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An Information Theory Approach to the Density of the Earth

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ABSTRACT

Information theory can develop a technique which takes experimentally determined numbers and produces a uniquely specified "best" density model satisfying those numbers. This technique does not depend on previous density models for starting conditions; it is self-starting. A model was generated using five numerical parameters: the mass of the earth, its moment of inertia, three zero-node torsional normal modes ($l = 2, 8, 26$). In order to determine the stability of the solution, six additional density models were generated, in each of which the period of one of the three normal modes was increased or decreased by one standard deviation. The superposition of the seven models is shown in Figure 4. It indicates that current knowledge of the torsional modes is sufficient to specify the density in the upper mantle but that the lower mantle and core will require smaller standard deviations before they can be accurately specified.

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AN INFORMATION THEORY APPROACH TO THE DENSITY OF THE EARTH

In information theory, an important consideration in determining probability distributions has been the maximization of missing information. This concept is important because, "If we wish to avoid introducing a bias, if we wish to avoid unwarranted preference for some inferences and unwarranted neglect of others, we must choose the compatible probability distribution with the largest value of the missing information," (Baierlein 1971, p. 87). The basis for this assertion is fully developed by Baierlein in Chapter 3 where it is shown that the missing information, MI, is

$$MI = \sum p \ln p$$

where p is a probability.

David P. Rubincam (private communication 1976) has proposed the novel idea that there is a relationship between the density of a body and a probability function. The justification is developed in this manner; consider that a solid body is composed of infinitesimal, incompressible ping-pong balls, each having equal mass. Then the density of an arbitrary unit volume of the solid is proportional to the number of ping-pong balls in the unit volume at that point. If one of the ping-pong balls is painted red (all others are white), the probability of finding that red ping-pong ball in a given unit volume of the body is related to the number

of ping-pong balls in that volume. Thus a relationship is developed between a probability function and a density function.

The probability of finding the red ball is large if there are many balls in the volume, small if there are few, and zero if there are none. Thus the most general relationship between the probability and density is

$$p(x,y,z) = \sum_{i=1}^{\infty} a_i \, [\rho(x,y,z)]^i$$

For the purposes of this paper, the series is truncated after one term and it is asserted that

$$p \propto \rho$$

This yields the extremal principle for the density of a solid body

$$I = \int k\rho \ln(k\rho)\, dv \qquad \delta I = 0 \qquad (1)$$

with $\underline{k}$ an unspecified constant. This equation with suitable integral constraints forms the basis for the remainder of this paper.

The constraints which will be applied to Equation (1) have the form of an integral involving the density

$$K_1 = \int \rho f_1(r)\, dv$$

$$K_2 = \int \rho f_2(r)\, dv \qquad (2)$$

where K_1 and K_2 are physical constants such as mass or moment of inertia. These would be incorporated into the variational equation with Lagrange multipliers, resulting in

$$I = \int k\rho \ln(k\rho)dv + \lambda_1 \int \rho f_1(r)dv + \lambda_2 \int \rho f_2(r)dv$$

The functional form of the density can then be determined from the Euler-Lagrange equation

$$\frac{\partial f}{\partial \rho} - \frac{d}{dr}\frac{\partial f}{\partial \rho'} = 0$$

where

$$\rho' = \frac{d\rho}{dr}$$

$$f = 4\pi r^2[k\rho \ln(k\rho) + \lambda_1 \rho f_1(r) + \lambda_2 \rho f_2(r)]$$

This yields

$$k \ln(k\rho) + k + \lambda_1 f_1(r) + \lambda_2 f_2(r) = 0$$

$$\rho = \left[\frac{1}{k} \exp(-1)\right] * \exp\left[-\frac{\lambda_1}{k} f_1(r) - \frac{\lambda_2}{k} f_2(r)\right]$$

By redefining the Lagrange multipliers,

$$\rho(r) = A \exp[-\lambda_1 f_1(r) - \lambda_2 f_2(r)] \qquad (3)$$

The number of constraints which may be incorporated is limited only by computational considerations and is not necessarily two as in the above example.

The values of the constants in Equation (3) are determined by substituting the density equation into the original constraint equations. There appears to be a difficulty here in that there are two equations and three unknowns, which leads to an indeterminate situation. However, in practice this is overcome by always making the first constraint conservation of total mass

$$K_1 = M = \int \rho \, dv$$

Thus $f_1(r) = 1$ and

$$\rho(r) = A \exp[- \lambda_1 - \lambda_2 f_2(r)]$$

By redefining A to include $\exp(-\lambda_1)$ the equation becomes, for two constraints

$$\rho(r) = A \exp[-\lambda_2 f_2(r)]$$

and we are back to two equations and two unknowns.

Rubincam's first computation of a density distribution utilized two constraints: mass and moment of inertia. The second constraint is placed into a spherically symmetric mode by noting

$$\int \rho(x^2 + y^2) \, dv = C_{\text{polar}}$$

$$\int \rho(y^2 + z^2) \, dv = C_{\text{equatorial}}$$

$$\int \rho(z^2 + x^2) \, dv = C_{\text{equatorial}}$$

$$\int \rho\, 2\, r^2\, dv = C_{polar} + 2\, C_{equatorial}$$

$$\int \rho\, r^2\, dv = 0.5(C_{polar} + 2\, C_{equatorial})$$

So that numerically

$$\int \rho\, dv = 5.976 \times 10^{27}\, gm$$

$$\int \rho\, r^2\, dv = 1.5(8.051 \times 10^{34}\, gm\, km^2)$$

with $f_2(r) = r^2$ this yields the two-parameter density model

$$\rho(r) = A \exp[-\lambda_2 r^2]$$

and $A = 12.14 \times 10^{15}\, gm/km^3$, $\lambda_2 = 3.535 \times 10^{-8}\, km^{-2}$.

This paper is concerned with extending Rubincam's two-parameter model (unpublished) by developing integral constraints based on the eigenvalues ω_L of the torsional normal modes of the earth. These constraints have the form

$$\omega_L^2 = \int \rho\, f_L(r)\, dv \qquad (4)$$

The precise form of f_L is expressed in Appendix 4, Equation (A4-1).

The function $f_L(r)$ depends on the transverse velocity $v_T(r)$, the displacement potential $W(r)$ of the mode (see Appendix 3), and its first derivative. The requirement for v_T can be satisfied by experimental seismology. Several recent

papers (Choudhury, Poupinet, Perrier 1975; Braile, Keller 1975; Huestis, Molnar, Oliver 1973; Niazi 1973) were used to determine the mantle velocity profile shown in Figure 1. Qamar (1973) and Niazi (1973) used core radii of 3477 km and 3485 km, respectively, in their velocity models. In Figure 1, the cut-off 3481 km was chosen. This discontinuity in v_I generates the core-mantle density discontinuities exhibited in the derived density models.

Since $f_L(r)$ depends on the displacement potential of the normal mode, a bootstrap method is required. Beginning with the two-parameter density model, the

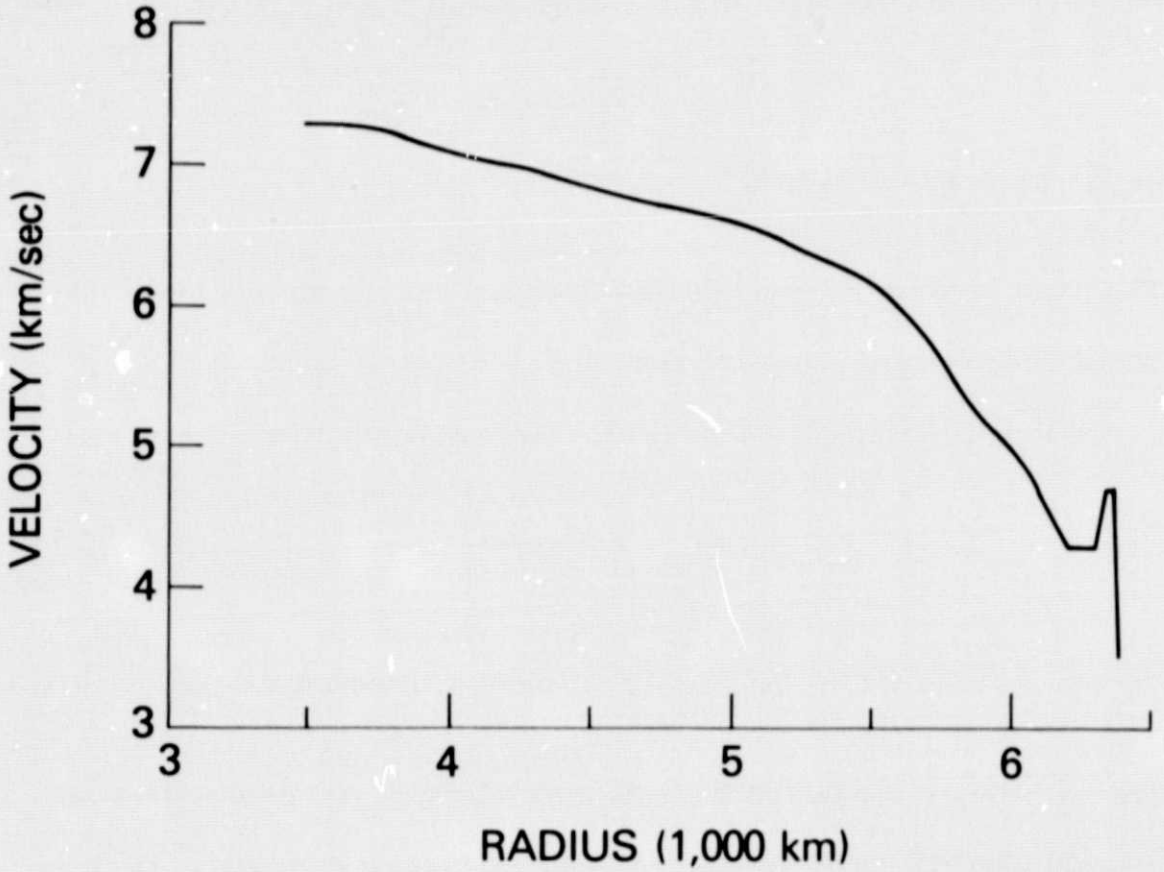

Figure 1. Transverse Velocity Profile of the Mantle

first normal mode is calculated using the Ritz method (see Appendix 3) and finite elements (see Appendix 2). The solution for the displacement potential is substituted into $f_L(r)$ and a preliminary three-parameter density is generated. This density model generates an improved displacement potential, which in turn is used to generate another three-parameter density model. The procedure is continued until the density model converges to a stable solution. No attempt was made to prove the uniqueness of the solution.

Then a second normal mode was added, and the three-parameter solution was boot-strapped into a four-parameter solution. And this was repeated for the addition of a third normal mode.

One feature of this calculation which has been temporarily ignored is the assumption used in deriving the form for the density function, Equation (3), that $f_L(r)$ is independent of the density. Obviously, the displacement potential depends on the density model. The normalization requirement (see Appendix 3) also introduces a density dependence. However, within the range of density models considered here, the displacement potentials varied quite slowly, indicating that the approximation assuming complete independence has some value.

The torsional normal modes chosen for this calculation are the zero-node oscillations of degree 2, 8, 26. The observed periods for these torsional normal modes are (Bullen 1975, p. 305)

Degree	Period (sec)
2	2642. ± 24.
8	735.0 ± 1.3
26	289.8 ± 0.3

Kanamori and Anderson (1977) have shown that these observations include a contribution due to dispersion. Therefore, in a non-dispersive model such as the present one, the values which should be used are

Degree	Period (sec)
2	2624. ± 26.
8	727.6 ± 2.8
26	285.7 ± 1.1

where the probable error has been increased to incorporate Kanamori and Anderson's statement that the amplitudes of the corrections are accurate to 20%.

The numerical improvement in the boot-strap procedure can be seen from the calculated periods for the three torsional modes as shown in Table 1. The density distributions for the four models represented in Table 1 are shown in Figure 2.

In order to determine the stability of the five-parameter solution, individual periods were increased or decreased by the probable error and six additional density models were derived. These perturbed solutions with the nominal

Table 1

Model	Mode (degree)	Calculated Period (sec)	Displacement of Calculated From Observed Period
Two-Parameter Modes: None	2	2535.	-89.
	8	724.3	-3.3
	26	286.5	+0.8
Three-Parameter Modes: 2	2	2624.	0.
	8	747.1	+19.5
	26	289.8	+4.1
Four-Parameter Modes: 2,8	2	2623.	-1.
	8	727.3	-0.3
	26	281.8	-3.9
Five-Parameter Modes: 2,8,26	2	2624.	0.0
	8	727.6	0.0
	26	285.7	0.0

solution are shown in Figure 3 with the legend indicating which period has been adjusted.

For each of the seven density models, the periods of all the zero-node torsional modes from L = 2 to L = 44 were calculated. These calculated periods were compared with the values listed by Bullen (1975) after applying a correction for dispersion. The calculated periods were within 1.5 standard deviations from the expected periods.

The general variation of all the models is indicated in Figure 4 where all seven of the five-parameter models are superimposed. This figure indicates that

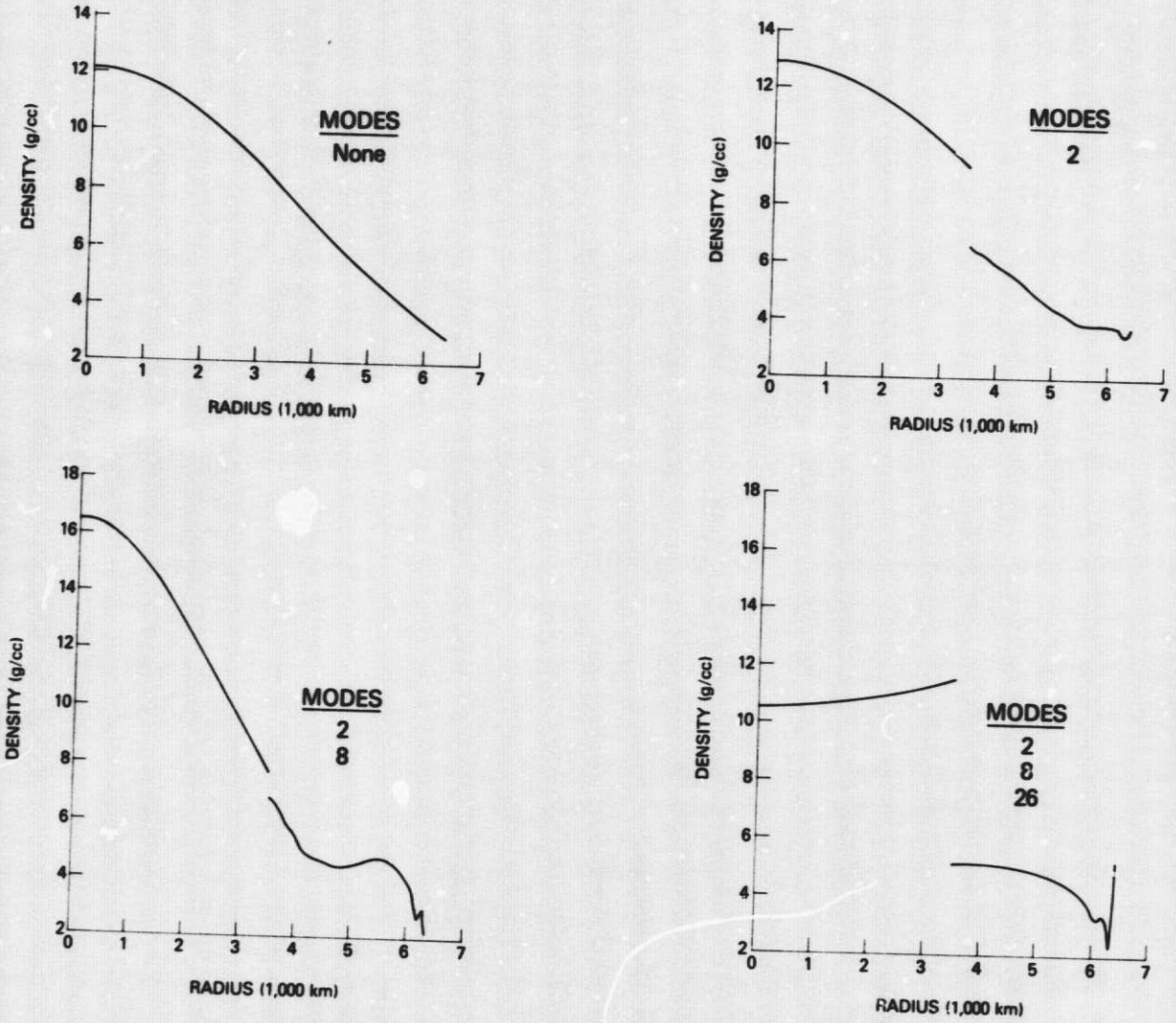

Figure 2. Solution Models Developed with Different Sets of Normal Modes

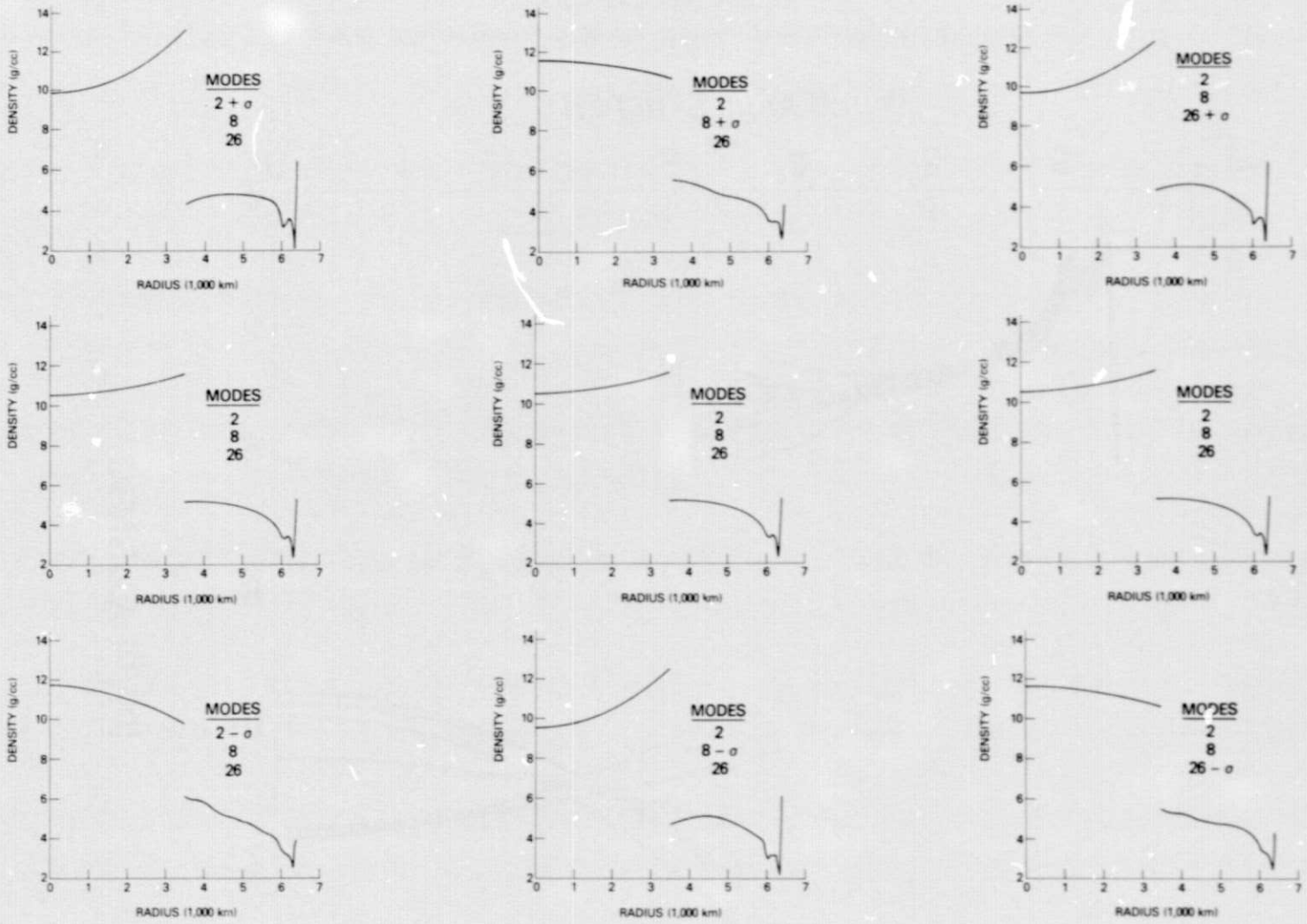

Figure 3. Comparison of Five-Parameter Models Having Shifted Periods with the Nominal Solution

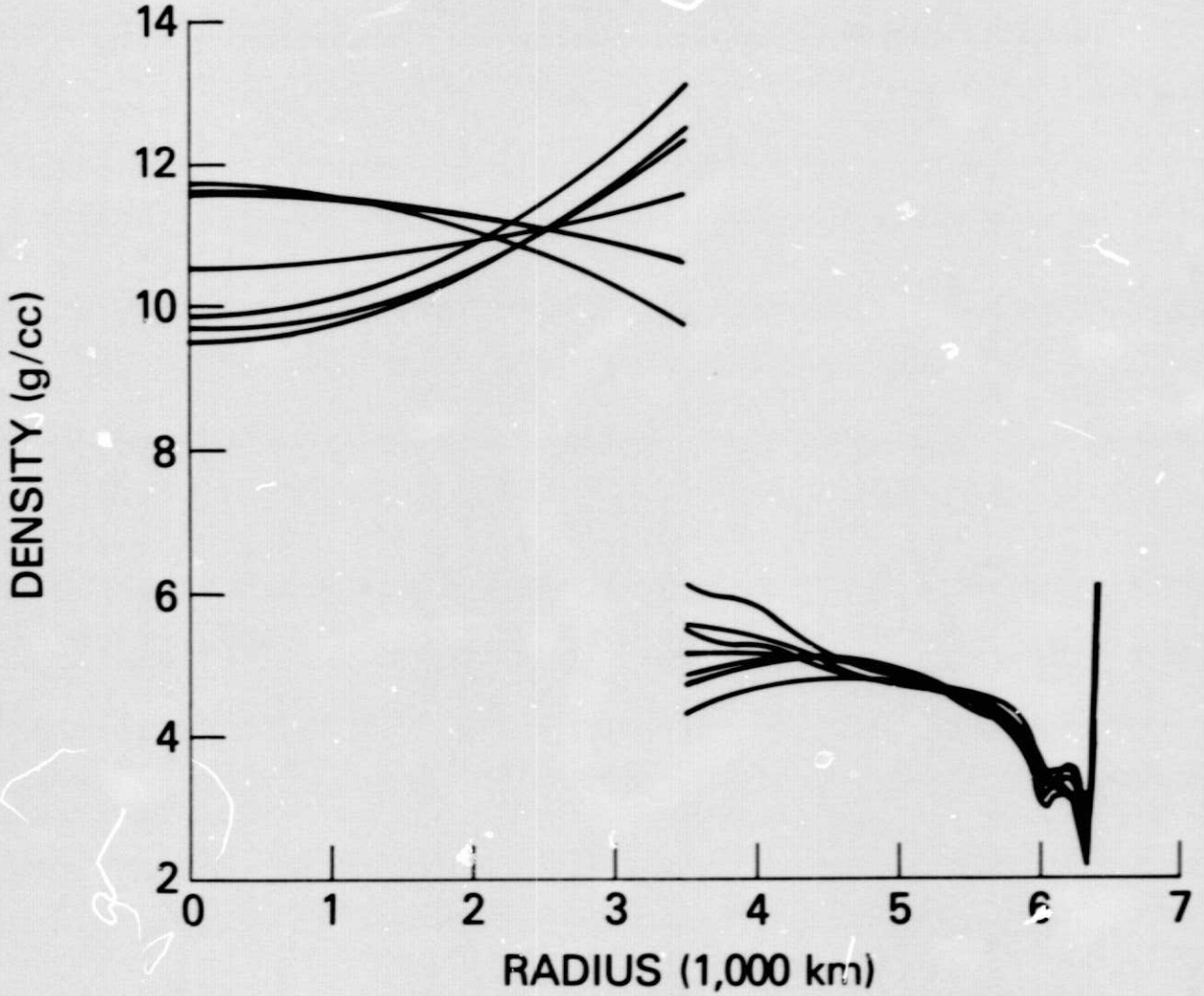

Figure 4. Superposition of Six Models Having Shifted Periods Plus the Nominal Solution

current knowledge of the periods of the torsional modes is adequate to specify the density distribution of the upper mantle, but not that of the lower mantle or core.

Currently accepted earth density models (Bullen, 1975) place a lower bound on the density at the earth's center at approximately 12 g/cc, with a probable value between 12 and 13 g/cc. The density then decreases as radius increases to reach approximately 9.5 to 10 g/cc at the outer core boundary. A discontinuity is present in the core at the inner core boundary. This is not present in any of the models in Figure 3 because none of the constraints contains information about the existence of the inner core. The addition to this calculation of spheroidal free oscillations would include the compressional velocity, and the discontinuity in this velocity profile would generate an inner core discontinuity. On the mantle side of the core-mantle boundary, the density is believed to be between 5.5 and 6 g/cc, and then it decreases to 3.3 g/cc in the upper mantle. In Figure 3, three of the perturbed solutions are very close to this currently accepted picture of the earth's interior. In considering the differences from current earth models, it is important to recall the simplicity of the present calculation and the small amount of experimental data included.

This information theory technique has taken experimentally derived numbers and constructed from them a nominal density model with additional perturbed models to indicate the general uncertainty in the solution. An improvement in the value of this technique would occur if the relative probable errors of the periods could be brought below 10^{-4}. In order to accomplish this reduction,

the error associated with the correction for dispersion would have to be reduced along with the observational error. With the probable errors sufficiently small, this technique would specify very precise earth density models, and comparison of these with existing models would provide a test of the assumptions upon which this technique is based.

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APPENDIX 1

SCALAR POTENTIALS FOR SECOND-RANK TANGENT TENSOR FIELDS

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SCALAR POTENTIALS FOR SECOND-RANK TANGENT TENSOR FIELDS

A major computational simplification can be achieved by replacing tensor differential equations with equivalent scalar differential equations. Backus (1966; 1967) suggested a technique for accomplishing this simplification in spherical systems. Backus' method has been followed here. More recently, Phinney and Burridge (Burridge 1969; Phinney and Burridge 1973) and James (1976) have developed similar techniques with more general applicability.

Backus' technique is similar to replacing a vector field by three scalar potentials. Defining the outward normal vector $\hat{n}$ and the tangential gradient operator

$$\vec{\nabla}_s = \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi}$$

then a general vector field can be written

$$\vec{A} = \hat{n}\, U(r,\theta,\phi) + \vec{\nabla}_s V(r,\theta,\phi) - \hat{n} \times \vec{\nabla}_s W(r,\theta,\phi)$$

in terms of three scalar potentials U, V, W.

Backus derives a scalar representation for a general second-rank tensor in a three-dimensional space. However, for brevity, the result for a symmetric tensor is given here, since these are of primary interest. Using a tensor

notation more in agreement with current practice, Backus' equation 4.13 (made symmetric) is

$$
\begin{aligned}
T_{rr} &= P \\
T_{ir} &= T_{ri} = Q_{,i} + \epsilon_{ij}\, g^{jk}\, R_{,k} \\
T_{ji} &= T_{ij} = [L + \ell(\ell+1)M]\, g_{ij} + 2\, M_{,ij} \\
&\quad + \epsilon_{in}\, g^{nk}\, N_{,kj} + \epsilon_{jn}\, g^{nk}\, N_{,ki}
\end{aligned}
\qquad \text{(A1-1)}
$$

where i, j, k, or n = θ or ϕ

P, Q, R, L, M, N are six scalar functions of position; ϵ_{ij} is the two-dimensional Levi-Civita tensor; g_{ij} the metric tensor restricted to i, j = θ, ϕ on a unit sphere. $R_{,i}$ represents tangential covariant differentiation and $F_{,ij}$ represents two applications of tangential covariant differentiation.

The tangential covariant derivative is calculated in the manner of an ordinary covariant derivative, except all terms introduced by the connection coefficients which are superscripted by "r" are eliminated and the derivatives are evaluated on a unit sphere so r = 1. This can be illustrated in an example by calculating $g^{ij} A_{,ij}$ for the basis described below, at the end of this appendix, Equation (A1-2)

$$
A_{,\theta} = \frac{1}{r}\frac{\partial}{\partial\theta} A = \frac{\partial}{\partial\theta} A \quad (r = 1)
$$

$$
A_{,\phi} = \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} A = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} A
$$

$$A_{,\theta\theta} = \frac{\partial^2}{\partial\theta^2} A - \Gamma^{i}_{\theta\theta} A_{,i}$$

where $i = \theta$ or ϕ (not r)

$$\Gamma^{\theta}_{\theta\theta} = 0$$

$$\Gamma^{\phi}_{\theta\theta} = 0$$

$$A_{,\phi\phi} = \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} A - \Gamma^{i}_{\phi\phi} A_{,i}$$

where $i = \theta$, or ϕ (not r)

$$\Gamma^{\theta}_{\phi\phi} = - \frac{\cot\theta}{r} = - \cot\theta$$

$$\Gamma^{\phi}_{\phi\phi} = 0$$

$$A_{,\phi\phi} = \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} A + \cot\theta \frac{\partial}{\partial\theta} A$$

$$g^{ij} A_{,ij} = A_{,\theta\theta} + A_{,\phi\phi}$$

$$= \frac{\partial^2}{\partial\theta^2} A + \cot\theta \frac{\partial A}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} A$$

If $A(r, \theta, \phi) \equiv A(r)\, Y_{\ell m}(\theta, \phi)$, where $Y_{\ell m}$ is a spherical harmonic, then

$$g^{ij} A_{,ij} = - \ell(\ell + 1) A$$

This identity has been used in the expression for T_{ij} in Equation (A1-1) where $\ell(\ell + 1)M$ has replaced $-g^{ij} M_{,ij}$. For a more general equation, the reversing substitution may be made.

In reading Backus (1967) it is worthwhile to note that while he indicates in his Equations 2.22 and 2.26 that the connection coefficients Γ^{i}_{jk} are equal to the Christoffel symbols of the second kind $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$, this equality in fact only occurs in holonomic bases. This becomes important because the most commonly used spherical coordinate system is anholonomic, namely

$$ds^2 = dx^{1\,2} + dx^{2\,2} + dx^{3\,2}$$

where

$$\begin{aligned} dx^1 &= dr \\ dx^2 &= r\, d\theta \\ dx^3 &= r \sin\theta\, d\phi \end{aligned} \qquad \text{(A1-2)}$$

and

$$g_{ij} = \delta_{ij}$$

In an anholonomic system (Misner, Thorne, Wheeler, 1973, p. 210)

$$\Gamma^{\ell}_{jk} = \left\{ \begin{matrix} \ell \\ jk \end{matrix} \right\} + \frac{1}{2}\, g^{\ell i}(c_{ijk} + c_{ikj} - c_{jki})$$

where c_{ijk} are the commutation coefficients of the basis. In the basis given in Equation (A1-2)

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = 0 \qquad \text{for all i,j,k}$$

while in general

$$\Gamma^{i}_{jk} \neq 0$$

APPENDIX 2

FINITE ELEMENTS IN THE CALCULUS OF VARIATIONS

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The general format of a problem in variational calculus requires the minimization of I, where

$$I = \int_{x_1}^{x_2} f(x,y,y') \, dx$$

and y is a function of x over the range of integration having a first derivative y'. The minimization of I is carried out with respect to all functions y(x) which satisfy the boundary conditions specified in the problem.

One numerical technique which can be used to put some order into the search for this minimum is to express y(x) as a sum of functions which satisfy the boundary conditions of the problem. These basis functions could be sines and cosines, or if intuition gives a general picture of the solution, one could use a set of functions having the general srtucture of the intuitive solution (ramps, steps, etc.). Then a well-documented technique known as the Ritz method can be used to solve for the coefficients in this series of functions. If the basis functions are chosen well, the Ritz solution will be very accurate.

The Ritz technique was followed in this paper. The basis functions used were finite element functions. A finite element basis function is zero over most of the range of the variational integral, and non-zero over a small, specific segment (the finite element). The form of the function over this segment is arbitrary,

but is usually limited for numerical reasons to simple polynomials. In this work, linear finite elements were used. This means that the n-th basis function is defined over $[a_n, b_n]$ which is a part of $[x_1, x_2]$ and where

$$|x_2 - x_1| > |b_n - a_n|$$

Then if c_n is in the segment $[a_n, b_n]$ (not necessarily the midpoint), the n-th basis function is

$$\begin{aligned} f_n(x) &= 0 && x_1 \leq x \leq a_n \\ &= \frac{x - a_n}{c_n - a_n} && a_n \leq x \leq c_n \\ &= \frac{b_n - x}{b_n - c_n} && c_n \leq x \leq b_n \\ &= 0 && b_n \leq x \leq x_2 \end{aligned}$$

At the ends of $[x_1, x_2]$, the finite element is modified to

$$\begin{aligned} f_1(x) &= \frac{b_1 - x}{b_1 - x_1} && x_1 \leq x \leq b_1 \\ &= 0 && b_1 \leq x \leq x_2 \end{aligned}$$

These linear finite elements have the advantages that they are continuous, they are easy to handle numerically, and when summed they approximate an arbitrary function with a continuous function made of linear segments whose slopes change at each a_n, c_n, b_n.

APPENDIX 3

A VARIATIONAL FORMULATION FOR THE TORSIONAL ELASTIC MODES OF THE EARTH

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A VARIATIONAL FORMULATION FOR THE TORSIONAL ELASTIC MODES OF THE EARTH

Backus (1967) has shown that for the elastic stress tensor, the six tensor potentials given in Appendix 1 divide into two groups: R and N; and P, Q, L and M. If P, Q, L, M are non-zero, while R, N are zero, one generates spheroidal normal modes. If R, N are non-zero, while P, Q, L, M are zero, one generates torsional modes. Since this paper deals with torsional modes, the general stress tensor will be specialized to P, Q, L, M equal to zero.

If one postulates that the general form of the displacements of a torsional normal mode is

$$U_r = 0$$

$$U_\vartheta = \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} W$$

$$U_\phi = - \frac{\partial}{\partial\theta} W$$

where W is a scalar function of position, then the equations of elasticity

$$e_{\ell m} = \frac{1}{2} (U_{\ell,m} + U_{m,\ell})$$

$$\tau_{\ell m} = \lambda\theta g_{\ell m} + 2\mu e_{\ell m}$$

where $\theta = g^{jk} e_{jk}$

(with the differentiation indicated by $A_{,m}$ now general covariant differentiation) yield a relationship developed by Backus between the tensor potentials R and N and the displacement potential W, namely

$$N = \frac{\mu}{r} W \qquad \text{(for } \ell \geq 2\text{; N has no } \ell = 1 \text{ component)}$$

$$R = \frac{\mu}{r}\left(r \frac{\partial}{\partial r} W - W\right) \qquad (\ell \geq 1)$$

The basic equation of motion for the torsional modes of a spherically symmetric earth is based upon the minimization of I, where

$$I = \int (T - V)\, dv$$

T is the kinetic energy

$$T = \frac{1}{2} \rho \dot{U}^2$$

and V is the elastic energy

$$V = \frac{1}{2} \tau^{\ell m} e_{\ell m}$$

(Note that the torsional modes in a spherically symmetric earth have no gravitational contribution to the potential energy; this contribution is present for the spheroidal modes.)

Assuming that the displacement has a harmonic time dependence $\exp(i\omega t)$ then the integral equation of motion becomes (with $W' = dW/dr$)

$$I_\ell = \int dr \left\{ \mu r^2 \left(W'^2 - \frac{2}{r} WW' + [\ell(\ell+1) - 1] \frac{W^2}{r^2} \right) - \omega^2 \rho r^2 W^2 \right\} \tag{A3-1}$$

where ℓ is the degree of the normal mode. This integral formulation of the equation of motion can be checked by evaluating the corresponding Euler-Lagrange equation

$$\frac{\partial f}{\partial W} - \frac{d}{dr} \frac{\partial f}{\partial W'} = 0$$

subject to the restriction that at the boundary surfaces

$$\frac{\partial f}{\partial W'} = 0$$

The Euler-Lagrange equation yields

$$\rho\omega^2 rW + r \frac{dR}{dr} + 3R + [2 - \ell(\ell+1)] \frac{\mu}{r} W = 0$$

which is in agreement with the sixth equation in Backus' equations 5.37 (1967).

The boundary conditions derived with the Euler-Lagrange equation become

$$r^2 R = 0$$

Since a boundary at r = 0 for a sphere is meaningless, this equation simplifies to

$$R = 0$$

Referring to Equation (A1-1) in Appendix 1, and recalling that P and Q are zero for torsional modes, this boundary condition is seen to be equivalent to the regular boundary conditions for elastic normal modes in a sphere

$$T_{rr} = T_{r\theta} = T_{r\phi} = 0$$

Thus, solutions to the torsional normal mode problem found by minimizing I_ℓ will automatically satisfy the relevant boundary conditions.

The advantage of this variational formulation is that one can use finite elements (see Appendix 2) and the Ritz method (Weinstock 1952, Section 7-6) to restate the integral variational equation as a matrix eigenvalue problem which can be solved extremely rapidly with modern computers. Although the differential equation will give the same answer, considerations of limited computer time dictated that the calculus of variations is the preferred mode of operation.

APPENDIX 4

INTEGRAL CONSTRAINTS BASED ON TORSIONAL NORMAL MODES

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Additional integral constraints can be obtained for the torsional normal modes. In Equation (A3-1) of Appendix 3, I_ℓ contains an arbitrary constant which can be chosen so that the minimization procedure leaves $I_\ell = 0$. Thus once the normal mode solution has been achieved one has an integral equation for the square of the eigenvalue

$$\omega^2 = \int dr\, \mu r^2 \left(W'^2 - \frac{2}{r} WW' + [\ell(\ell+1) - 1] \frac{W^2}{r^2} \right) \qquad \text{(A4-1)}$$

with the normalization equation

$$\int \rho r^2 W^2 dr = 1$$

subject to the proper boundary conditions. Note that $\mu = \rho v_T^2$, which gives the required functional form for the constraint equation (see Equation 2). Although this is a short-cut derivation of this equation, the end results are supported by Weinstock's (1952) discussion of eigenvalues of membranes in Chapter 9 and by his short discussion of the similar problem in a solid in Problem 9-25. The eigenvalue equation is put on a more general basis by the Theorem of Minimum Strain Energy [discussed by Sokolnikoff (1956, p. 389)] since it is apparent that Equation (A4-1) is only

$$\omega^2 = k \int V \, dv$$

where V is the elastic strain energy [see Equation (A3-1)] and the normalization equation is (where T is the kinetic energy)

$$\frac{k}{\omega^2} \int T \, dv = 1$$

where k is an arbitrary constant chosen to make the normalization equal to one.

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16. Abstract

Information theory can develop a technique which takes experimentally determined numbers and produces a uniquely specified "best" density model satisfying those numbers. This technique does not depend on previous density models for starting conditions; it is self-starting. A model was generated using five numerical parameters: the mass of the earth, its moment of inertia, three zero-node torsional normal modes (L = 2, 8, 26). In order to determine the stability of the solution, six additional density models were generated, in each of which the period of one of the three normal modes was increased or decreased by one standard deviation. The superposition of the seven models is shown in Figure 4. It indicates that current knowledge of the torsional modes is sufficient to specify the density in the upper mantle but that the lower mantle and core will require smaller standard deviations before they can be accurately specified.

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