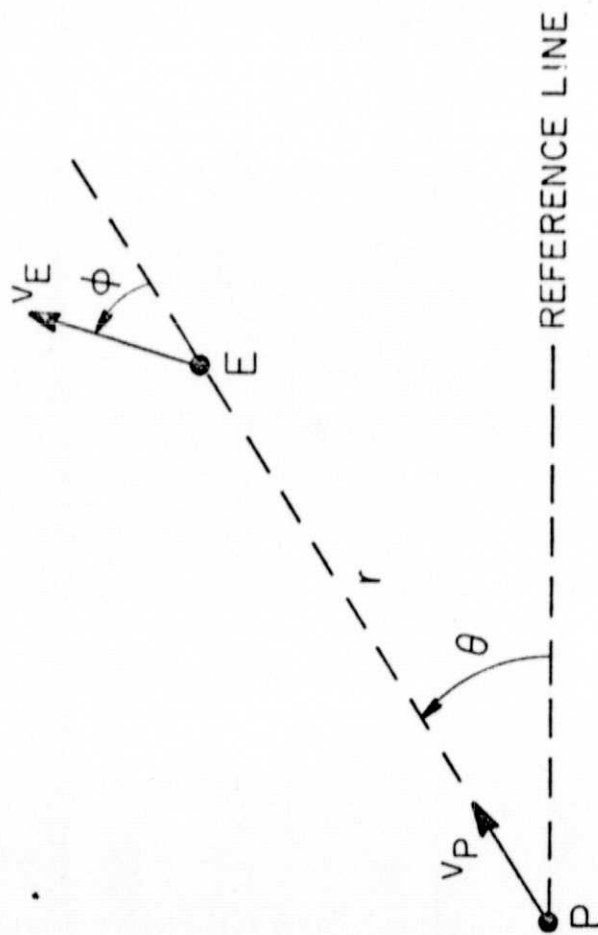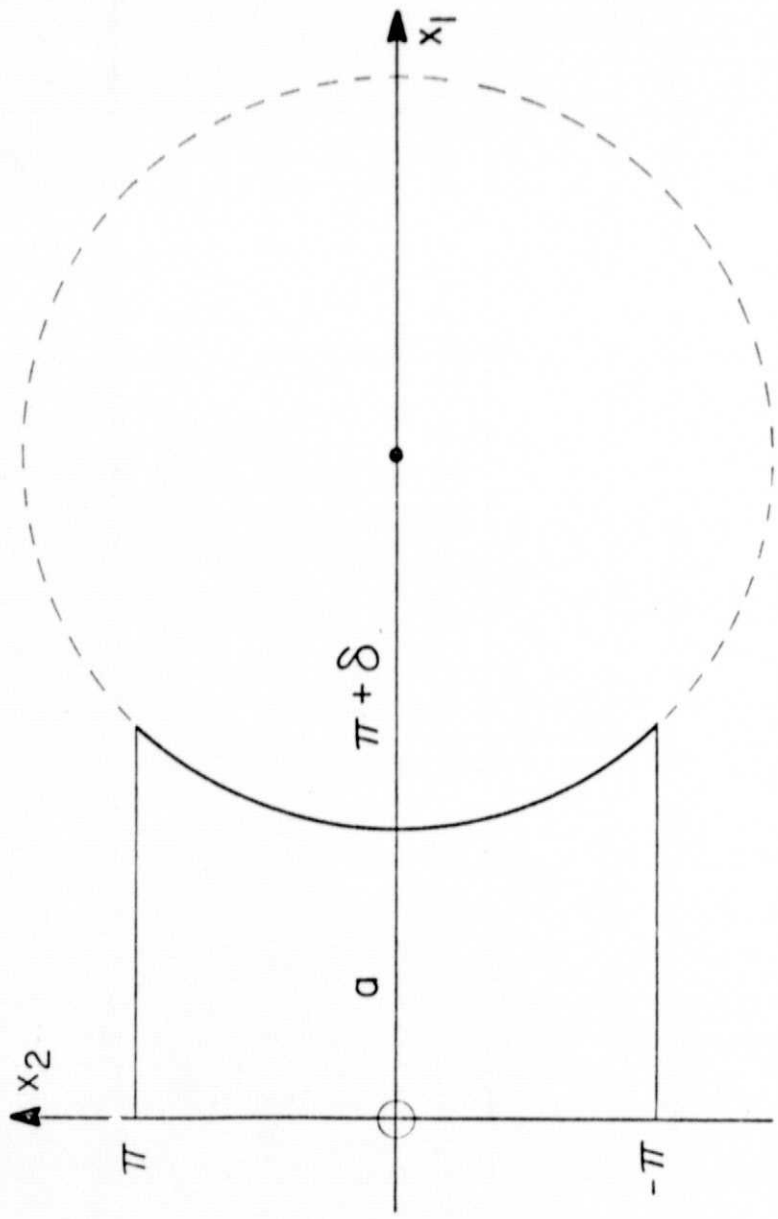FIG. I, ANTI-TARGET, AVOIDANCE SET AND SAFETY ZONE

FIG. 2, PURE PURSUIT

FIG. 3, AVOIDANCE SET