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A Well Posed Boundary Value Problem in Transonic Gas Dynamics

José M. Sanz

Research and Development Report

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ABSTRACT

A boundary value problem for the Tricomi equation is studied in connection with transonic gas dynamics. The transformed equation

$$\Delta u + \frac{1}{3Y} u_Y = 0$$

in canonical coordinates is considered in the complex domain of two independent complex variables. A boundary value problem is then set by prescribing the real part of the solution on the boundary of the real unit circle.

The Dirichlet problem in the upper unit semicircle with vanishing values of the solution at Y = 0 is solved explicitly in terms of the hypergeometric function for the more general Euler-Poisson-Darboux equation. An explicit representation of the solution is also given for a mixed Dirichlet and Neumann problem for the same equation and domain.

Reflection rules are given for these solutions which permit one to extend them from the upper to the lower unit semicircle. The transonic boundary value problem is solved by expressing the solution as a linear combination of these two types of solutions.

1. INTRODUCTION

The Tricomi equation

$$yu_{xx} + u_{yy} = 0 \tag{1.1}$$

is an example of a partial differential equation of mixed type, that is, an equation which is partly elliptic and partly hyperbolic. This equation has been widely studied because of its relation to transonic gas dynamics [10].

The Tricomi problem is a model of a well posed problem for equation (1.1). We recall [10] that it is formulated in a domain which is intersected by the parabolic line $y = 0$. In the elliptic half-plane the domain is bounded by a simple curve ending at two points on the parabolic line. In the hyperbolic half-plane the domain is bounded by two intersecting characteristics that start at these two points of the parabolic line. The Tricomi problem is well posed because the boundary values of the solution are prescribed on the whole elliptic boundary but only on one of these characteristics.

An important problem in transonic gas dynamics is the computation of the flow past a given airfoil with speed at infinity close to but smaller than the speed of sound. In this situation the flow over the upper side of the airfoil may become supersonic. The partial differential equations of motion are then of mixed elliptic and hyperbolic type.

The problem of calculating continuous transonic flow past such an airfoil was shown by C. Morawetz [20] to be not well posed.

Motivated by the method of complex characteristics, a transonic boundary value problem was developed in [9] for a certain domain obtained by analytic extension to complex values of the independent variables. Complex conjugate characteristic coordinates ξ and η were introduced. In these coordinates the stream function $\psi(\xi,\bar{\xi})$ is real in the subsonic domain but complex otherwise. The new boundary value problem is set by prescribing the real part of $\psi(\xi,\bar{\xi})$ on the boundary of the unit circle. This problem has, as yet, not been shown to be well posed, although its numerical solution was found successfully. Our contribution will be to show that a corresponding problem for the Tricomi equation is well posed.

Under the change of coordinates

$$\frac{9}{4} Y^2 = y^3 , \quad X = x$$

equation (1.1) can be written in the canonical form

$$\Delta u + \frac{1}{3Y} u_Y = 0 . \tag{1.2}$$

We consider this equation in the four-dimensional space where the variables X and Y are complex. Points in the upper unit semicircle of the real (X,Y)-plane correspond to points in the elliptic half-plane of the variables x and y, which we shall call the Tricomi variables. Points in the lower unit semicircle of the real (X,Y)-plane correspond to a complex surface in the space of the x and y coordinates.

A solution $u(X,Y)$ of (1.2), real in the upper unit semicircle, is continued analytically to the lower unit semicircle over a path which avoids the singular locus $Y = 0$. The boundary value problem in which we are interested consists in prescribing the real part of $u(X,Y)$ on the boundary of the real unit circle in the (X,Y)-plane. A solution of this boundary value problem will be given that can be described formally as follows:

Two types of solutions for equation (1.2) can be found. One of them is of the form

$$u_1(X,Y) = Y^{2/3}\, f(X,Y^2)\ , \tag{1.3}$$

where $f(X,Y^2)$ is an analytic function even in Y. The other is of the form

$$u_2(X,Y) = g(X,Y^2)\ , \tag{1.4}$$

where $g(X,Y^2)$ is again analytic and even in Y.

The Dirichlet problem in the upper unit semicircle with vanishing boundary values at $Y = 0$ can be solved by a solution of the type (1.3). The mixed Dirichlet-Neumann boundary value problem in the same domain, with vanishing normal derivative on $Y = 0$ and prescribed values of the solution on the remaining boundary, can be solved by a solution of the type (1.4). By analytic continuation, these solutions can be extended to the lower unit semicircle. The solution u_2 will still be real. It will be of the form

$$u_2(X,-Y) = g(X,Y^2) \ , \quad Y > 0 \ . \tag{1.5}$$

But the solution u_1 will be of the form

$$u_1(X,-Y) = |Y|^{2/3} e^{2\pi i/3} f(X,Y^2) \ , \quad Y > 0 \tag{1.6}$$

when a proper branch is chosen.

In Chapter 5 it will be shown that the solution u of our boundary value problem can be expressed in a unique way as

$$u(X,Y) = u_1(X,Y) + u_2(X,Y) \ , \tag{1.7}$$

where u_1 and u_2 are of the form (1.3) and (1.4) respectively. A uniquely invertible relationship between the boundary values of u and those of u_1 and u_2 is obtained from the evenness of f and g as functions of Y. Using (1.5) and (1.6) we can write this relationship as

$$u(X,Y) = |Y|^{2/3} f(X,Y^2) + g(X,Y^2) \ , \quad Y > 0 \ ,$$

and

$$\mathrm{Re}\{u(X,Y)\} = -\frac{1}{2} |Y|^{2/3} f(X,Y^2) + g(X,Y^2) \ , \quad Y < 0 \ .$$

This shows that the boundary values of u_1 and u_2 on the upper unit semicircle can be chosen so that the real part of u satisfies the prescribed boundary conditions on the whole perimeter of the unit circle.

2. AN APPLICATION

As we have mentioned in Chapter 1, our boundary value problem has been motivated by the problem of calculating shockless transonic airfoils by the method of complex characteristics described in [9]. In this chapter we give a description of the method, specifically of those aspects which motivate our problem.

For an understanding of the problem it is important to find a suitable set of coordinates in which it can be stated. We start by expressing the equations of motion in the systems of coordinates that are relevant. The boundary value problem of reference [9] is then stated. Afterwards we give a description of transonic paths of integration that are required. We consider the geometry primarily in the case of the Tricomi equation and the Tricomi gas. This is a hypothetical gas whose equation of state is such that, in pertinent coordinates, the equation for the stream function becomes the Tricomi equation (see [4]).

2.1. Equations of Motion and the Hodograph Transformation

Let us consider the equations of plane flow for an inviscid compressible fluid. We assume steady, isentropic and irrotational flow. Consider an ideal polytropic gas whose equation of state is given by $p = \rho^{\gamma}$, so the process is isentropic. The variables p and ρ are respectively the pressure and density, and the constant γ is the adiabatic exponent of the gas.

For steady, irrotational flow the Bernoulli law becomes

$$q^2 + \frac{2}{\gamma-1} c^2 = \frac{\gamma+1}{\gamma-1} c_*^2 . \tag{2.1}$$

In this equation q is the speed, c is the speed of sound, and c_* is the critical speed, which is constant. Let u and v be the components of the velocity vector in the (x,y)-plane. The continuity equation

$$(\rho u)_x + (\rho v)_y = 0 \tag{2.2}$$

and the irrotationality condition

$$u_y - v_x = 0 \tag{2.3}$$

permit us to introduce a stream function ψ and a velocity potential ϕ. Equations (2.2) and (2.3) can then be transformed, using the Bernoulli law (2.1), into second order partial differential equations for ϕ or ψ. We have, for instance, the equation

$$(c^2-u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2-v^2)\phi_{yy} = 0 . \tag{2.4}$$

The equation (2.4) is quasilinear. It can be reduced to a linear equation by use of the hodograph transformation. The hodograph transformation consists of introducing u and v as the independent variables. In the hodograph plane we use polar coordinates defined by

$$u + iv = q\, e^{i\theta} .$$

The equation (2.4) is transformed into the linear system

$$\phi_q = \frac{M^2-1}{\rho q}\psi_\theta \; , \tag{2.5}$$
$$\phi_\theta = \frac{q}{\rho}\psi_q \; ,$$

where $M = q/c$ is the local Mach number.

Equations (2.5) are known in the literature as the Chaplygin equations. They are of the hyperbolic type for supersonic flow, $M^2 > 1$, and of the elliptic type for subsonic flow, $M^2 < 1$. The ordinary differential equation for the characteristics of (2.5) is

$$(1-M^2)(dq)^2 + q^2(d\theta)^2 = 0 \; .$$

Introducing characteristic coordinates ξ,η, we obtain the canonical system

$$q\theta_\xi - \sqrt{M^2-1}\, q_\xi = 0 \; , \tag{2.6}$$
$$q\theta_\eta + \sqrt{M^2-1}\, q_\eta = 0$$

in the hyperbolic domain.

We can extend the concept of characteristics to subsonic flow by considering analytic extension of the function ψ into the four-dimensional domain of the complex variables q and θ, or x and y [10]. The solutions of (2.6) are then the complex characteristics and the characteristic coordinates ξ and η become independent complex variables. In terms of the complex characteristic coordinates we obtain the canonical equations

$$\phi_\xi = i\tau_+\psi_\xi \, , \qquad (2.7)$$
$$\phi_\eta = i\tau_-\psi_\eta$$

for ϕ and ψ where

$$\tau_\pm = \pm\frac{\sqrt{1-M^2}}{\rho} \, .$$

An analytic function of a characteristic coordinate is a new characteristic coordinate. Thus, there is a conformal mapping free that can be used to transform the characteristic coordinates conveniently. Introducing the modified speed h, defined by

$$\frac{dh}{h} = \sqrt{1-M^2}\,\frac{dq}{q} \, , \qquad (2.8)$$

we can write the equations of the characteristics as

$$\log h + i\theta = \eta, \qquad (2.9)$$
$$\log h - i\theta = \xi.$$

By use of a conformal transformation new characteristic coordinates $s = s(\xi)$ and $t = t(\eta)$ can be defined as

$$s = h(q)e^{-i\theta} \, , \quad t = h(q)e^{i\theta} \, . \qquad (2.10)$$

Then a further mapping $s = e^{f(\xi)}$, $t = e^{\overline{f(\bar{\eta})}}$ can be introduced to transform the flow onto the unit circle.

The system (2.7) can be transformed putting

$$X = \theta \, , \quad Y = \log h = \int_{c_*}^{q} \sqrt{1-M^2}\,\frac{dq}{q} \qquad (2.11)$$

into the linear equation in canonical form

$$(2.12) \qquad \psi_{XX} + \psi_{YY} + \frac{1}{\tau}\frac{d\tau}{dY}\psi_Y = 0$$

for ψ, with $\tau = \tau_+$. After another change of coordinates defined by

$$d\sigma = \rho \frac{dq}{q} ,$$

the Chaplygin system transforms into the equation

$$(2.12) \qquad K(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 ,$$

with

$$K(\sigma) = \tau^2 = \frac{1-M^2}{\rho^2} .$$

This may be viewed as a generalized Tricomi equation.

The characteristic coordinates s and t introduced in (2.10) have the property that, for $M < 1$, $s = \bar{t}$ if and only if q and θ are real. Two complex coordinates with this property are called conjugate coordinates [1]. More generally let x and y be any two complex variables, and let

$$z = x + iy , \qquad z^* = x - iy .$$

We see that $z^* = \bar{z}$ if and only if x and y are real. Therefore we call z and z^* conjugate coordinates.

Starting with the complex coordinates x and y there is a simple way of generating new conjugate coordinates. Let f(z) be an analytic function which is real on the real axis. Define $\xi = f(z)$ and $\eta = f(z^*)$. Then ξ and η are conjugate coordinates. For if x and y are real, then $z^* = \bar{z}$, so by the Schwarz reflection principle $\overline{f(z^*)} = \overline{f(\bar{z})} = f(z)$ and $\xi = \bar{\eta}$. The converse is also true, provided that f is one to one in the domain in question.

2.2. The Inverse Problem for Supercritical Airfoil Design

The use of the hodograph transformation is appropriate for solving inverse problems in transonic gas dynamics. By the inverse method a continuous flow can be computed in terms of hodograph coordinates. Consider the streamlines in the physical plane. If the streamline $\psi = 0$ is closed and not self intersecting, it can be taken as a profile which generates the flow.

A new idea is given in [9] to solve the problem of airfoil design. This idea is to map the unknown domain of the flow in the hodograph plane conformally onto the unit circle of the plane of one characteristic coordinate. The speed q on the airfoil is prescribed as a function q(s) of the arc length s. What is intended is to have the equations of motion solved for the stream function in such a way that the boundary of the unit circle, for subsonic points, is a streamline and furthermore its image in the hodograph plane acquires the prescribed values of q over the profile.

More general conjugate characteristic coordinates ξ and η can be introduced by means of the formulas

$$s = e^{f(\xi)} , \qquad t = e^{\overline{f(\bar{\eta})}} , \tag{2.13}$$

where the map function $f(\xi)$ is an analytic function defined in the unit circle $|\xi| < 1$, and s and t have been defined in (2.10). The characteristic coordinates ξ and η are conjugate in the subsonic domain, so points in the real subsonic domain correspond to points $\xi = \bar{\eta}$. Beyond the sonic line points $\xi = \bar{\eta}$ do not, however, correspond to real supersonic flow. The stream function $\psi(\xi,\bar{\xi})$ is thus real in the subsonic domain but complex otherwise.

A new boundary value problem was formulated in [9] for the stream function $\psi(\xi,\eta)$. Let ψ be a solution of the canonical system (2.7) that is allowed to have a pole together with a logarithmic singularity in the hodograph plane. The boundary condition

$$\mathrm{Re}\left\{\psi(\xi,\bar{\xi})\right\} = 0 , \qquad |\xi| = 1$$

is imposed. For this purpose a suitable branch of the analytic function $\psi(\xi,\eta)$ has to be specified.

Paths which are complex conjugates of each other are used to compute the solution in the real subsonic region [1,9,14]. Given the two initial characteristic planes $\xi = \xi_0$ and $\eta = \eta_0$, a path $\xi = \xi(\sigma)$ is defined in the plane $\eta = \eta_0$. The conjugate path $\eta(\tau)$ is defined in the characteristic plane $\xi = \xi_0$ as the set of points $\eta = \overline{\xi(\sigma)}$. Consider the rectangle formed by taking the Cartesian product of these paths. The diagonal of the rectangle is given by the set of points $(\xi,\eta) = (\xi,\bar{\xi})$ and they correspond to real subsonic points until the sonic line is reached.

To compute points below the sonic line special paths have to be defined that avoid the complex sonic surface $M^2=1$, where the canonical equations become singular. To achieve this the paths have to be separated so that they are no longer conjugate. An analytic continuation of the solution around the sonic line that is appropriate for the boundary value problem is defined by such integration paths.

The geometry in the space of the canonical coordinates X and Y, in which our boundary value problem for the Tricomi equation is formulated, is much simpler. In this space the complex characteristic coordinates are given by

$$\xi = X + iY,$$

and

$$\eta = X - iY .$$

The sonic surface corresponds to the complex plane $Y = 0$. For analytic continuation around $Y = 0$, a path that connects the points $(X,Y) = (0,1)$ and $(X,Y) = (0,-1)$ can simply be defined by

$$X = 0 , \qquad Y = e^{\theta i} , \qquad 0 \leq \theta \leq \pi .$$

The initial characteristic planes through the point $(0,1)$ are given by $\xi = i$ and $\eta = -i$. The corresponding ξ-path and η-path are respectively

$$\xi = iY = e^{\pi i/2}e^{i\theta} , \qquad 0 \leq \theta \leq \pi,$$

and

$$\eta = - iY = e^{3\pi i/2}e^{i\theta} , \qquad 0 \leq \theta \leq \pi .$$

In Figure 1 we present these paths. They make it possible to extend solutions of (1.2) from the upper unit semicircle in the X,Y plane through the complex domain into the lower unit semicircle.

3. GREEN'S AND NEUMANN'S FUNCTIONS AND THE POISSON FORMULAS

In this chapter we give closed formulas for the solutions of the Dirichlet and mixed Dirichlet and Neumann problems in the upper unit semicircle for the more general equation

$$u_{XX} + u_{YY} + \frac{k}{Y} u_Y = 0 \ , \tag{3.1}$$

with values of k to be specified. This is the Euler-Poisson-Darboux equation and is also known in the literature as the generalized axially symmetric potential equation [27].

For k a positive integer (3.1) is the equation of a potential with axial symmetry in space of dimension $n = k+2$. Weinstein has considered equation (3.1) as the equation for the axially symmetric potential in a fictitious space of dimension k+2, with k positive and real [25-28]. The parameter k will be referred to as the index of the equation.

A fundamental solution valid for values of $k > 0$ is given in Section 3.1. This fundamental solution is the Neumann function for the upper half-plane. In Section 3.2 we obtain the Green's function for the upper unit semicircle when $0 < k < 1$. In Section 3.3 we use this Green's function to obtain a Poisson formula for the unit semicircle; that is, we give a representation of the solution with vanishing boundary values on $Y = 0$ in terms of its values on the boundary of the semicircle. In Section 3.4 we obtain for $k > 0$ the representation of the solution when its values are prescribed on the boundary of the unit semicircle and the normal derivative vanishes on the X axis. We obtain these two representations independently. However, a "correspondence principle" [26] that will be used in Section 3.2 relates solutions of (3.1) with different indices and makes it possible to deduce the solution of the Dirichlet problem from that of the Neumann problem.

3.1 The Fundamental Solution

A fundamental solution for (3.1) with index $k > 0$ is given by

$$(3.2)\quad S(X,Y;\xi,\eta) = \eta^k \int_0^\pi \frac{\sin^{k-1}\alpha \, d\alpha}{[(X-\xi)^2 + (Y-\eta)^2 + 4Y\eta \sin^2 \frac{\alpha}{2}]^{k/2}} ,$$

where the singularity is at the point (ξ,η). This is a generalization of the Gauss formula for the gravitational potential of a circular wire in a three-dimensional space, to which it reduces if we make $k = 1$ ([10], p. 148). Similar formulas have been given by Weinstein [25-28], although there is some confusion in the literature.

Fundamental solutions have been known for the Tricomi equation since the time of his original paper [23].

It can be checked by direct calculation that (3.2) is a solution of equation (3.1). Moreover, as a function of the parameters ξ and η , (3.2) is a solution of the adjoint equation

$$v_{\xi\xi} + v_{\eta\eta} - \frac{k}{\eta} v_{\eta} + \frac{k}{\eta^2} v = 0 \ . \tag{3.3}$$

We introduce the notation

$$P_1^2 = (X-\xi)^2 + (Y-\eta)^2 \ , \quad Q_1^2 = (X-\xi)^2 + (Y+\eta)^2 \ ,$$
$$q^2 = \frac{P_1^2}{Q_1^2} \ . \tag{3.4}$$

With this notation (3.2) can be written in the form [25]

$$S(X,Y;\xi,\eta) = \left(\frac{\eta}{Y}\right)^{k/2} \int_0^1 t^{k-1} (1-t^2)^{(k-2)/2} \left(\frac{q^2}{1-q^2} + t^2\right)^{-k/2} dt. \tag{3.5}$$

This is a hypergeometric integral. It can be expressed as

$$S(X,Y;\xi,\eta) = \frac{2^{k-1}\,\eta^k}{Q_1^k} \frac{\left(\Gamma(\frac{k}{2})\right)^2}{\Gamma(k)} F\left(\frac{k}{2},\frac{k}{2};k;1-q^2\right) \tag{3.6}$$

by use of the relationship [18]

$$(3.7)\quad F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt,$$

valid for $\operatorname{Re} c > \operatorname{Re} b > 0$, $|\arg(1-z)| < \pi$.

The hypergeometric differential equation has three singular points, and at each of those points one of the two linearly independent solutions has a logarithmic singularity. Using known relations for the hypergeometric function [17], we can rewrite formula (3.6) as

$$(3.8)\quad S(X,Y;\xi,\eta) = \frac{2^k \eta^k}{Q_1^k} \left\{ F\left(\frac{k}{2},\frac{k}{2};1;q^2\right) \log \frac{1}{q} + G(X,Y;\xi,\eta) \right\},$$

where G is a regular function. The coefficient of the logarithmic term, namely,

$$(3.9)\qquad A(X,Y;\xi,\eta) = \frac{2^k \eta^k}{Q_1^k} F\left(\frac{k}{2},\frac{k}{2};1;q^2\right),$$

is the Riemann function of equation (3.1), which shows that (3.8) has the correct singularity for a fundamental solution.

3.2 The Green's Function for the Unit Semicircle

Let $z(k)$ be a solution of equation (3.1), where temporarily we let the index k be either positive or negative. The following "correspondence principle" has been introduced by Darboux and extensively used by Weinstein and Diaz [7,26]:

$$(3.10) \qquad z(k) = Y^{1-k}\, z(2-k)\ .$$

The proof is done by direct verification. For $0 < k < 2$ it relates solutions with positive index. Using this principle we have from (3.2) a new fundamental solution [28]

$$(3.11) \quad U(X,Y;\xi,\eta) = \eta Y^{1-k} \int_0^{\pi} \frac{\sin^{1-k}\alpha\, d\alpha}{[(X-\xi)^2+(Y-\eta)^2+4Y\eta\sin^2\frac{\alpha}{2}]^{(2-k)/2}}$$

The results of Section 3.1 applied to the solution of index $(2-k)$ show that (3.11) has the proper singularity. For $0 < k < 1$ the expression (3.11) is a Green's function relative to the upper half-plane, since it vanishes on the X axis.

We use the method of inversion to obtain the Green's function for the unit semicircle. Our idea is to extend Kelvin's transformation to equation (3.1). More precisely, it can be proved by direct calculation that if $u(X,Y)$ is a solution of (3.1), then so also is

$$u'(X,Y) = r^{-k}\, u(X',Y') \ ,$$

where

$$X' = \frac{X}{r^2}\ , \quad Y' = \frac{Y}{r^2}\ , \qquad r^2 = X^2 + Y^2\ .$$

Applying Kelvin's transformation to (3.11), we obtain a new solution of (3.1) with respect to the variables X and Y, namely,

$$(3.12) \qquad U'(X,Y;\xi,\eta)$$

$$= \eta Y^{1-k} r^{k-2} \int_0^{\pi} \frac{\sin^{1-k}\alpha\, d\alpha}{[(X'-\xi)^2+(Y'-\eta)^2+4Y'\eta\sin^2\frac{\alpha}{2}]^{(2-k)/2}}.$$

We recall that (3.11) is a solution of the adjoint equation (3.3) as a function of the variables ξ and η. Using the fact that if $u(X,Y)$ is a solution of (3.1), we find that

$$(3.13) \qquad v(X,Y) = Y^k\, u(X,Y)$$

is a solution of (3.3). Applying Kelvin's transformation, we obtain the new solution

$$(3.14) \qquad V(X,Y;\xi,\eta)$$

$$= \eta Y^{1-k} \rho^{k-2} \int_0^{\pi} \frac{\sin^{1-k}\alpha\, d\alpha}{[(X-\xi')^2+(Y-\eta')^2+4Y\eta'\sin^2\frac{\alpha}{2}]^{(2-k)/2}}\ ,$$

of (3.1), where

$$\xi' = \frac{\xi}{\rho^2}, \quad \eta' = \frac{\eta}{\rho^2}, \quad \rho^2 = \xi^2 + \eta^2 .$$

Formula (3.14) defines a fundamental solution of (3.1) with singularity at the point (ξ',η'), which is the inverse image of (ξ,η) with respect to the unit circle.

It can be verified that (3.11) and (3.14) take on the same values on the boundary of the unit semicircle. For that we have just to notice that for (X,Y) on the circumference of radius 1, the relation

$$\frac{R}{\rho} = \tilde{R}$$

holds, where

$$R^2 = (X-\xi)^2 + (Y-\eta)^2 , \qquad \tilde{R}^2 = (X-\xi')^2 + (Y-\eta')^2 .$$

Subtracting (3.14) from (3.11), we obtain the Green's function for the unit semicircle. Let us introduce the abridged notation

$$(3.15) \quad I_p(X,Y;\xi,\eta) = \int_0^{\pi} \frac{\sin^{1-k} \alpha \, d\alpha}{[(X-\xi)^2+(Y-\eta)^2+4Y\eta \sin^2 \frac{\alpha}{2}]^{p/2}} .$$

Then the Green's function for (3.1) in the unit semicircle, with index $0 < k < 1$, is

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$$(3.16) \quad G(X,Y;\xi,\eta) = \eta Y^{1-k}[I_{2-k}(X,Y;\xi,\eta) - r^{k-2} I_{2-k}(X,Y;\xi',\eta')] .$$

In the next section we shall insert this result into Green's formula to obtain a Poisson formula for the unit semicircle.

3.3 The Poisson Formula for the Unit Semicircle

The Green's function obtained in Section 3.2 can be used to solve the Dirichlet problem in the unit semicircle. We want to obtain the solution to (3.1) which takes given continuous values on the upper boundary ∂D of the unit semicircle D and which vanishes at $Y = 0$. We start from the representation formula [10]

$$(3.17) \quad u(X,Y) = \frac{1}{2\pi} \int_{\partial D} u(\xi,\eta) \frac{\partial}{\partial\nu} G_M(\xi,\eta;X,Y)\, ds ,$$

where G_M is the Green's function of the adjoint equation, $\partial/\partial\nu$ stands for the inner normal derivative, and s is the arc length.

For G_M we have the interchange rule

$$G_M(\xi,\eta;X,Y) = G_L(X,Y;\xi,\eta) ,$$

where G_L is the Green's function (3.16) of (3.1). Formula (3.17) can be applied to any domain such that the coefficients of equation (3.1) are regular up to the boundary. In particular, we can apply it to a domain D_ε obtained by deleting the strip $y < \varepsilon$ from the unit semicircle. By continuity we can then extend the formula to the case in which D is the unit semicircle itself. We can then write (3.17) as

$$(3.18)\quad u(X,Y) = -\frac{1}{2\pi}\int_0^\pi f(\phi)\frac{\partial}{\partial\rho}\Big\{Y^{1-k}\rho\sin\phi[I_{2-k}(X,Y;\xi,\eta) - \rho^{k-2} I_{2-k}(X,Y;\xi',\eta')]\Big\}\,d\phi\ ,$$

where $f(\phi)$ represents the values of the solution u on the boundary of the semicircle, and where use has been made of the fact that the solution vanishes at $Y = 0$. In this context we have introduced the notation

$$\xi = \rho\cos\phi\ ,\qquad \eta = \rho\sin\phi\ .$$

The kernel in (3.18) can be computed by a straightforward calculation. We obtain, using the notation (3.15),

$$(3.19)\quad \frac{\partial G}{\partial\rho} = (2-k)Y^{1-k}\sin\phi[I_{2-k}(X,Y;\xi,\eta) + 2\beta I_{4-k}(X,Y;\xi,\eta)]$$

at $\rho = 1$, where

$$\beta = X\cos\phi + Y\sin\phi - 1 - 2Y\sin\phi\sin^2\frac{\alpha}{2}\ .$$

Manipulating formula (3.19), we obtain

$$(3.20) \qquad \frac{\partial G}{\partial \rho} = (2-k)\ Y^{1-k} \sin\phi(r^2-1)\ I_{4-k}(X,Y;\xi,\eta)\ .$$

Therefore, the solution (3.18) can be written as

$$(3.21) \quad u(X,Y) = \frac{2-k}{2\pi}\int_0^{\pi} f(\phi)(1-r^2)Y^{1-k}$$

$$\cdot \sin\phi\left\{\int_0^{\pi}\frac{\sin^{1-k}\alpha\ d\alpha}{[(X-\xi)^2+(Y-\eta)^2+4Y\eta\sin^2\frac{\alpha}{2}]^{(4-k)/2}}\right\}d\phi,$$

when

$$\xi = \cos\phi\ , \qquad \eta = \sin\phi\ .$$

The hypergeometric integral under the bracket can be transformed into a hypergeometric function using results similar to those presented in Section 3.1. We obtain for this integral the formula

$$I_{4-k}(X,Y;\xi,\eta) = 2^{1-k}\frac{(\Gamma(1-\frac{k}{2}))^2}{\Gamma(2-k)}\ Q_1^{k-4}F(2-\frac{k}{2},\ 1-\frac{k}{2};\ 2-k;\ 1-q^2).$$

Making use of the quadratic transformation

$$F(2-\frac{k}{2},\ 1-\frac{k}{2};\ 2-k;z) = (1-z)^{-(1-k/4)}\ F\left(1-\frac{k}{2},\ -\frac{k}{4};\frac{3}{2}-\frac{k}{2};\frac{z^2}{4z-4}\right),$$

for the hypergeometric function ([18], p. 51), we obtain from (3.21) after further transformations, the representation

$$(3.22)\ u(X,Y) = cY^{1-k} \int_0^{\pi} \frac{f(\phi)[1-(X^2+Y^2)] \sin \phi}{(1+X^2+Y^2-2X \cos \phi)^{(4-k)/2}}$$

$$\cdot F\left(1 - \frac{k}{4}, \frac{3}{2} - \frac{k}{4}; \frac{3}{2} - \frac{k}{2}; \frac{4Y^2 \sin^2\phi}{(1+X^2+Y^2-2X \cos \phi)^2}\right) d\phi,$$

where

$$c = \frac{2-k}{2^k \pi} \frac{\left(\Gamma(1 - \frac{k}{2})\right)^2}{\Gamma(2-k)} .$$

This formula generalizes a result of Germain and Bader [8] that was given for the case $k = \frac{1}{3}$, and is the representation we have been seeking. It solves the Dirichlet problem for the Euler-Poisson-Darboux equation with index $0 < k < 1$ in the unit semicircle and with vanishing values of the solution at $Y = 0$.

3.4 The Mixed Dirichlet-Neumann Problem

The second problem we are interested in is the mixed boundary value problem in which the values of the solution are given on the upper boundary of the semicircle and the normal derivative vanishes at $Y = 0$. We ask that the solution be continuous up to the boundary. We have already found in (3.2) a Neumann's function for the upper half-plane, valid for $k > 0$. It can easily be checked that the normal derivative of this function vanishes at $Y = 0$.

In order to obtain a representation formula for the solution we have to devise a formula different from (3.17), since we are dealing with a Neumann condition at $Y = 0$. The difficulty is that we have an equation which is not self-adjoint, so we have to find a function that has simultaneously the properties of a Neumann's function for our equation and of a Green's function for the adjoint equation.

We start with the identity [10]

$$(3.23) \quad \iint_D (vL[u]-uM[v])\, dX\, dY + \int_{\partial D} [v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} + (a \frac{\partial X}{\partial \nu} + b \frac{\partial Y}{\partial \nu}) uv] ds = 0 ,$$

where

$$L[u] = u_{XX} + u_{YY} + a(X,Y)u_X + b(X,Y)u_Y + c(X,Y) = 0$$

and where $M[v] = 0$ is the adjoint equation.

This formula is valid if the coefficients a, b, and c are regular up to the boundary of the domain D. In our case we have

$$a = 0 , \quad b = \frac{k}{Y} , \quad c = 0 .$$

Formula (3.2) can be written as

$$N_L = \eta^k \; J_k(X,Y;\xi,\eta) \; ,$$

where we use the notation

$$J_p(X,Y;\xi,\eta) = \int_0^{\pi} \frac{\sin^{k-1} \alpha \; d\alpha}{[(X-\xi)^2+(Y-\eta)^2+4Y\eta \; \sin^2 \frac{\alpha}{2}]^{p/2}} \; .$$

Then,

$$N_M = Y^k \; J_k(X,Y;\xi,\eta)$$

is a fundamental solution for the adjoint equation

$$M[v] = v_{XX} + v_{YY} - \frac{k}{Y} v_Y + \frac{k}{Y^2} v = 0$$

By use of the Kelvin transformation we obtain, as in Section 3.3, the Green's function

$$(3.24) \quad G_M(X,Y;\xi,\eta) = Y^k[J_k(X,Y;\xi,\eta) - \rho^{-k}J_k(X,Y;\xi',\eta')] \; ,$$

where

$$\xi' = \frac{\xi}{\rho^2}, \quad \eta' = -\frac{\eta}{\rho^2}, \quad \rho^2 = \xi^2 + \eta^2 .$$

The function G_M vanishes on the whole boundary of the semi-circle, and the expression in square brackets in (3.24), is a solution of $L[u] = 0$ which has vanishing derivative with respect to Y at $Y = 0$.

We apply the identity (3.23) to a solution $u(X,Y)$ of (3.1) and to $v = G_M$. We take as domain of integration D_ε, the unit semicircle with a strip $0 \leq Y \leq \varepsilon$ deleted. Since G_M behaves like a Green's function on the upper boundary Γ of the semicircle, we obtain

$$2\pi u(\xi,\eta) = \int_\Gamma u(X,Y) \frac{\partial}{\partial \nu} G_M \, ds - \int_\gamma \left[G_M \frac{\partial u}{\partial Y} - u \frac{\partial G_M}{\partial Y} + \frac{k}{Y} uG_M\right] dX ,$$

where we note by γ the segment $Y = \varepsilon$ of D_ε. The second integral gives, after a cancellation and putting $J_k' = J_k(X,Y;\xi',\eta')$,

$$\int_\gamma \left[G_M \frac{\partial u}{\partial Y} - u \frac{\partial G_M}{\partial Y} + \frac{k}{Y} uG_M\right] dX =$$

$$= \int_\gamma \left[Y^k (J_k - \rho^{-k} J_k') \frac{\partial u}{\partial Y} - uy^k \frac{\partial}{\partial Y}(J_k - \rho^{-k} J_k')\right] dX .$$

If we now let $\varepsilon \to 0$, the whole integral over γ vanishes once we assume $\partial u/\partial Y$ bounded. Thus we obtain the formula

$$u(\xi,\eta) = -\frac{1}{2\pi}\int_0^\pi f(\phi)\,\frac{\partial}{\partial r}\,Y^k[J_k(X,Y;\xi,\eta) - \rho^{-k}J_k(X,Y;\xi',\eta')]\,d\phi,$$

or, equivalently,

$$u(X,Y) = -\frac{1}{2\pi}\int_0^\pi f(\phi)\,\frac{\partial}{\partial\rho}\,\eta^k[J_k(X,Y;\xi,\eta) - r^{-k}J_k(X',Y';\xi,\eta)]\,d\phi,$$

where

$$X' = \frac{X}{r^2}\,, \qquad Y' = \frac{Y}{r^2}\,, \qquad \text{and} \qquad r^2 = X^2 + Y^2\ .$$

Following the same steps as in Section 3.3 we are led to the formula

$$u(X,Y) = \frac{k}{2\pi}\int_0^\pi \sin^k\phi\ f(\phi)\left\{\int_0^\pi \frac{(1-r^2)\ \sin^{k-1}\alpha\ d\alpha}{[(X-\xi)^2+(Y-\eta)^2+4Y\eta\ \sin^2\frac{\alpha}{2}]^{(k+2)/2}}\right\}d\phi,$$

where

$$\xi = \cos\phi\ , \qquad \eta = \sin\phi\ .$$

This formula can be written, by using similar relations for the hypergeometric function, as

$$(3.25)\quad u(X,Y) = c\int_0^{\pi} \frac{f(\phi)\,[1-(X^2+Y^2)]\,\sin^k\phi}{(1+X^2+Y^2-2X\cos\phi)^{(k+2)/2}}$$

$$\cdot F\left(\frac{k}{4}+\frac{1}{2},\ \frac{k}{4}+1;\ \frac{k}{2}+\frac{1}{2};\ \frac{4Y^2\sin^2\phi}{(1+X^2+Y^2-2X\cos\phi)^2}\right)d\phi,$$

with

$$c = k\,\frac{2^{k-2}}{\pi}\,\frac{\left(\Gamma(\frac{k}{2})\right)^2}{\Gamma(k)}\ .$$

The result is valid for $k > 0$. It is the desired representation of the solution of the mixed Dirichlet-Neumann problem in the unit semicircle, with vanishing normal derivative on the X axis, in terms of the values of the solution on the boundary of the semicircle.

The correspondence principle (3.10) can be applied to the solution (3.25) with index $k' = 2-k$ and boundary values $\sin^{k-1}\phi\ f(\phi)$, $0 < k < 1$. We obtain from it the solution (3.22) of the Dirichlet problem with boundary values $f(\phi)$, as was mentioned earlier.

4. THE REFLECTION PRINCIPLE

We establish in this chapter reflection laws for solutions of the general Euler-Poisson-Darboux equation

$$(4.1) \qquad u_{XX} + u_{YY} + \frac{k}{Y} u_Y = 0 , \qquad 0 < k < 1 ,$$

which make it possible to extend them from the upper to the lower unit semicircle. They will be used in Chapter 5 to prove our main theorem about the transonic boundary value problem. However, they have an interest of their own. General theorems about reflection laws for solutions of elliptic partial differential equations with analytic conditions on the boundary are well known (Garabedian [11], Lewy [16]). The special feature of our case is that the boundary over which the reflection is going to be made is a singular line for the equation.

It is known (Hyman [12]) that a solution of (4.1) which is analytic in some neighborhood of a segment of the X axis is necessarily even in Y and is uniquely determined by its values on this segment. This is a local theorem based on power series expansion of the analytic solution, valid for k different from 0 or a negative integer. The result for $k = 1$ is classic [13]. However, this theorem does not apply to solutions with a branch singularity of the type

$$u(X,Y) = Y^{1-k} f(X,Y^2) \quad (4.2)$$

Our representation formula (3.22) is an example of this type of solution. We shall derive reflection laws that are valid in the large and permit one to reflect solutions of the type (4.2) for which the X axis is a branch line. We use analytic continuation of the solutions into the complex domain of the variables X and Y to do this.

Let the variables X and Y be extended to complex values

$$X = X_1 + iX_2 , \quad Y = Y_1 + iY_2 = |Y|e^{i\theta} .$$

We define the path γ by

$$\gamma = \left\{(X,Y: X = 0, Y = re^{i\theta}, 0 \leq \theta \leq \pi \right\},$$

where $0 < r < 1$. Consider the analytic continuation of the solutions (3.22) and (3.25) along this path γ. The explicit form of the solutions guarantees that the continuation can be performed. Two points A and B, with A in the upper unit semicircle and B in the lower, can be connected by a path of the form

$$\Gamma = \gamma_1 \cup \gamma \cup \gamma_2 , \quad (4.3)$$

where γ_1 is a path contained in the upper semicircle which connects the points A and (0,r), and γ_2 is the corresponding path which connects B and (0,-r) in the lower semicircle. By the monodromy theorem, any other path which does not wind around the X axis will give the same result.

We obtain in this fashion the following theorem: If $u(X,Y)$ is a solution to the Dirichlet problem for equation (4.1) in the upper unit semicircle with zero boundary values $u = 0$ at $Y = 0$, then it can be extended to the lower semicircle by the reflection law,

$$u(X,-Y) = e^{(1-k)\pi i}\, u(X,Y) , \qquad Y > 0 , \tag{4.4}$$

where analytic continuation has been made along the path (4.3). Similarly, the solution of the mixed Dirichlet and Neumann problem in the same domain, with vanishing normal derivative $u_Y = 0$ at $Y = 0$, can be extended to the lower semicircle according to the law

$$u(X,-Y) = u(X,Y) , \qquad Y > 0 . \tag{4.5}$$

Clearly, the reflection rules depend only on the form of the boundary condition at $Y = 0$ and are valid in more general domains.

5. SOLUTION OF THE BOUNDARY VALUE PROBLEM

We can now state and prove our main theorem. Consider the Tricomi equation

$$(5.1) \qquad yu_{xx} + u_{yy} = 0 ,$$

and the transformed equation

$$(5.2) \qquad u_{XX} + u_{YY} + \frac{1}{3Y} u_Y = 0 ,$$

in canonical coordinates X and Y. We introduce the class of analytic solutions of equation (5.1) which are single-valued in a neighborhood of the line $y = 0$. Let $u(X,Y)$ be a real solution of (5.2) in the upper unit semicircle

$$D_+ = \left\{(X,Y): X^2 + Y^2 < 1 , Y > 0\right\}$$

belonging to this class. Then in the complex domain of the extended variables X and Y the solution $u(X,Y)$ can be continued analytically along a path, as described in (4.3), into the lower unit semicircle,

$$D_- = \left\{(X,Y): X^2 + Y^2 < 1 , Y < 0\right\} .$$

We shall prove that the boundary value problem

$$\mathrm{Re}\left\{u(X,Y)\right\} = F(\theta) , \quad X = \cos\theta , \quad Y = \sin\theta , \quad 0 \le \theta \le 2\pi,$$

where $F(\theta)$ is periodic and continuous, is well posed within the specified class of solutions $u(X,Y)$.

We give first an explanation of the choice of the class of functions that we have made. If $u(x,y)$ is single valued and analytic, it admits a double power series expansion in a neighborhood of the origin of the form

$$u(x,y) = \sum a_{nm} x^n y^m .$$

Therefore, we can write

$$u(x,y) = \tilde{v}_1(x,y^3) + y\tilde{v}_2(x,y^3) + y^2\tilde{v}_3(x,y^3) ,$$

or

$$u(X,Y) = v_1(X,Y^2) + Y^{2/3} v_2(X,Y^2) + Y^{4/3} v_3(X,Y^2) ,$$

where v_1 , v_2 and v_3 are regular.

If we let the variable Y wind around the X-axis, through an angle of 2π or 4π, we obtain new solutions

$$u'(X,Y) = v_1(X,Y^2) + e^{4\pi i/3} Y^{2/3} v_2(X,Y^2) + e^{8\pi i/3}Y^{4/3}v_3(X,Y^2)$$

and

$$u''(X,Y) = v_1(X,Y^2) + e^{8\pi i/3}Y^{2/3}v_2(X,Y^2) + e^{16\pi i/3}Y^{4/3}v_3(X,Y^2)$$

of equation (5.2), as the equation itself remains unchanged.

We apply now the differential operator L defined by equation (5.2) to u, u' and u". This gives

$$L[v_1] + L[Y^{2/3}v_2] + L[Y^{4/3}v_3] = 0$$

$$L[v_1] + e^{4\pi i/3}\, L[Y^{2/3}v_2] + e^{8\pi i/3}\, L[Y^{4/3}v_3] = 0$$

$$L[v_1] + e^{8\pi i/3}\, L[Y^{2/3}v_2] + e^{16\pi i/3}\, L[Y^{4/3}v_3] = 0 \,.$$

Because the determinant of this linear system of equations is different from zero, it follows that each of the terms $v_1(x,y^3)$, $yv_2(x,y^3)$, and $y^2v_3(x,y^3)$ is a solution of the Tricomi equation separately. By the uniqueness theorem for the Cauchy problem for (5.1), we can conclude that

$$y^2v_3(x,y^3) \equiv 0 \,.$$

Therefore, $u(X,Y)$ admits a unique decomposition of the form

$$(5.3) \qquad u(X,Y) = v_1(X,Y^2) + Y^{2/3}v_2(X,Y^2) \,.$$

We proceed now to prove the main theorem. Let the boundary function $F(\theta)$ be defined by

$$u(X,Y) = h(X) \;; \quad Y \geq 0 \,, \qquad X^2 + Y^2 = 1 \,,$$

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and

$$\text{Re}\left\{u(X,Y)\right\} = k(X) \; ; \qquad Y \leq 0 \, , \qquad X^2 + Y^2 = 1 \, .$$

Let $u_1(X,Y)$ be a solution of the Dirichlet problem in the upper unit semicircle such that

$$u_1(X,Y) = f(X) \, , \qquad X^2 + Y^2 = 1 \, , \qquad Y > 0 \, ,$$

$$u_1(X,0) = 0 \, ,$$

with $f(X)$ continuous and vanishing at $X = \pm 1$. Let $u_2(X,Y)$ be a solution of the mixed Dirichlet-Neumann problem, in the same domain, with

$$u_2(X,Y) = g(x) \, , \qquad X^2 + Y^2 = 1 \, , \qquad Y > 0 \, ,$$

$$\frac{\partial u_2}{\partial Y}(X,0) = 0 \, .$$

Both $u_1(X,Y)$ and $u_2(X,Y)$ can be computed explicitly by means of the representation formulas (3.22) and (3.25) of Chapter 3.

By the reflection laws obtained in Chapter 4, we have

$$u_1(X,-Y) = e^{2\pi i/3} f(X) \, , \qquad Y > 0 \, , \qquad X^2 + Y^2 = 1 \, ,$$

and

$$u_2(X,-Y) = g(X) \, , \qquad Y > 0 \, , \qquad X^2 + Y^2 = 1 \, .$$

If we choose f and g so that

$$f(X) + g(X) = h(X)$$

$$f(X) \cos \frac{2}{3} \pi + g(X) = k(X) \; ,$$

that is, if we set

$$f(X) = \frac{2}{3} \left(h(X) - k(X)\right)$$

and

$$g(X) = \frac{h(X)}{3} + \frac{2}{3} k(X) \quad ,$$

then

$$u(X,Y) = u_1(X,Y) + u_2(X,Y) \tag{5.4}$$

is a solution of (5.2) which satisfies the prescribed boundary conditions.

The uniqueness of the solution of the boundary value problem follows immediately. The representation formulas (3.22) and (3.25) show that $u_1(X,Y)$ and $u_2(X,Y)$ have the form

$$u_1(X,Y) = Y^{2/3} v_1(X,Y^2)$$

and

$$u_2(X,Y) = v_2(X,Y^2)$$

respectively. The decomposition (5.4) is therefore of the form (5.3) and is unique. The boundary condition on u is seen to impose Dirichlet and Dirichlet-Neumann data on u_1 and u_2 respectively, which determine them uniquely, completing the proof. The continuous dependence on the boundary values is also a consequence of the explicit representation we have given of the solution.

We remark that the theorem could be extended to arbitrary domains symmetric with respect to the X axis by performing reflection in the manner described in this work, but resorting to a more general treatment of the Dirichlet and Dirichlet-Neumann problem for u_1 and u_2.

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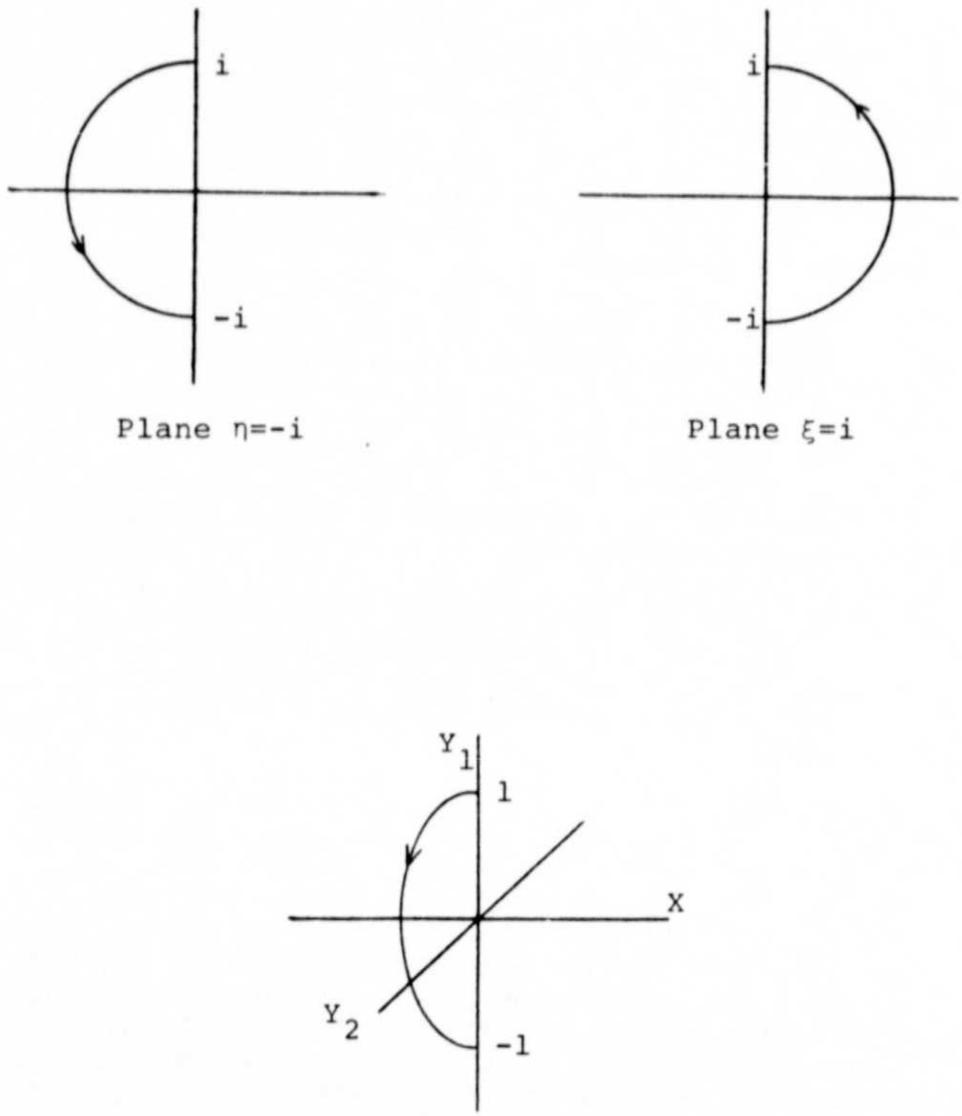

Figure 1.