NASA

Technical Memorandum 78049

(NASA-TM-78049) AMBIGUITY RESOLUTION FOR SATELLITE DOPPLER POSITIONING SYSTEMS (NASA) 30 p HC A03/MF A01 CSCL 22A
N78-19176
Unclas
G3/15 08723

Ambiguity Resolution for Satellite Doppler Positioning Systems

P. Argentiero
J. Marini

DECEMBER 1977

National Aeronautics and
Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

TM 78049

AMBIGUITY RESOLUTION FOR SATELLITE DOPPLER POSITIONING SYSTEMS

P. Argentiero

J. Marini

December 1977

GODDARD SPACE FLIGHT CENTER

Greenbelt, Maryland

ABSTRACT

The implementation of Satellite based Doppler positioning systems frequently requires the recovery of transmitter position from a single pass of Doppler data. The least squares approach to the problem yields conjugate solutions on either side of the satellite subtrack. It is important to develop a procedure for choosing the proper solution which is correct in a high percentage of cases.

This paper derives a test for ambiguity resolution which is the most powerful in the sense that it maximizes the probability of a correct decision. When systematic error sources are properly included in the least squares reduction process to yield an optimal solution, the test reduces to choosing the solution which provides the smaller valuation of the least squares loss function. When systematic error sources are ignored in the least squares reduction, the most powerful test is a quadratic form comparison with the weighting matrix of the quadratic form obtained by computing the pseudo-inverse of a reduced rank square matrix.

The paper provides a formula for computing the power of the most powerful test. A numerical example is included in which the power of the test is computed for a situation which may occur during an actual satellite aided search and rescue mission.

INTRODUCTION

A Search and Rescue Orbiting Satellite System has been suggested as a means for locating distress signals from downed aircraft, small boats, overland expeditions, etc.[1,2,3] Positioning with this system would be accomplished using emissions from Emergency Locator Transmitters (ELT), now available on most U.S. aircraft. An essential requirement of the system is that the location of ELT emissions be determined to within a few kilometers using data obtained during a single pass of a 700-1200 kilometer altitude satellite.

One specific mission proposal[5] involves the incorporation of search and rescue mission instruments on a TIROS-N satellite in a circular, 850 Km altitude, polar orbit. The instruments would be capable of detecting and locating ELT's operating at 121.5 and 243 MHz, as well as new and improved ELT's operating on the 406 MHz frequency authorized for ground to satellite use by the 1971 World Administrative Radio Conference on Space Telecommunications. The nominal launch data is Jan. 1981.

The requirements of such a mission lead to a certain estimation problem: that of determining transmitter position from a single pass of Doppler data and a satellite epoch state within or near the data span. In fact, it is very difficult to recover three position components from a single pass of Doppler observations. The usual procedure is to assume the transmitter to be on the surface of an ellipsoidal Earth and to use a conventional least squares filter to estimate longitude and latitude components of position. Also, the carrier frequency cannot be assumed as known. Hence it is necessary to recover the carrier frequency from the data along with transmitter longitude and latitude.

All least squares reduction processes are equivalent to finding a local minimum of a surface. In the case of single pass Doppler position determination the multidimensional surface in question has two local minima, representing solutions on each side of the satellite subtrack. Hence two solutions are available and they can be obtained by choosing initial values for the least squares iteration process on either side of the satellite subtrack. Within the context of a satellite aided search and rescue mission the ability to choose the correct least square solution can be critical. Unless one develops a decision procedure which is correct in a high percentage of cases, situations may occur in which it will be necessary to send rescue parties to two widely separated locations thus increasing the cost of each rescue. The ambiguity problem, the distinguishing of the correct from the spurious solution, is the subject of this paper.

The problem arises in other applications. An example is the proposed satellite-aided coastal surveillance in support of the recently imposed 200 mile fishing limit.[6] This system would require a single pass Doppler positioning capability. Also, Green[7] has analyzed the ambiguity problem in the course of a comprehensive treatment of the NIMBUS-TWERLE experiment. But he does not attempt to derive the optimal decision procedure. In addition, the technique recommended for computing the probability of a correct decision which relies on a cumulative F distribution is incorrect.

In what follows it is shown that the residual sets of the two solutions have different statistical properties and that this fact can be exploited to develop a simple quadratic form test which is the most powerful in the sense that it maximizes the probability of a correct decision.

A numerical example is given in which the power of the test is computed for a situation which may occur during an actual satellite aided search and rescue mission.

RESIDUAL STATISTICS

In this section we develop the statistical properties of the a-posteriori least squares residuals for both the correct and the spurious solution of the single pass Doppler positioning problem. The treatment is sufficiently general to account for systematic error sources which are left unadjusted in the reduction process.

Let $\tilde{x}$ be the three dimensional vector representing true values of transmitter longitude, latitude and transmitter carrier frequency. Let $\tilde{y}$ be an m dimensional set of correct or noiseless Doppler observations obtained from a single satellite pass. The actual observations are corrupted by noise. Hence

$$y = \tilde{y} + \nu,\ E(\nu) = \bar{0},\ E(\nu\nu^T) = Q \tag{1}$$

where Q is a full rank covariance matrix, and ν is normally distributed. In addition, there are other parameters $\tilde{z}$ (orbit epoch state, carrier frequency drift rate, etc.[8]) which influence the functional relationship between $\tilde{y}$ and $\tilde{x}$. If the estimates of these parameters are uncertain to a significant extent, this fact must be taken into account in obtaining an optimal estimate $\hat{x}$ of $\tilde{x}$. Let $\hat{z}$ be an estimate of $\tilde{z}$. Then

$$\hat{z} = \tilde{z} + \alpha, \ E(\alpha) = \overline{0}, \ E(\alpha\alpha^T) = P \tag{2}$$

where P is full rank and α normally distributed. The functional relationship between $\tilde{y}$, $\tilde{x}$ and $\tilde{z}$ can be expressed as

$$\tilde{y} = f(\tilde{x}, \tilde{z}) \tag{3}$$

Let x_N be a nominal estimate of $\tilde{x}$. Expand equation 3 in a first order Taylor series about x_N and $\hat{z}$ and include the results of equations 2 and 1. The result is

$$\delta y = A\delta\tilde{x} + \gamma \tag{4}$$

where

$$\gamma = B\alpha + \nu, \ E(\gamma) = \overline{0}, \ E(\gamma\gamma^T) = BPB^T + Q = \phi \tag{5a}$$

$$\delta y = y - f(x_N, \hat{z}) \tag{5b}$$

$$\delta\tilde{x} = \tilde{x} - x_N \tag{5c}$$

$$A = \frac{\partial f}{\partial \tilde{x}}, \ B = \frac{\partial f}{\partial \tilde{z}} \tag{5d}$$

Equation 4 can be used as an equation of condition for a standard least squares estimator. The result is

$$\hat{x} = x_N + (A^T\phi^{-1}A)^{-1}A^T\phi^{-1}\delta y \tag{6}$$

The left side of equation 6 can be used as a new nominal and the process can be repeated until a convergence criterion is satisfied. By employing the usual linearity assumptions one can show that the left side of eq. 6 is an optimal estimator of $\tilde{x}$ in the sense that the trace of its covariance is a minimum.[9]

We require that the first and second order statistics for the a-posteriori residuals of the least squares filter when the iteration process converges to a correct solution. Assume that the nominal value x_N used as an initial guess for the least squares iteration process is sufficiently close to $\tilde{x}$ that the first order Taylor series expansion represented by eq. 4 is valid. Then by combining eq. 4 with eq. 6 we obtain

$$\hat{x} = \tilde{x} + (A^T \phi^{-1} A)^{-1} A^T \phi^{-1} \gamma \tag{7}$$

The a posteriori residuals are defined as

$$R = y - f(\hat{x}, \hat{z}) \tag{8}$$

Assume that $\hat{x}$ is a good estimate of $\tilde{x}$. and rewrite equation 8 as

$$R = f(\tilde{x}, \tilde{z}) - f(\hat{x}, \hat{z}) + \gamma \tag{9}$$

Then a first order Taylor series expansion about $\hat{x}$ and $\hat{z}$ yields

$$R = [I - A(A^T \phi^{-1} A)^{-1} A^T \phi^{-1}] \gamma \tag{10}$$

Hence R is a linear function of a normal random variable and,

$$E(R) = \bar{0} \tag{11a}$$

$$COV(R) = E(RR^T) = \phi - A(A^T \phi^{-1} A)^{-1} A^T = C \tag{11b}$$

In fact the covariance matrix C of R is of reduced rank. This can be seen by focusing attention on eq. 10. Because of the interpretation of the least squares reduction process as a projection operator[9] it follows that the matrix which maps γ onto R is a projection operator on the complement space of the range of A. Since γ is a full rank random vector and since the range space of A is of rank 3, it follows that the M dimensional covariance matrix C is of reduced rank

M-3. These facts can be summarized in the following

> Theorem:
>
> Assume that a least squares process has been implemented to solve the single pass Doppler positioning problem and that the process has converged to a point near the true transmitter position. Assume also that the equation of condition (eq. 4) properly accounts for systematic error sources which are left unadjusted in the least squares process. Then equation 6 provides a minimum variance estimate of transmitter position. The a posteriori residuals of the process have first and second order moments given by eq. 11a., and 11b. The covariance matrix of the residuals is of reduced rank M-3 where M is the number of data points.

To obtain residual statistics for the spurious solution, let x' be the expected value of the least squares solution when the iterative process converges to the wrong side of the satellite subtrack. The spurious solution is obtained when the initial estimate x_N is within a certain radius of convergence of x'. Hence eq. 6 can be considered as the least squares estimator for the spurious solution but with the partial derivative matrices evaluated at x' rather than $\tilde{x}$. Perform the following first order Taylor series expansion

$$f(x_N, \hat{z}) - f(x', \tilde{z}) = A(x_N - x') + B(\hat{z}, \tilde{z}) \tag{12}$$

where the partial derivative matrices A and B are evaluated at x'. When eq. 12 is combined with eq. 6, the result is

$$\hat{x} = x' + (A^T \phi^{-1} A)^{-1} A^T \phi^{-1} [f(\tilde{x}, \tilde{z}) - f(x', \tilde{z}) + \gamma] \tag{13}$$

But since x' is interpreted as the expected value of the least squares estimator, we have

$$\hat{x} = x' + (A^T \phi^{-1} A)^{-1} A^T \phi^{-1} \gamma \tag{14}$$

By combining eq. 9 with eq. 14 and employing another Taylor series expansion about x', we obtain an expression for the residuals of the spurious solution as

$$R_S = f(\tilde{x}, \tilde{z}) - f(x', \tilde{z}) + [I - A(A^T \phi^{-1} A)^{-1} A^T \phi^{-1}] \gamma \tag{15}$$

Hence the first and second order statistics for the a-posteriori residuals of the spurious solution are

$$E(R_S) = f(\tilde{x}, \tilde{z}) - f(x', \tilde{z}) = U \tag{16a}$$

$$COV(R_S) = \phi - A(A^T \phi^{-1} A)^{-1} A^T \tag{16b}$$

The symbol U is the difference between the true Doppler signal received from a transmitter located at its actual position represented by $\tilde{x}$ and the two Doppler signals which would be received if the transmitter were at the position represented by x'.

The covariance matrix of the residuals for the spurious solution given by 16b has the same form as the covariance matrix for the residuals of the true solution as given by 11b. The difference is that the partial derivative matrices A and B defined in equation 5b. are evaluated at approximately mirror image points relative to the satellite subtrack. However, numerous simulations have shown that the matrix form given by equation 11b or 16b are nearly identical when evaluated at mirror image points. This is the result of the near symmetry of the function defined by equation 3 about the satellite subtrack. These facts are summarized as follows:

> Theorem:
>
> The covariance matrix of the a-posteriori residuals of the spurious solution is the same as that of the a-posteriori residuals for the correct solution. The expected value of the residuals associated with the spurious solution is the difference between the signal received if the transmitter were at $\tilde{x}$ and the signal received if the transmitter were at x'.

MOST POWERFUL TEST FOR AMBIGUITY RESOLUTION

Assume that a least squares solution to the single pass Doppler positioning problem has been implemented and that solutions on sides a and b of the satellite subtrack have been obtained. Let r_a and r_b be the residual sets of the two solutions. We require an algorithm which permits us to decide which of the two residual sets is associated with the correct solution. Furthermore, it is to be the most powerful test in the sense that among all possible such tests, it maximizes the probability of a correct decision.

The problem can be formulated in terms of a standard testing of a statistical hypothesis. Let the null hypothesis be that the vector r_a is a realization of a random vector R whose mean is $\overline{0}$ and whose covariance matrix is C as defined by eq. 11b. The null hypothesis is equivalent to the assumption that the correct solution is on side a. The alternative hypothesis is that r_a is a realization of a random vector R_S whose expected value is U as defined by equation 16a. and whose covariance matrix is C. The resultant test is to be the most powerful in the sense that it maximizes the probability of rejecting the null hypothesis when in fact the null hypothesis is false.

The derivation of the most powerful test would be quite straightforward if it were not for the fact that the random vectors R and R_S are of reduced rank. It is necessary to account for this fact by performing a linear transformation of variables by means of a matrix T of dimension (M-3) by M. The transformation defined by T must have the property that the composite mapping TC have a range space of dimension M-3. (One such transformation is defined by a matrix whose rows are the M-3 eigenvectors associated with the non-zero eigenvalues of C.) Define

$$\mathcal{L} = TR \tag{17a}$$

$$\mathcal{L}_S = TR_S \tag{17b}$$

$$V_a = Tr_a, \; V_b = Tr_b \tag{17c}$$

$$W = TU \tag{17d}$$

The null hypothesis is now that the vector V_a is a sample of the M-3 dimensional random vector $\mathcal{L}$ whose mean is $\overline{0}$ and whose covariance is

$$COV(\mathcal{L}) = TCT^T = D \tag{18}$$

where D is a full rank matrix. The alternative hypothesis is that V_a is a sample from the random vector $\mathcal{L}_S$ with mean W and covariance D.

Imagine an experiment whose first step is to choose either random variable $\mathcal{L}$ or random variable $\mathcal{L}_s$ with probability .5 for each choice. Next obtain a realization V from whichever random vector was chosen at the first step. The probability structure of the experiment can be described as follows: Define a two point random variable x as x = 0 if random vector $\mathcal{L}$ is chosen and x = 1 if random vector $\mathcal{L}_s$ is chosen. The outcome of the experiment can be described as a two tuple (V, x) where the value of x describes the chosen probability law and where V is the realization. From equations 10, 15 and 17a, 17b, it is seen that random vectors $\mathcal{L}$ and $\mathcal{L}_s$ are linear functions of normal random vectors. Since they are also full rank they are normally distributed. Hence the joint probability density function of the experiment outcome is

$$\alpha(V, x) = \frac{1}{2(2\pi)^{(M-3)/2}\,|D|^{1/2}} \exp^{-\frac{1}{2}\,[V^T D^{-1} V]} \tag{19}$$

if x = 0

$$\alpha(V, x) = \frac{1}{2(2\pi)^{(M-3)/2}\,|D|^{1/2}} \exp^{-\frac{1}{2}\,[(V-W)^T D^{-1} (V-W)]}$$

if x = 1.

The Neyman-Pearson Lemma[10] asserts that in the case of simple alternative hypotheses, a maximum likelihood criterion is the most powerful test for accepting or rejecting the null hypothesis. An application of Bayes' theorem[11] yields the conditional probability density function of x given a realization V_a of V as

$$f(x \mid V = V_a) = \frac{\exp^{-\frac{1}{2}[V_a^T D^{-1} V_a]}}{\exp^{-\frac{1}{2}[V_a^T D^{-1} V_a]} + \exp^{-\frac{1}{2}[(V_a - W)^T D^{-1} (V_a - W)]}} \tag{20}$$

if $x = 0$

$$f(x \mid V = V_a) = \frac{\exp^{-\frac{1}{2}[(V_a - W)^T D^{-1} (V_a - W)]}}{\exp^{-\frac{1}{2}[V_a^T D^{-1} V_a]} + \exp^{-\frac{1}{2}[(V_a - W)^T D^{-1} (V_a - W)]}}$$

if $x = 1$.

From equation 20 it follows that the maximum likelihood criterion is

$$f(0 \mid V = V_a) > f(1 \mid V = V_a) \tag{21}$$

But since we have alternative and mutually exclusive hypothesis:

$$f(1 \mid V = V_a) = f(0 \mid V = V_b) \tag{22}$$

From equations 20, 21, and 22 we obtain the result that the maximum likelihood test is to accept the null hypothesis if and only if

$$V_a^T D^{-1} V_a < V_b^T D^{-1} V_b \tag{23}$$

Equation 23 can be written as

$$r_a^T T^T D^{-1} T r_a < r_b^T T^T D^{-1} T r_b \tag{24}$$

But in fact,

$$T^T D^{-1} T = C^{\dagger} \tag{25}$$

where $C^{\dagger}$ denotes the unique pseudoinverse[9] of C. This can be seen by showing that the left side of equation 25 satisfies the four axioms of the pseudoinverse.

$$CC^{\dagger} C = C \tag{26a}$$

$$C^{\dagger} CC^{\dagger} = C^{\dagger} \tag{26b}$$

$$(CC^{\dagger})^T = CC^{\dagger} \tag{26c}$$

$$(C^{\dagger}C)^T = C^{\dagger}C \tag{26d}$$

Hence the criterion for accepting the null hypothesis is

$$r_a^T C^{\dagger} r_a < r_b^T C^{\dagger} r_b \tag{27}$$

A further simplification is possible.

Theorem:

Both residual set r_a and residual set r_b are in the range space of **C**.

Proof:

Residual sets r_a and r_b are realizations of random vectors R and R_S which are defined by equations 10 and 15. Hence it is sufficient to show the existence of random variables G and G_S such that

$$R = CG \tag{28a}$$

$$R_S = CG_S \tag{28b}$$

From equations 10 and 11b, it is clear that the required representation for G is $\phi^{-1}\gamma$. Also notice that from equations 13 and 14

$$(A^T\phi^{-1}A)^{-1}A^T\phi^{-1}[f(\tilde{x},\tilde{z}) - f(x',\tilde{z})] = \bar{0} \tag{29}$$

when x' is the expected value of the least squares estimator when the iteration process converges to the wrong side of the satellite subtrack. Hence equation 15 can be rewritten

$$R_S = [I - A(A^T\phi^{-1}A)^{-1}A^T\phi^{-1}][f(\tilde{x},\tilde{z}) - f(x',\tilde{z}) + \gamma] \tag{30}$$

The correct expression for G_S is

$$G_S = \phi[f(\widetilde{x},\widetilde{z}) - f(x',\widetilde{z}) + \gamma] \tag{31}$$

Theorem:

Let C be a matrix transformation whose form is that of 11b, and let X be a vector in the range space of C. Then

$$X^T C^{\dagger} X = X^T \phi^{-1} X \tag{32}$$

Proof:

Let F be such that

$$X = CF \tag{33}$$

where C has the form of equation 11b. The following set of equations proves the result

$$X^T C^{\dagger} X = F^T C C^{\dagger} C F \tag{34a}$$

$$F^T C C^{\dagger} C F = F^T C F \tag{34b}$$

$$F^T C F = F^T C \phi^{-1} C F \tag{34c}$$

$$F^T C \phi^{-1} C F = X^T \phi^{-1} X \tag{34d}$$

The above results imply that inequality 27 is satisfied if and only if

$$r_a^T \phi^{-1} r_a < r_b^T \phi^{-1} r_b \tag{35}$$

We have derived the fundamental result of the paper,

Theorem:

Assume that the least squares solution to the single pass Doppler positioning problem has been implemented and that solutions on either side of the satellite subtrack have been obtained. Let r_a and r_b be the residual sets of the two solutions. Assume also that systematic error sources have been properly modeled in the reduction process so that when ambiguity is properly resolved the resulting least squares estimator is optimal in a minimum variance sense. Then the most powerful test for ambiguity resolution is obtained by choosing the solution on side a of the satellite subtrack as the correct transmitter position if and only if inequality 35 is satisfied. The quadratic forms on the left and right sides are the valuations of the least squares loss function obtained at each of the two solutions and, hence, are available as an automatic byproduct of the least squares reduction process. If no systematic error sources are present and if the data noise is uncorrelated and of constant variance, the most powerful test reduces to choosing the solution whose residuals provide the smaller square sum. Once the residuals are available the probability of the test providing a correct answer can be obtained by computing

$$P = \frac{\exp^{-\frac{1}{2}\left[R_a^T Q^{-1} R_a\right]}}{\exp^{-\frac{1}{2}\left[R_a^T Q^{-1} R_a\right]} + \exp^{-\frac{1}{2}\left[R_b^T Q^{-1} R_b\right]}} \tag{36}$$

where the required probability is P if $P > .5$ and $1-P$ if $P < .5$.

An entirely different approach which also derives the above result is given in the appendix.

If the systematic error sources defined by equation 2 are neglected in the least squares reduction process, the residual statistics are different from those given by equations 11a, 11b and 16a, and 16b. Hence for this situation the above theorem is invalid. The most powerful test for this case can be obtained by noticing that if the uncertainties implied by equation 2 are ignored in the least squares reduction process, the least squares estimator becomes

$$\hat{x} = x_N + (A^T Q^{-1} A)^{-1} A^T Q^{-1} \delta y \tag{37}$$

By the same reasoning that obtained equation 10 from equation,6, an equation for the residuals of the correct solution can be obtained as

$$R = [I - A(A^T Q^{-1} A)^{-1} A^T Q^{-1}]\gamma \tag{38}$$

Define

$$H = I - A(A^T Q^{-1} A)^{-1} A^T Q^{-1} \tag{39}$$

Then residual statistics for the correct solution become

$$E(R) = 0 \tag{40a}$$

$$COV(R) = H\phi H^T = C_0 \tag{40b}$$

By repeating the logic which obtained equation 15 from equation 6 we obtain an expression for the residuals of the spurious solution

$$R_S = f(\tilde{x}, \tilde{z}) - f(x', \tilde{z}) + H\gamma \tag{41}$$

Hence

$$E(R_S) = f(\tilde{x}, \tilde{z}) - f(x', \tilde{z}) = U \tag{42a}$$

$$COV(R_S) = H\phi H^T = C_0 \tag{42b}$$

The reasoning employed from equation 17 to equation 27 is again applicable and we have the following result, Theorem:

Assume that the least squares solution to the single pass Doppler positioning problem has been implemented and that solutions on either side of the satellite subtrack have been obtained. Let r_a and r_b be the residual sets of the two solutions. Assume that the uncertainty associated with parameters defined by equation 2 have been ignored in the least squares reduction process. Then the most powerful test for ambiguity resolution is obtained by choosing the solution on side a of the satellite subtrack as the correct transmitter position if and only if

$$r_a^T C_0^{\dagger} r_a < r_b C_0^{\dagger} r_b \qquad (43)$$

where C_0 is defined by equation 39b.

It would be inconsistant to ignore systematic error sources in a least squares data reduction and then include their effects in an effort to resolve ambiguity. Hence it is unlikely that the test implied by inequality 43 would be implemented in practice. However, the test is of use in providing a bound on the probability of ambiguity resolution in the presence of different types and magnitudes of systematic error sources.

COMPUTING THE POWER OF THE TEST

Suppose that an optimal least squares estimator defined by equation 6 has been implemented to determine transmitter position from a single pass of Doppler data. The inequality 35 provides the most powerful test for ambiguity resolution. The power of the test is the probability of inequality 35 being satisfied when the correct solution is on side a.

From equations 10 and 15 it follows that

$$\pm U = r_a - r_b \tag{44}$$

With regard to determining the power of the test it will be seen that the sign ambiguity is irrelevant. We will assume the positive sign. Inequality 35 can now be rewritten as

$$r_a^T \phi^{-1} r_a < (r_a - U)^T \phi^{-1} (r_a - U) \tag{45}$$

Expand inequality 45 to obtain

$$2 U^T \phi^{-1} r_a < U^T \phi^{-1} U \tag{46}$$

If the correct solution is on side a, the probability of satisfying inequality 45 is the probability that a standard normal random variable satisfies inequality

$$x < \frac{U^T \phi^{-1} U}{2(U^T \phi^{-1} C \phi^{-1} U)^{1/2}} \tag{47}$$

where C is defined by equation 11b. But from equation 29, and 11b it follows that

$$C \phi^{-1} U = U \tag{48}$$

Hence inequality 47 is equivalent to

$$x < \frac{(U^T \phi^{-1} U)^{1/2}}{2} \tag{49}$$

The power of the test can be obtained by computing the right side of inequality 49 and consulting a tabulation of the cumulative normal distribution for the probability of the standard normal variable not exceeding the number in question. If no systematic error sources are present and if the data noise is uncorrelated and of constant variance, inequality 49 reduces to

$$x < \frac{(U^T U)^{1/2}}{2\sigma} \tag{50}$$

where σ is the standard deviation of the data noise.

NUMERICAL EXAMPLE

To illustrate the application of equation (50), an example that is typical of the satellite-orbit search and rescue system to be tested using the TIROS-N satellite[12] was chosen. A circular satellite orbit with a radius of 7200 kilometers and an inclination of 98.7° was simulated. The latitude of the transmitter was about 36°N, and the distance from the satellite subtrack was chosen so that the satellite elevation was 20° at the time of closest approach. The range rates that would be measured in such a configuration using one minute intervals between observations over a span of three minutes were calculated, and are listed in the second column of both Table I and Table II. The first column in each of these tables, the time of observation, is given relative to the time of closest approach of the satellite to the transmitter, which occurs at the second range-rate observation.

In Table I, noise was generated using a standard deviation of about 2.9 meters per second, which, at a transmitter frequency of 406MHz, corresponds to a determination of the received frequency to within a bin width of 10 Hz. The noise values obtained are listed in the second column of Table I. The simulated range rate observations were formed by adding the range rates in the first column to the corresponding noise values in the second. Using these simulated observations a least squares position location algorithm was used to solve both for the minimum near the transmitter position, and also for the false minimum on the opposite side of the satellite subtrack. The residuals of the least squares fits to these minima are listed in the fifth and fourth columns of Table I respectively. The vector U appearing in equation (50), which is defined in equation (44) as the difference between the residuals at the false minimum and those at the minimum near the true position, is listed in the last column. The fundamental test, equation (35), in this case gives the correct results since the sum of one squares of the fourth column, 23.684, exceeds that of the fifth column, 9.027. Using the residual difference in equation (50), the right side of (50) is equal to 0.438 which, consulting a table of the cumulative normal distribution, yields a value of 67% for the power of the test, a low value.

In Table II the process above has been repeated, except that in this case the receiver resolution is taken to be 1 Hz at 406 MHz. The power of the test increases to 99.99%. Clearly the power is a strong function of data noise standard deviation. It is also a function of Beacon latitude, satellite elevation at TCA, and the data distribution along the satellite pass.

Summary

The use of Satellite Based, single pass Doppler positioning systems creates a certain estimation problem – that of determining transmitter position from a single pass of Doppler data and a satellite epoch state within or near the data span. A least squares approach is usually employed to solve the problem. However, all least squares reduction processes are equivalent to finding a local minimum of a surface. In the case of single pass Doppler position determination the multidimensional surface in question has two minima, representing solutions on each side of the satellite subtrack. In many applications it is important to have a procedure for choosing the proper solution which is correct in a high percentage of cases.

It is shown that the residual sets of the correct and the spurious solutions have different statistical properties and that this fact can be exploited to develop a simple quadratic form test for ambiguity resolution. This test is the most powerful in the sense that it maximizes the probability of a correct decision. When systematic error sources are properly included in the data reduction process to yield an optimal solution, the test reduces to choosing the solution which provides the smaller valuation of the least squares loss function. When systematic error sources are ignored in the least squares reduction, the most powerful test is a quadratic form comparison with the weighting matrix of the quadratic form obtained by computing the pseudo-inverse of a reduced rank square matrix.

The paper provides a formula for computing the power of the most powerful test. The power of the test is a function of the data noise, the number and distribution of the data points, transmitter latitude, and the maximum elevation of the data points, transmitter latitude, and the maximum elevation of the satellite pass relative to the transmitter position. A numerical example is included in which the power of the test is computed for a situation which could occur during an actual satellite aided search and rescue mission.

Acknowledgment

We thank Dr. T. Englar of Business and Technological Systems Inc. for his aid during the course of this research.

References

1. Committee on Aeronautical and Space Sciences, United States Senate, 93rd Congress. First Session, "Toward a Better Tomorrow with Aeronautical and Space Technology," U.S. Government Printing Office, 1973.

2. Search and Rescue Satellite Project. Phase 1, Preliminary System Design Review." Canadian Astronautics Limited, Ontario, Canada, Dec., 1976.

3. Koch, D., Error Analysis for Satellite-Aided Search and Rescue," NASA-X-932-76-245, Oct. 1976.

4. Marini, J., Initial Position Estimates for Satellite-Aided Search and Rescue," NASA-X-932-76-245, Oct. 1976.

5. Schmid, P., Lynn, J., Vonbun, F. O., Single Pass Doppler Positioning for Search and Rescue Satellite Missions." Presented at IEEE Position Location and Navigation Symposium, San Diego, California, Nov. 1-3, 1976.

6. Satellite Data Collection Newsletter, Oct. 1977, No. 4.

7. Green, T., "Satellite Doppler Data Processing for Platform Navigation," IEEE Transactions on Geoscience Electronics, Vol. GE. 13, Nov., 1974.

8. Argentiero, P., Garza-Robles, R., "Efficient Estimation Algorithms for a Satellite-Aided Search and Rescue Mission," Presented at AAS/AIAA Astrodynamics Conference, Grand Teton National Park, Wyoming, Sept. 7-9, 1977.

9, Deutsch, R., "Estimation Theory," Prentice-Hall, Inc. 1965.

10. Davenport, W., Root, W., "Random Signals and Noise," McGraw-Hill, Inc. 1958.

11. Cramer, H., "Mathematical Methods of Statistics," Princeton University Press, 1945.

12. Aviation Week and Space Technology, Jan. 24, 1977; p. 24.

13. Mood, A., "Introduction to the Theory of Statistics," McGraw-Hill Book Co., New York, 1950; pp. 84, 178.

Appendix

Alternate Derivation

The probability that the true position x of the transmitter lies in a given region A when a particular series y of range rates are observed is given by [13]

$$P[\tilde{x} \text{ in } A | y] = \frac{1}{p(y)} \int_A p(y|x)p(x)dx \tag{A1}$$

where p(x) is the marginal probability density of x, p(y|x) is the conditional probability density of y given x, and p(y) is the marginal probability density of y

$$p(y) = \int_S p(y|x)p(x)dx \tag{A2}$$

where the integration is over the entire range S of x, that is, over both the entire surface of the earth and the possible values of frequency.

The observed range rates are modelled as the sum of the true range rates f(x) and a noise component ν whose covariance is the matrix Q (see equation 1). It follows that the conditional probability of y given $\tilde{x}$ is [13]

$$p(y|\tilde{x}) = \frac{1}{(2\pi)^{M/2}\sqrt{|Q|}} \exp - \frac{1}{2}[y - f(\tilde{x})]^T Q^{-1} [y - f(\tilde{x})] \tag{A3}$$

Substituting (A3) into (A1)

$$P[\tilde{x} \text{ in } A | y] = \frac{1}{(2\pi)^{M/2} \sqrt{|Q|}\, p(y)} \int_A p(x) \exp -\frac{1}{2}[y-f(x)]^T Q^{-1} [y-f(x)]\, dx \quad \text{(A4)}$$

The exponential function in (A4) has two maxima. Assume that the area A encompasses only one of these, a maximum at $x = x_A$. An approximate expression for the integral in (A4) can be obtained by expanding f(x) linearly about x_A. Let

$$x = x_A + \Delta x \quad \text{(A5)}$$

$$f(x) \doteq f(x_A) + A\Delta x \quad \text{(A6)}$$

The exponent in (A4) then becomes

$$-\frac{1}{2}[y - f(x_A) - A\Delta x]^T Q^{-1} [y - f(x_A) - A\Delta x]$$

$$= -\frac{1}{2}[y - f(x_A)]^T Q^{-1} [y - f(x_A)] - \frac{1}{2}\Delta x^T A^T Q^{-1} A \Delta x \quad \text{(A7)}$$

The cross product terms

$$\Delta x^T A^T Q^{-1} [y - f(x_A)] + [y - f(x_A)]^T Q^{-1} A \Delta x$$

are absent from the right hand side of (A7) because the condition that x_A is a maximum of the exponential function is equivalent to requiring that these terms vanish.

Also, the quantities $y - f(x_A)$ appearing in (A7) can be recognized as the residuals R_A of the weighted least squares fit to y of $f(x_A)$. Assuming that p(x) is a slowly varying function, (A4) becomes

ORIGINAL PAGE IS OF POOR QUALITY

$$P[\tilde{x} \text{ in } A|y] = \frac{p(x_A)\exp - \frac{1}{2} R_A^T Q^{-1} R_A}{(2\pi)^{M/2}\sqrt{|Q|}\,p(y)} \int_A \exp - \frac{1}{2} \Delta x^T A^T G^{-1} A \Delta x \, d\Delta x \qquad \text{(A8)}$$

The integral in (A8) can be approximated by extending the limits of integration to infinity, giving for (A8)

$$P[\tilde{x} \text{ in } A|y] \doteq \frac{p(x_A)}{|Q|^{1/2}|A^T Q^{-1} A|^{1/2} p(y)} \exp - \frac{1}{2} R_A^T Q^{-1} R_A \qquad \text{(A9)}$$

The probability p[x in B/y] is given by a similar expression, with B replacing A throughout. Because of the symmetry of the two maxima x_a and x_b, the coefficients of the exponential factors are equal, whence equation (36) is obtained in the form

$$P[\tilde{x} \text{ in } A|y] = \frac{\exp - \frac{1}{2} R_A^T Q^{-1} R_A}{\exp - \frac{1}{2} R_A^T Q^{-1} R_A + \exp - \frac{1}{2} R_B^T Q^{-1} R_B} \qquad \text{(A10)}$$

by dividing (A9) by $1 = P[\tilde{x} \text{ in } A|y] + P[\tilde{x} \text{ in } B|y]$.

Table I
Simulated Data, 10 Hz Resolution

Time (minutes)	Range Rate m/s	Noise m/s	Residuals of False Min. (m/s)	Residuals of True Min. (m/s)	Residual Difference (m/s)
-1	-1550.382	0.605	1.137	0.704	0.433
0	0	0.728	-3.120	-1.925	-1.195
+1	1547.107	2.859	3.303	2.039	1.264
+2	2876.796	-5.109	-1.321	-0.818	-0.503

Table II
Simulated Data, 1 Hz Resolution

Time (minutes)	Range Rate m/s	Noise m/s	Residuals of False Min. (m/s)	Residuals of True Min. (m/s)	Residual Difference (m/s)
-1	-1550.382	-0.225	0.476	0.039	0.437
0	0	-0.189	-0.303	-0.105	-1.198
1	1547.107	0.021	1.377	0.111	1.266
2	2876.796	-0.299	-0.550	-0.045	-0.505

BIBLIOGRAPHIC DATA SHEET

1. Report No. TM 78049	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle AMBIGUITY RESOLUTION FOR SATELLITE DOPPLER POSITIONING SYSTEMS		5. Report Date December 1977
		6. Performing Organization Code 932
7. Author(s) P. Argentiero and J. Marini		8. Performing Organization Report No.
9. Performing Organization Name and Address Goddard Space Flight Center Greenbelt, Md.		10. Work Unit No.
		11. Contract or Grant No.
		13. Type of Report and Period Covered TM Dec. 1977
12. Sponsoring Agency Name and Address		
		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract The implementation of Satellite based Doppler positioning systems frequently requires the recovery of transmitter position from a single pass of Doppler data. The least squares approach to the problem yields conjugate solutions on either side of the satellite subtract. It is important to develop a procedure for choosing the proper solution which is correct in a high percentage of cases.

This paper derives a test for ambiguity resolution which is the most powerful in the sense that it maximizes the probability of a correct decision. When systematic error sources are properly included in the least squares reduction process to yield an optimal solution, the test reduces to choosing the solution which provices the smaller valuation of the least squares loss function. When systematic error sources are ignored in the least squares reduction, the most powerful test is a quadratic form comparison with the weighting matrix of the quadratic form obtained by computing the pseudo-inverse of a reduced rank square matrix.

The paper provides a formula for computing the power of the most powerful test. A numberical example is included in which the power of the test is computed for a situation which may occur during an actual satellite aided search and rescue mission.

17. Key Words (Selected by Author(s))	18. Distribution Statement
Ambiguity resolution Doppler positioning	

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
None	Unclassified		

*For sale by the National Technical Information Service, Springfield, Virginia 22151.

GSFC 25-44 (10/77)