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ANALYTICAL AND COMPUTATIONAL MATHEMATICS, INC.

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A SINGULARITY FREE
ANALYTICAL SOLUTION
OF ARTIFICIAL SATELLITE MOTION
WITH DRAG

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A Singularity Free Analytical Solution Of Artificial Satellite Motion With Drag

by

G. Scheifele, A. Mueller and S. Starke

GENERAL

This report is broken down into three parts.

PART I (by G. Scheifele) gives the connection between the existing Delaunay-Similar (DS) and Poincaré-Similar (PS) satellite theories in the *true anomaly* version for the J_2 perturbation and the new drag approach. It also gives an overall description of the concept of the approach. The necessary expansions and the procedure to arrive at the computer program for the canonical forces is then outlined in detail.

PART II (by A. Mueller) describes the procedure for the analytical integration of the equations developed in PART I. In addition, some numerical results are given.

PART III (by S. Starke) describes and documents the computer program for the algebraic multiplication of the fourier series which creates the FORTRAN coding in an automatic manner.

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PART I

THE EQUATIONS OF MOTION FOR THE DRAG PROBLEM AND THEIR EXPANSION INTO FOURIER SERIES

PART I

THE EQUATIONS OF MOTION FOR THE DRAG PROBLEM AND THEIR EXPANSION INTO FOURIER SERIES

by

G. Scheifele

1. INTRODUCTION

The *objective* of the theory described in this report is to arrive at a *fully analytical theory* for the motion of an artificial satellite which is perturbed by the J_2 term of the zonal geopotential expansion and by a drag force which is tangential to the orbit and proportional to the square of the velocity magnitude. The density function

$$C(x_1, x_2, x_3, t) \qquad (I.1)$$

does not have to be specified for developing the theory at hand. This is probably one of the most significant features of the theory. It is achieved by postponing the process of creating the final computer source - language coding, which is FORTRAN - compilable, to the very last stage. In this last stage (step No. 2 below) this coding for the final expressions is created in an automatic manner by a minicomputer. The corresponding program is written in BASIC language which allows easy alphanumeric string manipulation. This program may also be executed on a large computer if a BASIC compiler is available (as for instance on the UNIVAC 1110).

The two-body *elements* used are of Poincaré-type (PS-elements in the true anomaly). These elements have been derived from the DS (Delaunay-Similar) elements which were first presented in Reference 1. The corresponding *regular* set of elements (PS-elements) has been derived in Reference 2, and the corresponding differential equations for the perturbed motion are given in Reference 3.

The drag theory is built on top of the *J_2-theory in PS elements* which is the subject of References 9 and 10. Only the first order short period and secular perturbation of the J_2-theory have been taken into account. The long period effects of J_2 and of the higher harmonics have been omitted because they are of the same order of magnitude as the uncertainities in the drag force and density model. Some coupling of drag and J_2 is obtained implicitly because the total energy is used as a canonical variable, and is evaluated for the drag forces.

In general, the theory described here is very similar to the one carried out by Lane (Reference 7) and Lane and Cranford (Reference 8) except for the fact that those approaches are based on the classical Brouwer-Hori approach (Reference 6) which uses classical elements (time as independent variable), while here the new PS formalism of perturbed two-body motion is used (true anomaly as the independent variable). In addition, the density model used here can be different from the ones used by Lane and by Brouwer-Hori and is not a fixed input to the theory.

No reference is made to noncanonical approaches to the solution of the drag problem. The author considers these noncanonical approaches to be not adequate because they do not make use of the powerful tools which are typically provided by all hamilton mechanics approaches to orbital mechanics problems.

Canonical treatment of the drag effects is done by using the transformation rules for the "canonical forces" as outlined in References 4 and 5. In order to allow an analytical integration of these canonical forces they must be expanded into fourier series with respect to the true longitude σ_1 which is one of the canonical angle variables in the PS-theory. These expansions are carried out in two steps:

1st Step. Manual computation of the fourier expansions of individual expressions which are not too complicated to expand. These are for instance the positive and negative powers of r (as they may occur in (I.1)), expansions of expressions of the type r^2v, r^2v^3 and of the derivatives of the cartesian coordinates and of the time with respect to the PS elements. These expansions are carried out in such a way that terms of order magnitude $O(e^4)$ (e=eccentricity) are maintained. Subsequently each of these expansions has been tested out on a minicomputer to verify the expected convergence properties.

2nd Step. On a WANG 2200 minicomputer a program was established which automatically discards higher order terms, carries out the multiplication logic of a product of up to four fourier series, and then produces a FORTRAN compilable alphanumeric output. This "custom tailored" algebraic processor is described and documented in PART III of this report.

This two step procedure has the advantages that, first the errors are avoided which are created by manually multiplying fourier series, secondly the theory is flexible in the sense that step number two can be reproduced at any later time to either include a different model for the density and drag force or to increase or reduce the number of terms accounted for in the expansions which were carried out in the first step.

The resulting theory is of relatively concise form and the resulting FORTRAN program compiles on the UNIVAC 1110 EXEC 8 system and it is executable in the interactive mode.

Some *numerical results* are given in PART II, where the analytical integration procedure is outlined explicitly.

We can *conclude* that the use of the new Poincaré-Similar elements in the true anomaly version is very well suited for developing a concise near earth satellite theory that accounts for a suitable density model. An attempt will be made to further reduce the coding of the resulting FORTRAN program and to incorporate some accurate density models.

2. THE BASIC SET OF PS ELEMENTS USED

The true anomaly DS elements are transformed into the singularity free PS elements by the following canonical transformation

$$\begin{array}{ll} \text{(momentas)} & \text{(coordinates)} \\ \sigma_5 = \rho_1 = \Phi & \sigma_1 = \phi+g+h \\ \sigma_6 = \rho_2 = \sqrt{2(\Phi-G)}\cos(g+h) & \sigma_2 = -\sqrt{2(\Phi-G)}\sin(g+h) \\ \sigma_7 = \rho_3 = \sqrt{2(G-H)}\cos h & \sigma_3 = -\sqrt{2(G-H)}\sin h \\ \sigma_8 = \rho_4 = L & \sigma_4 = \ell \end{array} \tag{I.2}$$

The DS-elements are interpreted as follows:

- G: total angular momentum
- H: x_3-component of angular momentum
- L: total energy
- Φ: related to two-body energy

canonically conjugated elements

- g: argument of perigee
- h: argument of ascending node
- ℓ: time element
- ϕ: canonical true anomaly

With q being

$$q = G - \frac{1}{2}\Phi + \frac{\mu}{2\sqrt{2L}} = -\frac{1}{2}(\sigma_2^2+\rho_2^2) + \frac{1}{2}\rho_1 + \frac{\mu}{2\sqrt{2\rho_4}} \tag{I.3}$$

the transformation from the time t to the new independent variable τ reads

$$\frac{dt}{d\tau} = \frac{r^2}{q} \tag{I.4}$$

The hamiltonian for the perturbed two-body motion is then

$$F = \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} + \frac{r^2}{q} V \quad , \tag{I.5}$$

(V=perturbing potential)
and the initial conditions must be chosen such that F will initially vanish.

From Reference 2 we can record the following abbreviations and expressions which will be used later.

$$G - H = \frac{1}{2}(\sigma_3^2+\rho_3^2) \tag{I.6}$$

$$G = \rho_1 - \frac{1}{2}(\sigma_2^2+\rho_2^2) \tag{I.7}$$

$$\frac{H}{G} = \cos I \tag{I.8}$$

$$e^2 = \frac{L}{\mu^2}\left[(\frac{2\mu}{\sqrt{2L}} - \frac{1}{2}(\sigma_2^2+\rho_2^2)\right](\sigma_2^2+\rho_2^2) \tag{I.9}$$

$$e = \sqrt{1 - \frac{2L}{\mu}p} \tag{I.10}$$

$$p = \frac{1}{\mu}\left[- \frac{1}{2}(\sigma_2^2+\rho_2^2) + \frac{\mu}{\sqrt{2L}}\right]^2 \tag{I.11}$$

$$Q = \frac{\sqrt{L}}{\mu}\sqrt{\frac{2\mu}{\sqrt{2L}} - \frac{1}{2}(\sigma_2^2+\rho_2^2)} \tag{I.12}$$

$$Z_1 = \rho_2 \cos\sigma_1 - \sigma_2 \sin\sigma_1 \tag{I.13}$$

$$Z_2 = \rho_2 \sin\sigma_1 + \sigma_2 \cos\sigma_1 \tag{I.14}$$

$$\sqrt{1-e^2} = 1 - \frac{\sqrt{2L}}{2\mu}(\sigma_2^2+\rho_2^2) \tag{I.15}$$

$$e \sin\phi = Q\, Z_2 \tag{I.16}$$

$$e \cos\phi = Q\, Z_1 \tag{I.17}$$

$$R = \sigma_3 \cos\sigma_1 + \rho_3 \sin\sigma_1 \tag{I.18}$$

By using the above relations the transformations from cartesian coordinates and velocity components to PS variables can be written as follows:

$$x_1 = -\frac{r}{2G}\sigma_3 R + r\cos\sigma_1 \tag{I.19}$$

$$x_2 = -\frac{r}{2G}\rho_3 R + r\sin\sigma_1 \tag{I.20}$$

$$x_3 = \frac{r}{\sqrt{G}}\sqrt{1-\frac{1}{4G}(\sigma_3^2+\rho_3^2)}\; R = \frac{rR}{2G}\sqrt{2(G+H)} \tag{I.21}$$

$$\dot{x}_1 = \dot{r}\,(\cos\sigma_1 - \frac{1}{2G}\sigma_3 R) - r(\frac{G}{r^2}\sin\sigma_1 + \frac{\sigma_3}{2G}\dot{R}) \tag{I.23}$$

$$\dot{x}_2 = \dot{r}\,(\sin\sigma_1 - \frac{1}{2G}\rho_3 R) + r(\frac{G}{r^2}\cos\sigma_1 - \frac{\rho_3}{2G}\dot{R})$$

$$\dot{x}_3 = (\dot{r}\, R + r\,\dot{R})\,\frac{1}{\sqrt{G}}\sqrt{1-\frac{1}{4G}(\sigma_3^2+\rho_3^2)} = \frac{(rR)^{\cdot}}{2G}\sqrt{2(G+H)} \tag{I.24}$$

where r, $\dot{r}$ and $\dot{R}$ are given by

$$\frac{p}{r} = 1 + Q\,(\rho_2\cos\sigma_1 - \sigma_2\sin\sigma_1) \tag{I.25}$$

$$\dot{r} = \frac{Q}{p}\,(V\,\frac{r^2}{q} + G)\,(\sigma_2\cos\sigma_1 + \rho_2\sin\sigma_1) \tag{I.26}$$

$$\dot{R} = (\rho_3\cos\sigma_1 - \sigma_3\sin\sigma_1)\,\frac{G}{r^2} \tag{I.27}$$

The last transformation equation that needs to be given here is the one for the transformation of the physical time t :

$$t = \sigma_4 + \frac{\mu}{(2L)^{3/2}}(E-\phi-\frac{r}{p}\sqrt{1-e^2}\,e\sin\phi) \tag{I.28}$$

with

$$E - \phi = -2 \operatorname{arctg} \frac{e \sin\phi}{1+\sqrt{1-e^2}+e \cos\phi} \qquad (I.29)$$

and where $\sqrt{1-e^2}$, $e \sin\phi$, $e \cos\phi$ are taken from (I.15), (I.16), and (I.17).

3. TRANSFORMATION OF THE CANONICAL FORCES

The formalism of transforming additional forces which can not be derived from a potential function under canonical variable transformations has first been given by Brouwer and Hori (Reference 6). It has been generalized to include canonical systems which are based on independent variables different from time in Reference 4. A thorough and exhaustive description of this extension can be found in Reference 5 (Chapter VIII).

Let us go back to the original formulation of the perturbed two-body problem in cartesian coordinates where the canonical equations of motion read

$$\frac{dx_k}{dt} = \frac{\partial F^c}{\partial p_k} - X_k \quad , \qquad (I.30)$$

$$(k=1,2,3,4)$$

$$\frac{dp_k}{dt} = -\frac{\partial F^c}{\partial x_k} + P_k \qquad (I.31)$$

F^c is the hamiltonian of the perturbed two-body motion in the extended phase space:

$$F^c = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{\mu}{r} + V + p_4 \qquad (I.32)$$

where μ is the gravitational parameter, V is the perturbing potential of the J_2 part of the geopotential expansion and p_4 is the negative total energy:

$$p_4 = -\left[\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{\mu}{r} + V\right] \quad , \qquad (I.33)$$

$$V = \varepsilon \frac{1}{r^3}\left[(\frac{x_3}{r})^2 - \frac{1}{3}\right] \quad , \qquad (I.34)$$

$$\varepsilon = \frac{3}{2} J_2 \mu R_\oplus^2 \quad . \qquad (I.35)$$

($R_\oplus$: mean radius of earth).

The canonical forces of the x-type, X_0, X_1, X_2, X_3, are absent at this initial stage and the canonical forces P_1, P_2, P_3 are the original forces given by the drag:

$$P_j = C(x_k,t)v\, v_j \quad , \quad (j=1,2,3) \qquad (I.36)$$

$C(x_k,t)$ is the product of the density with the ballistic number of the body. In this report, C is restricted to be a function of r only, namely the Laurent series given in (I.1).

According to the rules given in References 4 and 5, a canonical force P_4 must be introduced which is defined by

$$P_4 = \sum_{i=1}^{3} (\frac{\partial F^c}{\partial x_i} X_i - \frac{\partial F^c}{\partial p_i} P_i) \qquad (I.37)$$

The left hand part of this expression is zero and for the right hand part we obtain

$$P_4 = - p_1P_1 - p_2P_2 - p_3P_3 = C\, v^3 \qquad (I.38)$$

(observe that (I.30) and (I.31) imply $p_k = v_k$).

At this point we are ready to transform the *independent variable* from t to τ as posted in (I.4). Again, according to References 4 and 5 we can proceed as follows:

$$F = \frac{r^2}{q} F^c \qquad (I.39)$$

$$\tilde{P}_k = \frac{r^2}{q} p_k , \quad \tilde{X}_k = 0 , \quad (k=1,2,3,4). \tag{I.40}$$

In the next step we will give the transformation of the canonical forces under the canonical transformation which leads from the cartesian variables to the PS-elements. The transformation scheme can be described by

$$\begin{aligned} x_k &\to \sigma_k , & p_k &\to \rho_k \\ \tilde{X}_k &\to T_k , & \tilde{P}_k &\to U_k , \end{aligned} \tag{I.41}$$

and the explicit formulas are

$$T_j = \sum_{k=1}^{4} \left(\tilde{X}_k \frac{\partial p_k}{\partial \rho_j} + \tilde{P}_k \frac{\partial x_k}{\partial \rho_j} \right) , \tag{I.42}$$

$$(j=1,2,3,4)$$

$$U_j = \sum_{k=1}^{4} \left(\tilde{X}_k \frac{\partial p_k}{\partial \sigma_j} + \tilde{P}_k \frac{\partial x_k}{\partial \sigma_j} \right) \tag{I.43}$$

They reduce to

$$T_j = \sum_{k=1}^{4} \tilde{P}_k \frac{\partial x_k}{\partial \rho_j} , \quad U_j = \sum_{k=1}^{4} \tilde{P}_k \frac{\partial x_k}{\partial \sigma_j} , \tag{I.44}$$

and by inserting the expressions (I.40) and (I.36) for $\tilde{P}_k$ we obtain

$$T_j = \frac{r^2}{q} C(r) \left(v \sum_{k=1}^{3} v_k \frac{\partial x_k}{\partial \rho_j} - v^3 \frac{\partial t}{\partial \rho_j} \right) \tag{I.45}$$

$$(j=1,2,3,4)$$

$$U_j = \frac{r^2}{q} C(r) \left(v \sum_{k=1}^{3} v_k \frac{\partial x_k}{\partial \sigma_j} - v^3 \frac{\partial t}{\partial \sigma_j} \right) , \tag{I.46}$$

(Observe the relation $x_4 = t$).

These equations form the basis for all of the subsequent theory which will deal with the explicit expressions and with the expansions of T_j and U_j .

The final form of the equations of motion is

$$\frac{d\sigma_k}{d\tau} = \frac{\partial F}{\partial \rho_k} - T_k$$

$$(k=1,2,3,4) \qquad (I.47)$$

$$\frac{d\rho_k}{d\tau} = -\frac{\partial F}{\partial \sigma_k} + U_k$$

For the case $T_k = U_k = 0$ (no drag) t - equations (I.47) have been solved. This is the J_2-perturbed solution of the motion of an artificial satellite described in References 10 and 9. The corresponding equations have been programmed and tested out (Program PSANS). This J_2-solution is not discussed here and we will proceed to the expansion of the canonical forces T_1, T_2, T_3, T_4, U_1, U_2, U_3, U_4 into fourier series.

4. FOURIER EXPANSION OF THE CANONICAL FORCES

To enable an integration by quadrature of the canonical system (I.47), we have to expand the canonical foces T_k and U_k into fourier series with respect to the true longitude σ_1. The process to arrive at this expansion is rather lengthy, and we will only outline the major steps and give the results. No intermediate calculations will be given here.

Throughout the expansions, terms of order magnitude $O(e^4)$ will be maintained. In some cases where derivatives of the expanded expressions will be needed, the terms had to be expanded to order magnitude $O(e^5)$.

Let us list some additional abbreviations for quantities which occur very often:

$$s^2 = \frac{1}{2}(\sigma_2^2 + \rho_2^2) = O(e^2) \tag{I.48}$$

$$\beta^2 = \frac{\sqrt{2\rho_4}}{\mu} \quad , \qquad \beta = \frac{(2\rho_4)^{1/4}}{\sqrt{\mu}} \tag{I.49}$$

$$\zeta_1 = \beta Z_1 = \beta(\rho_2 \cos\sigma_1 - \sigma_2 \sin\sigma_1) = O(e) \tag{I.50}$$

$$\zeta_2 = \beta Z_2 = \beta(\rho_2 \sin\sigma_1 + \sigma_2 \cos\sigma_1) = O(e) \tag{I.51}$$

$$\eta^2 = s^2\beta^2 = O(e^2) \tag{I.53}$$

The quantities ζ_1, ζ_2 and η are dimensionless (their physical dimension is of unity).

For the velocity magnitude v the expression

$$\rho_4 = -\frac{1}{2}v^2 + \frac{\mu}{r} - V \quad , \tag{I.53}$$

which stems from $F=0$ is used, giving

$$v^2 = 2(-\rho_4 + \frac{\mu}{r} - V) \tag{I.54}$$

The expression for r in PS variables is

$$r = \frac{p}{1+QZ_1} \tag{I.55}$$

v and r , as well as products of mixed powers of v and r will be expanded later.

Let us first concentrate on the terms

$$\sum_{k=1}^{3} v_k \frac{\partial x_k}{\partial \rho_j} \quad , \quad \sum_{k=1}^{3} v_k \frac{\partial x_k}{\partial \sigma_j} \qquad (j=1,2,3,4) \tag{I.56}$$

which occur in (I.45) and (I.46). After having inserted the PS elements into the expressions (I.56) and by making extensive use of the formulas (I.19) to (I.27), a considerable number of cancellations will occur. The result is an astonishingly simple final result for the above eight expressions. The explicit algebra to arrive at these simplified expressions takes several pages of hand-computation even if presented in a compressed form. These derivations are not presented here, we should like, however, to point out that the resulting formulas have been tested for correctness with a computer program that compares the original expressions with the ones obtained after the lengthy calculations. By taking into account the results of these algebraic manipulations we arrive at a new form of the canonical equations, which will serve as a starting basis for the fourier expansions:

$$T_j = \frac{r^2}{q} C(\vec{x}) \left\{ v \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\sigma_3 \\ 0 \end{bmatrix} + v \frac{\partial r}{\partial \rho_j} \left[\frac{QZ_2}{\beta^2 p}(1-s^2\beta^2) \right] - v^3 \frac{\partial t}{\partial \rho_j} \right\}$$

$$(j=1,2,3,4) \qquad \text{(I.57)}$$

$$U_j = \frac{r^2}{q} C(\vec{x}) \left\{ v \begin{bmatrix} G \\ 0 \\ \frac{1}{2}\rho_3 \\ 0 \end{bmatrix} + v \frac{\partial r}{\partial \sigma_j} \left[\frac{QZ_2}{\beta^2 p}(1-s^2\beta^2) \right] - v^3 \frac{\partial t}{\partial \sigma_j} \right\}$$

The expressions which need to be expanded into fourier series are now

$$r^2 v,\ r^2 v^3,\ C(\vec{x}),\ \frac{\partial r}{\partial \sigma_j},\ \frac{\partial r}{\partial \rho_j},\ \frac{\partial t}{\partial \sigma_j},\ \frac{\partial t}{\partial \rho_j} \qquad \text{(I.58)}$$

We will now proceed to these expansions. Expanding r and its powers is straightforward. As an example of these binomial expansions we will give the one for r :

$$r = p(1-Z_1Q + Z_1^2Q^2 - Z_1^3Q^3 + Z_1^4Q^4 - Z_1^5Q^5) + O(e^6) \qquad (I.59)$$

The expansions for Q are found by binomial expansion too, resulting in

$$Q = \beta\sqrt{1-\frac{s^2\beta^2}{2}} = \beta(1 - \frac{1}{4}s^2\beta^2 - \frac{1}{32}s^4\beta^4) + O(e^6) \qquad (I.60)$$

$$Q^2 = \beta^2(1 - \frac{1}{2}s^2\beta^2) \qquad \text{(exact)} \qquad (I.61)$$

$$Q^3 = \beta^3(1 - \frac{3}{4}s^2\beta^2 + \frac{3}{32}s^4\beta^4) + O(e^6) \qquad (I.62)$$

$$Q^4 = \beta^4(1 - s^2\beta^2 + \frac{1}{4}s^4\beta^4) \qquad \text{(exact)} \qquad (I.63)$$

$$Q^5 = \beta^5(1 - \frac{5}{4}s^2\beta^2 + \frac{15}{32}s^4\beta^4) + O(e^6) \qquad (I.64)$$

The fourier expansions of Z_1^k however, are becoming rather complicated since Z_1^4, for instance, will create twelve terms which are powers of σ_2, ρ_2, $\sin\sigma_1$, $\cos\sigma_1$, which each have to be expanded into individual fourier polynomials. For this reason it was decided to expand all the expressions in (I.58) into power series with the general term

$$\beta^\ell \rho_2^n \sigma_2^m Z_1^k Z_2^j \sin\sigma_1 \cos\sigma_1 \quad , \qquad (I.65)$$

and to multiply these power series at the end in an automatic way with each other, establishing a table for the fourier series of expressions of the type (I.65). By observing the relations

$$Z_1^2 = 2s^2 - Z_2^2 \qquad (I.66)$$

$$Z_1^3 = 2s^2Z_1 - Z_1Z_2^2 \qquad (I.67)$$

$$Z_1^4 = 4s^4 - 4s^2Z_2^2 + Z_2^4 \quad , \qquad (I.68)$$

the highest power of Z_2 which will occur can be restricted to $j=1$.

If these general guidelines are followed, the expressions for the quantities in (I.58) can be expanded with respect to the eccentricity to obtain power series with the general term (I.65). Where these expansions are more or less straightforward no intermediate steps are given. Here again, the formula manipulation to arrive at these expressions is generally very tedious and lengthy. For this reason, the resulting expressions have been programmed and their expected convergence properties have been tested out numerically by comparing with the exact expressions (I.65). The dimensionless quantities η^2, ζ_1 and ζ_2 are used.

Expression for $r^2 v$:

$$r^2 v = p^2 \sqrt{2\rho_4}\left[(1+\eta^2)^2 - \frac{1}{4}\zeta_1(4+15\eta^2) + \frac{1}{4}\zeta_1^2(2+27\eta^2) + \frac{1}{2}\zeta_1^3 - \frac{17}{8}\zeta_1^4 - \frac{V}{2\rho_4}(1-3\zeta_1)\right] + O(e^5) + O(V\cdot e^2) \tag{I.69}$$

Some remarks are in order concerning the last term in this expression. By evaluating the drag on a J_2 orbit the order of magnitude of the potential V will be of order magnitude J_2, i.e. 10^{-3}. Since we are only taking into account terms of order magnitude $O(e^4)$, and we are limiting the eccentricity to the interval $0<e<0.1$, terms which have V as a common factor need only to be expanded to order magnitude $O(e)$, thus giving four significant digits in the limit case $e=0.1$. The expression for V can be obtained from the condition that the hamiltonian must vanish in extended phase space:

$$\frac{V}{\rho^4} = 1 + 2Z_1\ Q - \rho_1^2\ \beta^4 - 2Z_1\ Q\ \rho_1^2\ \beta^4\ . \tag{I.70}$$

Because this expression contains information beyond the third digit only, it can be simplified by using

$$Q = \beta \sqrt{1 - \frac{1}{2}\eta^2} = \beta + O(e^2), \qquad (I.71)$$

i.e. replacing Q by β, resulting in

$$\frac{V}{\rho_4} = 1 - \rho_1^2 \beta^4 + 2\zeta_1(1 - \rho_1^2 \beta^4) + O(e^2) \quad . \qquad (I.72)$$

By observing in addition

$$\sqrt{2\rho_4} = \beta^2 \mu \qquad (I.73)$$

and neglecting $O(e^5)$ terms we obtain the final expression:

$$r^2 v = \frac{1}{8} p^2 \mu \beta^2 \left\{ 4\left[2(1 + \eta^2)^2 - 1 + \rho_1^2 \beta^4\right] - 2 \zeta_1(2 + 15\eta^2 + 2\rho_1\beta^4) + 2\zeta_1^2(2 + 27\eta^2) + 4\zeta_1^3 - 17\zeta_1^4 \right\} + O(e^5) \qquad (I.74)$$

Expression for $r^2 v^3$:

Similarily, we obtain for $r^2 v^3$:

$$r^2 v^3 = \frac{1}{4}\beta^2 \mu p^2 \rho_4 \left\{ 4\rho_1^2 \beta^4(1 - \rho_1^2\beta^4 + 8\eta^2 + 8\eta^4) + 2\zeta_1\rho_1^2\beta^4(2 + 2\rho_1^2\beta^4 - \eta^2) + 2\zeta_1^2\left[\rho_1^2\beta^4(-4 + 6\eta^2) - 2 - 15\eta^2\right] + 4\zeta_1^3(1 + 2\rho_1^2\beta^4) - 9\zeta_1^4\rho_1^2\beta^4 \right\} + O(e^5) \qquad (I.75)$$

Expressions for $\frac{\partial r}{\partial \rho_j}$ *and* $\frac{\partial r}{\partial \sigma_j}$:

The expressions for the derivatives of r are found by evaluating equation (I.59). This equation was given to the *fifth* order accuracy because taking the partial derivatives will reduce the order by one. Using the expanded versions of the Q - powers, which are given in (I.60) to (I.64) we can write a general expression for $\frac{\partial r}{\partial \rho_j}$:

$$\frac{\partial r}{\partial \rho_j} = \frac{\partial p}{\partial \rho_j}\left[1 - Z_1\ \beta(1-\frac{1}{4}\eta^2) + Z_1^2\ \beta^2(1-\frac{1}{2}\eta^2) - Z_1^3\ \beta^3 + Z_1^4\ \beta^4\right]$$

$$- p\ \frac{\partial Q}{\partial \rho_j}\left[Z_1 - 2Z_1^2\ \beta(1-\frac{1}{4}\eta^2) + 3Z_1^3\ \beta^2 - 4Z_1^4\ \beta^3\right] \qquad (I.76)$$

$$- p\ \frac{\partial Z_1}{\partial \rho_j}\left[\beta(1-\frac{1}{4}\eta^2-\frac{1}{32}\eta^4) - 2Z_1\ \beta^2(1-\frac{1}{2}\eta^2)\right.$$

$$\left. + 3Z_1^2\ \beta^3(1-\frac{3}{4}\eta^2) - 4Z_1^3\ \beta^4 + 5Z_1^4\ \beta^5\right] + O(e\ ) \quad ,$$

$$(j=1,2,3,4)$$

and an equivalent expression holds for $\frac{\partial r}{\partial \sigma_j}$.

The *eight* partial derivatives of β (they are needed for the Q - derivatives), of Q, of p and of Z_1 which occur in the above expression are listed below. Derivatives of these expressions which are not listed are vanishing.

$$\frac{\partial \beta}{\partial \rho_4} = \frac{\beta}{4\rho_4} \qquad (I.77)$$

$$\frac{\partial Q}{\partial \rho_2} \simeq -\ \rho_2\ (\frac{1}{4}\beta^3 + \frac{1}{16}\ s^2\beta^5) \qquad (I.78)$$

$$\frac{\partial Q}{\partial \sigma_2} \simeq -\ \sigma_2\ (\frac{1}{4}\beta^3 + \frac{1}{16}\ s^2\beta^5) \qquad (I.79)$$

$$\frac{\partial Q}{\partial \rho_4} \simeq \frac{\beta}{4L}\ (1-\frac{3}{4}\eta^2-\frac{5}{32}\eta^4) \qquad (I.80)$$

$$\frac{\partial p}{\partial \rho_2} = -\frac{2\rho_2}{\mu\beta^2}(1-\eta^2) \tag{I.81}$$

$$\frac{\partial p}{\partial \sigma_2} = -\frac{2\sigma_2}{\mu\beta^2}(1-\eta^2) \tag{I.82}$$

$$\frac{\partial p}{\partial \rho_4} = -\frac{\mu}{2\rho_4^2}(1-\eta^2) \tag{I.83}$$

$$\frac{\partial Z_1}{\partial \sigma_1} = -Z_2 \tag{I.84}$$

$$\frac{\partial Z_1}{\partial \rho_2} = \cos\sigma_1 \tag{I.85}$$

$$\frac{\partial Z_1}{\partial \sigma_2} = -\sin\sigma_1 \tag{I.86}$$

Inserting these expressions into (I.76) yields, after the truncation of higher order terms and collection of the terms has been carried out, the following expressions:

$$\frac{\partial r}{\partial \rho_1} = 0, \quad \frac{\partial r}{\partial \rho_3} = 0, \quad \frac{\partial r}{\partial \sigma_3} = 0, \quad \frac{\partial r}{\partial \sigma_4} = 0 \tag{I.87}$$

$$\begin{aligned}\frac{\partial r}{\partial \rho_2} = &-\frac{2\rho_2}{\mu\beta^2}\left[1 - \eta^2 - \zeta_1(1-\tfrac{5}{4}\eta^2) + \zeta_1^2(1-\tfrac{3}{2}\eta^2) - \zeta_1^3 + \zeta_1^4\right] \\ &+ p\ \rho_2\beta^2\left[\zeta_1(\tfrac{1}{4} + \tfrac{1}{16}\eta^2) - \tfrac{1}{2}\zeta_1^2 + \tfrac{3}{4}\zeta_1^3 - \zeta_1^4\right] \\ &- p\ \beta\cos\sigma_1\left[1 - \tfrac{1}{4}\eta^2 - \tfrac{1}{32}\eta^4 - 2\zeta_1(1-\tfrac{1}{2}\eta^2) + 3\zeta_1^2(1-\tfrac{3}{4}\eta^2) \right. \\ &\left. - 4\zeta_1^3 + 5\zeta_1^4\right] + O(e^5).\end{aligned} \tag{I.88}$$

$$\frac{\partial r}{\partial \rho_4} = -\frac{1}{2}\frac{\mu}{\rho_4^2}\left[1 - \eta^2 - \zeta_1(1-\frac{5}{4}\eta^2) + \zeta_1^2(1-\frac{3}{2}\eta^2) - \zeta^3 + \zeta^4\right] \tag{I.89}$$

$$- \frac{p}{4\rho_4}\left[\zeta_1(1-\frac{3}{4}\eta^2) - 2\zeta_1^2(1-\eta^2) + 3\zeta_1^3 - 4\zeta_1^4\right] + O(e^5)$$

$$\frac{\partial r}{\partial \sigma_1} = p\ \zeta_2\left[1 - \frac{1}{4}\eta^2 - \frac{1}{32}\eta^4 - 2\zeta_1(1-\frac{1}{4}\eta^2) + 3\zeta_1^2(1-\frac{3}{4}\eta^2)\right. \tag{I.90}$$

$$\left. - 4\zeta_1^3 + 5\zeta_1^4\right] + O(e^5)$$

$$\frac{\partial r}{\partial \sigma_2} = -\frac{\sigma_2}{\mu\beta^2}\left[1 - \eta^2 - \zeta_1(1-\frac{5}{4}\eta^2) + \zeta_1^2(1-\frac{3}{2}\eta^2) - \zeta_1^3 + \zeta_1^4\right]$$

$$+ p\ \sigma_2\beta^2\left[\zeta_1(\frac{1}{4} + \frac{1}{16}\eta^2) - \frac{1}{2}\zeta^2 + \frac{3}{4}\zeta_1^3 - \zeta_1^4\right] \tag{I.91}$$

$$+ p\ \beta\sin\sigma_1\left[1 - \frac{1}{4}\eta^2 - \frac{1}{32}\eta^4 - 2\zeta_1(1-\frac{1}{2}\eta^2) + 3\zeta_1^2(1-\frac{3}{4}\eta^2)\right.$$

$$\left. - 4\zeta_1^3 + 5\zeta_1^4\right] + O(e^5)$$

Expressions for $\frac{\partial t}{\partial \sigma_k}$ *and* $\frac{\partial t}{\partial \rho_k}$:

Let us first record the time equation (I.28) again, and write it by using the new abbreviations. For this purpose we may first list some basic relations

$$\sqrt{1-e^2} = 1 - \frac{\sqrt{2\rho_4}}{2\mu}(\sigma_2^2+\rho_2^2) = 1 - \eta^2 \quad , \tag{I.92}$$

$$e\sin\sigma_1 = Q\ Z_2 \quad , \quad e\cos\sigma_1 = Q\ Z_1 \tag{I.93}$$

Thus the time equation may now be written in the form

$$t = \sigma_4 + \frac{1}{2\rho_4\beta^2}\left[(E-\phi) - \frac{r}{p}(1-\eta^2)\,Q\,Z_2\right] \quad , \qquad (I.94)$$

where $(E-\phi)$ stands for

$$E - \phi = -2\arctan\frac{Z_2Q}{2-\eta^2+Z_1Q} \quad . \qquad (I.95)$$

Let us introduce the quantity Γ

$$\Gamma = \frac{1}{1+\frac{1}{2}(Z_1Q-\eta^2)} \qquad (I.96)$$

and first expand $E - \phi$. Since this is the difference between the true and the eccentric anomaly it is of the order of the eccentricity. This implies that the argument of the arctan function is small, allowing us to expand it into a power series about the point zero. The result is

$$E - \phi = -Z_2\beta(1-\frac{1}{4}\eta^2)\,\Gamma + \frac{1}{12}Z_2^3\,\beta^3\,\Gamma^3 - \frac{1}{80}Z_2^5\,\beta^5\,\Gamma^5 + O(e^5) \qquad (I.97)$$

In the above and all the subsequent expansions for t we have to be careful to take into account terms up to and including $O(e^5)$, because of the derivatives to be taken later. Inserting (I.97) into the time equation (I.94) yields

$$t = \sigma^4 + \frac{1}{2\rho_4\beta^2}\left[-\zeta_2(1-\frac{1}{4}\eta^2)\,\Gamma + \frac{1}{12}\zeta_2^3\,\Gamma^3 - \frac{1}{80}\zeta_2^5\,\Gamma^5 - \frac{r}{p}(1-\eta^2)\,Z_2Q\right] + O(e^6) \qquad (I.98)$$

The last term $\frac{r}{p}(1-\eta^2)Z_2Q$ is expanded by inserting $\frac{r}{p}$ from (I.59) and then using equations (I.60) to (I.64) for the powers of Q. The result is, after adequate truncations to the order needed:

$$\frac{r}{p}(1-\eta^2)Z_2Q = \zeta_2\Big[1 - \zeta_1(1-\tfrac{3}{2}\eta^2) + \zeta_1^2(1-\tfrac{7}{4}\eta^2) - \zeta_1^3 + \zeta_1^4 - \tfrac{5}{4}\eta^2 + \tfrac{1}{4}\eta^4\Big] + O(e^6) \qquad \text{(I.99)}$$

We may now write down a general expression for the derivatives $\frac{\partial t}{\partial \rho_j}$ which equally applies to the derivatives $\frac{\partial t}{\partial \sigma_j}$:

$$\begin{aligned}
\frac{\partial t}{\partial \rho_j} = \frac{\partial \sigma_4}{\partial \rho_j} - \frac{3\mu}{(2\rho_4)^{5/2}}\Big\{ &- \zeta_2\,\Gamma(1-\tfrac{1}{4}\eta^2) + \tfrac{1}{12}\zeta_2^3\,\Gamma^3 \\
&- \zeta_2\Big[1 - \tfrac{5}{4}\eta^2 - \zeta_1(1-\tfrac{3}{2}\eta^2) + \zeta_1^2 - \zeta_1^3\Big]\Big\} \\
+ \frac{\mu}{(2L)^{3/2}}\Big\{ &\zeta_2\frac{\partial \zeta_1}{\partial \rho_j}\Big[1 - \tfrac{3}{2}\eta^2 - 2\zeta_1(1-\tfrac{7}{4}\eta^2) + 3\zeta_1^2 - 4\zeta_1^3\Big] \\
&- \frac{\partial \zeta_2}{\partial \rho_j}\Big[1 - \tfrac{5}{4}\eta^2 + \tfrac{1}{4}\eta^4 - \zeta_1(1-\tfrac{3}{2}\eta^2) + \zeta_1^2(1-\tfrac{7}{4}\eta^2) - \zeta_1^3 + \zeta_1^4 \\
&+ \Gamma(1-\tfrac{1}{4}\eta^2) - \tfrac{1}{4}\zeta_2^2\,\Gamma^3(1-\tfrac{3}{4}\eta^2) + \tfrac{1}{16}\zeta_2^4\,\Gamma^5\Big] \\
&+ \zeta_2\frac{\partial(\eta^2)}{\partial \rho_j}\left(\tfrac{5}{4} - \tfrac{1}{2}\eta^2 - \tfrac{3}{2}\zeta_1 + \tfrac{7}{4}\zeta_1^2 + \tfrac{1}{4}\Gamma - \tfrac{1}{16}\zeta_2^2\Gamma^3\right) \\
&- \zeta_2\frac{\partial \Gamma}{\partial \rho_j}\Big[1 - \tfrac{1}{4}\eta^2 - \tfrac{1}{4}\zeta_2^2\,\Gamma^2(1-\tfrac{3}{4}\eta^2)\Big]\Big\}
\end{aligned} \qquad \text{(I.100)}$$

$(j=1,2,3,4)$

In the further procedure the powers of ζ_2^n are reduced to $n=1$ by using the relation

$$\zeta_2^2 = 2\,\eta^2 - \zeta_1^2 \tag{I.101}$$

which is equivalent to (I.66). The following expressions need to be inserted in (I.100). The results of their expressions are given below.

$$\Gamma \approx 1 + \frac{1}{2}\,\eta^2 + \frac{1}{4}\,\eta^4 - \frac{1}{2}\,\zeta_1(1+\frac{3}{4}\eta^2+\frac{15}{32}\eta^4) + \frac{1}{4}\,\zeta_1^2(1+\eta^2) - \zeta_1^3(\frac{1}{8}+\frac{5}{32}\eta^2) + \frac{1}{16}\,\zeta_1^4 - \frac{1}{32}\,\zeta_1^5 \tag{I.102}$$

$$\Gamma^2 \approx 1 + \eta^2 + \frac{3}{4}\,\eta^4 - \zeta_1(1+\frac{5}{4}\eta^2) + \zeta_1^2(\frac{3}{4}+\frac{9}{8}\eta^2) - \frac{1}{2}\,\zeta_1^3 + \frac{5}{16}\,\zeta_1^4 \tag{I.103}$$

$$\Gamma^3 \approx 1 + \frac{3}{2}(\eta^2+\eta^4) - \zeta_1(\frac{3}{2}+\frac{21}{8}\eta^2) + \zeta_1^2(\frac{3}{2}+3\eta^2) - \frac{5}{4}\,\zeta_1^3 + \frac{15}{16}\,\zeta_1^4 \tag{I.104}$$

$$\Gamma^4 \approx 1 + 2\,\eta^2 + \frac{5}{2}\,\eta^4 - \zeta_1(2+\frac{9}{2}\eta^2) + \zeta_1^2(\frac{5}{2}+\frac{25}{4}\eta^2) - \frac{5}{2}\,\zeta_1^3 + \frac{35}{16}\,\zeta_1^4 \tag{I.105}$$

$$\zeta_2^4\Gamma^5 \approx 4\,\eta^2 - 4\,\zeta_1^2\,\eta^2 + \zeta_1^4 \tag{I.106}$$

$$\frac{\partial \Gamma}{\partial \rho_j} \approx -\frac{1}{2}\frac{\partial \zeta_1}{\partial \rho_j}\Big[1 + \frac{3}{4}\eta^2 + \frac{15}{32}\eta^4 - \zeta_1(1+\eta^2)$$

$$+ 6\,\zeta_1^2\left(\frac{1}{8} + \frac{5}{32}\eta^2\right) - \frac{1}{2}\zeta_1^3 + \frac{5}{16}\zeta_1^4\Big] \qquad (I.107)$$

$$-\frac{1}{2}\frac{\partial(\eta^2)}{\partial \rho_j}\Big[\zeta_1\left(\frac{3}{4} + \frac{15}{16}\eta^2\right) - \frac{1}{2}\zeta_1^2 + \frac{5}{16}\zeta_1^3 - 1 - \eta^2\Big]$$

Last, the derivatives of ζ_1, ζ_2 and η^2 need to be listed. The ones which vanish are omitted in the list.

$$\frac{\partial \zeta_1}{\partial \rho_1} = -\zeta_2 \;,\quad \frac{\partial \zeta_1}{\partial \rho_2} = \beta\cos\sigma_1 \;,\quad \frac{\partial \zeta_1}{\partial \sigma_2} = -\beta\cos\sigma_1 \qquad (I.108)$$

$$\frac{\partial \zeta_1}{\partial \rho_4} = \frac{\zeta_1}{4\rho_4} \;,\quad \frac{\partial \zeta_2}{\partial \sigma_1} = \zeta_1 \;,\quad \frac{\partial \zeta_2}{\partial \rho_2} = \beta\sin\sigma_1 \qquad (I.109)$$

$$\frac{\partial \zeta_2}{\partial \sigma_2} = \beta\cos\sigma_1 \;,\quad \frac{\partial \zeta_2}{\partial \rho_4} = \frac{\zeta_2}{4\rho_4} \qquad (I.110)$$

$$\frac{\partial(\eta^2)}{\partial \rho_2} = \rho_2\,\beta^2 \;,\quad \frac{\partial(\eta^2)}{\partial \sigma_2} = \sigma_2\,\beta^2 \;,\quad \frac{\partial(\eta^2)}{\partial \rho_4} = \frac{\eta^2}{2\rho_4} \qquad (I.111)$$

At this stage, all the information is provided to insert the necessary quantities into (I.100). However, at this stage too, the individual expansions for the Γ^n and for the derivatives have been tested out numerically before proceeding with the algebra. After a considerable amount of explicit computations the following final expansions for the partial derivatives of the time with respect to the PS variables result.

$$\frac{\partial t}{\partial \sigma_1} = \frac{-1}{4\rho_4\beta^2}\left\{6\ \eta^2(1-\eta^2) + \zeta_1(4-13\eta^2) - \zeta_1^2(6-21\eta^2)\right.$$

$$\left. +8\ \zeta_1^3 - 10\ \zeta^4\right\} + 0(e^5) \qquad \text{(I.112)}$$

$$\frac{\partial t}{\partial \sigma_2} = \frac{-1}{16\rho_4\beta}\left\{\zeta_2\ \sin\sigma_1\left[12\ (1 - \eta^2) - \zeta_1(20-29\eta^2)\right.\right.$$

$$\left. + 28\ \zeta_1^2 - 36\ \zeta_1^3\right]$$

$$+ \cos\sigma_1\left[16 - 12\ \eta^2 + 2\ \eta^4 - 4\ \zeta_1(3-4\eta^2)\right. \qquad \text{(I.113)}$$

$$\left. + \zeta_1^2(12-19\eta^2) - 12\ \zeta_1^3 + 12\ \zeta_1^4\right]$$

$$\left. - \zeta_2\ \sigma_2\ \ \beta\left[8 - 5\ \eta^2 - 10\ \zeta_1 + 12\ \zeta_1^2\right]\right\} + 0(e^5)$$

$$\frac{\partial t}{\partial \sigma_3} = 0 \qquad \text{(I.114)}$$

$$\frac{\partial t}{\partial \sigma_4} = 1 \qquad \text{(I.115)}$$

$$\frac{\partial t}{\partial \rho_1} = 0 \qquad \text{(I.116)}$$

$$\frac{\partial t}{\partial \rho_2} = \frac{1}{16}\frac{1}{\rho_4 \beta}\Bigg\{\zeta_2 \cos\sigma_1 \left[12\ (1-\eta^2) - \zeta_1(20-29\eta^2) + 28\ \zeta_1^2 - 36\ \zeta_1^3\right]$$

$$- \sin\sigma_1 \left[16 - 12\ \eta^2 + 2\ \eta^4 - 4\ \zeta_1(3-4\eta^2)\right.$$

$$\left. + \zeta_1^2\ (12 - 19\ \eta^2) - 12\ \zeta_1^3 + 12\ \zeta_1^4\right] \qquad \text{(I.117)}$$

$$+ \zeta_2\ \rho_2\ \beta \left[8 - 5\ \eta^2 - 10\ \zeta_1 + 12\ \zeta_1^2\right]\Bigg\} + O(e^5)$$

$$\frac{\partial t}{\partial \rho_3} = 0 \qquad \text{(I.118)}$$

$$\frac{\partial t}{\partial \rho_4} = \frac{1}{16}\frac{\zeta_2}{\rho_4^2 \beta^2}\left\{20 - 7\ \eta^2 - \zeta_1\ (12-6\eta^2) + 8\ \zeta_1^2 - 4\ \zeta_1^3\right\}$$

$$+ O(e^5) \qquad \text{(I.119)}$$

Now all the necessary derivatives and expressions are expanded which will be needed to evaluate (I.57) except for the density function C . To test out the theory and the general procedure outline in the next section we will insert a constant density for the preliminary numerical tests. Various expansions of simple representations of the density function are currently being investigated. The general form of the resulting expression for a spherically symmetric density function is an expression which is very similar to the ones for the partial derivatives of r , and only powers of ζ_1 will occur. If latitude and time dependent density functions are considered, the resulting expressions become more complicated, and will be dependent on powers of ζ_2 .

5. AUTOMATIC GENERATION OF THE FINAL EQUATIONS

As it was outlined in the introduction, this theory is of a "dynamic" character. This allows for example a continuous updating of the density model. It is of course, necessary to develop such a function into an expansion in powers of ζ_1, ζ_2, $\sin\sigma_1$, $\cos\sigma_1$ by *explicit manual computation*. Also the expansion must be complete to a certain order of the eccentricity; in our case $O(e^4)$ terms must be included. Then, before the expansion may be used for an input to the automatic algebraic multiplier its expected numerical convergence behavior must be tested out.

Let us here record the expressions (I.57) for the canonical forces

$$T_j = \frac{r^2}{q} C(\vec{x}) \left\{ v \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\sigma_3 \\ 0 \end{bmatrix} + v \frac{\partial r}{\partial \rho_j} \left[\frac{QZ_2}{\beta^2 p}(1-s^2\beta^2) \right] - v^3 \frac{\partial t}{\partial \rho_j} \right\}$$

$$(j=1,2,3,4) \qquad \text{(I.120)}$$

$$U_j = \frac{r^2}{q} C(\vec{x}) \left\{ v \begin{bmatrix} G \\ 0 \\ \frac{1}{2}\rho_3 \\ 0 \end{bmatrix} + v \frac{\partial r}{\partial \sigma_j} \left[\frac{QZ_2}{\beta^2 p}(1-s^2\beta^2) \right] - v^3 \frac{\partial t}{\partial \sigma_j} \right\}$$

and denote the U_j terms by

$$T_{j+4} = U_j \qquad (j=1,2,3,4). \qquad \text{(I.121)}$$

The expressions which were given in the previous section are

$$r^2 v,\ r^2 v^3,\ C(\vec{x}),\ \frac{\partial r}{\partial \sigma_j},\ \frac{\partial r}{\partial \rho_j},\ \frac{\partial t}{\partial \sigma_j},\ \frac{\partial t}{\partial \rho_j} \qquad \text{(I.122)}$$

In addition, the expression for the middle term is

$$u = \frac{QZ_2}{\beta^2 p}(1-s^2\beta^2) \qquad \text{(I.123)}$$

which can be simplified to

$$u = \frac{1}{4\beta^2 p} \zeta_2 \ (4-5\eta^2) + O(e^5) \qquad (I.124)$$

needs to be considered.

By denoting the left hand terms by T_k^{left}, the middle terms by T_k^{mid} and the right hand terms by T_k^{right} we can write the following multiplication table

$$T_k^{left} = (r^2 v)\cdot(C)\cdot\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\sigma_3 \\ 0 \\ G \\ 0 \\ -\frac{1}{2}\rho_3 \\ 0 \end{bmatrix} \qquad (k=1,2,3,4,5,6,7,8) \qquad (I.125)$$

$$T_2^{mid} = (r^2 v)\cdot(\frac{\partial r}{\partial \rho_2})\cdot(u)\cdot(C) \qquad (I.126)$$

$$T_4^{mid} = (r^2 v)\cdot(\frac{\partial r}{\partial \rho_4})\cdot(u)\cdot(C) \qquad (I.127)$$

$$T_5^{mid} = (r^2 v)\cdot(\frac{\partial r}{\partial \sigma_1})\cdot(u)\cdot(C) \qquad (I.128)$$

$$T_6^{mid} = (r^2 v)\cdot(\frac{\partial r}{\partial \sigma_2})\cdot(u)\cdot(C) \qquad (I.129)$$

$$T_2^{right} = (r^2 v^3)\cdot(\frac{\partial t}{\partial \rho_2})\cdot(C) \qquad (I.130)$$

$$T_4^{right} = (r^2 v^3)\cdot(\frac{\partial t}{\partial \rho_4})\cdot(C) \qquad (I.131)$$

$$T_5^{right} = (r^2 v^3)\cdot(\frac{\partial t}{\partial \sigma_1})\cdot(C) \qquad (I.132)$$

$$T_6^{right} = (r^2v^3)\cdot(\frac{\partial t}{\partial\sigma_2})\cdot(C) \tag{I.133}$$

$$T_8^{right} = (r^2v^3)\cdot(C) \tag{I.134}$$

The terms which are not listed here are absent, as can be seen by inspection of the expressions developed in the last section.

It is of interest to note, that due to(I.125), T_3 and T_7 are equal up to a different common factor; T_5 is the only canonical force which gets contributions from all three terms in (I.123).

Of special interest is the force T_1. It is the only vanishing one. The consequence is that the variable σ_1 (true longitude) is *not affected by the drag perturbation.* This is a peculiarity of the PS-theory. Even though this seems strange on a first look, it must be emphasized that it does by no means imply that there would be no in-track effect. This in-track or timing effect of the moving body within the orbit is accounted for by T_4, which determines the motion of the time element σ_4. This interesting observation reflects the fact, that the geometry of the motion is fully separated from the dynamics within the orbit, which is typical for the new canonical variable formulation of the DS and PS theories in the true or eccentric anomaly case.

The ten products displayed in (I.125) to (I.134) are carried out with the automatic algebraic multiplier which is documented in PART III of this report. The task to be performed by this multiplier is twofold:

(1) It will determine if a term is of order magnitude $O(e^5)$ or higher. If this is the case, the term is discarded. Practical experience show that about 85% of the terms occuring in these products will be discarded.

(2) If a non-discardable term is encountered, it is identified, and a pointer is established which refers to one of the elements listed in the "Table" below (T1, T2, ···, T27). The pertinent factors as well as the pointer are then temporarily stored and saved on tape in such a manner that the resulting statements are FORTRAN compilable.

The "TABLE":

A careful inspection of the expressions occuring in (I.125) to (I.134) and of the expansions of their product terms which were given in Section 3 shows that the only terms of order magnitude $O(e^4)$ and smaller are the ones given in the following "TABLE". The expressions for the right hand sides are computed in a separate FORTRAN subroutine and are transferred by means of a common block.

TABLE

order magnitude O(1) terms:

$$T22 = 1$$

$$T26 = \sin\sigma_1$$

$$T27 = \cos\sigma_1$$

order magnitude O(e) terms:

$$\zeta_1 = T1 = \beta(-\sigma_2 \sin\sigma_1 + \rho_2 \cos\sigma_1) \qquad (I.135)$$

$$\zeta_1 \sin\sigma_1 = T2 = \beta(-\frac{1}{2}\sigma_2 + \frac{1}{2}\rho_2 \sin 2\sigma_1 + \frac{1}{2}\sigma_2 \cos 2\sigma_1) \qquad (I.136)$$

$$\zeta_1 \cos\sigma_1 = T3 = \beta(\frac{1}{2}\rho_2 - \frac{1}{2}\sigma_2 \sin 2\sigma_1 + \frac{1}{2}\rho_2 \cos 2\sigma_1) \qquad (I.137)$$

$$\zeta_2 = T23 = \beta(\rho_2 \sin\sigma_1 + \sigma_2 \cos\sigma_1) \qquad (I.138)$$

$$\zeta_2 \sin\sigma_1 = T24 = \beta(\frac{1}{2}\rho_2 + \frac{1}{2}\sigma_2 \sin 2\sigma_1 - \frac{1}{2}\rho_2 \cos 2\sigma_1) \qquad (I.139)$$

$$\zeta_2 \cos\sigma_1 = T25 = \beta(\frac{1}{2}\sigma_2 + \frac{1}{2}\rho_2 \sin 2\sigma_1 + \frac{1}{2}\sigma_2 \cos 2\sigma_1) \qquad (I.140)$$

order magnitude O(e²) terms:

$$\zeta_1 \zeta_2 = T4 = \beta^2\left[\frac{1}{2}(\rho_2^2-\sigma_2^2)\sin 2\sigma_1 + \rho_2\sigma_2 \cos 2\sigma_1\right] \qquad (I.141)$$

$$\zeta_1 \zeta_2 \sin\sigma_1 = T5 = \beta^2\left[-\frac{1}{2}\rho_2\sigma_2 \sin\sigma_1 + \frac{1}{4}(\rho_2^2-\sigma_2^2)\cos\sigma_1 + \frac{1}{2}\rho_2\sigma_2 \sin 3\sigma_1 - \frac{1}{4}(\rho_2^2-\sigma_2^2)\cos 3\sigma_1\right] \qquad (I.142)$$

$$\zeta_1 \zeta_2 \cos\sigma_1 = T6 = \beta^2\left[\frac{1}{4}(\rho_2^2-\sigma_2^2)\sin\sigma_1 + \frac{1}{2}\rho_2\sigma_2 \cos\sigma_1 + \frac{1}{4}(\rho_2^2-\sigma_2^2)\sin 3\sigma_1 + \frac{1}{2}\rho_2\sigma_2 \cos 3\sigma_1\right] \qquad (I.143)$$

$$\zeta_1^2 = T7 = \beta^2\left[\frac{1}{2}(\rho_2^2-\sigma_2^2) - \rho_2\sigma_2 \sin 2\sigma_1 + \frac{1}{2}(\rho_2^2-\sigma_2^2)\cos 2\sigma_1\right] \qquad (I.144)$$

$$\zeta_1^2 \; sin\sigma_1 = T8 = \beta^2 \Big[\frac{1}{4}(\rho_2^2+3\sigma_2^2)\; sin\sigma_1 - \frac{1}{2}\rho_2\sigma_2\; cos\sigma_1 + \frac{1}{4}(\rho_2^2-\sigma_2^2)\; sin3\sigma_1 + \frac{1}{2}\rho_2\sigma_2\; cos3\sigma_1\Big] \quad (I.145)$$

$$\zeta_1^2 \; cos\sigma_1 = T9 = \beta^2 \Big[-\frac{1}{2}\rho_2\sigma_2\; sin\sigma_1 + \frac{1}{4}(3\rho_2^2+\sigma_2^2)\; cos\sigma_1 - \frac{1}{2}\rho_2\sigma_2\; sin3\sigma_1 + \frac{1}{4}(\rho_2^2-\sigma_2^2)\; cos3\sigma_1\Big] \quad (I.146)$$

order magnitude $O(e^3)$ *terms:*

$$\zeta_1^2\,\zeta_2 = T10 = \beta^3 \Big[\frac{1}{4}\rho_2(\rho_2^2+\sigma_2^2)\; sin\sigma_1 + \frac{1}{4}\sigma_2(\rho_2^2+\sigma_2^2)\; cos\sigma_1 + \frac{1}{4}\rho_2(\rho_2^2+3\sigma_2^2)\; sin3\sigma_1 + \frac{1}{4}\sigma_2(3\rho_2^2-\sigma_2^2)\; cos3\sigma_1\Big] \quad (I.147)$$

$$\zeta_1^2\,\zeta_2\; sin\sigma_1 = T11 = \beta^3 \Big[\frac{1}{8}\rho_2(\rho_2^2+\sigma_2^2) - \frac{1}{4}\sigma_2(\rho_2^2-\sigma_2^2)\; sin2\sigma_1 - \frac{1}{2}\rho_2\sigma_2^2\; cos2\sigma_1 + \frac{1}{8}\sigma^2(3\rho_2^2-\sigma_2^2)\; sin4\sigma_1 - \frac{1}{8}\rho_2(\rho_2^2-3\sigma_2^2)\; cos4\sigma_1\Big] \quad (I.148)$$

$$\zeta_1^2\,\zeta_2\; cos\sigma_1 = T12 = \beta^3 \Big[\frac{1}{8}\sigma_2(\rho_2^2+\sigma_2^2) + \frac{1}{4}\rho_2(\rho_2^2-\sigma_2^2)\; sin2\sigma_1 + \frac{1}{2}\rho_2^2\sigma_2\; cos2\sigma_1 + \frac{1}{8}\rho_2(\rho_2^2-3\sigma_2^2)\; sin4\sigma_1 + \frac{1}{8}\sigma_2(3\rho_2^2-\sigma_2^2)\; cos4\sigma_1\Big] \quad (I.149)$$

$$\zeta_1^3 = T13 = \beta^3 \Big[-\frac{3}{4}\sigma_2(\rho_2^2+\sigma_2^2)\; sin\sigma_1 + \frac{3}{4}\rho_2(\rho_2^2+\sigma_2^2)\; cos\sigma_1 - \frac{1}{4}\sigma_2(3\rho_2^2-\sigma_2^2)\; sin3\sigma_1 + \frac{1}{4}\rho_2^2(\rho_2-3\sigma_2^2)\; cos3\sigma_1\Big] \quad (I.150)$$

$$\zeta_1^3\; sin\sigma_1 = T14 = \beta^3 \Big[-\frac{3}{8}\sigma_2(\rho_2^2+\sigma_2^2) + \frac{1}{4}\rho_2(\rho_2^2+3\sigma_2^2)\; sin2\sigma_1 + \frac{1}{2}\sigma_2^3\; cos2\sigma_1 + \frac{1}{8}\rho_2(\rho_2^2-3\sigma_2^2)\; sin4\sigma_1 + \frac{1}{8}\sigma_2(3\rho_2^2-\sigma_2^2)\; cos4\sigma_1\Big] \quad (I.151)$$

$$\zeta_1^3\; cos\sigma_1 = T15 = \beta^3 \Big[\frac{3}{8}\rho_2(\rho_2^2+\sigma_2^2) - \frac{1}{4}\sigma_2(3\rho_2^2+\sigma_2^2)\; sin2\sigma_1 + \frac{1}{2}\rho_2^3\; cos2\sigma_1 - \frac{1}{8}\sigma_2(3\rho_2^2-\sigma_2^2)\; sin4\sigma_1 + \frac{1}{8}\rho_2(\rho_2^2-3\sigma_2^2)\; cos4\sigma_1\Big] \quad (I.152)$$

order magnitude $O(\epsilon^4)$ *terms:*

$$\zeta_1^3\ \zeta_2 = T16 = \beta^4\left\{\frac{1}{4}\ (\rho_2^4-\sigma_2^4)\ sin2\sigma_1 + \frac{1}{2}\ \rho_2\sigma_2(\rho_2^2+\sigma_2^2)\ cos2\sigma_1 + \left[\frac{1}{8}\ (\rho_2^4+\sigma_2^4) - \frac{3}{4}\ \rho_2^2\sigma_2^2\ sin4\sigma_1 + \frac{1}{2}\ \rho_2\sigma_2(\rho_2^2-\sigma_2^2)\ cos4\sigma_1\right\} \quad (I.153)$$

$$\zeta_1^3\ \zeta_2\ sin\sigma_1 = T17 = \beta^4\left\{-\frac{1}{4}\ \rho_2\sigma_2(\rho_2^2+\sigma_2^2)\ sin\sigma_1 + \frac{1}{8}\ (\rho_2^4-\sigma_2^4)\ cos\sigma_1 + \frac{1}{2}\ \rho_2\sigma_2^3\ sin3\sigma_1 - \left[\frac{1}{16}\ (\rho_2^4-3\sigma_2^4) + \frac{3}{8}\ \rho_2^2\sigma_2^2\right] cos3\sigma_1 + \frac{1}{4}\ \rho_2\sigma_2(\rho_2^2-\sigma_2^2)\ sin5\sigma_1 - \left[\frac{1}{16}\ (\rho_2^4+\sigma_2^4) - \frac{3}{8}\ \rho_2^2\sigma_2^2\right] cos5\sigma_1\right\} \quad (I.154)$$

$$\zeta_1^3\ \zeta_2\ cos\sigma_1 = T18 = \beta^4\left\{\frac{1}{8}\ (\rho_2^4-\sigma_2^4)\ sin\sigma_1 + \frac{1}{4}\ \rho_2\sigma_2(\rho_2^2+\sigma_2^2)\ cos\sigma_1 + \left[\frac{1}{16}\ (3\rho_2^4-\sigma_2^4) - \frac{3}{8}\ \rho_2^2\sigma_2^2\right] sin3\sigma_1 + \frac{1}{2}\ \rho_2^3\sigma_2\ cos3\sigma_1 + \left[\frac{1}{16}\ (\rho_2^4+\sigma_2^4) - \frac{3}{8}\ \rho_2^2\sigma_2^2\right] sin5\sigma_1 + \frac{1}{4}\ \rho_2\sigma_2(\rho_2^2-\sigma_2^2)\ cos5\sigma_1\right\} \quad (I.155)$$

$$\zeta_1^4 = T19 = \beta^4\left\{\frac{3}{8}\ (\rho_2^2+\sigma_2^2)^2 - \rho_2\sigma_2(\rho_2^2+\sigma_2^2)\ sin2\sigma_1 + \frac{1}{2}\ (\rho_2^4-\sigma_2^4)\ cos2\sigma^1 - \frac{1}{2}\ \rho_2\sigma_2(\rho_2^2-\sigma_2^2)\ sin4\sigma_1 + \left[\frac{1}{8}\ (\rho_2^4+\sigma_2^4) - \frac{3}{4}\ \rho_2^2\sigma_2^2\right] cos4\sigma_1\right\} \quad (I.156)$$

$$\zeta_1^4\ sin\sigma_1 = T20 = \beta^4\left\{\left[\frac{1}{8}(\rho_2^4+5\sigma_2^4) + \frac{3}{4}\ \rho_2^2\sigma_2^2\right] sin\sigma_1 - \frac{1}{2}\ \rho_2\sigma_2(\rho_2^2+\sigma_2^2)\ cos\sigma_1 + \left[\frac{1}{16}\ (3\rho_2^4-5\sigma_2^4) + \frac{3}{8}\ \rho_2^2\sigma_2^2\right] sin3\sigma_1 + \frac{1}{4}\ \rho_2\sigma_2(\rho_2^2+3\sigma_2^2)\ cos3\sigma_1 + \left[\frac{1}{16}\ (\rho_2^4+\sigma_2^4) - \frac{3}{8}\ \rho_2^2\sigma_2^2\right] sin5\sigma_1 + \frac{1}{4}\ \rho_2\sigma_2\ (\rho_2^2-\sigma_2^2)\ cos5\sigma_1\right\} \quad (I.157)$$

$$\zeta_1^4 \; cos\sigma_1 \qquad = T21 = \beta^4\Big\{-\frac{1}{2}\,\rho_2\sigma_2(\rho_2^2+\sigma_2^2)\; sin\sigma_1 + \Big[\frac{1}{8}\,(5\rho_2^4+\sigma_2^4)$$

$$+\frac{3}{4}\,\rho_2^2\sigma_2^2\Big]cos\sigma_1 - \frac{1}{4}\,\rho_2\sigma_2\,(3\rho_2^2+\sigma_2^2)\; sin3\sigma_1$$

$$+\Big[\frac{1}{16}\,(5\rho_2^4-3\sigma_2^4) - \frac{3}{8}\,\rho_2^2\sigma_2^2\Big]cos3\sigma_1 \qquad (I.158)$$

$$-\frac{1}{4}\,\rho_2\sigma_2(\rho_2^2-\sigma_2^2)\; sin5\sigma_1 + \Big[\frac{1}{16}\,(\rho_2^4+\sigma_2^4)$$

$$-\frac{3}{8}\,\rho_2^2\sigma_2^2\Big]cos5\sigma_1\Big\}$$

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PART II

ANALYTICAL INTEGRATION PROCEDURE

PART II

ANALYTICAL INTEGRATION PROCEDURE

by

A. Mueller

1. INTRODUCTION

The subsequent considerations are concerned with the analytical integration of the equations of motion defined in PART I. These equations have the form

$$\frac{d\sigma_i}{d\tau} = \frac{\partial F}{\partial \rho_i} - T_i \qquad i=1,2,3,4 \qquad \text{(II.1)}$$
$$\frac{d\rho_i}{d\tau} = -\frac{\partial F}{\partial \sigma_i} + T_{i+4}$$

where F is the hamiltonian which includes the two-body and J_2 potential and the drag perturbation is included in the canonical forces T_i . Because of the coupling considerations given in PART I we will assume that the integration may be separated into two integrals.

$$\int \frac{d\sigma_i}{d\tau}\, d\tau = \int \frac{\partial F}{\partial \rho_i}\, d\tau - \int T_i \, d\tau$$
$$\text{(II.2)}$$
$$\int \frac{d\rho_i}{d\tau}\, d\tau = - \int \frac{dF}{\partial \sigma_i}\, d\tau + \int T_{i+4} \, d\tau$$

The integration of the part due to the hamiltonian has already been treated in Reference 9 and we proceed to the integration of the canonical forces.

2. CANONICAL FORCES

As described in PART I, the canonical forces T_i have been expanded about the eccentricity so that these forces may be written as a fourier series in the fast variable σ_1, which in turn will facilitate the integration. However, to analytically integrate the forces, they must be expressed as a function of the independent variable and the initial conditions. As is standard in perturbation theory, this is done by inserting the two-body expressions for the motion of the elements into the equations for the canonical forces. These two-body expressions are obtained from the integration of the canonical differential equations due to the two-body part of the hamiltonian. It is known that this integration is *a canonical transformation in itself*, thus to be consistent, the canonical forces must undergo another transformation.

The transformation in the elements is defined by the following equations

$$\sigma_1(\tau) = \tau + \sigma_1(0)$$

$$\sigma_4(\tau) = \frac{\mu}{(2\rho_4)^{3/2}}\,\tau + \sigma_4(0) \qquad \text{(II.3)}$$

$$\sigma_k(\tau) = \sigma_k(0) \text{ for } k=2,3$$

$$\rho_k(\tau) = \rho_k(0) \text{ for } k=1,2,3,4$$

As defined in Reference 5 the canonical transformation of the forces is given by

$$T_j(\tau,\sigma_i) = \sum_{k=0}^{4}\left[T_k \frac{\partial \rho_k}{\partial \rho_j(\tau)} + T_{k+4}\frac{\partial \sigma_k}{\partial \rho_j(\tau)}\right] \qquad \text{(II.4a)}$$

and the conjugate forces

$$T_{j+4}(\tau,\sigma_i) = \sum_{k=0}^{4} \left[T_k \frac{\partial \rho_k}{\partial \sigma_j(\tau)} + T_{k+4} \frac{\partial \sigma_k}{\partial \sigma_j(\tau)} \right] \quad \text{(II.4b)}$$

Considering (II.3) one finds that the transformation reduces to the following

$$\begin{aligned} T_j(\tau,\sigma_k) &= T_j \quad \text{for} \quad j=1,2,3,5,6,7,8 \\ T_4(\tau,\sigma_k) &= T_4 + \frac{3\mu}{(2\rho_4)^{5/2}} \tau \; T_8 \end{aligned} \quad \text{(II.5)}$$

The canonical forces are now in a form to be easily integrated.

3. INTEGRALS

Let us define the right hand integral of (II.2) as

$$R_i(\tau,\sigma_k) = \int T_i(\tau,\sigma_k)\, d\tau \quad \text{(II.6)}$$

Due to the fact that the canonical forces have been expressed in a fourier series, the indefinite integral R_i may be expressed by a summation of terms which are multiples of the following integrals

$$\int d\tau = \tau$$

$$\int \cos\left[n\sigma_1(\tau)\right] d\tau = + \frac{1}{n} \sin\left[n\sigma_1(\tau)\right] \quad \text{(II.7a)}$$

$$\int \sin\left[n\sigma_1(\tau)\right] d\tau = - \frac{1}{n} \cos\left[n\sigma_1(\tau)\right]$$

$$\rightarrow \int \tau \, d\tau = \frac{\tau^2}{2}$$

$$\rightarrow \int \tau \cos\left[n\sigma_1(\tau)\right] \, d\tau = -\frac{1}{n^2}\left\{n\tau \sin\left[n\sigma_1(\tau)\right] + \cos\left[n\sigma_1(\tau)\right]\right\}$$

$$\rightarrow \int \tau \sin\left[n\sigma_1(\tau)\right] \, d\tau = -\frac{1}{n^2}\left\{n\tau \cos\left[n\sigma_1(\tau)\right] - \sin\left[n\sigma_1(\tau)\right]\right\} \qquad \text{(II.7b)}$$

Forms marked by the arrow appear only in the integral R_4 and are a result of the independent variable appearing in (II.5) as factor.

4. PERTURBATIONS

Partial coupling effects may be introduced by evaluating the integrals with the elements $\sigma_k(\tau)$ which are predicted from the J_2 satellite theory. The procedure by which the integrals are evaluated and the perturbations due to the J_2 and drag are summed requires four steps

1. given $\tau=o$, $\sigma_k(o) \rightarrow$ (II.6) $\rightarrow R_i(o)$

2. given τ , $\sigma_k(o) \rightarrow J_2 \rightarrow \sigma_k(\tau)$

3. given $\sigma_k(\tau)$, $\tau \rightarrow$ (II.6) $\rightarrow R_i(\tau)$

5. given $R_k(\tau)$ and $R_k(o)$ update $\sigma_k(\tau)$, $\rho_k(\tau)$

$$\sigma_k(\tau) = \sigma_k(\tau) - \left[R_k(\tau) - R_k(o)\right]$$

$$\rho_k(\tau) = \rho_k(\tau) + \left[R_{k+4}(\tau) - R_{k+4}(\tau)\right]$$

The algorithm requires no iteration process to determine the value of the true anomaly as a function of both drag and the J_2 perturbations as in Reference 8. This is due to the fact that the anomaly is a canonical element and is only perturbed by the oblateness potential.

5. THE MEAN MOTION

In the integration of (II.6) we considered the energy ρ_4 to be fixed. However the drag force dissipates energy and thus the mean motion is changing with the independent variable. In the expression for $T_4(\tau,\sigma_k)$ in (II.5) the error in the mean motion is multiplied by τ again, so that the resulting error for considering the mean motion fixed is of order $\gamma^2\tau^3$, γ being the relative magnitude of the drag force. This error may not be neglected for large τ.

Let us assume that the energy dissipated by drag is a linear function of the independent variable. For large values of τ this dissipated energy ΔL can be expressed as

$$\Delta L = \left[\frac{R_8(\tau) - R_8(o)}{\tau}\right] \tau = \left[\frac{\Delta L}{\Delta \tau}\right] \tau$$

This is true because the secular term dominates the periodic terms for large τ in $R_8(\tau)$.

Also if the dissipated energy ΔL is small compared to the total energy L then one may express the mean motion term in (II.5) as

$$\frac{3\mu\tau}{(2(L-\Delta L))^{5/2}} = \frac{3\mu}{(2L)^{5/2}} \left(\tau + \frac{5}{2L}\left[\frac{\Delta L}{\Delta \tau}\right] \tau^2\right)$$

This new term introduces integrals of the form

$$\int \tau^2 \, d\tau = \tau^3/3$$

$$\int \tau^2 \cos\left[n\sigma_1(\tau)\right] d\tau = \frac{1}{n^3}\left\{ n^2\tau^2 \sin\left[n\sigma_1(\tau)\right] + 2n\tau \cos\left[n\sigma_1(\tau)\right] - 2 \sin\left[n\sigma_1(\tau)\right]\right\}$$

$$\int \tau^2 \sin\left[n\sigma_1(\tau)\right] d\tau = \frac{1}{n^3}\left\{ -n^2\tau^2 \cos\left[n\sigma_1 \ \tau\right] + 2n\tau \sin\left[n\sigma_1(\tau)\right] + 2 \cos\left[n\sigma_1(\tau)\right]\right\}$$

These terms are premultiplied by factors of order γ^2 . Also it should be noted that the procedure to evaluate the drag perturbations on the solution remains unchanged except for the fact that perturbation of the energy element must be known before the perturbation of the time element may be computed. Also the additional terms due to the mean motion in the perturbation of the time element do not require additional FORTRAN coding to be developed by the algebraic manipulator. From (II.5) one can see that the expression for T_8 may be used to compute the additional terms.

The procedure outlined in this section corresponds to what is commonly known as "the second integration of the mean motion".

6. NUMERICAL EXAMPLES

To demonstrate the accuracy of the expansions and the integration procedure, the analytical solution has been compared to a numerically integrated solution for several test orbits. Both the numerical and analytical techniques assume a drag model of the form

$$|\vec{F}| = \frac{1}{2} C_d \rho \frac{A}{M} v^2$$

where

$$C_d = 2.2$$

$$\frac{M}{A} = 100 \text{ lbs/ft}^2$$

$$\rho = .5\text{x}10^{-9} \text{ kg/m}^3$$

The density ρ , is to be a constant. The value considered is the approximate density of the earth's atmosphere at the very low height of 175 km. The value of the ballistic number, $\frac{M}{A}$, is the average for the Shuttle. Also, both numerical and analytical models neglect the inertial velocity of the earth's atmosphere.

Both the numerical and analytical solutions will include the perturbation due to the oblateness of the earth.

Three different orbit cases (Table 1) with eccentricities of e=0, e=.015, e=.1, were chosen for integration. In Table 2 the position difference between the numerical and analytical solutions are compared after 20 revolutions. Also displayed is the position difference if drag is neglected in the analytical solution.

In all three cases the analytical theory accounts very well for the large perturbation due to drag. The large position differences when drag is neglected is from the "in track" error. The results in Table 2 confirm that the basic analytical formulation is sound.

	a(km)	e	I	ω	Ω	M
Case 1	6678	0.0	0	0	0	20°
Case 2	6678	0.15	30°	0	0	20°
Case 3	7300	0.1	30°	0	0	20°

TABLE 1

Orbit Cases (initial conditions)

Model	e=0	e=.015	e=.1
	Position difference		
Neglect Drag	1481 km	1506 km	1920 km
With Drag	.97 km	1.01 km	2.18 km

TABLE 2

Differences of analytical vs numerical integration

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PART III

THE ALGEBRAIC MULTIPLIER

PART III

THE A .GEBRAIC MULTIPLIER

by

S. Starke

1. INTRODUCTION

The algebraic multiplier[†] was programmed in BASIC on a Wang 2200-T minicomputer (16k bytes). Extensive use was made of the alphanumeric character string manipulation offered by BASIC.

The program documented below has the capability of multiplying up to four fourier series with respect to σ_1 and each individual fourier series may be multiplied by a common factor AK$. The trigonometric terms are stored in a separate alphanumeric array ZK$(J) , and the corresponding factors are given in array elements EK$(J).

The program *input* consists of fourier series, which are stored on tape, and up to ten fourier series can be loaded into core sequentially to allow automatic (buffered) operation of the multiplier without manual interaction.

During *execution* products of the type

Z1$(I)*Z2$(J)*Z3$(K)*Z4$(L)

are searched for terms which are not of higher order. If a low order term is encountered, the program will identify the term by pointing to the "Table" which is stored in memory. Subsequently, ASCII Coding for a FORTRAN compatible statement will be created and saved on an output tape. With a separate telecommunication program, the output tapes are *dumped* to the UNIVAC 1110 computer where they are compiled after some minor modifications have been made (Subroutine headings, dimension statements, common-bloacks).

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† Multiplier and manipulator will be used interchangeably.

For the following description of the program the following reference should be consulted.

"System 2200 A/B Reference Manual,
Wang Laboratories, Inc., 1974".

2. PROGRAM DESCRIPTION

The A variables and arrays (Analysis)

A1 = A1(1) + A2(1) + A3(1) + A4(1) = power of the Z_1 term
A2 = A1(2) + A2(2) + A3(2) + A4(2) = power of the Z_2 term
A3 = A1(3) + A2(3) + A3(3) + A4(3) = power of the ρ_2 term
A4 = A1(4) + A2(4) + A3(4) + A4(4) = power of the σ_2 term

A5 Not used
A6 = A1(6) + A2(6) + A3(6) + A4(6) = power of sin σ_1 term
A7 = A1(7) + A2(7) + A3(7) + A4(7) = power of cos σ_1 term

In all cases the A1(5)→A4(5) terms are the power of the individual term being multiplied. Therefore, if these terms are added we have the power of the resulting product. This is usually the primary test for excluding a term.

C, C1, C2, C3, C4

C1→C4 are the numerical constants associated with each of the four terms with C being the product of C1→C4. The general form of the term being analyzed is

$$C\, Z_1^k\, Z_2\, \rho_2\, \sigma_2\, \sin\sigma_1\, \cos\sigma_1$$

this term is held in the T1$( )→T4$( ) arrays.

T$(27)13 this array holds pointers if a term is present after a particular multiplication. An "*" indicates that a term is present. The first two characters of this array tell if a Z_2 or a $\frac{\sin}{\cos}$ is present. Distinction on the power of Z_1 is made by the entry point and exit point of the table.

The general form of the arrays being manipulated is

A1\${T1\$(1)E1\$(1) + T1\$(2)E1\$(2) + · · · + T1\$(10)E1\$(10)} = FACTOR 1

A2\${T2\$(1)E2\$(1)·····································}

⋮ ⋮

A4\${··} = FACTOR 4

A1$ contains a character string of up to 20 characters.
⋮ They are never changed just carried along.
A4$

T1$(n) contains the terms being multiplied.
⋮
T4$(n) n=1,2,···,10

E1$(n) these terms are considered constants for the
⋮ T1$(n)→T4$(n) arrays. They may have up to 20 char-
⋮ acters. They are never manipulated only carried
E4$(n) along.

N(1)···N(4) this array contains the number of terms in the T1$( ) → T4$( ) and E1$( ) → E4$( ) arrays.

I,J,K,L these are the loop counters for the nested Do-Loop (FOR : NEXT Loops) used by the multiplier. Once in the multiplier section these values must never be changed.

M Loop counter used within the multiplier. This value may be reused as necessary.

R1,R2,R3,R4

these are the bookkeeping variables for determining how many terms were examined, higher order ···, etc.

H;H(9);H$(9):H1$(9,10):H2$(9,10)

this is the density buffer. It can contain 9 terms of the type being multiplied.

H: buffer pointer (H=1→9)

H(H): array holding the (N(4) value)

H$(H): array holding the A4$ values.

H1$(H,M): array holding the T4$(M) values. (M=1→H(H)≤10)

H2$(H,M): array holding the E4$(M) values. (M=1→H(H)≤10)

As can be seen, the density buffer will only load into the factor 4 array.

B;B(3);B1$(3,10);B2$(3,10):B$(3);B1

This is the factor buffer. It will hold up to 3 factors of the type being multiplied.

B	buffer pointer when loading buffer B=1→3
B(B1)	array holding the N(2) values
B1$(B1,M)	array holding the T2$(M) values (M=1→B(B1)≤ 3)
B2$(B1,M)	array holding the E2$(M) values (M=1→B(B1)≤ 3)
B$(B1)	array holding the A2$ value
B1	buffer pointer when unloading buffer (B1=1→3) As can be seen this buffer can only be loaded into the factor 2 array
M1( )	this array holds the extra term generated if a power of Z_2 is found.
M9	a flag if an extra term was generated. M9=0 if no term was generated, otherwise M9 points to the extra term stored in M1( )

All the other variables are less important as they are used for general bookkeeping.

SUBROUTINES (as they appear in the listing)

'1 to '4	general print routines used to print out factors 1 to factor 4
'37	subroutine to print the input variables
'16	used to load the table into memory
'17 to '20	loads, respectively, factor 1 to factor 4 into memory
'22	loads the density buffer from tape
'23	loads the factor buffer from tape
'30	initial entry into the multiplier (the starting point)
'10	re-entry into the multiplier if it has stopped because of an error (usually an end of tape error). This is the manual re-entry point.
'11	loads/resets the density buffer into factor 4

'12 loads/resets the factor buffer into factor 2

'35(G$,Z) used to analyze the T1$( )→T4$( ) arrays. The position being analyzed is given by Z ; the string being analyzed is given in G$. If a value is present its power is returned in A .

'36 analyzes the G$ string for the presence of a sin or cos. If a sin is found A is set to 1, if a cos is found B is set to 1.

'41 subroutine to make the substitution for a power of Z_2 . The extra term is temporarily stored in the M1 array.

'42 subroutine to read back the extra term stored in the M1 array by '41.

'43(FO) routine to create a FORTRAN COMPATIBLE statement from the input data. It works in conjunction with '44. The resulting statement is in F8$ and F9$.

'44 routine to pack the A1$→A4$ and E1$( )→E4$( ) arrays into F8$ and F9$ inserting an "*" when necessary and deleting all unnecessary blanks.

The storing of data onto tape is also done by '43 and '44. The F1$8 string contains the character used for generating the continuation cards. When a "Z" is encountered it is reset to an @(in HEX a 1 added to @ = A).

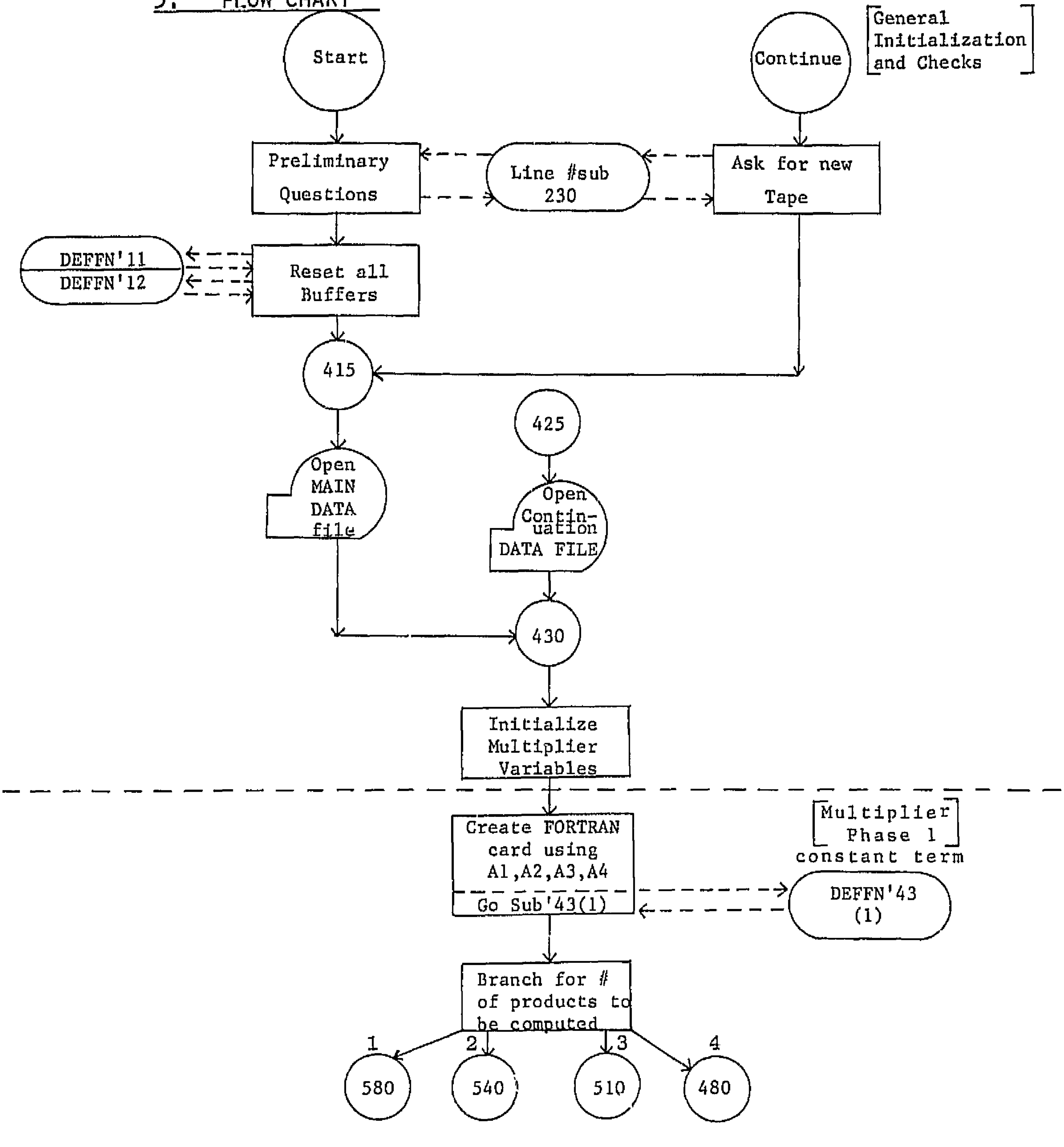
General Starting Procedures

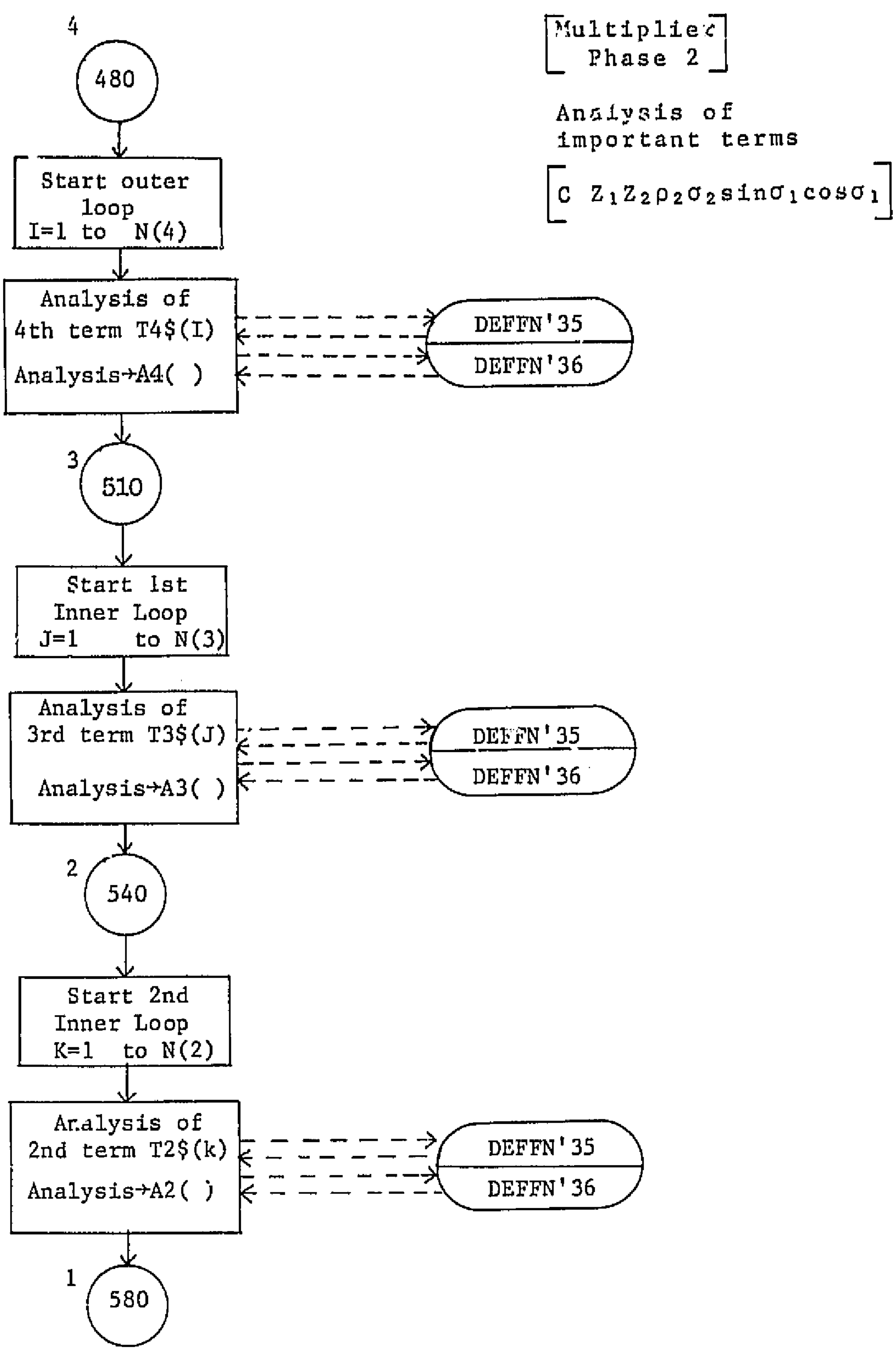
1. load program "MULTFF"
2. load Density buffer (S.F. key 22)
3. load factor buffer (S.F. key 23)
4. load factor 1 (S.F. key 1)
5. load factor 3 (S.F. key 3)
6. press special function key "START" (S.F. key 30)
7. when told insert "TABLE" tape and press the continue button on the console <u>not</u> the S.F. key marked "CONTINUE"

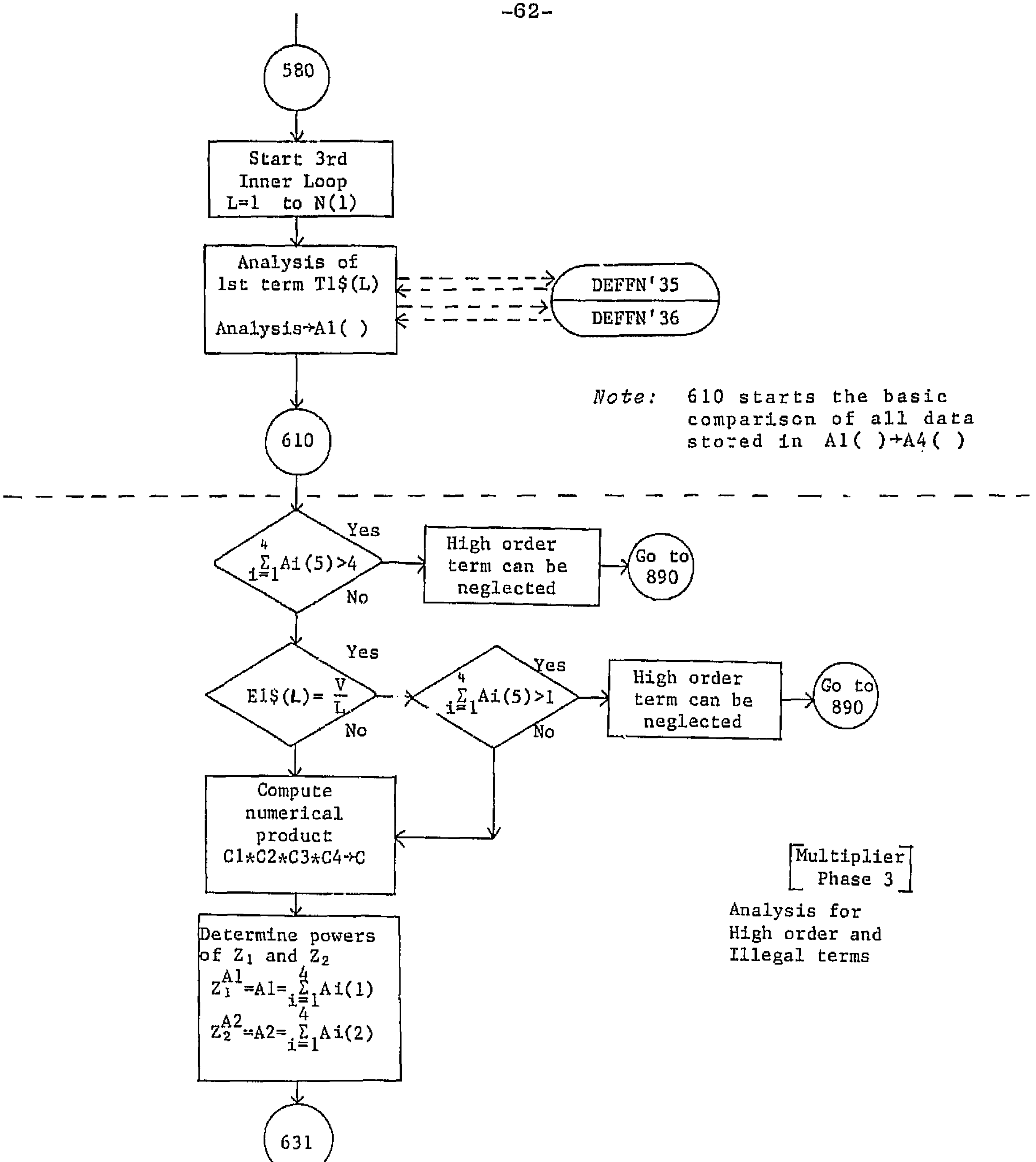
8. when prompted check paper, insert a free data tape and press the continue button on the console <u>not</u> the S.F. key marked "CONTINUE"
9. if the program has an end of tape error (Error code 49)
   a) remove the full tape
   b) insert a fresh tape
   c) press S.F. key "CONTINUE" (S.F. key 10)

In general the machine will issue a message if the operator must supply some type of information.

3. FLOW CHART

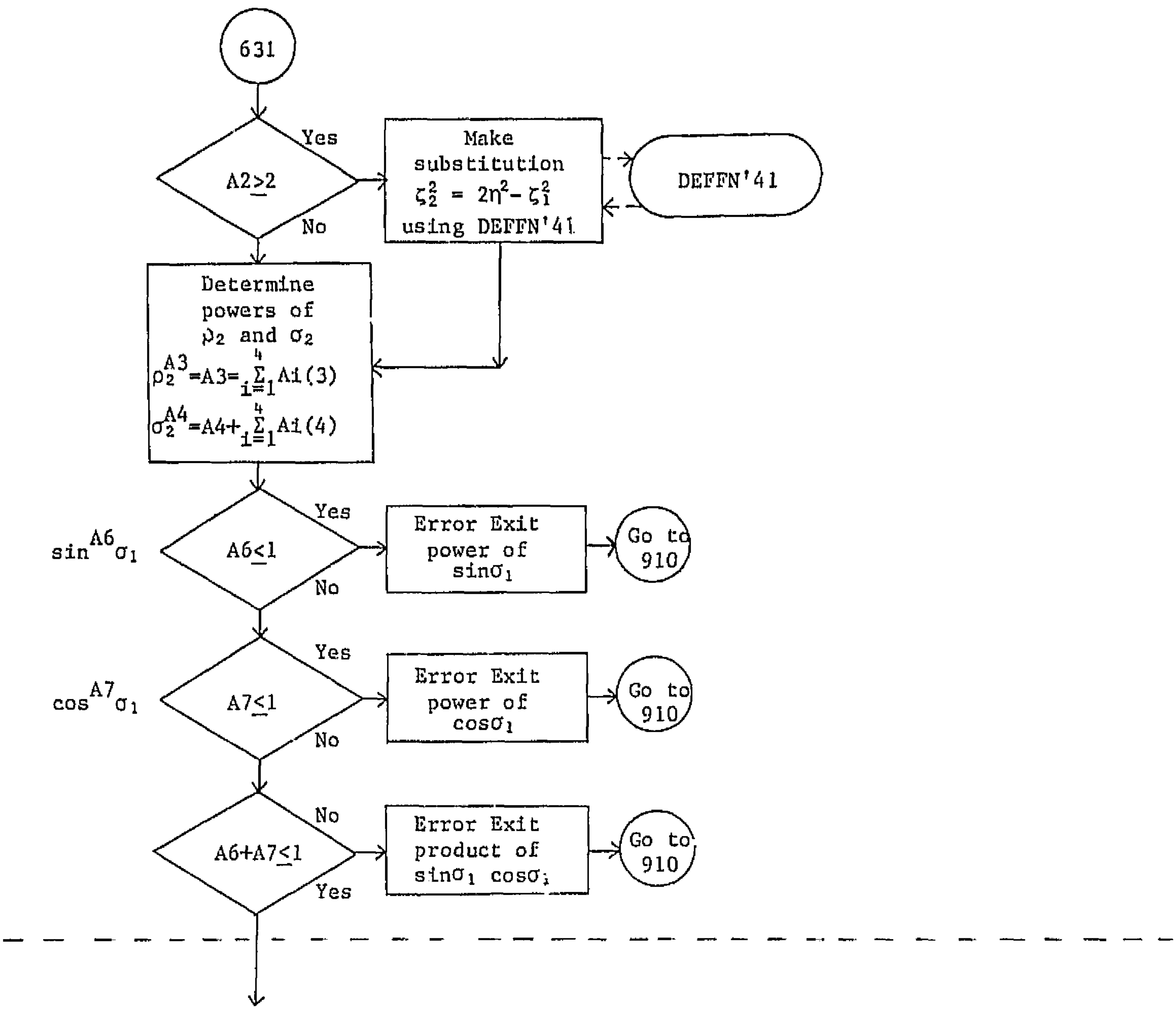

Start
Continue
General Initialization and Checks
Preliminary Questions
Line #sub 230
Ask for new Tape
DEFFN'11
DEFFN'12
Reset all Buffers
415
425
Open MAIN DATA file
Open Contin- uation DATA FILE
430
Initialize Multiplier Variables
Create FORTRAN card using A1,A2,A3,A4
Go Sub'43(1)
Multiplier Phase 1
constant term
DEFFN'43 (1)
Branch for # of products to be computed
1
2
3
4
580
540
510
480

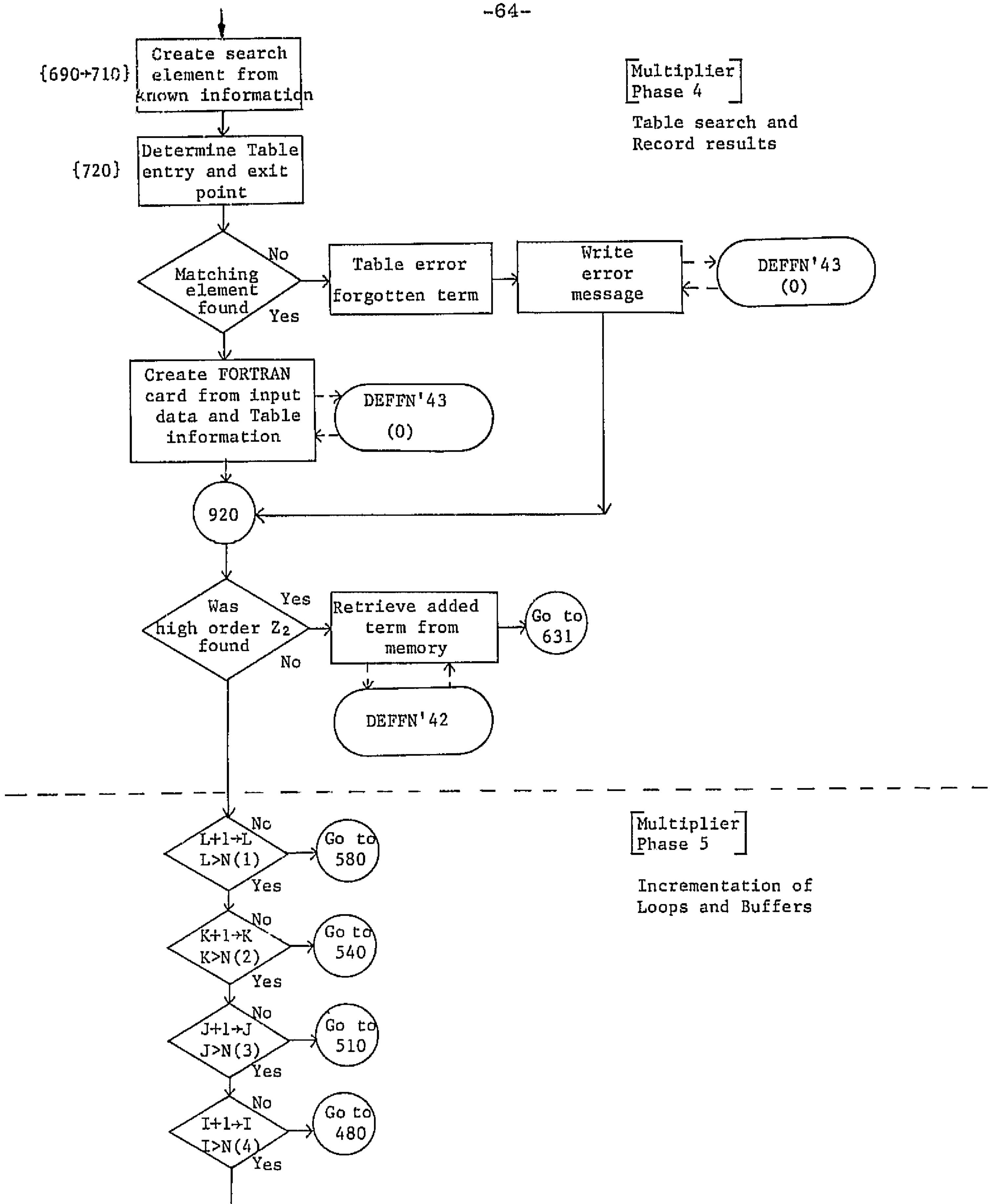

Multiplier
Phase 2
Analysis of
important terms
C $Z_1Z_2\rho_2\sigma_2\sin\sigma_1\cos\sigma_1$
4
480
Start outer
loop
I=1 to N(4)
Analysis of
4th term T4$(I)
Analysis→A4( )
DEFFN'35
DEFFN'36
3
510
Start 1st
Inner Loop
J=1 to N(3)
Analysis of
3rd term T3$(J)
Analysis→A3( )
DEFFN'35
DEFFN'36
2
540
Start 2nd
Inner Loop
K=1 to N(2)
Analysis of
2nd term T2$(k)
Analysis→A2( )
DEFFN'35
DEFFN'36
1
580

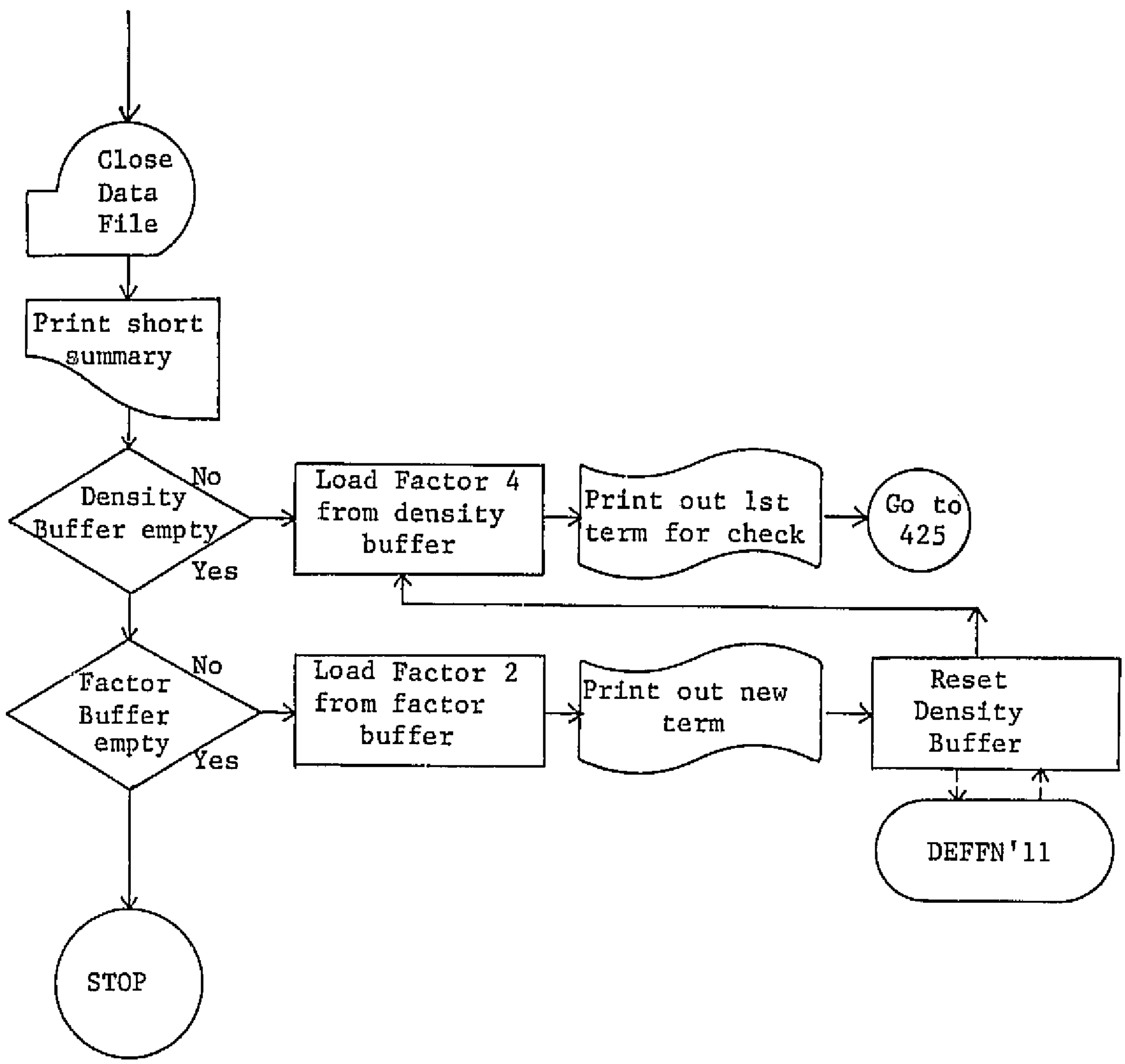


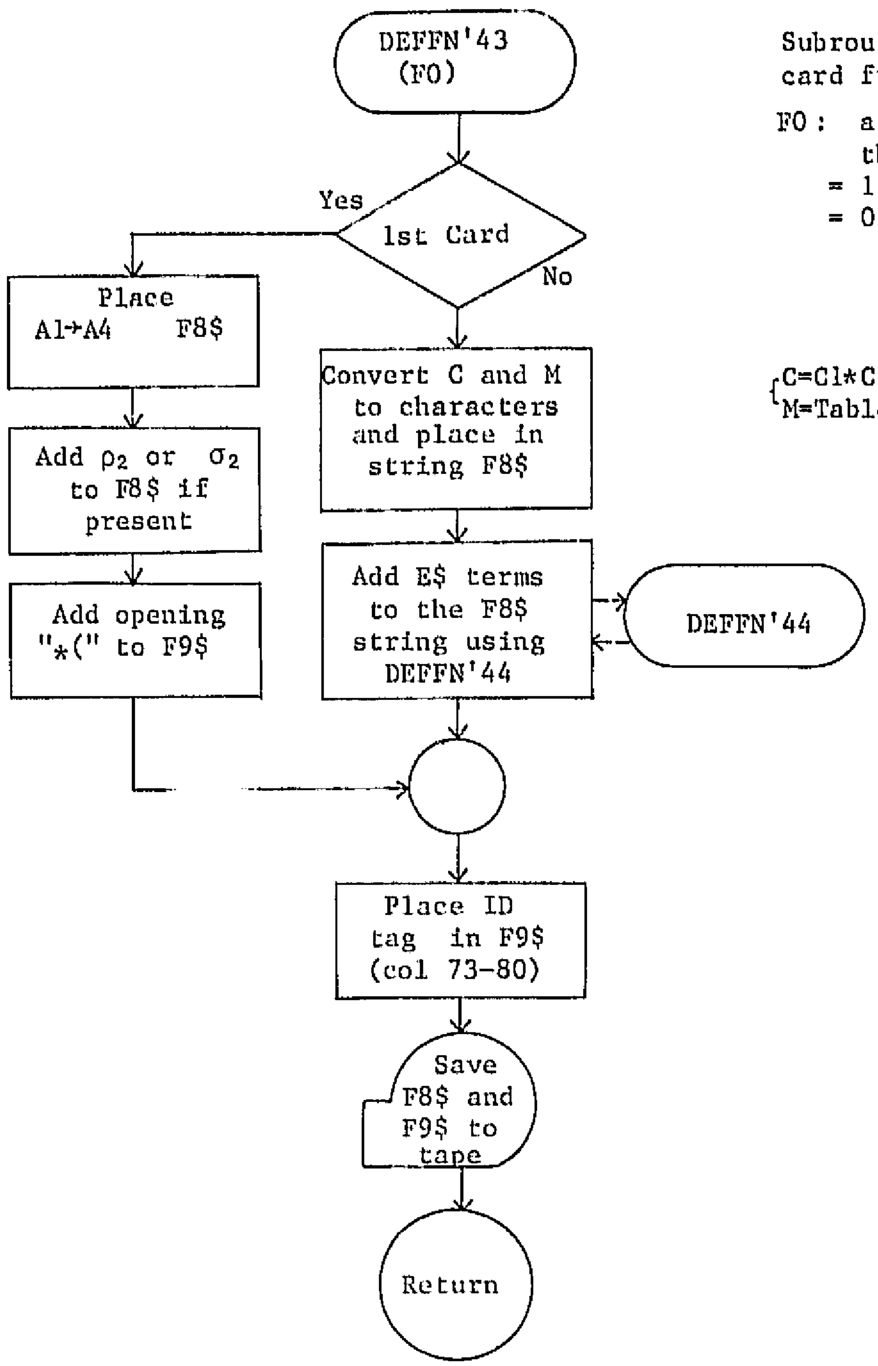
[Multiplier Phase 3]

Analysis for High order and Illegal terms


631
A2≥2
Yes
No
Make substitution $\zeta_2^2 = 2\eta^2 - \zeta_1^2$ using DEFFN'41
DEFFN'41
Determine powers of ρ_2 and σ_2
ρ_2^{A3}=A3=$\sum_{i=1}^{4}$Ai(3)
σ_2^{A4}=A4+$\sum_{i=1}^{4}$Ai(4)
$\sin^{A6}\sigma_1$
A6≤1
Error Exit power of $\sin\sigma_1$
Go to 910
$\cos^{A7}\sigma_1$
A7≤1
Error Exit power of $\cos\sigma_1$
Go to 910
A6+A7≤1
Error Exit product of $\sin\sigma_1$ $\cos\sigma_1$
Go to 910


Multiplier Phase 4
Table search and Record results
{690→710}
Create search element from known information
{720}
Determine Table entry and exit point
Matching element found
No
Yes
Table error forgotten term
Write error message
DEFFN'43 (0)
Create FORTRAN card from input data and Table information
DEFFN'43 (0)
920
Was high order Z2 found
Yes
No
Retrieve added term from memory
Go to 631
DEFFN'42
Multiplier Phase 5
Incrementation of Loops and Buffers
L+1→L L>N(1)
No
Yes
Go to 580
K+1→K K>N(2)
No
Yes
Go to 540
J+1→J J>N(3)
No
Yes
Go to 510
I+1→I I>N(4)
No
Yes
Go to 480


Close
Data
File
Print short
summary
Density
Buffer empty
No
Yes
Load Factor 4
from density
buffer
Print out 1st
term for check
Go to
425
Factor
Buffer
empty
No
Yes
Load Factor 2
from factor
buffer
Print out new
term
Reset
Density
Buffer
DEFFN'11
STOP

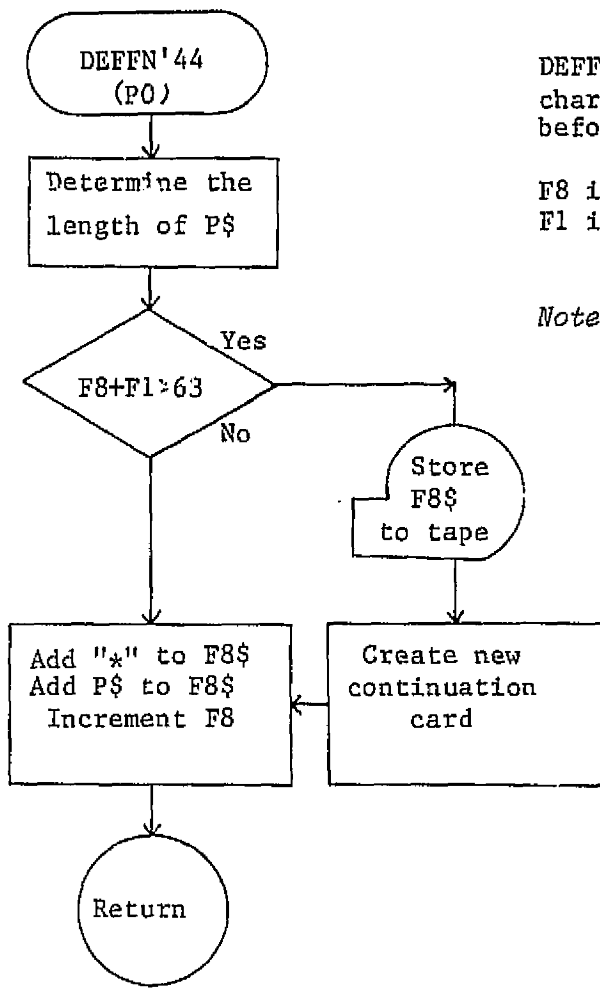


Subroutine creates a FORTRAN card from the given input

F0 : a flag telling if this is the 1st card of the multiplication
= 1 create a card using A1→A4
= 0 create a card using E1$( )→E4$( )

{C=C1*C2*C3*C4
M=Table location}

DEFFN'44 inserts an "*" between the character strings and omits all unnecessary blanks, it also checks to see if an overflow of the F8 string will occur. If an overflow occurs the information is stored to tape, a continuation card is generated and the process continues.



DEFFN'44 adds P$ to the F8$ character string and insert an "*" before insertion.

F8 is the current length of F8$
F1 is the length of P$

Note: F8 always points to the NEXT available location of F8$

VARIABLES USED IN ALGEBRAIC MULTIPLIER

SYSTEM 2200 VARIABLE CHECK-OFF LIST

PROGRAM NAME ______________________ DATE ______________

VERSION ______________________ PROGRAMMER ______________

SYSTEM ____

N \ M	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Δ	X	X	X						X	X	X	X	X	/		X						X				X
1	X	X	X			X												X								
2	X		X															X								
3	X		X															X								
4	X		X															X								
5																										
6	X																									
7	X																									
8						X																				
9													X										X			
0																										

NUMERIC SCALARS
FORMAT = MN

N \ M	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Δ		X						X						X												
1	X												X													
2	X																									
3	X																									
4	X																									
5																										
6																										
7																										
8																										
9																										
0																										

NUMERIC ARRAYS
FORMAT = MN(

N \ M	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Δ	X	X					X									X	X	X		X						X
1	X					X	X																			
2	X																									
3	X																									
4	X																									
5																										
6																										
7																										
8						X																X				
9						X														X		X				
0																										

ALPHA NUMERIC SCALARS
FORMAT = MN$

N \ M	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Δ		X				X		X																		
1		X			X			X												X		X				
2		X			X			X												X						
3					X															X						
4					X															X						
5																										
6																										
7																										
8																										
9																										
0																						X				

ALPHA NUMERIC ARRAYS
FORMAT = MN$(

NOTE:
0 = NON COMMON
1 = COMMON DEFINED BY THIS MODULE
2 = COMMON DEFINED BY PREVIOUS MODULE

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Listing of BASIC Program for Algebraic Multiplier

```
10 REM ** ALGEBRAIC MULTIPLIER WRITTEN BY S. STARKE **
20 REM **        VERSION=5 ADAPTED FOR JSC BY G.S. MARCH 77**
30 % A##= ####################
40 % T##(#)= ####################  E##(#)= ####################
50 DIM H(9),H$(9)20,H1$(9,5)20,H2$(9,5)2:P=0 REM -DENSITY BUFFER-
60 DIM B(2),B$(2)20,B1$(2,9)20,B2$(2,9)20:B=0:REM -FACTOR BUFFER-
65 DIM O8$64,O9$16
70 DIM T$(27)13,A1$20,A2$20,A3$20,A4$20,Z$1,G$20,G1$20,F$(1)10
71 DIM V9$10,V8$8:INIT(FF) V9$
75 DIM A$20,B$20,M1(4,3),F8$64,F9$16,P$20,F1$6,T9$11
80 DIM T1$(10)20,T2$(10)20,T3$(10)20,T4$(10)20,N(4)
90 DIM E1$(10)2,E2$(10)2,E3$(10)2,E4$(10)2,B$5
100 DIM A1(7),A2(7),A3(7),A4(7):SELECT PRINT 005(64):PRINT HEX(03):END
120 DEFFN'00:P=4:PRINT HEX(0D):GOSUB '1:GOSUB '2:GOSUB '3:GOSUB '4:RETUR
N
130 DEFFN'1
140 PRINTUSING 30,1,A1$:FOR I=1TO N(1):GOSUB '37(1,T1$(I),E1$(I)):NEXT I
:PRINT " N(1) =";N(1),HEX(0D0A):RETURN
150 DEFFN'2
160 PRINTUSING 30,2,A2$:FOR I=1 TO N(2):GOSUB '37(2,T2$(I),E2$(I)):NEXT
I:PRINT " N(2) =";N(2),HEX(0D0A):RETURN
170 DEFFN'3
180 PRINTUSING 30,3,A3$:FOR I=1 TO N(3):GOSUB '37(3,T3$(I),E3$(I)):NEXT
I:PRINT " N(3) =";N(3),HEX(0D0A):RETURN
190 DEFFN'4
200 PRINTUSING 30,4,A4$:FOR I=1 TO N(4):GOSUB '37(4,T4$(I),E4$(I)):NEXT
I:PRINT " N(4) =";N(4),HEX(0D0A)
210 RETURN
220 REM - PRINT UTILITY ROUTINES-
230 INIT(20) F$():INPUT "FILE NAME",STR(F$(1),2,8):SELECT PRINT 015(72):
PRINTUSING 240,STR(F$(1),2,8)
231 STR(F$(1),1,1)=HEX(C2):STR(F$(1),LEN(F$(1))+1)=STR(V9$,1,10-LEN(F$(1
))):STR(F$(1),10,1)=HEX(0D):RETURN
240 % FILE NAME = "########"
250 DEFFN'37(A,A$,B$):PRINTUSING 40,A,I,A$,A,1,B$:RETURN
270 REM -LOAD MEMORY-
280 DEFFN'16:STOP "TABLE CASSETTE ?":DATA LOAD "TABLE":FOR I=1TO 27:DATA
LOAD T$(I):NEXT I:RETURN
290 STOP "POSITION TAPE USING < DATA LOAD 'NAME' >":RETURN
300 DEFFN'17:PRINT "EQU 1":GOSUB 290:DATA LOAD N(1),A1$:FOR I=1TO N(1):D
ATA LOAD T1$(I),E1$(I):NEXT I:PRINT "LOADED":RETURN
310 DEFFN'18:PRINT "EQU 2":GOSUB 290:DATA LOAD N(2),A2$:FOR I=1TO N(2):D
ATA LOAD T2$(I),E2$(I):NEXT I:PRINT "LOADED":RETURN
320 DEFFN'19:PRINT "EQU 3":GOSUB 290:DATA LOAD N(3),A3$:FOR I=1TO N(3):D
ATA LOAD T3$(I),E3$(I):NEXT I:PRINT "LOADED":RETURN
330 DEFFN'20:PRINT "EQU 4":GOSUB 290:DATA LOAD N(4),A4$:FOR I=1TO N(4):D
ATA LOAD T4$(I),E4$(I):NEXT I:PRINT "LOADED":RETURN
```

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```
335 DEFFN'22:H=0:PRINT "DENSITY BUFFER";:GOSUB 290
340 H=H+1:DATA LOAD H(H),H$(H):FOR I=1TO H(H):DATA LOAD H1$(H,I),H2$(H,I
):NEXT I:PRINT H,:IF H=9THEN 360:DATA LOAD "+":GOTO 340
345 DEFFN'23:B=0:PRINT "FACTOR BUFFER";:GOSUB 290:INPUT "LOAD WITH",W9:I
F W9<2THEN 350:W9=2
350 B=B+1:DATA LOAD B(B),B$(B):FOR I=1TO B(B):DATA LOAD B1$(B,I),B2$(B,I
):NEXT I:PRINT B,:IF B>=W9THEN 360:DATA LOAD "+":GOTO 350
360 PRINT "BUFFER LOADED":RETURN
370 REM
380 REM -MULTIPLIER-
390 REM
400 DEFFN'30:REM - START -:SELECT PRINT 005(64):PRINT HEX(03):GOSUB '16:
P=4:GOSUB '11:GOSUB '12
410 STOP "PAPER ?":GOSUB 230:STOP "CORRECT TAPE ?"
412 K9=0:K8=0:K7=0:Q3=0
415 DATA SAVE BT(H) F$():SELECT PRINT 015(72):GOSUB '00:GOTO 430
420 DEFFN'10:REM - RESTART -:RETURN CLEAR :SELECT PRINT 015(72):V8$=STR(
F$(1),2,8):IF POS(F$(1)=FF)=0 THEN 422
421 V8$=STR(F$(1),2,POS(F$(1)=FF)-1)
422 GOSUB 230:PRINT "( CONTINUATION OF :";V8$;" )":STOP "CORRECT TAPE ?"
:GOTO 415
425 DATA SAVE OPEN "$CONT$"
430 M9,A1,A2,A3,A4,R1,R2,R3,R4=0:I,J,K,L,M,C1,C2,C3,C4=1
440 MAT A1=ZER:MAT A2=ZER:MAT A3=ZER:MAT A4=ZER:INIT(20) F8$,F9$:0$="$CO
NT$":F1$="      A"
450 SELECT PRINT 005(64):PRINT "DO NOT DISTURB -  PROGRAM WORKING - ":SE
LECT PRINT 015(72):GOSUB '43(1)
470 ON P GOTO 580,540,510,480
480 FOR I=1 TO N(4)
490 FOR M=1 TO 4:GOSUB '35(T4$(I),5+2*M):A4(M)=A:A4(5)=A4(5)+A:NEXT M:CO
NVERT STR(T4$(I),1,NUM(T4$(I)))TO C4
500 GOSUB '36(T4$(I)):A4(6)=A:A4(7)=B
510 FOR J=1 TO N(3)
520 FOR M=1 TO 4:GOSUB '35(T3$(J),5+2*M):A3(M)=A:A3(5)=A3(5)+A:NEXT M:CO
NVERT STR(T3$(J),1,NUM(T3$(J)))TO C3
530 GOSUB '36(T3$(J)):A3(6)=A:A3(7)=B
540 FOR K=1 TO N(2)
550 FOR M=1 TO 4:GOSUB '35(T2$(K),5+2*M):A2(M)=A:A2(5)=A2(5)+A:NEXT M
560 CONVERT STR(T2$(K),1,NUM(T2$(K))) TO C2
570 GOSUB '36(T2$(K)):A2(6)=A:A2(7)=B
580 FOR L=1 TO N(1)
590 FOR M=1 TO 4:GOSUB '35(T1$(L),5+2*M):A1(M)=A:A1(5)=A1(5)+A:NEXT M:GO
SUB '36(T1$(L)):A1(6)=A:A1(7)=B
600 CONVERT STR(T1$(L),1,NUM(T1$(L))) TO C1
610 IF A1(5)+A2(5)+A3(5)+A4(5)>4 THEN 890
612 IF E1$(L)="(1.-R1*R1*B**4)" THEN 615:GOTO 616
615 IF A1(5)+A2(5)+A3(5)+A4(5)>1 THEN 890
616 C=C1*C2*C3*C4
620 A1=A1(1)+A2(1)+A3(1)+A4(1)
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```
630 A2=A1(2)+A2(2)+A3(2)+A4(2)
631 IF A2<2THEN 640:GOSUB '41
640 A3=A1(3)+A2(3)+A3(3)+A4(3)
650 A4=A1(4)+A2(4)+A3(4)+A4(4)
660 A6=A1(6)+A2(6)+A3(6)+A4(6):IF A6<2THEN 670:G1$="'SIN'":Z$="S":GOTO 9
10
670 A7=A1(7)+A2(7)+A3(7)+A4(7):IF A7<2THEN 680:G1$="'COS'":Z$="C":GOTO 9
10
680 IF A6+A7<2 THEN 690:G1$="'SIN*COS'":Z$="2":GOTO 910
690 R$=" ":IF A2=0 THEN 700:STR(R$,1,1)="2"
700 IF A6=0 THEN 710:STR(R$,2,1)="S":GOTO 720
710 IF A7=0 THEN 720:STR(R$,2,1)="C"
720 A=A1:Z=0:ON A GOTO 730,730,730:Z=3:IF A=4THEN 730:A=27/6:Z=0
725 REM -- TABLE SEARCH --
730 FOR M=6*A-5 TO 6*A-2:IF STR(T$(M),1,2)=STR(R$,1,2)THEN 740:NEXT M:G1
$="TABLE ERROR":Z$="*":GOTO 910
740 Z$=STR(T$(M),3,1):GOSUB '43(0):IF Z$="*"THEN 920:GOTO 900
880 %(##,##,##,##)
890 R1=R1+1:GOTO 920:REM -- HIGH ORDER --
900 R2=R2+1:GOTO 920:REM -- NO CONST. --
910 R3=R3+1:PRINT "-- ERROR -  UNEXPECTED  ";G1$;"  AT ";:PRINTUSING 88
0,L,K,J,I:M=99:GOSUB '43(0)
920 R4=R4+1:IF M9<=0 THEN 930:GOSUB '42:GOTO 631
930 MAT A1=ZER:C1=1:NEXT L:IF P=1THEN 1011
940 MAT A2=ZER:C2=1:NEXT K:IF P=2THEN 1011
950 MAT A3=ZER:C3=1:NEXT J:IF P=3THEN 1011
960 MAT A4=ZER:C4=1:NEXT I
970 REM -- MULTIPLICATION FINISHED -- PRINT RESULTS --
980 IF K9=0 THEN 1011
990 GOSUB '47
1011 STR(F8$,1)=F1$:STR(F8$,7)="     )":DATA SAVE F8$,F9$:DATA SAVE END
1012 PRINTUSING 1450,F8$::PRINTUSING 1460,F9$:SELECT PRINT 015(72)
1013 PRINT HEX(0D);" TOTAL TERMS = ";R4;":HO= ";R1;":NC= ";R2;":UX= ";R3
;":CONST. = ";R4-R1-R2-R3,HEX(0D),"*****",HEX(0D0A)
1020 GOSUB 1070:IF H>=10THEN 1040:PRINTUSING 240,Q$:PRINT :PRINTUSING 30
,4,A4$:PRINT HEX(0D0A):GOTO 425
1040 PRINTUSING 240,Q$:PRINT :GOSUB '2:GOSUB '11:PRINTUSING 30,4,A4$:PRI
NT HEX(0D0A):GOTO 425
1050 REM ------- END OF MULTIPLIER SECTION -- -----
1060 DEFFN'11:H=0:REM -- -- DENSITY LOAD AND RESET - ---
1070 H=H+1:IF H>=10THEN 1090:N(4)=H(H):FOR M=1TO H(H):T4$(M)=H1$(H,M):E4
$(M)=H2$(H,M):NEXT M:A4$=H$(H):RETURN
1080 DEFFN'12:B1=0:REM ----- FACTOR LOAD AND RESET -----
1090 B1=B1+1:IF B1>=W9+1THEN 1110:N(2)=B(B1):FOR M=1TO B(B1):T2$(M)=B1$(
B1,M):E2$(M)=B2$(B1,M):NEXT M:A2$=B$(B1):GOSUB '4 :RETURN
1110 SELECT PRINT 005(64):STOP "PRODUCT FINISHED. LOAD NEW EQU AND - 'ST
ART' -":END
1120 DEFFN'35(G$,Z):A=VAL(STR(G$,Z,1))-48:IF A>=0 THEN 1130:A=0
1130 IF Z<=7THEN 1140:A=A/2
1140 RETURN
1150 DEFFN'36(G$):A,B=0:IF STR(G$,14,3)<>"SIN"THEN 1160:A=1
1160 IF STR(G$,17,3)<>"COS" THEN 1170:B=1
1170 RETURN
1220 DEFFN'41:REM POWER OF Z2 FOUND:M9=M9+1
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```
1240 M1(M9,1)=2*C:C=-C
1250 M1(M9,2)=A1:A1=A1+2
1260 A2,M1(M9,3)=A2-2:RETURN
1270 DEFFN'42:REM *READ EXTRA PRODUCT*
1280 C=M1(M9,1):A1=M1(M9,2):A2=M1(M9,3)
1290 Q3=1:M9=M9-1:RETURN
1300 DEFFN'43(F0):IF F0=1THEN 1380:STR(F8$,1)=F1$:CONVERT C TO STR(F8$,7
,6),(+####.#)
1310 STR(F8$,13)="*T":CONVERT M TO STR(F8$,15,2),(##):F8=LEN(F8$)+1
1320 SELECT PRINT 005(64):GOSUB '44(E1$(L)):GOSUB '44(E2$(K)):GOSUB '45(
E3$(J)):GOSUB '44(E4$(I))
1330 CONVERT LTO STR(F9$,9,1),(#):CONVERT KTO STR(F9$,10,1),(#)
1340 CONVERT JTO STR(F9$,11,1),(#):CONVERT ITO STR(F9$,12,1),(#)
1350 STR(F9$,13)=Z$:STR(F9$,14)=STR(F$(1),3,1)
1360 CONVERT B1TO STR(F9$,15,1),(#):CONVERT HTO STR(F9$,16,1),(#)
1370 ADDC(F1$,01):IF STR(F1$,6,1)>HEX(5B)THEN 1390:F1$="      @":GOTO 139
0
1380 F8=7:SELECT PRINT 015(72):GOSUB '44(A1$):STR(F8$,7)=STR(F8$,8):F8=F
8-1:GOSUB '44(A2$)
1381 GOSUB '44(A3$):GOSUB '44(A4$)
1382 STR(F9$,LEN(F9$)+1)=" * (":Q0$=F8$:Q9$=F9$:GOTO 1398
1390 ON K9+1 GOTO 1392,1394,1396
1392 K9=K9+1:Q0$=F8$:STR(Q9$,9,8)=STR(F9$,9,8):INIT(20)F8$,F9$:GOTO 1399
1394 K9=K9+1:STR(Q8$,29,22)=STR(F8$,7,22):INIT(20)F8$,F9$:GOTO 1399
1396 K9=K9+1:STR(Q8$,51,14)=STR(F8$,7,14):STR(Q9$,1,8)=STR(F8$,21,8)
1398SELECT PRINT 015(72):PRINTUSING 1450,Q8$,:PRINTUSING 1460,Q9$:DATA S
AVE Q8$,Q9$,INIT(20) F8$,F9$,Q8$,Q9$:K9=0
1399 RETURN
1400 DEFFN'45(P$):IF Q3=0 THEN 1403:P$="Q3":Q3=0
1403 F1=LEN(P$):IF F1=1THEN 1440:IF F8+F1>64THEN 1430
1410 STR(F8$,F8)="*":STR(F8$,F8+1)=P$:F8=LEN(F8$)+1:GOTO 1440
1430 Q8$=F8$:Q9$=F9$: GOSUB '47:PRINT HEX(07):STR(F8$,6,1)="$":F8=7:GOTO
 1410
1440 RETURN
1450 %################################################################
1460 %#################
1500DEFFN'47:SELECT PRINT 015(72):PRINTUSING 1450,Q8$,:PRINTUSING 1460,Q
9$:DATA SAVE Q8$,Q9$:INIT(20) F8$,F9$,Q8$,Q9$:K9=0:RETURN
1600 DEFFN'44(P$)
1603 F1=LEN(P$):IF F1=1THEN 1640:IF F8+F1>64THEN 1630
1610 STR(F8$,F8)="*":STR(F8$,F8+1)=P$:F8=LEN(F8$)+1:GOTO 1640
1630 Q8$=F8$:Q9$=F9$: GOSUB '47:PRINT HEX(07):STR(F8$,6,1)="$":F8=7:GOTO
 1610
1640 RETURN
2300 DEFFN'24:PRINT "EQU 1":GOSUB 290:DATA SAVE N(1),A1$:FOR I=1TO N(1):
DATA SAVE T1$(I),E1$(I):NEXT I:PRINT "SAVED":RETURN
2310 DEFFN'25:PRINT "EQU 2":GOSUB 290:DATA SAVE N(2),A2$:FOR I=1TO N(2):
DATA SAVE T2$(I),E2$(I):NEXT I:PRINT "SAVED":RETURN
2320 DEFFN'26:PRINT "EQU 3":GOSUB 290:DATA SAVE N(3),A3$:FOR I=1TO N(3):
DATA SAVE T3$(I),E3$(I):NEXT I:PRINT "SAVED":RETURN
2330 DEFFN'28:PRINT "EQU 4":GOSUB 290:DATA SAVE N(4),A4$:FOR I=1TO N(4):
DATA SAVE T4$(I),E4$(I):NEXT I:PRINT "SAVED":RETURN
3335 DEFFN'5:H=0:PRINT "DENSITY BUFFER":GOSUB 290
3340 H=H+1:DATA SAVE H(H),H$(H):FOR I=1TO H(H):DATA SAVE H1$(H,I),H2$(H,
I):NEXT I:PRINT H,:DATA SAVE END :IF H=9THEN 3360:DATA SAVE OPEN "+":GOT
O 3340
3360 RETURN
5000 PRINT B2$(K8,1),B2$(K8,2),B2$(K8,3),B2$(K8,4),B2$(K8,6)
6000 PRINT B1$(K8,1),B1$(K8,2),B1$(K8,3),B1$(K8,4),B1$(K8,5)
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REFERENCES

1. Scheifele, G.: Généralisation des éléments de Delaunay en mécanique céleste. Application au mouvement d'un satellite artificiel, C.R. Acad. Sc. Paris 271, 729, 1970.

2. Scheifele, G.: True Anomaly PS-Elements, ACM Memo 104, November 1974.

3. Mueller, Alan: The Development of the Poincaré-Similar Elements with the True Anomaly as the Independent Variable, NASA/JSC Internal Note No. 76-FM-60, August 1976.

4. Scheifele, G.: On Nonclassical Canonical Systems, Celestial Mechanics, Vol. 2, pp. 296-310, 1970.

5. Stiefel, E. and Scheifele, G.: Linear and Regular Celestial Mechanics, Springer Verlag Berlin, Heidelberg, New York, 1971.

6. Brouwer, D. and Hori, G.: Theoretical Evaluation of Atmospheric Drag Effects in the Motion of an Artifical Satellite, Astron. Journal, Vol.66, No. 5, pp. 193-225, 1961.

7. Lane M.H.: The Development of an Artifical Satellite Theory Using a Power-Law Atmospheric Density Representation. AIAA Paper No. 65-35, January 1965.

8. Lane, M.H. and Cranford, K.H.: An Improved Analytical Drag Theory for the Artifical Satellite Problem, AIAA Paper No. 69-925, August 1969.

9. Bond, V.R. and Scheifele, G.: An Analytical Singularity-Free Solution to the J_2 Perturbation Problem, (to be published).

10. Scheifele, G. and Graf, O.: Analytical Satellite Theories Based on a New Set of Canonical Elements, AIAA Paper No. 74-838, August 1974.