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Evasion in the Plane

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Abstract

We consider dynamical systems subject to control by two agents one of whom desires that no trajectory of the system emanating from outside a given set, intersects that set no matter what the admissible actions of the other agent. Constructive conditions sufficient to yield a feedback control for the agent seeking avoidance were given earlier. These are employed here to deduce an evader control for the planar pursuit-evasion problem with bounded normal accelerations.

1. Introduction

A problem of collision avoidance arises whenever two, or more, objects move in space. Here we consider the case of two objects moving in the same plane, e.g., two ships. One object (evader) is capable of determining the relative position and velocity of the other object (pursuer). The pursuer may be active or passive; that is, he may desire collision, or he may be unable to measure the evader's relative position and velocity and thereby cause collision through inadvertence. Each controls his motion by means of his normal acceleration whose values are constrained. The evader desires to maneuver so as to avoid collision no matter what the actions of the pursuer.

The problem outlined above belongs to the following class of problems. There is given a dynamical system subject to control by two agents, one of whom desires that no trajectory of the system, emanating from outside a prescribed set, intersects that set no matter what the admissible actions of the other agent. Such problems have been discussed in Refs. 1-9, among others. There the treatment is within the framework of differential games, Refs. 6 and 7, either as games of kind (qualitative games) or games of degree (quantitative games). In the former approach, the players seek a saddlepoint for time of collision or for miss-distance, Refs. 1-3 and 8, 9,and in the latter,barriers are sought which separate regions in which collision can be brought about from regions in which avoidance can be assured, Refs. 4 and 5. These techniques usually require numerical integration. Furthermore, only <u>necessary</u> conditions are employed so that avoidance cannot be assured. In Ref. 10 we propose an alternative approach, namely, the <u>constructive</u> utilization of conditions <u>sufficient</u> to guarantee avoidance. Before discussing the planar avoidance problem in Sec. 3, we state the general avoidance problem and the results of Ref. 10. The more general case in which each player has his own target on which he desires termination is treated in Ref. 11.

2. General Problem Statement and Results

Let

$$p^i(\cdot) : R^n \times R \to \text{the nonempty subsets of } R^{d_i}, \quad i = 1, 2$$

be feedback controls (strategies) belonging to given classes of possibly set-valued functions, $\mathcal{U}_i$, with control values u^i ranging in prescribed sets, U_i (which may depend on state and time); that is, given $(x, t) \in R^n \times R$

$$u^i \in p^i(x, t) \subseteq U_i \subseteq R^{d_i}, \quad i = 1, 2 \quad .$$

Let

$$f(\cdot) : R^n \times R \times R^{d_1} \times R^{d_2} \to R^n$$

be a prescribed function, and for given $p^i(\cdot) \in \mathcal{U}_i$, $i = 1, 2$, define a set-valued function $F(\cdot)$ by

$$F(x, t) \triangleq \{z \in R^n \mid z = f(x, t, u^1, u^2), u^i \in p^i(x, t)\}$$
$$= f(x, t, p^1(x, t), p^2(x, t)).$$

Then a dynamical system, e.g. Refs. 12 and 13, is defined by the relation

$$\dot{x} \in F(x, t) . \tag{1}$$

Given $(x_0, t_0) \in \Delta$, where Δ is an open set (or the closure of an open set) in $R^n \times R$, solutions of (1) are absolutely continuous functions on intervals of R

$$x(\cdot) : [t_0, t_1] \to R^n \quad , \quad x(t_0) = x_0 \tag{2}$$

such that

$$\dot{x}(t) \in f(x(t),\ t,\ p^1(x(t),\ t),\ p^2(x(t),\ t)) \tag{2}$$

a.e. $[t_o,\ t_1]$.

Now let there be given an <u>anti-target</u>, T , in Δ , that is a given set into which no solution of (1) must enter for some $p^1(\cdot) \in U_1$ and all $p^2(\cdot) \in U_2$. Consider a closed subset, A , of Δ such that $A \supset T$ and consider also the closure, Δ_ε , of an open subset of Δ such that

$$\Delta_\varepsilon \supset A \quad \text{and} \quad \partial\Delta_\varepsilon \cap \partial A \cap \text{int}\ \Delta = \phi\ .$$

We call A the <u>avoidance set</u> and

$$\Delta_A \triangleq \Delta_\varepsilon \setminus A$$

the <u>safety zone</u>. If a solution avoids A then it cannot enter T , and if a strategy $p^1(\cdot)$ is used in Δ_A that guarantees avoidance of A for all $p^2(\cdot)$, then a solution originating outside of A cannot reach A .

For given $p^1(\cdot) \in U_1$, let K denote the set of all trajectories of (1) for all $(x_o,\ t_o) \in \Delta_A \times R$ and all $p^2(\cdot) \in U_2$. Then, given system (1) and sets U_1 and U_2 , a prescribed set A is <u>avoidable</u> if there is a $p^1(\cdot) \in U_1$ and $\Delta_A \times R \neq \phi$ such that

$$K \cap A = \phi\ . \tag{3}$$

Note that (3) implies global avoidance, that is, avoidance for all $(x_o,\ t_o) \in (\Delta \setminus A) \times R$. Avoidance set A may be any set containing anti-target T ; often it is different from T .

The following theorem and corollary are proved in Ref. 10.

<u>Theorem</u> A given set A is avoidable if there exist a nonempty set Δ_A and two functions, a strategy $p^1(\cdot) \in U_1$ and a C^1 function $V(\cdot) : S \to R$, $S(\text{open}) \supset \overline{\Delta_A \times R}$, such that for all $(x,\ t) \in \Delta_A \times R$

(i) $V(x, t) > V(x', t') \quad \forall\, x' \in \partial\Lambda\ , \quad \forall\, t' \geqslant t,$

and $\forall\, u^1 \in \tilde{p}^1(x, t)$

(ii) $\dfrac{\partial V(x, t)}{\partial t} + \nabla_x V(x, t)\ f(x, t, u^1, u^2) \geqslant 0$

$\forall\, u^2 \in U_2$, where $\tilde{p}^1(\cdot)$ is the restriction of $p^1(\cdot)$ to $\Delta_\Lambda \times R$.

Let

$$H(x, t, u^1, u^2) \triangleq \frac{\partial V(x, t)}{\partial t} + \nabla_x V(x, t)\ f(x, t, u^1, u^2)\ .$$

Then the Theorem has a

<u>Corollary</u> Given $(x, t) \in \Delta_\Lambda \times R$, if there is a $(\tilde{u}^1, \tilde{u}^2) \in U_1 \times U_2$ such that

(i) $H(x, t, \tilde{u}^1, \tilde{u}^2) = \max\limits_{u^1 \in U_1}\ \min\limits_{u^2 \in U_2}\ H(x, t, u^1, u^2)$

and

(ii) $H(x, t, \tilde{u}^1, \tilde{u}^2) \geqslant 0$

then condition (ii) of the Theorem is met. Furthermore, $\tilde{u}^1 \in \tilde{p}^1(x, t)$, provided the resulting $p^1(\cdot) \in \mathcal{U}_1$.

Note that the Corollary is constructive in that it may permit construction of $\tilde{p}^1(\cdot)$.[3]

3. <u>Avoidance in the Plane</u>

Consider two agents, called pursuer P and evader E , moving in a plane. Let $\vec{v}_P$ and $\vec{v}_E$ be the velocities (relative to an inertial reference frame) of P and E , respectively. We suppose that their speeds, $v_P = |\vec{v}_P|$ and $v_E = |\vec{v}_E|$, are constants and that $v_E > v_P$.

Referring to Figure 1, the kinematic equations of motion are[4]

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[3]Usually, $\tilde{u}^1 = \tilde{p}^1(x, t)$ a.e., that is, except on discontinuity manifolds.

[4]Note that here, unlike in Ref. 5, position is relative to the pursuer.

$$\dot{r} = v_E \cos\beta - v_P \sin\theta$$

$$\dot{\theta} = \frac{1}{r}(v_E \sin\beta - v_P \cos\theta) - u_P \tag{4}$$

$$\dot{\beta} = \frac{-1}{r}(v_E \sin\beta - v_P \cos\theta) + u_E$$

where

$$u_E \triangleq \dot{\theta}_E \quad , \quad u_P \triangleq \dot{\theta}_P$$

are the controls of E and P , respectively; that is, E and P control their motions by means of their normal acceleration components. These are constrained; namely,

$$| u_E | \leqslant \bar{u}_E \text{ (given)} \quad , \quad | u_P | \leqslant \bar{u}_P \text{ (given)} \quad . \tag{5}$$

Evader, E , wishes to avoid having pursuer, P , approach more closely than a given distance $\bar{r}$; that is, the anti-target

$$T = \{(r, \theta, \beta) \in \Delta \mid r \leqslant \bar{r}\} \tag{6}$$

with[4]

$$\Delta = \{(r, \theta, \beta) \mid r \in R_+ \ , \ \theta \in R \quad , \quad | \beta | \leqslant \pi\} \ ,$$

There arises now the question of selecting an avoidance set, A . To allow E maneuverability, one wants r "sufficiently" large when $\dot{r} = \dot{r}_{min}$, but when $\dot{r} = \dot{r}_{max}$ one can allow $r = \bar{r}$, where[5]

$$\dot{r}_{min} = - v_E - v_P \quad (\theta = \frac{\pi}{2} \pm 2n\pi \ , \ \beta = \pm \pi)$$

$$\dot{r}_{max} = v_E + v_P \quad (\theta = - \frac{\pi}{2} \pm 2n\pi \ , \ \beta = 0)$$

This is accomplished, for instance, by

$$A = \{(r, \theta, \beta) \in \Delta \mid r - \bar{r} \leqslant b(1 + \sin\theta) + c\beta^2\} \tag{7}$$

for given constants $b \geqslant 0$, $c > 0$.

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[4] As will be seen subsequently, it suffices to consider $| \beta | \leqslant \pi$.

[5] Here, n is any integer.

To satisfy condition (i) of the Theorem we choose $V(\cdot)$ such that

$$V(r, \theta, \beta, t) = r - \bar{r} - b(1 + \sin\theta) - c\beta^2 . \tag{8}$$

To apply condition (ii) of the Theorem we form

$$\begin{aligned} H(r, \theta, \beta, t, u_E, u_P) = {} & (v_E \cos\beta - v_P \sin\theta) \\ & - b \cos\theta \left[\frac{1}{r} (v_E \sin\beta - v_P \cos\theta) - u_P \right] \\ & - 2c\beta \left[\frac{1}{r} (v_P \cos\theta - v_E \sin\beta) + u_E \right] , \end{aligned} \tag{9}$$

First we check to see what is required to assure $H \geq 0$ for all u_P, satisfying constraints (5), and for all $(r, \theta, \beta) \in \Delta_A$.[6] At $\beta = 0$

$$\begin{aligned} H &= (v_E - v_P \sin\theta) + \frac{b\, v_P}{r} \cos^2\theta + b\, u_P \cos\theta \\ &\geq v_E - v_P \sin\theta + b\, u_P \cos\theta \geq v_E - \sqrt{v_P^2 + b^2 u_P^2} . \end{aligned}$$

Thus, to satisfy condition (ii) of the Theorem we make the conservative choice

$$b < \frac{\sqrt{v_E^2 - v_P^2}}{\bar{u}_P} \tag{10}$$

Next we impose the conditions of the Corollary. Since u_E and u_P are separated in H, it follows readily that

$$\begin{aligned} \tilde{u}_E &= -\bar{u}_E & &\text{for } \beta > 0 \\ \tilde{u}_E &= \bar{u}_E & &\text{for } \beta < 0 \\ \tilde{u}_E &\in [-\bar{u}_E, \bar{u}_E] & &\text{for } \beta = 0 \end{aligned} \tag{11}$$

and

[6] Δ_A has not been defined yet; it depends on Δ_ε. For instance, one might let $\Delta_c = \{(r, \theta, \beta) \in \Delta \mid r \leq \bar{r} + b(1 + \sin\theta) + c\beta^2 + \varepsilon ,\ \ \varepsilon = \text{constant} > 0\}$.

$$\tilde{u}_p = \bar{u}_p \qquad \text{for} \quad \cos\theta < 0$$

$$\tilde{u}_p = -\bar{u}_p \qquad \text{for} \quad \cos\theta > 0 \tag{12}$$

$$\tilde{u}_p \in [-\bar{u}_p, \bar{u}_p] \quad \text{for} \quad \cos\theta = 0 \quad .$$

Now we investigate conditions on $\bar{u}_E$ which assure satisfaction of (ii) of the Corollary; namely, in view of (11) and (12), for all $(r, \theta, \beta) \in \Delta_A$ and all (u_E, u_p) satisfying (5)

$$\begin{aligned} \min_{u_p} \max_{u_E} H = {} & v_E \cos\beta - v_p \sin\theta + \frac{b}{r}(v_p \cos\theta - v_E \sin\beta)\cos\theta \\ & + \frac{2c\beta}{r}(v_E \sin\beta - v_p \cos\theta) \\ & + 2\,c\,|\beta|\,\bar{u}_E - b\,|\cos\theta|\,\bar{u}_p \geq 0 \quad . \end{aligned} \tag{13}$$

To obtain a conservative estimate for the required value of $\bar{u}_E$ we rewrite (13) as

$$\begin{aligned} \bar{u}_E \geq \max_{(r,\,\theta,\,\beta) \in \Delta_A} \frac{1}{2\,c\,|\beta|} [\, & v_p \sin\theta - v_E \cos\beta \\ & + \frac{b}{r}(v_E \sin\beta - v_p \cos\theta)\cos\theta \\ & + \frac{2\,c\,\beta}{r}(v_p \cos\theta - v_E \sin\beta) + b\,|\cos\theta|\,\bar{u}_p \end{aligned}$$

whence

$$\begin{aligned} \bar{u}_E \geq \max_{\beta \in [0,\,\pi]} [\, \frac{v_p}{\bar{r}} + \frac{1}{2\,c\,\beta} (& \sqrt{v_p^{\,2} + b^2\,\bar{u}_p^{\,2}} \\ & - v_E \cos\beta + \frac{b\,v_E}{\bar{r}} \sin\beta)] \quad . \end{aligned} \tag{14}$$

Letting

$$g(\beta) \triangleq \sqrt{v_p^{\,2} + b^2\,\bar{u}_p^{\,2}} - v_E \cos\beta + \frac{b\,v_E}{\bar{r}} \sin\beta$$

$$h(\beta) \triangleq \frac{1}{2\,c\,\beta}$$

we rewrite (14) as

$$\bar{u}_E \geq \frac{v_P}{\bar{r}} + \max_{\beta \in [0,\, \pi]} g(\beta)\, h(\beta) \quad . \tag{15}$$

An even more conservative bound is then found by replacing

$$\max_{\beta \in [0,\, \pi]} g(\beta)\, h(\beta)$$

by

$$\max g(\beta) \quad \max h(\beta) \quad \text{for} \quad \beta \in \{\beta \in [0,\, \pi] \mid g(\beta) \geq 0\} \quad .$$

In this connection we utilize condition (10) whence

$$g(0) = -\, v_E + \sqrt{v_P^{\,2} + b^2\, \bar{u}_P^{\,2}} < 0$$

so that $\hat{\beta} \in (0,\, \pi]$ for $g(\hat{\beta}) = 0$. We arrive at the very conservative bound

$$\bar{u}_E \geq \frac{v_P}{\bar{r}} + \frac{1}{2c} \left[v_E \sqrt{1 + (b/\bar{r})^2} + \sqrt{v_P^{\,2} + b^2\, \bar{u}_P^{\,2}} \right] \left[\cos^{-1} \frac{1}{v_E} \sqrt{\frac{v_P^{\,2} + b^2\, \bar{u}_P^{\,2}}{1 + (b/\bar{r})^2}} - \cos^{-1} \frac{1}{\sqrt{1 + (b/\bar{r})^2}} \right]^{-1} . \tag{16}$$

To reiterate, given pursuer and evader speeds v_P and v_E , respectively, pursuer control bound $\bar{u}_P$, missdistance $\bar{r}$, and constants $b \geq 0$, $c > 0$, with b subject to (10), the use of evader control (11) with $\bar{u}_E$ satisfying (16) guarantees collision avoidance;[7] of course, the evader needs to implement such a control only on Δ_A .

As assumed in the definition of Δ , $\beta \in [-\pi,\, \pi]$ since

$$\dot{\beta}\big|_{\beta = \pm \pi} = \frac{v_P \cos\theta}{r} \mp \bar{u}_E$$

and by (16)

$$\bar{u}_E \geq \frac{v_P}{\bar{r}} \; , \qquad \bar{r} < r$$

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[7] Provided, of course, U_1 admits piecewise continuous functions.

whence it follows that

$$\dot{\beta}|_{\beta = \pi} < 0 \quad , \quad \dot{\beta}|_{\beta = -\pi} > 0 \quad .$$

To illustrate the aforegoing results consider

$$v_E = 300 \text{ m/s} \quad , \quad v_P = 225 \text{ m/s}$$

$$\bar{u}_P = 1 \text{ rad/s} \quad , \quad \bar{r} = 3000 \text{ m}$$

Then (10) becomes

$$b \in [0,\ 198.4)$$

For example, with $c = 6 \times 10^4$ and

(i) $b = 100$, $\bar{u}_E \geqslant 0.083$

(ii) $b = 10$, $\bar{u}_E \geqslant 0.081$

(iii) $b = 1$, $\bar{u}_E \geqslant 0.081$

Finally, we can draw these conclusions:

(i) The bound on $\bar{u}_E$ given by (16) is quite insensitive to changes in the value of b , and it can be decreased by increasing the value of c (that is, by increasing the size of the avoidance set Λ).

(ii) For given β , the contour of Λ is "nearly" circular (more so at $\beta = \pi$ than at $\beta = 0$).

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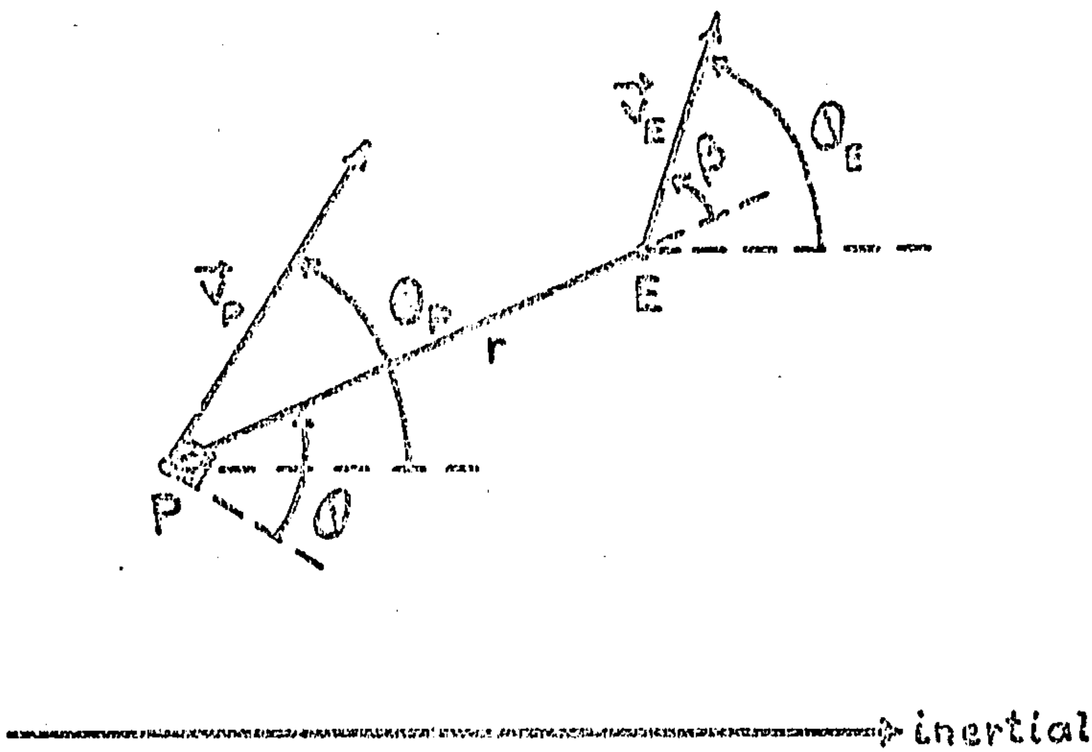

Figure 1, Coordinate System