NASA

Technical Memorandum 78090

Theory of Flux Anisotropies in a Guiding Center Plasma

(NASA-TM-78090) THEORY OF FLUX ANISOTROPIES IN A GUIDING CENTER PLASMA (NASA) 19 p HC A02/MF A01 CSCL 03E

N78-21028

Unclas

G3/90 12572

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FEBRUARY 1978

National Aeronautics and
Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

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ABSTRACT

Assuming time stationarity of the one particle distribution function f on the scale of the bounce motion of particles in a magnetic field $\underline{B}$, we expand the Vlasov Equation through $O(\epsilon)$ in the adiabatic parameter ϵ, which is the ratio of particle gyroradius to scale length of the magnetic field. Since f is directly proportional to particle flux $d\Phi/dWd\Omega$ differential in kinetic energy W and solid angle Ω, f is in principle measurable in space experiments, and our analysis is tailored to be explicitly applicable to space problems. To O(1), f is gyrotropic; its first velocity moment is (if non-vanishing) parallel to $\underline{B}$, and hence macroscopic parallel flow is included in this term. The $O(\epsilon)$ contribution is non-gyrotropic and macroscopic flow $\perp$ to $\underline{B}$ plus additional parallel flow results from these terms. The degree of non-gyrotropy and hence the amount of cross-field macroscopic flow depend on the perpendicular component of the electric field $\underline{E}$, on curvature and shear in the magnetic field, and on the spatial gradient f_0, pitch angle derivative $\partial f_0/\partial\delta$ and speed derivative $\partial f_0/\partial v$ of the lowest order distribution function f_0. We also show that the usual expression for the electric field $\underline{E}$ which produces plasma co-rotation in an axisymmetric system such as a dipole also holds for any non-axisymmetric but rigidly rotating magnetic field pattern, provided the observed magnetic field is used in place of the dipole field.

CONTENTS

THEORY OF FLUX ANISOTROPIES IN A GUIDING CENTER PLASMA

I. INTRODUCTION

We often consider charged particles interacting with the electric and magnetic fields $\underline{E}$ and $\underline{B}$ in a magnetospheric environment as behaving adiabatically: time variations are sufficiently slow and spatial variations sufficiently long that individual particles undergo nearly periodic motions. Under adiabatic conditions the Vlasov Equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m}\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c}\right) \cdot \frac{\partial f}{\partial \underline{v}} = 0, \tag{1}$$

which governs the distribution function $f(\underline{r}, \underline{v}, t)$ of such particles, can be expanded asymptotically in the adiabatic parameter.

In this paper (Section II) we carry out a formal adiabatic expansion of the Vlasov Equation in a manner paralleling that of Hastie et al. [1967], but we tailor it to be more explicitly applicable to space problems, such as the one in the following paper. We do so assuming that f is stationary on the bounce time scale of particles moving in $\underline{B}$. We find that f is constrained in form: the dependence of f on λ, the gyrophase angle of $\underline{v}$ with respect to $\underline{B}$, is related to the spatial gradient ∇f_0 of the lowest order f, to the derivatives $\partial f_0/\partial v$ and $\partial f_0/\partial \delta$ with respect to speed and pitch angle respectively, and to the magnitude, curvature, and shear of $\underline{B}$ as well as the magnitude and direction of $\underline{E}$. This λ-dependence of f is called gyro-anisotropy and leads, for example, to a net particle flow perpendicular to $\underline{B}$.

Given values of $\underline{E}$ and $\underline{B}$ and of the derivatives of f_0, our result can be used to calculate f and its velocity space moments for comparison with charged particle measurements. Measurement of the flux of particles $d\Phi/dWd\Omega$ differential in energy and solid angle is equivalent to measuring f because of the well known relationship

$$\frac{d\Phi}{dWd\Omega} = \frac{2fW}{m^2} \tag{2}$$

Conversely we may deduce the values of one or more of the sources of gyro-anisotropy, i.e., of the λ-dependence, by examining the phase space variation of measured fluxes. In a subsequent paper, to which this paper is a formal theoretical prelude, we apply this latter approach to Pioneer measurements of energetic protons in Jupiter's magnetosphere.

The electric field $\underline{E}$ which enforces plasma "corotation" is the most generally agreed upon $\underline{E}$ to which magnetospheric particles are subjected. In Section III we derive for use in our f equation the functional dependence on $\underline{r}$ and t of the corotational $\underline{E}$ for the case of a magnetic rotor whose magnetic field has no symmetry axis. We require that the magnetic field pattern, whatever it might be, rotate rigidly, so that the field at any time looks like that at an earlier time, but rotated. Our work here is an extension of previous theoretical work of Mestel [1961, 1968].

II. EXPANSION OF VLASOV EQUATION

Let us define a triad of locally orthogonal unit vectors $\hat{e}_1(\underline{r}, t)$, $\hat{e}_2(\underline{r}, t)$, and $\hat{e}_3(\underline{r}, t)$ with $\hat{e}_1 = \underline{B}/|\underline{B}|$ and $\hat{e}_2$ and $\hat{e}_3$ oriented arbitrarily (in this paper) in the plane transverse to $\underline{B}$. Because we usually measure particle fluxes as functions of the energy and of the direction of the velocity vector, it is convenient to introduce the particle pitch angle

$$\delta = \cos^{-1} \frac{\underline{v} \cdot \hat{e}_1}{|\underline{v}|} \tag{3}$$

and its gyrophase angle

$$\lambda = \tan^{-1} \frac{\underline{v} \cdot \hat{e}_3}{\underline{v} \cdot \hat{e}_2} \tag{4}$$

and to write the Vlasov Equation using v, δ, and λ as variables.

$$\frac{\partial f(v, \delta, \lambda, \underline{r}, t)}{\partial t} + \underline{v} \cdot \nabla f - \frac{1}{v \sin \delta} \underline{v} \cdot (\nabla \hat{e}_1) \cdot \underline{v} \frac{\partial f}{\partial \delta}$$

$$+ \frac{\cos \lambda}{v \sin \delta} \underline{v} \cdot [(\nabla \hat{e}_3) \cdot \underline{v} - \tan \lambda (\nabla \hat{e}_2) \cdot \underline{v}] \frac{\partial f}{\partial \lambda} + \frac{q}{m} \left[E_1 \left(\cos \delta \frac{\partial f}{\partial v} - \frac{\sin \delta}{v} \frac{\partial f}{\partial \delta} \right) \right.$$

$$+ E_2 \left(\sin \delta \cos \lambda \frac{\partial f}{\partial v} + \frac{\cos \delta \cos \lambda}{v} \frac{\partial f}{\partial \delta} - \frac{\sin \lambda}{v \sin \delta} \frac{\partial f}{\partial \lambda} \right)$$

$$\left. + E_3 \left(\sin \delta \sin \lambda \frac{\partial f}{\partial v} + \frac{\cos \delta \sin \lambda}{v} \frac{\partial f}{\partial \delta} + \frac{\cos \lambda}{v \sin \delta} \frac{\partial f}{\partial \lambda} \right) \right] - \frac{q |\underline{B}|}{mc} \frac{\partial f}{\partial \lambda} = 0 \tag{5}$$

In Equation (5), E_1, E_2, and E_3 are the components of $\underline{E}$ in the directions of our unit vectors. Note that the spatial gradient ∇f in (5) is at constant (v, δ, λ, t) and differs from that at constant $\underline{v}$ because the reference directions ($\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$), with respect to which δ and λ are defined, change with $\underline{r}$. However, f, even though expressed as a function of v, δ, λ, and $\underline{r}$ from here on, is still the density in $\underline{r}$, $\underline{v}$ space.

In the adiabatic expression we write

$$f = f_0 + f_1 + \ldots$$

where successive terms become smaller by the ratio

$$\frac{f_{n+1}}{f_n} = O(\epsilon) = O\left(\frac{r_G}{L}\right)$$

of the particle gyroradius $r_G = v/(q|\underline{B}|/mc) = v/\omega$ to the characteristic scale length L of the magnetic field. Note that ϵ depends on particle energy as $W^{1/2}$. In Jupiter's dayside magnetosphere between 20 and 40 Jovian radii from the planet ϵ is typically $\sim.01$ for a 1 MeV proton. There are magnetospheric regions, however, where $\underline{B}$ is very weak as in the Earth's tail near the neutral sheet or L is very small as in the current sheet region in Jupiter's dawn meridian (Smith et al. [1976], Goertz et al. [1976]) and adiabatic theory is invalid for a 1 MeV proton.

We make the small electric field assumption that $\underline{E} = O(\epsilon)$ for both the parallel E_1 and perpendicular components E_2 and E_3. The smallness of E_1 is necessary in order that particles do not freely accelerate along $\underline{B}$, thereby destroying any semblance of periodicity and adiabaticity. Whether E_2 and E_3 are small of $O(\epsilon)$ depends on the ratio of the electric field drift speed $cE_\perp/B$ to the particle speed v. In the following paper we take $E_\perp$ to be the electric field which produces corotation and find that this ratio is about $10^{-3}\rho$ for a 1 MeV proton, where ρ is the distance from Jupiter's spin axis in Jovian radii. Since $\rho < 50$, $cE_\perp/Bv$ does not exceed .05, and hence it is indeed permissible to choose $\underline{E}$ of $O(\epsilon)$ for the purposes of the following paper.

The relative asymptotic sizes of terms in (5) can now be estimated. A handy device here is to regard the coefficient m/q as carrying an intrinsic weight of ϵ. All terms in (5) are at largest of O(1) except the first and last, and to lowest ($1/\epsilon$) order

$$\frac{\partial f_0}{\partial t} - \omega\frac{\partial f_0}{\partial \lambda} = 0. \qquad (6)$$

Equation (6) expresses the result that any gyrophase dependence of f_0 gyrates about $\underline{B}$ at the gyro-frequency ω. We demand stationarity on the gyro-timescale so that $\partial f_0/\partial t = O(1)$, $\partial f_0/\partial\lambda \equiv 0$, and $f_0 = f_0(v, \delta, \underline{r}, t)$.

We now demand further that $\partial f/\partial t = O(\epsilon)$, i.e., that f be stationary on the time scale of the bounce motion. With this restriction there can be no particle bunches moving along $\underline{B}$. The O(1) contribution to (5) thus becomes

$$\omega \frac{\partial f_1}{\partial\lambda} = \underline{v} \cdot \nabla f_0 - \frac{1}{v \sin\delta} (\underline{v} \cdot \nabla \hat{e}_1) \cdot \underline{v} \frac{\partial f_0}{\partial\delta}$$

$$+ \frac{q}{m} \left[E_1 \left(\cos\delta \frac{\partial f_0}{\partial v} - \frac{\sin\delta}{v} \frac{\partial f_0}{\partial\delta} \right) \right.$$

$$\left. + \left(E_2 \cos\lambda + E_3 \sin\lambda \right) \left(\sin\delta \frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v} \frac{\partial f_0}{\partial\delta} \right) \right] \tag{7}$$

which may be integrated to yield

$$f_1 = f_1(v, \delta, \lambda = 0, \underline{r}) + \frac{v}{\omega} \int_0^\lambda d\lambda' \left\{ \hat{v} \cdot \nabla f_0 - \frac{1}{\sin\delta} (\hat{v} \cdot \nabla \hat{e}_1) \cdot \hat{v} \frac{\partial f_0}{\partial\delta} \right.$$

$$+ \frac{q}{mv} \left[E_1 \left(\cos\delta \frac{\partial f_0}{\partial v} - \frac{\sin\delta}{v} \frac{\partial f_0}{\partial\delta} \right) \right.$$

$$\left.\left. + \left(E_2 \cos\lambda' + E_3 \sin\lambda' \right) \left(\sin\delta \frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v} \frac{\partial f_0}{\partial\delta} \right) \right] \right\} \tag{8}$$

where $\hat{v} = \vec{v}/v$. The integration in (8) can be carried out by noting

$$\hat{v} = \hat{e}_1 \cos\delta + \sin\delta (\hat{e}_2 \cos\lambda' + \hat{e}_3 \sin\lambda')$$

The result is

$$
\begin{aligned}
f_1 = {} & f_1(v, \delta, \lambda = 0, \underline{r}) + \frac{v}{\omega} \lambda \Bigg(\Bigg\{ (\hat{e}_1 \cdot \nabla f_0) \cos\delta \\
& - \frac{1}{2} \sin\delta \frac{\partial f_0}{\partial \delta} [(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3] \\
& + \frac{qE_1}{mv} \left(\cos\delta \frac{\partial f_0}{\partial v} - \frac{\sin\delta}{v} \frac{\partial f_0}{\partial \delta} \right) \Bigg\} \\
& + \Bigg\{ (\hat{e}_3 \cdot \nabla f_0) \sin\delta - \frac{\partial f_0}{\partial \delta} \Bigg[\cos\delta\, (\hat{e}_1 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3 \\
& + \sin\delta \, \frac{(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2}{4} \Bigg] \\
& + \frac{qE_3}{mv} \left(\sin\delta \frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v} \frac{\partial f_0}{\partial \delta} \right) \Bigg\} \\
& + \sin\lambda \Bigg[(\hat{e}_2 \cdot \nabla f_0) \sin\delta - \frac{\partial f_0}{\partial \delta} \cos\delta\, (\hat{e}_1 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2 \\
& + \frac{qE_2}{mv} \left(\sin\delta \frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v} \frac{\partial f_0}{\partial \delta} \right) \Bigg] \\
& + \cos\lambda \Bigg[- (\hat{e}_3 \cdot \nabla f_0) \sin\delta + \frac{\partial f_0}{\partial \delta} \cos\delta\, (\hat{e}_1 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3 \\
& - \frac{qE_3}{mv} \left(\sin\delta \frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v} \frac{\partial f_0}{\partial \delta} \right) \Bigg] \\
& + \frac{\sin 2\lambda}{4} \sin\delta \frac{\partial f_0}{\partial \delta} [- (\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3] \\
& + \frac{\cos 2\lambda}{4} \sin\delta \frac{\partial f_0}{\partial \delta} [(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2] \Bigg)
\end{aligned}
\qquad (9)
$$

Since λ is an angle variable, f must be periodic in it. f_0, being independent of λ, is clearly so. Everything in Equation (9) for f_1 is periodic in λ with the exception of the linear terms. We enforce periodicity by constraining f_0 further so that the coefficient of λ vanishes:

$$(\hat{e}_1 \cdot \nabla f_0) \cos \delta - \frac{1}{2} \sin \delta \frac{\partial f_0}{\partial \delta} [(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3]$$

$$+ \frac{qE_1}{mv} \left(\cos \delta \frac{\partial f_0}{\partial v} - \frac{\sin \delta}{v} \frac{\partial f_0}{\partial \delta} \right) \equiv 0 \tag{10}$$

Now note that

$$0 = \nabla \cdot \underline{B} = \nabla \cdot (\hat{e}_1 B) = \hat{e}_1 \cdot \nabla B + [(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3] B \tag{11}$$

and thus

$$(\hat{e}_1 \cdot \nabla f_0) \cos \delta + \frac{1}{2} \sin \delta \frac{\hat{e}_1 \cdot \nabla B}{B} \frac{\partial f_0}{\partial \delta} + \frac{qE_1}{mv} \left(\cos \delta \frac{\partial f_0}{\partial v} - \frac{\sin \delta}{v} \frac{\partial f_0}{\partial \delta} \right) \equiv 0 \tag{12}$$

Let us denote $\hat{e}_1 \cdot \nabla$ by $\partial/\partial s$. Equation (12), being linear in f_0, is satisfied by any arbitrary function of the characteristic constants which label flow lines in the 3-dimensional s, δ, v space. These constants are obtained by solving the ordinary differential equations

$$\frac{mv}{qE_1} dv = ds = \frac{\cos \delta}{\sin \delta} \left(\frac{1}{2B} \frac{\partial B}{\partial s} - \frac{qE_1}{mv^2} \right)^{-1} d\delta \tag{13}$$

From the first pair of these equations comes the parallel energy

$$K = \frac{mv^2}{2} - q \int E_1 \, ds \tag{14}$$

as one constant, and from the second pair we find (using 14) the magnetic moment

$$M = \frac{mv^2 \sin^2 \delta}{2B} \tag{15}$$

to be a second. The constraint (10) therefore dictates that on any given field line $f_0 = f_0(K, M)$. Because M is double-valued in δ, separate distributions may be specified for particles moving parallel and antiparallel to $\underline{B}$ (Hastie et al. [1967]). If the particles are trapped in a magnetic mirror, the stationarity of f requires that these two distributions be the same. However, there are situations, such as circulating particles in a toroidal device or particles precipitating at a steady rate in a magnetosphere, where the symmetry need not exist. In addition, the form of the dependence of f_0 on K and M may differ on different field lines. If we label a field line by the parameters α and β (Northrop [1961], Stern [1970]) then $f_0 = f_0(K, M, \alpha, \beta)$.

Note further that the remaining λ-independent terms in (9) can be lumped together by redefining the arbitrary function f_1. Thus, through $O(\epsilon)$,

$$\begin{aligned} f = {} & f_0(K, M, \alpha, \beta) + f_1(v, \delta, \underline{r}) \\ & + \frac{v}{\omega}\Bigg\{\sin\lambda\Bigg[(\hat{e}_2 \cdot \nabla f_0)\sin\delta - \frac{\partial f_0}{\partial \delta}\cos\delta\,(\hat{e}_1 \cdot \nabla \hat{e}_1)\cdot \hat{e}_2 \\ & \quad + \frac{qE_2}{mv}\left(\sin\delta\,\frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v}\,\frac{\partial f_0}{\partial \delta}\right)\Bigg] \\ & + \cos\lambda\Bigg[-(\hat{e}_3 \cdot \nabla f_0)\sin\delta + \frac{\partial f_0}{\partial \delta}\cos\delta\,(\hat{e}_1 \cdot \nabla \hat{e}_1)\cdot \hat{e}_3 \\ & \quad - \frac{qE_3}{mv}\left(\sin\delta\,\frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v}\,\frac{\partial f_0}{\partial \delta}\right)\Bigg] \end{aligned}$$

$$+ \sin 2\lambda \frac{\sin\delta}{4} \frac{\partial f_0}{\partial \delta} [-(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3]$$

$$+ \cos 2\lambda \frac{\sin\delta}{4} \frac{\partial f_0}{\partial \delta} [(\hat{e}_2 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3 + (\hat{e}_3 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2] \Bigg\} \tag{16}$$

Equation (16) is the principal result of this paper. Given $f_0 + f_1$ and the magnetic and electric fields $\underline{B}$ and $\underline{E}$, it allows one to calculate the non-gyrotropy (which contributes, for example, to the cross-field particle flow). The existence of f_1 means that the gyrotropic part of f may have a small $O(\epsilon)$ component which depends in an arbitrary fashion on s, v, and δ and still maintain $\partial f/\partial t = 0 + O(\epsilon)$. If the time variation of f were limited yet further, restrictions on f_1, similar to those imposed on f_0 here, would arise (Hastie et al. [1967]). It should also be noted that because of the redefinition of f_1, $f_0 + f_1$ is no longer $f(\lambda = 0)$ but is the λ average of f.

It is instructive to examine velocity space moments of (16) to see the descendants of each term. The density

$$n = \int d^3v f = \int d^3v (f_0 + f_1) = \int d^3v f_0 + O(\epsilon) \tag{17}$$

does not involve any λ-dependent terms, because they integrate to zero. To the macroscopic flow velocity

$$\langle \underline{v} \rangle = \frac{1}{n} \int d^3v\, \hat{e}_1 v \cos\delta\, (f_0 + f_1)$$

$$+ \frac{1}{n\omega} \int d^3v\, v^2 \sin\delta \Bigg\{ \hat{e}_2 \cos^2\lambda \Bigg[-(\hat{e}_3 \cdot \nabla f_0) \sin\delta$$

$$+ \frac{\partial f_0}{\partial \delta} \cos\delta\, (\hat{e}_1 \cdot \nabla \hat{e}_1) \cdot \hat{e}_3 - \frac{qE_3}{mv} \left(\sin\delta \frac{\partial f_0}{\partial v} + \frac{\cos\delta}{v} \frac{\partial f_0}{\partial \delta}\right) \Bigg]$$

$$+ \hat{e}_3 \sin^2 \lambda \Big[(\hat{e}_2 \cdot \nabla f_0) \sin \delta - \frac{\partial f_0}{\partial \delta} \cos \delta \, (\hat{e}_1 \cdot \nabla \hat{e}_1) \cdot \hat{e}_2$$

$$+ \frac{qE_2}{mv} \left(\sin \delta \frac{\partial f_0}{\partial v} + \frac{\cos \delta}{v} \frac{\partial f_0}{\partial \delta} \right) \Big] \Big\} \tag{18}$$

on the other hand, $f_0 + f_1$ contributes the parallel (to $\underline{B}$) component and the $\cos \lambda$ and $\sin \lambda$ terms in f are responsible for the perpendicular fluxes in the $\hat{e}_2$ and $\hat{e}_3$ directions respectively. In order that a non-zero $\langle \underline{v} \rangle \cdot \hat{e}_1$ exist, $(f_0 + f_1)$ for particles with pitch angles $0 < \delta < \pi/2$ must be different from $(f_0 + f_1)$ for particles with pitch angles $\pi/2 < \delta < \pi$.

An experimental detector frequently samples only a portion of the 4π steradians through which $\underline{v}$ may vary. For example, in the Pioneer experiment analyzed in the following paper, the proton detector scans in θ through 2π radians in a plane to which $\underline{B}$ is arbitrarily inclined. An experimental anisotropy is then obtained from the magnitude and direction of the quantity

$$\int_0^\infty dv\, v^2 \int_0^{2\pi} d\theta \, f[v, \delta(\theta), \lambda(\theta)] \, \underline{v}(\theta).$$

The $\cos 2\lambda$ and $\sin 2\lambda$ terms in (16), which vanish upon integration over all angles of $\underline{v}$, contribute to this "reduced flow velocity" provided the pitch angle distribution of particles is non-uniform so that $\partial f_0 / \partial \delta \neq 0$.

The perpendicular flow terms in (18) can be cast into a more familiar form by some straightforward integrations, integrations by parts, and vector manipulation. Details are omitted because we are interested only in the fate of terms in (18).

$$\langle \underline{v} \rangle_1 = \underset{(\underline{E})}{\frac{c\underline{E} \times \underline{B}}{B^2}} + \frac{c}{nq} \Big[\underset{(\nabla f_0)}{\frac{\hat{e}_1}{B} \times \nabla(MB)} - \underset{(\hat{e}_1 \cdot \nabla \hat{e}_1)}{M\hat{e}_1 \times (\hat{e}_1 \cdot \nabla \hat{e}_1)} \Big]$$

$$+\frac{mc}{nqB}\,\hat{e}_1 \times (\hat{e}_1 \cdot \nabla\hat{e}_1)\int d^3v\, f_0 v^2 \cos^2\delta$$

$$(\hat{e}_1 \cdot \nabla\hat{e}_1) \qquad (19)$$

where $\underline{M} \equiv -\hat{e}_1 M = -\hat{e}_1 (m/2)\int d^3v\, f_0 v^2 \sin^2\delta$ is the magnetic moment per unit volume. Indicated in parentheses under each term is its ancestry in (16) or (18). With a little more manipulation we obtain

$$\langle \underline{v} \rangle_\perp = \underset{(\underline{E})}{\frac{c\underline{E} \times \underline{B}}{B^2}} + \underset{(\nabla f_0)}{\frac{M}{n}\frac{c}{qB}\,\hat{e}_1 \times \nabla B}$$

$$+ \underset{(\hat{e}_1 \cdot \nabla\hat{e}_1)}{\frac{mc}{q}\langle v_\parallel^2 \rangle\, \hat{e}_1 \times (\hat{e}_1 \cdot \nabla\hat{e}_1)} + \underset{(\nabla f_0,\, \hat{e}_1 \cdot \nabla\hat{e}_1)}{\frac{c}{nq}(\nabla \times \underline{M})_\perp} \qquad (20)$$

or

$$\langle \underline{v} \rangle_\perp = \langle \underline{v}_{drift} \rangle + \frac{c}{nq}(\nabla \times \underline{M})_\perp \qquad (21)$$

where $\langle \underline{v}_{drift} \rangle$ is the average drift velocity of guiding centers. As shown by Northrop [1961], the right hand side of (21) also equals

$$\frac{c}{nqB}\,\hat{e}_1 \times (\nabla \cdot \underline{\underline{P}}) + \frac{c\underline{E} \times \underline{B}}{B^2} + \frac{mc}{qB^2}\,\hat{e}_1 \times \frac{d\langle \underline{v} \rangle}{dt},$$

a result which also follows from the moment equation

$$\frac{d\langle \underline{v} \rangle}{dt} = -\frac{\nabla \cdot \underline{\underline{P}}}{nm} + \frac{q}{nm}\left(\underline{E} + \frac{1}{c}\langle \underline{v} \rangle \times \underline{B}\right),$$

$\underline{\underline{P}}$ being the pressure tensor.

Although it might have been tempting at first sight of (16) to make off-hand identications, such that ∇B drifts arise from terms containing ∇f_0, and the curvature drifts from those containing $(\hat{e} \cdot \nabla \hat{e}_1)$, this is only partially true in that parts of all these terms also lead to the $(\nabla \times \underline{M})_\perp$ term. Thus when looking at (16), it is difficult to think in terms of guiding center motion.

III. THE ELECTRIC FIELD IN A RIGIDLY COROTATING MAGNETOSPHERE

As a preliminary to our using Equation (16) in following papers, we present here a short derivation of the form which $\underline{E}$ must have in order that the magnetospheric plasma corotate. The source of $\underline{B}$ may be finite in size and arbitrary in shape. It is rotating about a fixed axis which we take to define the z-axis of a cylindrical "laboratory" system, so that $\underline{\Omega} = \hat{e}_z \Omega$. By rigid corotation we mean that all physical quantities are time invariant when seen by an observer moving with the velocity $\underline{\Omega} \times \underline{r}$. However, if the magnetic field is not axisymmetric with respect to $\underline{\Omega}$, a $\partial\underline{B}/\partial t$ will exist at every fixed spatial point $\underline{r}$ in the laboratory coordinate system and have a value (Goldstein [1950], Backus [1956], Birmingham and Northrop [1968])

$$\frac{\partial \underline{B}}{\partial t} = \underline{\Omega} \times \underline{B} - (\underline{\Omega} \times \underline{r}) \cdot \nabla \underline{B}$$

$$= \underline{\Omega} \times \underline{B} - \Omega \frac{\partial \underline{B}}{\partial \phi} \tag{23}$$

Maxwell's Curl Equation hence becomes

$$\nabla \times \underline{E} = -\frac{1}{c}\left(\underline{\Omega} \times \underline{B} - \Omega \frac{\partial \underline{B}}{\partial \phi}\right) \tag{24}$$

Further

$$\underline{E} = -\frac{\underline{u} \times \underline{B}}{c} \tag{25}$$

where $\underline{u}$ is the flow velocity of the plasma, assumed to be ideally conducting.

Next we postulate that

$$\underline{u} = k(\underline{r})\,\hat{e}_1 + a(\underline{r})\,\rho\hat{\phi} \tag{26}$$

so that the plasma flow consists of a component of amplitude (ρa), to be determined from (24), in the "corotational" direction ($\hat{\phi}$ may have a field-aligned component) plus the arbitrary field-aligned portion $k(\underline{r})$.

Inserting (25) and (26) into (24) and performing the vector differentiations one obtains the equation

$$\frac{\partial}{\partial\phi}(a\underline{B}) - a\hat{z} \times \underline{B} - \rho\hat{\phi}\,\underline{B} \cdot \nabla a = \Omega\frac{\partial \underline{B}}{\partial\phi} - \underline{\Omega} \times \underline{B}. \tag{27}$$

Equation (27) is trivially satisfied by a = const. = Ω. The "corotational" electric field $\underline{E}$ thus is

$$\underline{E} = -\frac{\rho\Omega}{c}\hat{\phi} \times \underline{B} \tag{28}$$

Note that since $k(\underline{r})$ in (26) is still arbitrary it could be chosen to be

$$k = -\rho\Omega\,\hat{\phi} \cdot \hat{e}_1$$

so that $\underline{u}$ is totally perpendicular to $\underline{B}$. Our result (27) has the same form as Mestel's [1961] solution for the case of a rotating azimuthally symmetric magnetosphere whose rotation and magnetic symmetry axes are parallel. The form of the expression for $\underline{E}$ thus is unchanged by generalizing to this non-axisymmetric geometry. In applications such as in the following paper to Jupiter's magnetosphere, the electric field has not been measured and it is desirable to postulate one in order to reduce the number of unknowns. Although (28) undoubtedly is not exact, it is a stage better than assuming the $\underline{E}$ which one would have for a magnetic dipole field.

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BIBLIOGRAPHIC DATA SHEET

1. Report No. TM 78090	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Theory of Flux Anisotropies in a Guiding Center Plasma		5. Report Date February 1978
		6. Performing Organization Code 695
7. Author(s) T. J. Birmingham, T. G. Northrop		8. Performing Organization Report No.
9. Performing Organization Name and Address NASA/Goddard Space Flight Center Laboratory for Extraterrestrial Physics Planetary Magnetospheres Branch Greenbelt, Maryland 20771		10. Work Unit No.
		11. Contract or Grant No.
		13. Type of Report and Period Covered Technical Memorandum
12. Sponsoring Agency Name and Address		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

Assuming time stationarity of the one particle distribution function f on the scale of the bounce motion of particles in a magnetic field $\underline{B}$, we expand the Vlasov Equation through $O(\varepsilon)$ in the adiabatic parameter ε, which is the ratio of particle gyroradius to scale length of the magnetic field. Since f is directly proportional to particle flux $d\Phi/dWd\Omega$ differential in kinetic energy W and solid angle Ω, f is in principle measurable in space experiments, and our analysis is tailored to be explicitly applicable to space problems. To $O(1)$, f is gyrotropic; its first velocity moment is (if non-vanishing) parallel to $\underline{B}$, and hence macroscopic parallel flow is included in this term. The $O(\varepsilon)$ contribution is non-gyrotropic and macroscopic flow $\perp$ to $\underline{B}$ plus additional parallel flow results from these terms. The degree of non-gyrotropy and hence the amount of cross-field macroscopic flow depend on the perpendicular component of the electric field $\underline{E}$, on curvature and shear in the magnetic field, and on the spatial gradient f_0, pitch angle derivative $\partial f_0/\partial \delta$ and speed derivative $\partial f_0/\partial v$ of the lowest order distribution function f_0. We also show that the usual expression for the electric $\underline{E}$ which produces plasma co-rotation in an axisymmetric system such as a dipole also holds for any non-axisymmetric but rigidly rotating magnetic field pattern, provided the observed magnetic field is used in place of the dipole field.

17. Key Words (Selected by Author(s)) Flux Anisotropies Energetic Particles	18. Distribution Statement

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
U	U	15	

*For sale by the National Technical Information Service, Springfield, Virginia 22161.

GSFC 25-44 (10/77)