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FINAL REPORT

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THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES

PART A: DISCRETE MODEL AND MODAL CONTROL

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THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES

PART A: DISCRETE MODEL AND MODAL CONTROL

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ABSTRACT

Attitude control techniques for the pointing and stabilization of very large, inherently flexible spacecraft systems are investigated. The attitude dynamics and control of a long, homogeneous flexible beam whose center of mass is assumed to follow a circular orbit is analyzed. In this study, first order effects of gravity-gradient are included, whereas external perturbations and related orbital station keeping maneuvers are neglected. A mathematical model which describes the system rotations and deflections within the orbital plane has been developed by treating the beam as a number of discretized mass particles connected by massless, elastic structural elements. The uncontrolled dynamics of this system are simulated and, in addition, the effects of the control devices are considered. The concept of distributed modal control, which provides a means for controlling a system mode independently of all other modes, is examined. The effect of varying the number of modes in the model as well as the number and location of the control devices are also considered.

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NOMENCLATURE

a	=	location of mass m_a (or m_1) from the origin point of the main body
b	=	location of mass m_b from the origin point of the main body
EI	=	bending stiffness of uniform beam
F	=	control vector (includes torques and forces)
f	=	scalar actuator variables; also, feedback control gains
$f^{(c)}$	=	control force vector acting on the discrete mass system
$f^{(r_i)}$	=	residual coupling coefficent
I_1, I_2, I_3	=	principal moments of inertia of the main body
K	=	stiffness matrix of the system
L	=	length of the beam ($L = 2\ell$)
M	=	mass matrix of the system
M_c	=	mass of the main body
m	=	mass of the end mass
m_i	=	generalized mass for mode i; also refers to end masses
m_0	=	mass of the interior mass located in between m_1 and m_2
N	=	number of modes
P	=	number of actuators
q_i	=	modal coordinates
R	=	radius of the orbit
$\bar{r}_i$	=	radius vector from center of earth to mass, m_i
T	=	kinetic energy of the system; also, control transformation matrix which provides transformation from discrete actuator variables to distributed actuator variables ($f = Tu$)

T_c	=	orbital energy of the system
T_r	=	rotational energy of the system
T_t	=	translation energy of the system
T_θ	=	torque acting on the main body
U	=	elastic energy of the system
u_i	=	independent distributed actuator variables
V	=	potential energy due to gravity forces
v_1, v_2	=	deflections of the end masses ($m_1 = m_2 = m$)
v_a, v_b	=	deflection of masses, m_a and m_b, respectively
X	=	state vector of the system (discrete coordinates)
x, z	=	local vertical coordinates in the orbit plane (z - along the local vertical with the origin at the system center of mass)
θ	=	pitch angle of the main body
$\omega_1, \ldots, \omega_n$	=	eigen values (modal frequencies)
ω_0	=	orbital angular velocity
$\phi^{(1)}, \ldots, \phi^{(N)}$	=	eigen vectors (mode shapes)
Φ	=	coordinate transformation matrix which provides transformation from discrete coordinates to modal coordinates
ϕ_1, ϕ_2	=	relative angular motions of both end masses relative to the undeflected configuration of the beam
ξ, ζ	=	coordinate system with origin at the interior point of the beam, m_0 (parallel to the x, z axes)

LIST OF ILLUSTRATIONS

1. INTRODUCTION

In the recently completed "Outlook for Space Study"[1] a number of proposed new space missions were shown to require large-scale, light-weight space structures. Three representative proposed future missions utilizing such systems are:

(1) ocean data systems involving a 100 m. wide structure for the purpose of collecting data on the state of the oceans, pollution (both in the atmosphere and the oceans), and salinity.

(2) electronic mail systems requiring a 50 m. diameter antenna-receiver system to be placed in a synchronous equatorial orbit.

(3) a space-based solar power collector system, also operating in a synchronous equatorial orbit where the arrays for collecting the incident solar energy would have the dimensions on the order of kilometers.

It is evident that a complete new technology must be considered and developed so that these structures can be delivered into orbit (using the Shuttle Transportation System), deployed, and then fully assembled in a space environment. Because of their inherent size, the testing of such systems in a ground environment is not practical. Modeling techniques and scaling algorithms must be developed so that the performance of these systems can be accurately predicted prior to launch and assembly.

In many circumstances, it will be necessary to control the shapes of the antenna or collector surfaces to within centimeters or even millimeters by using a variety of sensor-actuator systems to be positioned throughout the flexible members.[1]

To address some of these unprecedented problems a special Industry Workshop on Large Space Structures[2] was held at NASA Langley Research Center in Feb. 1976. Among the principal conclusions was that technology development is most critically needed in: the definition of large space structural configurations; improved modeling and scaling techniques; and the interaction between the control system and structural dynamic responses. It was also stated that some of these development techniques would utilize improved computerized analytical models. Concern was also expressed over possible resonant interactions between some of the flexible structural modes (with frequencies greatly reduced as compared with more conventional structures) and the frequencies associated with the attitude control systems. Analytic areas requiring model development also include the precision determination of gravity-gradient forces and moments acting on such large structures.[2]

The AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft[3] held at Virginia Polytechnic Institute and State University, June 1977 provided a review of the state of the art in this area. It reflected the tremendous strides made in modeling and analysis of spacecraft in the last two decades. Yet, the next two decades are likely to place a severe strain on the current state of the art if some of the advanced concepts, such as solar power satellites or space colonies, are to be converted into reality.

In this regard, this symposium raised as many questions as it answered and, in the process, it pointed to areas of future research.

It was stated in Ref. 2 (Page 12) that to assure the availability of the large space structures technology to support future missions a comprehensive research and development program must be defined. Three of the objectives of such a program are:[2]

(1) development of active surface control techniques and systems which can measure and correct surface deformations to within <u>millimeters</u> of <u>accuracy</u>.

(2) development of attitude control techniques for the pointing and stabilization of very large, inherently flexible space structures.

(3) development of analysis and simulation techniques which can extend subscale ground test experience to high-confidence predictions of full-scale performance in the space environment.

The present study represents a preliminary contribution to the accomplishment of the second stated objective.

In this report, Chapter 2 deals with the development of a mathematical model which describes the rotation and deflection of a long, thin, flexible beam in the orbital plane. It is assumed that the beam is represented by a main body and a series of discretized particles and the model prevously developed by Meirovitch and Nelson[4] is used as a starting point. This model is extended to include first order effects of gravity-gradient torques.

The discrete models are developed under the assumptions, first, that the beam is represented by a massless rod connected by two end masses, and, secondly, that the beam is represented by four discrete particles, two at the ends and two located at points taken half-way between the center of the undeflected beam and the end points. The equations are developed and linearized for the cases of small amplitude rotations and deflections. These equations are then cast in the proper form for the application of the mode control concept (Chapter 4) as expounded in a recent contract report by Rockwell International.[5]

A more general case of a flexible beam system using three discretized masses without a central body is considered in Chapter 3. The nonlinear equations of motion are developed using the Lagrangian formulation. These equations are then linearized assuming small amplitude deformations about two equilibrium positions : (1) alignment along the local vertical and (2) alignment along the local horizontal or orbit tangent. Stability conditions for system motion taken about these equilibrium positions are obtained.

The mode control concept[5] for controlling a spacecraft by independently controlling motions of the rigid body and the vibrational modes is presented in Chapter 4. Development of the mode control concept is based on two coordinate transformations.

In Chapter 5, the mode control concept described in Chapter 4 is applied to the linearized equations developed in Chapter 3 for the three-mass system. The uncontrolled dynamics, as well as the dynamics of the system with two actuators (equal to the number of vibrational modes), or one actuator, are simulated numerically using closed form expressions.

The concluding comments from this study and suggestions for future work are summarized in Chapter 6.

2. MAIN BODY WITH TWO FLEXIBLE BEAMS

2.1 Equations of Motion-Torque Free System

2.1.1 Each beam modelled by an end mass

The configuration of the system considered is shown in Fig. 2.1. The system center of mass is assumed to move in a circular orbit and the system is constrained to move in the orbital plane. The system consists of a main body (M_c) with two beams attached to the main body. Each beam, modelled by an end mass (m_i), is attached to the main body as shown in Fig. 2.1. The equations of motion for the system will be derived by use of the Lagrangian formulation. One generalized coordinate of the system will be the angle θ which represents the rigid body pitch angle. The other generalized coordinates will be the deflections v_1 and v_2 of the end masses, m_1 and m_2, respectively, relative to the undeformed (but rotated) body z axis.

The total kinetic energy of the system, in terms of the rotational and translational energies, is

$$T = T_r + T_t + \text{const. due to circular orbital motion} \tag{2.1}$$

The rotational kinetic energy of the main body is

$$T_r = \frac{1}{2} I_2 \dot{\theta}^2 \tag{2.2}$$

and the translational energy, due to the end masses, can be written as[6,7]

$$T_t = \frac{m}{2} \Sigma (\overline{V}_i \cdot \overline{V}_i) - \frac{m^2}{2\overline{M}} (\Sigma\overline{V}_i \cdot \Sigma\overline{V}_i) \tag{2.3}$$

where

$$\overline{V}_i = \dot{\overline{r}}_i\big|_{body} + \overline{\omega} \times \overline{r}_i \tag{2.4}$$

$$\overline{M} = M_c + 2m \;;\; m_1 = m_2 = m \tag{2.5}$$

The coordinates of the two masses m_1 and m_2 with respect to point 0 (Fig. 2.1) are given in terms of the position vectors,

$$\bar{r}_1 = v_1\hat{j} + a\,\hat{k} \tag{2.6}$$

$$\bar{r}_2 = -\,v_2\hat{j} - a\,\hat{k} \tag{2.7}$$

After substitution of Eqs. (2.2) - (2.7) into Eq. (2.1), the resulting expression for the kinetic energy is

$$\begin{aligned} T = {} & \frac{1}{2}\ I_2{}'\ \dot{\theta}^{\,2} \\ & + \frac{m}{2}\ [\ \dot{v}_1{}^2 + \dot{v}_2{}^2 + (v_1{}^2 + v_2{}^2)\ \dot{\theta}^2 + 2\dot{\theta}\ a(\dot{v}_1 + \dot{v}_2)] \\ & - \frac{m^*}{2}\ [\ (\dot{v}_1 - \dot{v}_2)^2 + (v_1 - v_2)^2\ \dot{\theta}^2\ ] + \text{const.} \end{aligned} \tag{2.8}$$

where

$$I_2{}' = I_2 + 2\ ma^2 \text{ and } m^* = m^2/\overline{M}$$

Initially the effect of gravity-gradient forces will be neglected so that the potential energy consists of the elastic energy, plus a constant due to the assumed circular orbital motion of the mass center.

$$V = U + \text{const.} = \frac{k}{2}\ (v_1{}^2 + v_2{}^2) + \text{const.} \tag{2.9}$$

where k is the stiffness of each beam ($k = 3EI/a^3$).

The Lagrangian equations of motion for the system have the form

$$(d/dt)\ (\partial L/\partial \dot{q}_i) - (\partial L/\partial q_i) = 0 \tag{2.10}$$

where q_i assumes the values: θ, v_1, and v_2. The Langrangian has the form

$$L = T - V = T - U + \text{const.}$$

where T is the kinetic energy and U is the elastic potential energy of the system.

Considering only the linear terms, the equations of motion can be represented in matrix form as

$$\begin{bmatrix} I_2' & ma & ma \\ ma & m-m^* & m^* \\ ma & m^* & m-m^* \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \begin{Bmatrix} \theta \\ v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \qquad (2.11)$$

If the control forces (including the control torques also) acting on the system are represented by, F, then Eq. (2.11) is reduced to the standard form

$$M \ddot{X} + K X = F \qquad (2.12)$$

For the special case, when the deflections of the end masses are assumed to be antisymmetric $(v_1 = v_2 = v)$, Eqs. (2.11) and (2.12) reduce to the following equation:

$$\begin{bmatrix} I_2' & 2ma \\ 2ma & 2m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{v} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2k \end{bmatrix} \begin{Bmatrix} \theta \\ v \end{Bmatrix} = \begin{Bmatrix} T_\theta \\ F_v \end{Bmatrix} \qquad (2.13)$$

2.1.2 Each beam modelled by two masses

The mathematical model for the system considered now consists of a main body and two flexible beams which are attached to the main body. Each beam is modelled by two masses as shown in Fig. 2.2. For simplicity, it is assumed also that the elastic motion of the beams is antisymmetric, that is $v_a = v_a'$ and $v_b = v_b'$.

With these restrictions, the kinetic energy expression takes the form

$$T = \frac{1}{2} I_2 \dot{\theta}^2 + m \left[(\dot{v}_a + \dot{\theta}a)^2 + \dot{\theta}^2 (v_a{}^2 + v_b{}^2) + (\dot{v}_b + \dot{\theta}b)^2\right] + \text{const.} \tag{2.14}$$

The elastic strain energy is

$$U = k_{11}v_a{}^2 + 2k_{12}v_a v_b + k_{22}v_b{}^2 \tag{2.15}$$

For small displacements, the equations of motion can be written as

$$\begin{bmatrix} I_2'' & 2ma & 2mb \\ 2ma & 2m & 0 \\ 2mb & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{v}_a \\ \ddot{v}_b \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k_{11} & 2k_{12} \\ 0 & 2k_{12} & 2k_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ v_a \\ v_b \end{Bmatrix} = \begin{Bmatrix} T_\theta \\ F_{v_a} \\ F_{v_b} \end{Bmatrix} \tag{2.16}$$

where

$$I_2'' = I_2 + 2m\,(a^2 + b^2)$$

Eq. (2.16) is in the form of Eq. (2.12) where $X = (\theta\ v_a\ v_b)^T$ and $F = (T_\theta\ F_{v_a}\ F_{v_b})^T$.

The stiffness matrix [s] for the beam modelled by n masses can be obtained from the flexibility matrix $[\alpha]$ using the relation $[s] = [\alpha]^{-1}$. The displacements (v_i) in terms of the flexibility influence[8] coefficients (α_{ij}) and the forces (f_i) which are associated with the displacements can be written as

$$\begin{aligned} v_1 &= \alpha_{11} f_1 + \alpha_{12} f_2 + \ldots + \alpha_{1n} f_n \\ &\vdots \\ v_n &= \alpha_{n1} f_1 + \alpha_{n2} f_2 + \ldots + \alpha_{nn} f_n \end{aligned}$$

In matrix form, we can write

$$\{v\} = [\alpha]\ \{f\} \tag{2.17}$$

For the case where each beam is modelled by two masses, the stiffness matrix becomes ($b = a/2$)

$$[s] = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = \frac{48\ EI}{7\ a^3} \begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix} \qquad (2.18)$$

2.2 Equations of Motion - Gravitational Effects

2.2.1 Each beam modelled by an end mass

The configuration of the system with main body and two flexible beams attached to the main body is shown in Fig. 2.3. Here, each beam is modelled by an end mass and the elastic motion is assumed to result only from antisymmetric bending ($v_1 = v_2 = v$). The kinetic energy of the system can be written :

$$T = \frac{1}{2}\ (I_2 + 2ma^2)\ \dot{\theta}^2 + m\ [\ \dot{v}^2 + v^2\dot{\theta}^2 + 2\dot{\theta}\ a\dot{v}\ ]$$

$$+ \text{ const. due to circular orbital motion} \qquad (2.19)$$

The elastic strain energy is represented by

$$U = kv^2 \qquad (2.20)$$

The potential energy due to gravity forces is expressed as

$$V = V_0 - \mu m\ (\frac{1}{|\bar{r}_1|} + \frac{1}{|\bar{r}_2|})\ ;\ \text{where } \mu = GM_{earth} \qquad (2.21)$$

where the potential due to the main body[9] is

$$V_0 = -\frac{3}{4}\ \omega_0^{\ 2}\ (I_1 - I_3)\ \cos 2\theta$$

and the radial distances of m_1 and m_2 from the center of the earth are

$$|\bar{r}_1| = [r_0^{\ 2} + a^2 - 2\ r_0\ a \cos\ \{\pi - (\theta + {}^{v}/_{a})\ \}\ ]^{\frac{1}{2}}$$

$$|\bar{r}_2| = [r_0^{\ 2} + a^2 - 2\ r_0\ a \cos\ (\theta + {}^{v}/_{a})\ ]^{\frac{1}{2}}$$

By use of the binomial theorem and neglecting terms of order $(^a/r_0)^3$ and higher, one can develop for the gravitational potential, V, the expression

$$V = -\frac{3}{4}\omega_0^2 (I_1 - I_3) \cos 2\theta - \mu \frac{2m}{r_0}$$
$$- \omega_0^2 \frac{ma^2}{2} \{1 + 3 \cos 2 (\theta + {}^v/_a)\} \qquad (2.22)$$

Lagrange's equations of motion can be obtained and then linearized as

$$\begin{bmatrix} I_2' & 2ma \\ 2ma & 2m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{v} \end{Bmatrix} + \begin{bmatrix} 3\omega_0^2 (I_1' - I_3) & 6\omega_0^2 ma \\ 6\omega_0^2 ma & 6\omega_0^2 m+2k \end{bmatrix} \begin{Bmatrix} \theta \\ v \end{Bmatrix} = \begin{Bmatrix} T_\theta \\ F_v \end{Bmatrix}$$
$$(2.23)$$

where
$$I_1' = I_1 + 2ma^2$$

2.2.2 Each beam modelled by two masses

The equations of motion, with the assumptions stated for the case where each beam is modelled by two masses ($m_a = m_b = m$), can be developed in a manner similar to that used in Section 2.2.1. Here only the final matrix form of the linearized equations is presented:

$$\begin{bmatrix} I_2'' & 2ma & 2mb \\ 2ma & 2m & 0 \\ 2mb & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{v}_a \\ \ddot{v}_b \end{Bmatrix}$$

$$+ \begin{bmatrix} 3\omega_0^2(I_1'' - I_3) & 6\omega_0^2 ma & 6\omega_0^2 mb \\ 6\omega_0^2 ma & 6\omega_0^2 m+2k_{11} & 2k_{12} \\ 6\omega_0^2 mb & 2k_{12} & 6\omega_0^2 m+2k_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ v_a \\ v_b \end{Bmatrix} = \begin{Bmatrix} T_\theta \\ F_{v_a} \\ F_{v_b} \end{Bmatrix} \qquad (2.24)$$

where
$$I_1'' = I_1 + 2m (a^2 + b^2)$$

3. THREE-MASS SYSTEM

3.1 Equations of Motion-Local Vertical

The long beam modelled by three masses as illustrated in Fig. 3-1 consists of two end masses ($m_1 = m_2 = m$) and an interior (point) mass, m_0. The system center of mass is assumed to move in a circular orbit and the system is constrained to move in the orbital plane. The equations of motion are derived using the Lagrangian formulation. The generalized coordinates ϕ_1 and ϕ_2 represent the relative angular motions of both end masses relative to the undeflected orientation of the beam.

3.1.1 Expression for kinetic energy

The total kinetic energy of the system can be written as

$$T = T_r + T_t + T_c \tag{3-1}$$

The rotational and orbital energies are

$$T_r = 0;\ T_c = \frac{1}{2}\,\overline{M}\,R^2\omega_0{}^2 \tag{3.2}$$

and the translational energy is obtained as[6,7]

$$T_t = \frac{1}{2}\,m\,\Sigma\,(\overline{V}_i \cdot \overline{V}_i) - \frac{m^2}{2\overline{M}}\,(\Sigma\,\overline{V}_i \cdot \Sigma\,\overline{V}_i) \tag{3.3}$$

where $\overline{M} = 2m + m_0$, ω_0 = orbital angular velocity and R = orbital radius

The velocities $\overline{V}_1$ and $\overline{V}_2$ can be obtained from Eq. (2.4) as (Fig. 3-1)

$$\overline{V}_1 = \omega_{y_1}\,\hat{j} \times (\xi_1\,\hat{i} + \zeta_1\,\hat{k})$$

$$= \omega_{y_1}\,\hat{j} \times (\ell_1\,\sin\phi_1\hat{i} + \ell_1\cos\phi_1\,\hat{k}) \tag{3.4}$$

$$\overline{V}_2 = \omega_{y_2}\ \hat{j} \times (-\ \xi_2\ \hat{i} - \zeta_2 \hat{k})$$

$$= \omega_{y_2}\ \hat{j} \times (-\ \ell_2 \sin\phi_2 \hat{i} - \ell_2 \cos\phi_2 \hat{k}) \tag{3.5}$$

where

$$\omega_{y_1} = \omega_0 + \dot{\phi}_1;\ \omega_{y_2} = \omega_0 + \dot{\phi}_2 \tag{3.6}$$

By substitution of Eqs. (3.2) - (3.6) in Eq. (3.1), we find that

$$T = \frac{1}{2} M^* \ [\{(\omega_0 + \dot{\phi}_1)\ \ell_1 \sin\phi_1 + (\omega_0 + \dot{\phi}_2)\ \ell_2 \sin\phi_2\}^2$$

$$+\ \{(\omega_0 + \dot{\phi}_1)\ \ell_1 \cos\phi_1 + (\omega_0 + \dot{\phi}_2)\ \ell_2 \cos\phi_2\}^2$$

$$+ \overline{m}_0\ \{(\omega_0 + \dot{\phi}_1)^2\ \ell_1{}^2 + (\omega_0 + \dot{\phi}_2)^2\ \ell_2{}^2\}]$$

$$+ \overline{M}\ R^2 \omega_0{}^2/2 \tag{3.7}$$

where $M^* = m^2/\overline{M}$, the system reduced mass, and $\overline{m}_0 = m_0/m$. It should be noted that the last term in Eq. (3.7) is a constant for the case of a circular orbit and does not affect the equations of motion.

3.1.2 Expression for potential energy

The potential energy V of the system in an inverse square force field is given by

$$V = -\ \mu \{ \frac{m}{|\overline{r}_1|} + \frac{m}{|\overline{r}_2|} + \frac{m_0}{|\overline{r}_0|} \} \tag{3.8}$$

where $\mu = GM$ is the earth's gravitational constant, and $|\overline{r}_i|$ is the distance of m_i from the center of the spherical earth. Then $|\overline{r}_i|$ may be expressed, in terms of $R = |\overline{R}|$ and the coordinates of m_i in the local vertical system with origin at the center of the beam, as

$$|\overline{r}_1| = [(R + \zeta_1 + \zeta_{cm})^2 + (\xi_1 + \xi_{cm})^2]^{\frac{1}{2}} \tag{3.9}$$

$$|\bar{r}_2| = [(R - \zeta_2 + \zeta_{cm})^2 + (-\xi_2 + \xi_{cm})^2]^{1/2} \quad (3.10)$$

$$|\bar{r}_0| = [(R + \zeta_{cm})^2 + \xi_{cm}^2]^{1/2} \quad (3.11)$$

where the coordinates of the system center of mass measured with respect to point, m_0, on the beam are given by

$$\xi_{cm} = -m(\xi_1 - \xi_2)/\bar{M} \; ; \; \zeta_{cm} = -m(\zeta_1 - \zeta_2)/\bar{M} \quad (3.12)$$

The expression for V, after omitting the terms of order $1/R^3$ and higher in the binomial expansion of $1/|\bar{r}_i|$, becomes

$$V = -\omega_0^2 M^* [\{(\ell_1 \cos\phi_1 + \ell_2 \cos\phi_2)^2 - \frac{1}{2}(\ell_1 \sin\phi_1 + \ell_2 \sin\phi_2)^2\}$$
$$+ \bar{m}_0 \{(\ell_1^2 \cos^2\phi_1 + \ell_2^2 \cos^2\phi_2) - \frac{1}{2}(\ell_1^2 \sin^2\phi_1 + \ell_2^2 \sin^2\phi_2)\}] \quad (3.13)$$

3.1.3 Expression for elastic energy

The elastic energy[10] is obtained by assuming that the end masses move as cantilevers with respect to the reference point, m_0, on the beam (Fig. 3.2). Thus the elastic energy in terms of elastic deformations ξ_1 and ξ_2 is expressed as

$$U \cong \frac{1}{2}[k_1 \xi_1^2 + k_2 \xi_2^2]$$
$$= \frac{1}{2}[k_1 \ell_1^2 \sin^2\phi_1 + k_2 \ell_2^2 \sin^2\phi_2] \quad (3.14)$$

For the later approximate modelling of the elastic deflections of a free-free beam it will be convenient to assume that $\ell_1 = \ell_2 = \ell$ such that m_0 and the system center of mass will be coincident when $\phi_1 = \phi_2 = 0$ (see Fig. 3.2).

3.1.4 Lagrange's equations of motion

The general equations of motion are developed using the Lagrangian formulation for the variables ϕ_i, $i = 1,2$. The Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) - \frac{\partial T}{\partial \phi_i} + \frac{\partial V}{\partial \phi_i} + \frac{\partial U}{\partial \phi_i} = T_{\phi_i} \tag{3.15}$$

The equations of motion are obtained, using Eqs. (3.7), (3.13) and (3.14) in Eq. (3.15), as

$$\begin{aligned}
&M^* [\{\ddot{\phi}_1 \ell_1 \sin\phi_1 + (\omega_0+\dot{\phi}_1)\, \ell_1 \cos\phi_1\, \dot{\phi}_1 + \ddot{\phi}_2\, \ell_2 \sin\phi_2 \\
&\quad + (\omega_0+\dot{\phi}_2)\, \ell_2 \cos\phi_2\, \dot{\phi}_2\}\, \ell_1 \sin\phi_1 \\
&\quad + \{(\omega_0+\dot{\phi}_1)\, \ell_1 \sin\phi_1 + (\omega_0+\dot{\phi}_2)\, \ell_2 \sin\phi_2\}\ell_1 \cos\phi_1 \dot{\phi}_1 \\
&\quad + \{\ddot{\phi}_1\, \ell_1 \cos\phi_1 - (\omega_0+\dot{\phi}_1)\, \ell_1 \sin\phi_1\, \dot{\phi}_1 + \ddot{\phi}_2\, \ell_2 \cos\phi_2 \\
&\qquad - (\omega_0+\dot{\phi}_2)\, \ell_2 \sin\phi_2\, \dot{\phi}_2\}\, \ell_1 \cos\phi_1 \\
&\quad - \{(\omega_0+\dot{\phi}_1)\, \ell_1 \cos\phi_1 + (\omega_0+\dot{\phi}_2)\, \ell_2 \cos\phi_2\}\, \ell_1 \sin\phi_1\, \dot{\phi}_1\} \\
&\qquad + \frac{m_0}{m}\, \{\ddot{\phi}_1\, \ell_1^{\,2}\}] \\
&-M^* [\{(\omega_0+\dot{\phi}_1)\, \ell_1 \sin\phi_1 + (\omega_0+\dot{\phi}_2)\, \ell_2 \sin\phi_2\}\, \ell_1 \cos\phi_1\, (\omega_0+\dot{\phi}_1) \\
&\qquad - \{(\omega_0+\dot{\phi}_1)\, \ell_1 \cos\phi_1 + (\omega_0+\dot{\phi}_2)\, \ell_2 \cos\phi_2\}\, \ell_1 \sin\phi_1\, (\omega_0+\dot{\phi}_1)] \\
&+ 2\omega_0^{\,2} M^* [\{\ell_1 \cos\phi_1 + \ell_2 \cos\phi_2\}\, \ell_1 \sin\phi_1 \\
&\qquad + \frac{1}{2}\{\ell_1 \sin\phi_1 + \ell_2 \sin\phi_2\}\, \ell_1 \cos\phi_1 \\
&\qquad + \frac{m_0}{m}\left(\frac{3}{2}\right) \ell_1^{\,2} \sin\phi_1 \cos\phi_1] \\
&+ k_1\, \ell_1^{\,2} \sin\phi_1 \cos\phi_1 = T_{\phi_1}
\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
& M^* [\{\ddot{\phi}_1 \ell_1 \sin\phi_1 + (\omega_0+\dot{\phi}_1) \ell_1 \cos\phi_1 \dot{\phi}_1 + \ddot{\phi}_2 \ell_2 \sin\phi_2 \\
& \quad + (\omega_0+\dot{\phi}_2) \ell_2 \cos\phi_2 \dot{\phi}_2\} \ell_2 \sin\phi_2 \\
& \quad + \{(\omega_0+\dot{\phi}_1) \ell_1 \sin\phi_1 + (\omega_0+\dot{\phi}_2) \ell_2 \sin\phi_2\} \ell_2 \cos\phi_2 \dot{\phi}_2 \\
& \quad + \{\ddot{\phi}_1 \ell_1 \cos\phi_1 - (\omega_0+\dot{\phi}_1) \ell_1 \sin\phi_1 \dot{\phi}_1 + \ddot{\phi}_2 \ell_2 \cos\phi_2 \\
& \quad\quad - (\omega_0+\dot{\phi}_2) \ell_2 \sin\phi_2 \dot{\phi}_2\} \ell_2 \cos\phi_2 \\
& \quad\quad - \{(\omega_0+\dot{\phi}_1) \ell_1 \cos\phi_1 + (\omega_0+\dot{\phi}_2) \ell_2 \cos\phi_2\} \ell_2 \sin\phi_2 \dot{\phi}_2 \\
& \quad\quad + \frac{m_0}{m} \{\ddot{\phi}_2 \ell_2{}^2\}] \\
& - M^* [\{(\omega_0+\dot{\phi}_1) \ell_1 \sin\phi_1 + (\omega_0+\dot{\phi}_2) \ell_2 \sin\phi_2\} \ell_2 \cos\phi_2 (\omega_0+\dot{\phi}_2) \\
& \quad - \{(\omega_0+\dot{\phi}_1) \ell_1 \cos\phi_1 + (\omega_0+\dot{\phi}_2) \ell_2 \cos\phi_2\} \ell_2 \sin\phi_2 (\omega_0+\dot{\phi}_2)] \\
& + 2\, \omega_0{}^2 M^* [\{\ell_1 \cos\phi_1 + \ell_2 \cos\phi_2\} \ell_2 \sin\phi_2 \\
& \quad + \frac{1}{2} \{\ell_1 \sin\phi_1 + \ell_2 \sin\phi_2\} \ell_2 \cos\phi_2 \\
& \quad + \frac{m_0}{m} (\frac{3}{2}) \ell_2{}^2 \sin\phi_2 \cos\phi_2] \\
& + k_2 \ell_2{}^2 \sin\phi_2 \cos\phi_2 = T_{\phi_2} \qquad (3.17)
\end{aligned}
$$

3.1.5 Linearized equations of motion

Eqs. (3.16) and (3.17) are linearized about the local vertical for small amplitude motions such that $\sin\phi_i \approx \phi_i$ and $\cos\phi_i \approx 1$. Also as indicated in section 3.1.3, we now assume, in order to model a free-free beam, $\ell_1 = \ell_2 = \ell$ and $k_1 = k_2 = k$.

By letting $\phi_1 = v_1/\ell$ and $\phi_2 = v_2/\ell$, the linearized equations of motion can be represented as

$$\begin{bmatrix} M^*(1+\bar{m}_0) & M^* \\ M^* & M^*(1+\bar{m}_0) \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} 3\omega_0^2 M^*(2+\bar{m}_0) + k & 0 \\ 0 & 3\omega_0^2 M^*(2+\bar{m}_0) + k \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{v_1} \\ F_{v_2} \end{Bmatrix} \quad (3.$$

The control forces that are assumed to act on m_1 and m_2 are represented by F_{v_1} and F_{v_2}, respectively. For the special case of a two-mass system with $m_0 = 0$, Eq. (3.18) reduces to

$$\begin{bmatrix} \frac{m}{2} & \frac{m}{2} \\ \frac{m}{2} & \frac{m}{2} \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} 3\omega_0^2 m+k & 0 \\ 0 & 3\omega_0^2 m+k \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{v_1} \\ F_{v_2} \end{Bmatrix}$$

The characteristic equation for this two-mass system is obtained as

$$s^2 + (3\omega_0^2 + k/m) = 0$$

This equation shows that the system is forced to oscillate in an anti-symmetric mode if the flexible beam is modelled only by the two end masses.

3.1.6 Stability analysis

The characteristic equation for the three-mass system from Eq. (3.18) is

$$(M^*)^2\ (2\bar{m}_0 + \bar{m}_0^{\,2})s^4$$
$$+\ 2M^*\ (1+\bar{m}_0)\ \{3\omega_0^{\,2}M^*\ (2+\bar{m}_0) + k\}s^2$$
$$+\ \{3\omega_0^{\,2}M^*\ (2+\bar{m}_0) + k\ \}^2 = 0 \tag{3.19}$$

The magnitudes of the two natural frequencies, ω_1 and ω_2, as a function of the central mass, m_0, with $\bar{M} = 2m + m_0$ = constant, as obtained by solving Eq. (3.19), is given in Table 3.1.

m_0	ω_1	ω_2
$\frac{1}{3}\bar{M}$	$[3\omega_0^{\,2}+(3k/\bar{M})]^{\frac{1}{2}}$	$[3\{3\omega_0^{\,2}+(3k/\bar{M})\}]^{\frac{1}{2}}$
$\frac{1}{2}\bar{M}$	$[3\omega_0^{\,2}+(2k/\bar{M})]^{\frac{1}{2}}$	$[2\{3\omega_0^{\,2}+(2k/\bar{M})\}]^{\frac{1}{2}}$
$\frac{3}{5}\bar{M}$	$[3\omega_0^{\,2}+(5k/3\bar{M})]^{\frac{1}{2}}$	$[\frac{5}{3}\{3\omega_0^{\,2}+(5k/3\bar{M})\}]^{\frac{1}{2}}$
$\frac{2}{3}\bar{M}$	$[3\omega_0^{\,2}+(3k/2\bar{M})]^{\frac{1}{2}}$	$[\frac{3}{2}\ \{3\omega_0^{\,2}+(3k/2\bar{M})\}]^{\frac{1}{2}}$

Table 3.1 Variation of ω_1 and ω_2 with m_0

The following system parameters and initial conditions are assumed for numerical study: $L = 2\ell = 100$m; orbital altitude = 463 km (250 nautical miles); $EI = 7.707197\text{x}10^3$ N-m^2; $k = 3EI/\ell^3$; $\bar{M} = 1000$ kg; $v_1(0) = 0.01$m and $v_2(0) = \dot{v}_1(0) = \dot{v}_2(0) = 0$. Structural parameters are taken for a beam made of wrought aluminum (2014 T6) and cylindrical in shape. The outer diameter of the beam is 50mm and the thickness is 5mm. The variation of the two natural frequencies, ω_1 and ω_2, with m_0 is shown in Fig. 3.2. Also shown in the figure are the variations of ω_1 and ω_2 with m_0 when $k = 0$, indicated by ω_{10} and ω_{20}.

We observe that ω_{10} remains constant at $\sqrt{3}\,\omega_0$ (which is the frequency at which a rigid dumbell satellite will oscillate), but ω_{20} decreases with an increase of m_0. For the general case, ω_1 increases with m_0, but ω_2 decreases up to a certain value of m_0 but then increases with an increase of m_0.

3.2 Equations of Motion - Local Horizontal

3.2.1 Linearized equations of motion

In order to develop the small amplitude equations of motion for the case where the beam is nominally aligned along the local horizontal, we begin by replacing ϕ_i in Eqs. (3.16) and (3.17) by $(\pi/2 + \phi_i)$. After appropriate simplification and subsequent linearization the equations of motion may be represented in matrix form as:

$$\begin{bmatrix} M^*(1+\bar{m}_0) & M^* \\ M^* & M^*(1+\bar{m}_0) \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} -\{3\omega_0^{\,2}M^*\,(1+\bar{m}_0)-k\} & -3\omega_0^{\,2}M^* \\ -3\omega_0^{\,2}M^* & -\{3\omega_0^{\,2}M^*\,(1+\bar{m}_0)-k\} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{v_1} \\ F_{v_2} \end{Bmatrix} \tag{3.20}$$

For the two-mass system with $m_0 = 0$ Eq. (3.20) reduces to

$$\begin{bmatrix} \frac{m}{2} & \frac{m}{2} \\ \frac{m}{2} & \frac{m}{2} \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} -(\frac{3}{2}\omega_0^{\,2}m-k) & -\frac{3}{2}\,\omega_0^{\,2}m \\ -\frac{3}{2}\,\omega_0^{\,2}m & -(\frac{3}{2}\,\omega_0^{\,2}m-k) \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{v_1} \\ F_{v_2} \end{Bmatrix}$$

The characteristic equation for the two-mass system is

$$s^2 + (k/m) - 3\omega_0^2 = 0$$

The system behaves like a dumbell for $k > 3\omega_0^2 m$ and is unstable for $k \leq 3\omega_0^2 m$.

3.2.2 Stability analysis

The characteristic equation for the three-mass system can be developed from Eq. (3.20) as

$$\begin{aligned}&(M^*)^2(2\bar{m}_0+\bar{m}_0^2)s^4 \\ &\quad + [6(M^*)^2\omega_0^2 - 2M^*(1+\bar{m}_0)\{3\omega_0^2 M^*(1+\bar{m}_0) - k\}]s^2 \\ &\quad + [\{3\omega_0^2 M^*(1+\bar{m}_0) - k\}^2 - 9\omega_0^4(M^*)^2] = 0 \qquad (3.21)\end{aligned}$$

The variation of ω_1 and ω_2 with m_0 is shown in Table 3.2 with $\bar{M} = 2m + m_0 =$ constant.

m_0	ω_1	ω_2
$\frac{1}{3}\bar{M}$	$[(3k/\bar{M}) - 3\omega_0^2]^{\frac{1}{2}}$	$[3\{(3k/\bar{M}) - \omega_0^2\}]^{\frac{1}{2}}$
$\frac{1}{2}\bar{M}$	$[(2k/\bar{M}) - 3\omega_0^2]^{\frac{1}{2}}$	$[2\{(2k/\bar{M}) - 1.5\omega_0^2\}]^{\frac{1}{2}}$
$\frac{3}{5}\bar{M}$	$[(5k/3\bar{M}) - 3\omega_0^2]^{\frac{1}{2}}$	$[\frac{5}{3}\{(5k/3\bar{M}) - 1.8\omega_0^2\}]^{\frac{1}{2}}$
$\frac{2}{3}\bar{M}$	$[(3k/2\bar{M}) - 3\omega_0^2]^{\frac{1}{2}}$	$[\frac{3}{2}\{(3k/2\bar{M}) - 2\omega_0^2\}]^{\frac{1}{2}}$

Table 3.2. Variation of ω_1 and ω_2 with m_0.

The variation of the two natural frequencies, ω_1 and ω_2, for the three-mass system with m_0 is shown in Fig. 3.3. For the assumed system numerical parameters, both ω_1 and ω_2 decrease with an increase of m_0.

4. MODE CONTROL CONCEPT

A concept is presented in a recent Rockwell International Report (Ref. 5) for controlling a flexible space structure by independently controlling motions of the structures's rigid body and vibrational modes. The mode control concept leads to criteria for locating actuators and algorithms for their combined use. Development of the mode control concept is based on two coordinate transformations. Discrete displacements and discrete forces are transformed to modal coordinates and distributed actuator variables, thereby completely uncoupling the system equations. In this chapter, the mode control concept is outlined briefly for our specific application but more complete details are available in Ref. 5.

4.1 Dynamic Equations

A flexible space structure is modelled as many rigid bodies interconnected by massless, elastic structural elements. Then small amplitude motions are described by the linear differential equation

$$M \ddot{X} + K X = f^{(c)} \tag{4.1}$$

where X is an $N \times 1$ vector of discrete coordinates measuring the angular and translational displacements of each body relative to its inertially fixed rest position, M is the real $N \times N$ mass matrix, K is a real, symmetric $N \times N$ stiffness matrix, and $f^{(c)}$ is an $N \times 1$ vector of the control forces.

4.2 Eigen-Analysis

The eigenvalues (modal frequencies), ω_1, - - - , ω_N, and associated N x 1 eigenvectors (mode shapes), $\phi^{(1)}$, - - -, $\phi^{(N)}$, are obtained from the homogenous part of Eq. (4.1). The orthogonality properties of the modes provide that

$$\phi^{(i)T} M \phi^{(j)} = \begin{cases} 0 & i \neq j \\ m_i & i = j \end{cases} \tag{4.2}$$

$$\phi^{(i)T} K \phi^{(j)} = \begin{cases} 0 & i \neq j \\ m_i \omega_i^2 & i = j \end{cases} \tag{4.3}$$

where m_i is referred to as the generalized mass for mode i. Next, a coordinate transformation is performed utilizing an N x N matrix, Φ, constructed from the N eigenvectors

$$\Phi = [\phi^{(1)} \,\vdots\, \phi^{(2)} \,\vdots\, \ldots\ldots \,\vdots\, \phi^{(N)}]$$

A transformation is introduced by assuming that a given displacement profile may be expressed in terms of a series of the shape functions (mode shapes) multiplied by time-dependent weighting factors (modal coordinates q_i); i.e. $X = \phi^{(1)} q_1 + \phi^{(2)} q_2 + \cdots + \phi^{(N)} q_N$, or

$$X = \Phi q \tag{4.4}$$

Substitution of Eq. (4.4) into Eq. (4.1) and premultiplication by Φ^T produces

$$\Phi^T M \Phi \ddot{q} + \Phi^T K \Phi q = \Phi^T f^{(c)} \tag{4.5}$$

Finally, the transformed equation of motion is obtained with the aid of the orthogonality conditions, Eqs. (4.2) and (4.3), as

$$\ddot{q} + \lceil \omega_i^2 \rfloor \, q = \lceil {}^1/m_i \rfloor \, \Phi^T f^{(c)} \tag{4.6}$$

The advantages of the modal formulation are that the left-hand side of each scalar equation associated with Eq. (4.6) is uncoupled in q_i and criteria can be easily developed for truncation purposes. Generally, $f^{(c)}$ has low frequency components. Thus a few low frequency modal equations can be used to predict the response of a dynamic model which has been modelled initially with a larger number of discrete coordinates.

4.3 Actuation

4.3.1 Number of actuators equal to the number of modes

The discrete control forces, $f^{(c)}$, are transformed into generalized control forces, $f^{(g)}$, by

$$f^{(g)} = \lceil {}^1/m_i \rfloor \, \phi^T f^{(c)} \tag{4.7}$$

Independent actuation of all of the N modal equations can be achieved if N actuators are used in such a way so as to produce a generalized force in any given mode without forcing the other modes. Then we obtain $f^{(g)} = u$ if we let

$$f^{(c)} = M \, \Phi \, u \tag{4.8}$$

where u_i represent the independent actuator variables. After substitution of Eq. (4.8) into Eq. (4.6) one arrives at

$$\ddot{q} + \lceil \omega_i^2 \rfloor \, q = u \tag{4.9}$$

The mode control implementation for this case, where the number of independent actuators is equal to the number of modes of the system, is shown in Fig. 4.1.

4.3.2 Number of actuators less than the number of modes

In most instances, independent control of all the N modes of a dynamic model using N actuators is impractical since N is usually a very large number. Additionally, control is often required of only a few of the lower frequency modes. For these reasons, it is necessary to modify the foregoing procedure to establish a means of independently controlling only a selected small number of the system's modal coordinates.

Here an expample is illustrated taking the number of actuators (P) as 2 and the number of modes (N) of the system as 3 such that P < N. [The general theory is available in Ref. 5.] The control force for this case can be written as

$$f^{(c)} = \begin{Bmatrix} f_1^{(c)} \\ f_2^{(c)} \\ f_3^{(c)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = F \; f \tag{4.10}$$

where f_1 and f_2 represent the scalar actuator variables. The generalized control force, Eq. (4.7), is

$$f^{(g)} = \left[\;\; ^1/m_i \;\; \right] \Phi^T F \, f \tag{4.11}$$

A transformation of the form

$$f = T \, u \tag{4.12}$$

is required to produce independent generalized control forces.

Here T provides a transformation from discrete actuator variables to distributed actuator variables such that u_i produces independent actuation of the i^{th} modal equation. Eqs. (4.11) and (4.12) are combined to yield

$$f^{(g)} = \lceil {}^{1}/m_i \rfloor \Phi^T F T u \tag{4.13}$$

Eq. (4.13) can be written, with N = 3 and P = 2, as

$$\begin{Bmatrix} f_1^{(g)} \\ f_2^{(g)} \\ f_3^{(g)} \end{Bmatrix} = \begin{bmatrix} {}^{1}/m_1 & 0 & 0 \\ 0 & {}^{1}/m_2 & 0 \\ 0 & 0 & {}^{1}/m_3 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{21} & \phi_{31} \\ \phi_{12} & \phi_{22} & \phi_{32} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{4.14}$$

Expanding Eq. (4.14), we obtain

$$\begin{Bmatrix} f_1^{(g)} \\ f_2^{(g)} \\ f_3^{(g)} \end{Bmatrix} = \begin{bmatrix} \frac{1}{m_1} (\phi_{11} T_{11} + \phi_{21} T_{21}) & \frac{1}{m_1} (\phi_{11} T_{12} + \phi_{21} T_{22}) \\ \frac{1}{m_2} (\phi_{12} T_{11} + \phi_{22} T_{21}) & \frac{1}{m_2} (\phi_{12} T_{12} + \phi_{22} T_{22}) \\ \frac{1}{m_3} (\phi_{13} T_{11} + \phi_{23} T_{21}) & \frac{1}{m_3} (\phi_{13} T_{12} + \phi_{23} T_{22}) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{4.15}$$

The transformation matrix, T, is obtained by noting that for independent control of the first two modes, we need

$$f_1^{(g)} = u_1; \; f_2^{(g)} = u_2 \tag{4.16}$$

After a comparison of Eqs. (4.15) and (4.16) for $f_1^{(g)}$ and $f_2^{(g)}$, we can obtain T. Knowing T, the residual coupling coefficients, $f^{(r_1)}$ and $f^{(r_2)}$, for the third mode can be obtained by rewriting Eq. (4.15) using Eq. (4.16) as

$$\begin{Bmatrix} f_1^{(g)} \\ f_2^{(g)} \\ f_3^{(g)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f^{(r_1)} & f^{(r_2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{4.17}$$

Thus, $f^{(r_1)}$ and $f^{(r_2)}$ are obtained from the last row of Eq. (4.15) after the determination of T.

An interesting (and useful) result of this example is that, for the general case, only modes i=1 to P are needed to determine T, and only $\phi^{(i)}$ and T are needed to obtain the residual coupling elements for q_i (i = P+1, ..., N). Thus, for the case where P<N, independent control of modes i = 1 to P is possible and the response of the modes, N>P, depends on the residual coupling due to the P actuators. The mode control concept described in this chapter will be applied to a long beam in space modelled by the three-mass system with the number of actuators equal to or less than the number of modes of the system.

5. MODAL CONTROL OF THE THREE-MASS SYSTEM

5.1 Three-Mass System-Local Vertical

The modal control of the three-mass system (Fig. 3.1) is now considered. The linearized equations of motion [Chapter 3, Eq. (3.18)] are

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{v_1} \\ F_{v_2} \end{Bmatrix} \tag{5.1}$$

where

$$a = M^* (1+\overline{m}_0) \; ; \quad b = M^*$$

$$c = 3\omega_0^2 M^* (2+\overline{m}_0) + k$$

The uncontrolled dynamics of the system is considered first and then the mode control concept outlined in Chapter 4 is applied to obtain the controlled system response.

5.1.1 Uncontrolled motion

Using the Laplace transform method with $F = 0$, Eq. (5.1) can be written as

$$\begin{Bmatrix} v_1(s) \\ v_2(s) \end{Bmatrix} = \frac{1}{(s^2+\overline{c}a)^2 - (\overline{c}b)^2} \begin{bmatrix} s^2+\overline{c}a & \overline{c}b \\ \overline{c}b & s^2+\overline{c}a \end{bmatrix} \begin{Bmatrix} sv_1(0)+\dot{v}_1(0) \\ sv_2(0)+\dot{v}_2(0) \end{Bmatrix}$$

where

$$\overline{c} = c/(a^2-b^2)$$

The solution to Eq. (5.2) is obtained, under the assumption that $\dot{v}_1(0) = \dot{v}_2(0) = v_2(0) = 0$, but $v_1(0) \neq 0$, as

$$v_1(t) = \frac{1}{2} \; [\cos \omega_1 t + \cos \omega_2 t] \; v_1(0) \tag{5.3}$$

$$v_2(t) = \frac{1}{2} \; [\cos \omega_1 t - \cos \omega_2 t] \; v_1(0) \tag{5.4}$$

where

$$\omega_1 = [c/(a+b)]^{1/2} \quad ; \quad \omega_2 = [c/(a-b)]^{1/2}$$

The motion of the center of the beam, m_0, is obtained from Eq. (3.12), and noting that $\xi_i = \ell \sin \phi_i = v_i$,

$$\xi_{cm} = -(m/\bar{M}) \cos \omega_2 t \; v_1(0); \; \zeta_{cm} = 0 \tag{5.5}$$

The uncontrolled motion of the system with the assumed system parameters and initial conditions stated in Section 3.1.6 is obtained as

$$v_1(t) = 5 \; (\cos 0.023635t + \cos 0.040937t) \text{ mm} \tag{5.6}$$

$$v_2(t) = 5 \; (\cos 0.023635t - \cos 0.040937t) \text{ mm} \tag{5.7}$$

$$\xi_{cm} = - \frac{10}{3} \cos 0.040937t \text{ mm} \tag{5.8}$$

The time response of the beam end deflections, v_1 and v_2, is shown in Fig. 5.1. When $m_0 = 0$, the deflections v_1 and v_2 are the same indicating that the system behaves like a dumbell satellite. The presence of m_0 produces the second frequency, ω_2, and the deflections due to ω_2 are superimposed on v_1 and v_2 as seen from Eqs. (5.3) and (5.4).

5.1.2 Number of actuators equal to the number of modes

Considering the homogeneous part of Eq. (5.1), the eigen values are

$$\lambda_1 = \omega_1^{\,2} = c/(a+b); \; \lambda_2 = \omega_2^{\,2} = c/(a-b) \tag{5.9}$$

The coordinate transformation matrix, Φ, is determined from the eigen-vectors as

$$\Phi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{5.10}$$

The generalized mass matrix, from Eq. (4.2), is

$$\lceil\!\!\diagdown \; m_i \; \diagdown\!\!\rfloor = \Phi^T M \Phi = \begin{bmatrix} 2(a+b) & 0 \\ 0 & 2(a-b) \end{bmatrix} \tag{5.11}$$

The modal equation given by Eq. (4.9) for this case becomes

$$\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{5.12}$$

We assume a proportional displacement and rate feedback for u_1 and u_2 in the form:

$$u_1 = - f_1 \, q_1 - f_2 \, \dot{q}_1 \tag{5.13}$$

$$u_2 = - f_3 \, q_2 - f_4 \, \dot{q}_2 \tag{5.14}$$

The equation for q_1 with control u_1 is written as

$$\ddot{q}_1 + f_2 \, \dot{q}_1 + (\lambda_1 + f_1) \, q_1 = 0 \tag{5.15}$$

The solution of Eq. (5.15) is

$$q_1(t) = e^{-\frac{f_2}{2}t} \left[\cos \omega_{12}t + \frac{f_2}{2}\ \frac{1}{\omega_{12}}\ \sin \omega_{12}t\right] q_1(0) \tag{5.16}$$

where

$$\omega_{12} = [\lambda_1 + f_1 - (f_2{}^2/4)]^{\frac{1}{2}}\ ;\ q_1(0) = v_1(0)/2 \tag{5.17}$$

similarly,

$$q_2(t) = e^{-\frac{f_4}{2}t} \left[\cos \omega_{34}t + \frac{f_4}{2}\ \frac{1}{\omega_{34}} \sin \omega_{34}t\right] q_2(0) \tag{5.18}$$

where

$$\omega_{34} = [\lambda_2 + f_3 - (f_4{}^2/4)]^{\frac{1}{2}}\ ;\ q_2(0) = v_1(0)\ /2 \tag{5.19}$$

The discrete coordinates, v_1 and v_2, are related to the modal coordinates, q_1 and q_2, using Eqs. (4.4) and (5.10),

$$v_1 = q_1 + q_2;\ v_2 = q_1 - q_2 \tag{5.20}$$

The control forces are obtained from Eq. (4.8) as

$$\begin{aligned} F_{v_1} &= \frac{-m}{2}\ \Big[\Big(\frac{f_1+f_3}{3}\Big) v_1 + \Big(\frac{f_1-f_3}{3}\Big) v_2 \\ &\quad + \Big(\frac{f_2+f_4}{3}\Big) \dot{v}_1 + \Big(\frac{f_2-f_4}{3}\Big) \dot{v}_2\Big] \end{aligned} \tag{5.21}$$

$$\begin{aligned} F_{v_2} &= \frac{-m}{2}\ \Big[\Big(\frac{f_1-f_3}{3}\Big) v_1 + \Big(\frac{f_1+f_3}{3}\Big) v_2 \\ &\quad + \Big(\frac{f_2-f_4}{3}\Big) \dot{v}_1 + \Big(\frac{f_2+f_4}{3}\Big) \dot{v}_2\Big] \end{aligned} \tag{5.22}$$

As a special case, when $f_2 = f_4$ and also, $\omega_{12} = \omega_{34}$, we note that $q_1(t) = q_2(t)$ and that the displacement $v_2(t) \equiv 0$ and $v_1 = 2q_1$ for the assumed initial conditions. This condition is achieved if $f_3 = f_1 + \lambda_1 - \lambda_2$ and $f_2 = f_4$. Thus, by properly selecting the feedback control gains, it may be possible to control a portion of the beam such that it will not be subject to any deflections.

The uncontrolled motion of the system, obtained by setting $f_i = 0$, $i = 1, 2, 3, 4$ in Eqs. (5.16) and (5.18) and in turn in Eqs. (5.20), is illustrated in Fig. 5.1. The dynamic response of the controlled system with $f_i = 1$ is shown in Fig. 5.2. It is observed that the tip deflection amplitude, v_1, is reduced to 0.01 mm from an initial value of 10 mm within 12 secs. The initial control forces F_{v_1} and F_{v_2} are calculated to be −2.22N and −1.11N, respectively. The time history of the control forces is shown in Fig. 5.3. The initial amplitude of the control forces can be reduced by reducing the feedback gains. Fig. 5.4 illustrates the time response of the system with $f_i = 0.1$, a value which is 1/10 of the previously considered value for f_i. In this case, the time required to reach a deflection amplitude of 0.01 mm from an initial value of 10 mm for the tip deflection v_1 is about 120 secs. Thus a reduction in the control force increases the time constants of the system proportionally, as expected for a linear system.

5.1.3 Number of actuators less than the number of modes

Following Section 4.3.2, we obtain the equations for q_1 and q_2 as

$$\ddot{q}_1 + \lambda_1 q_1 = u_1 \tag{5.23}$$

$$\ddot{q}_2 + \lambda_2 q_2 = \{(a+b)/(a-b)\} u_1 \tag{5.24}$$

Eq. (5.23) can be controlled independently as in Section 5.1.2. Using $u_1 = -f_1 q_1 - f_2 \dot{q}_1$, Eq. (5.24) becomes

$$\ddot{q}_2 + \lambda_2 q_2 = -g (f_1 q_1 + f_2 \dot{q}_1) \tag{5.25}$$

where

$$g = (a+b)/(a-b)$$

(Note that as long as $m_0>0$, $a>b$, see Eq. (5.1).) Using Laplace transform techniques, the response of the mode q_2 due to the residual coupling of the actuator P_1 is

$$q_2(t) = q_2(0) \cos \omega_2 t + \frac{g f_2 q_1(0)}{\omega_2} \sin \omega_2 t$$

$$\frac{-g}{f_2} \left[A_{11} \sin (\omega_2 t + \phi_1^*) + A_{12} e^{-\frac{f_2 t}{2}} \sin (\omega_{12} t + \phi_2^*)\right] q_1(0) \tag{5.26}$$

where

$$A_{11} = \frac{1}{\omega_2} \sqrt{\frac{(f_1 - \omega_2{}^2)^2 + (1 + \frac{f_1}{f_2})^2 \omega_2{}^2}{(\frac{f_2{}^2}{4} + \omega_{12}{}^2 - \omega_2{}^2)^2 + (f_2\omega_2)^2}}$$

$$A_{12} = \frac{1}{\omega_{12}} \sqrt{\frac{\{\frac{f_2{}^2}{4} - \omega_{12}{}^2 - (1+ \frac{f_1}{f_2}) \frac{f_2}{2} + f_1\}^2 + \omega_{12}{}^2\{(1 + \frac{f_1}{f_2}) - f_2\}^2}{(\frac{f_2{}^2}{4} + \omega_{12}{}^2 - \omega_2{}^2)^2 + (f_2\omega_2)^2}}$$

$$\phi_1^* = \tan^{-1}\left\{\frac{(1+\frac{f_1}{f_2})\,\omega_2}{f_1-\omega_2^2}\right\} - \tan^{-1}\left\{\frac{f_2\,\omega_2}{\frac{f_2^2}{4}+\omega_{12}^2-\omega_2^2}\right\}$$

$$\phi_2^* = \tan^{-1}\left\{\frac{\omega_{12}\,\{(1+\frac{f_1}{f_2})-f_2\}}{\frac{f_2^2}{4}-\omega_{12}^2-(1+\frac{f_1}{f_2})\frac{f_2}{2}+f_1}\right\} - \tan^{-1}\left\{\frac{-f_2\,\omega_2}{\frac{f_2^2}{4}-\omega_{12}^2+\omega_2^2}\right\}$$

For large t, $q_1(t)$ is completely damped out [Eq. (5.16)] but $q_2(t)$ oscillates at the frequency, ω_2 [Eq. (5.26)].

The control force, F_{v_1}, is obtained as

$$F_{v_1} = -m\,[f_1(v_1+v_2)+f_2\,(\dot{v}_1+\dot{v}_2)] \qquad (5.27)$$

The time response of the system with the number of actuators less than the number of modes is shown in Fig. 5.5. The figure shows that the v_2 response becomes oscillatory as time increases even though its initial value was zero. The maximum amplitude of the oscillation for v_2 reaches a value of 10 mm. It should be noted that as time increases both v_1 and v_2 oscillate at the same frequency - i.e. that of the (second) uncontrolled mode.

6. CONCLUDING COMMENTS

6.1 Summary of the Conclusions

As a consequence of the present analysis and numerical results, the following conclusions can be made;

1. The equations of motion are developed for a large space structure system consisting of a main body with two attached flexible beams, where each beam is modelled by an end mass or by two masses. (It is assumed that the system center of mass moves in a circular orbit and the elastic displacement of the beams are anti-symmetric.) It is seen that the stiffness matrices will contain contributions due to both the elastic strain energy coefficients and also the gravitational restoring effects.

2. The three-mass system is stable about the local vertical for small amplitude motions but it behaves like a rigid dumbell satellite when the central mass is removed. It is also found that the two-mass system is stable about the local horizontal system if the stiffness of the beam is greater than a certain value of the restoring effect produced by the gravitational forces on the end masses ($k > 3\omega_0^2 m$).

3. Closed-form analytical solutions are obtained (the local vertical system) for the tip displacements of the long beam which is modelled by the three masses.

4. The modal control concept is very useful for independent control of the system's lower frequency modes.

For the three-mass system where only two modes are present, it is possible to keep one portion of the beam without any displacement by properly selecting the feedback control gains for the assumed initial conditions of the system.

5. When the three-mass system contains only one actuator, the modal control produces a complete control of the first mode, but the second mode is not controlled. This indicates the disadvantage of the mode control concept when all the modes can not be completely controlled.

6.2 Recommendations for Future Work

As a result of this investigation the following topics are suggested for future work:

1. In Chapter 3, the long beam is modelled by three discrete masses. For a better understanding of the system response, the modelling of the beam by a greater number of discrete mass points should be considered.

2. The study could be extended to consider the three dimensional motion of the beam by following the approach described in this report for planar motion.

3. Optimal control theory could be used to obtain control laws by minimizing certain cost functionals involving the state variables and elements of the control vector.

4. The mode control concept could be applied to consider the control of a continuum model of the beam and the system response obtained from this model could be compared with the system response obtained here for the discretized model of the long beam.

5. The deformation of the beam due to externally induced solar radiation effects should be considered both during steady-state and controlled responses.

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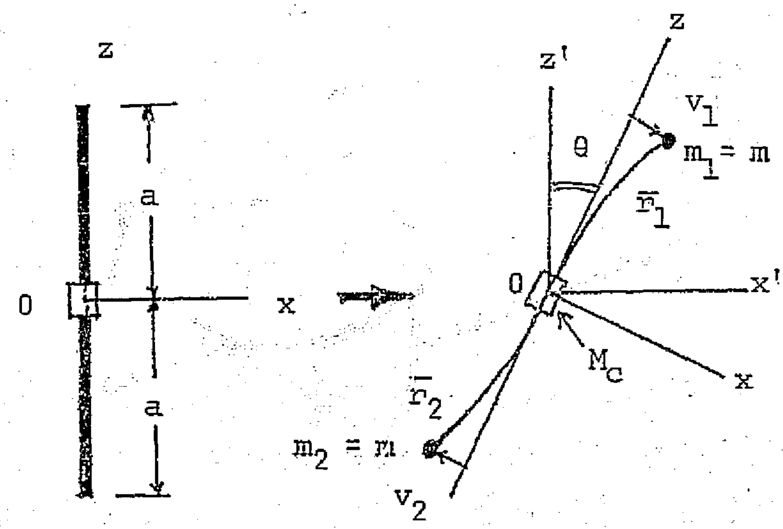


Fig. 2.1. System configuration with main body and each beam modelled by an end mass.

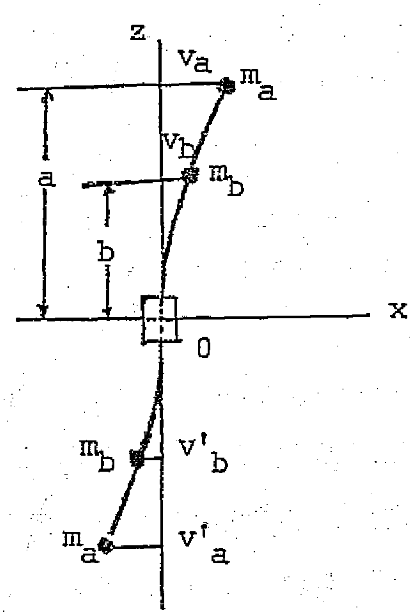


Fig. 2.2. System configuration with main body and each beam modelled by two masses.

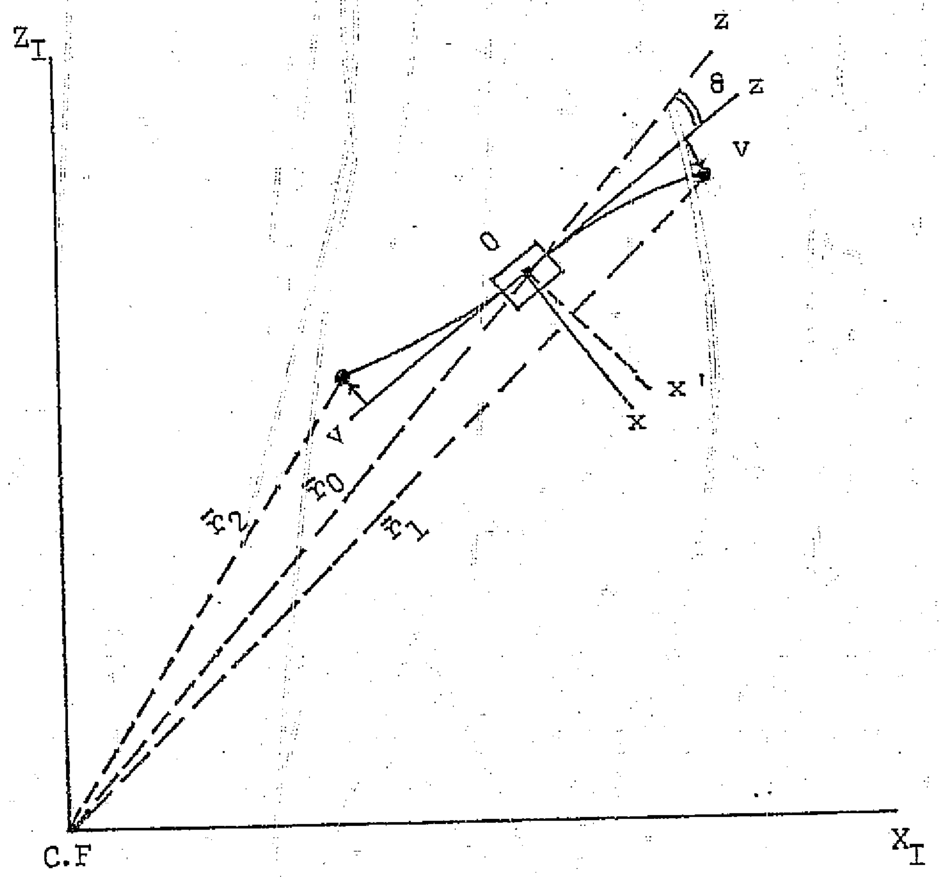


Fig. 2.3. Effect of gravitational forces on the system with main body and each beam modelled by an end mass.

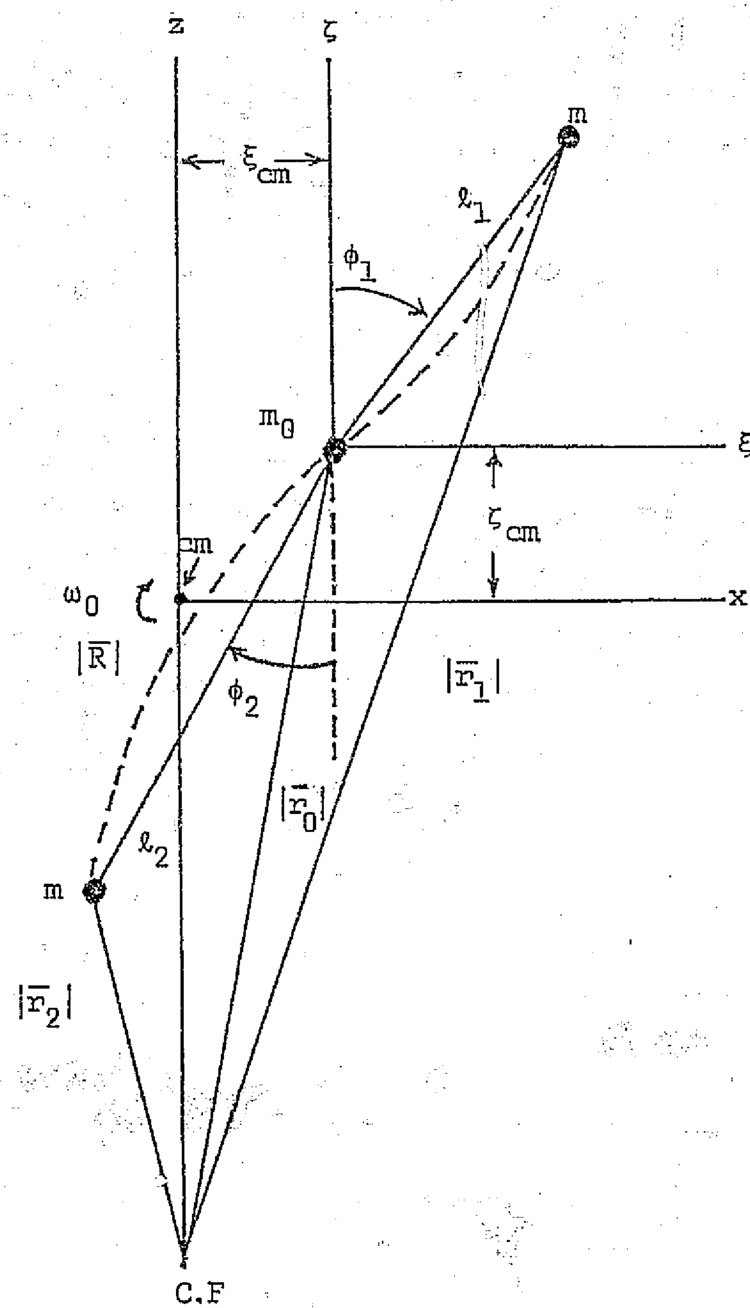
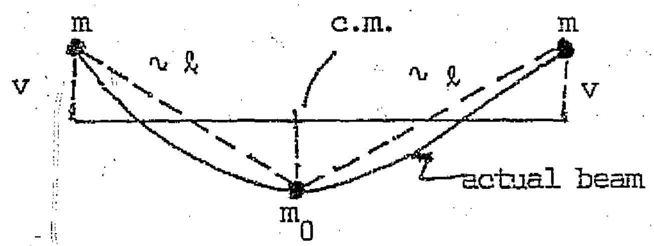
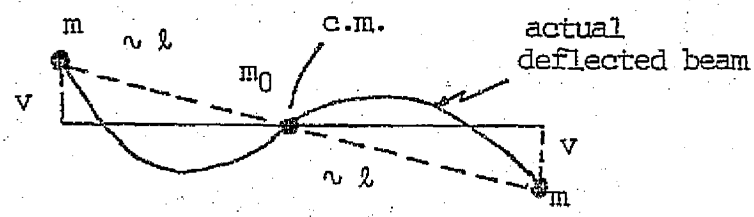


Fig. 3.1. Three-mass system configuration

FIRST MODE



SECOND MODE



$v_i/\ell << 1$

Fig. 3.2. Possible approximation of a free-free beam by three discrete particles (first two modes shown).

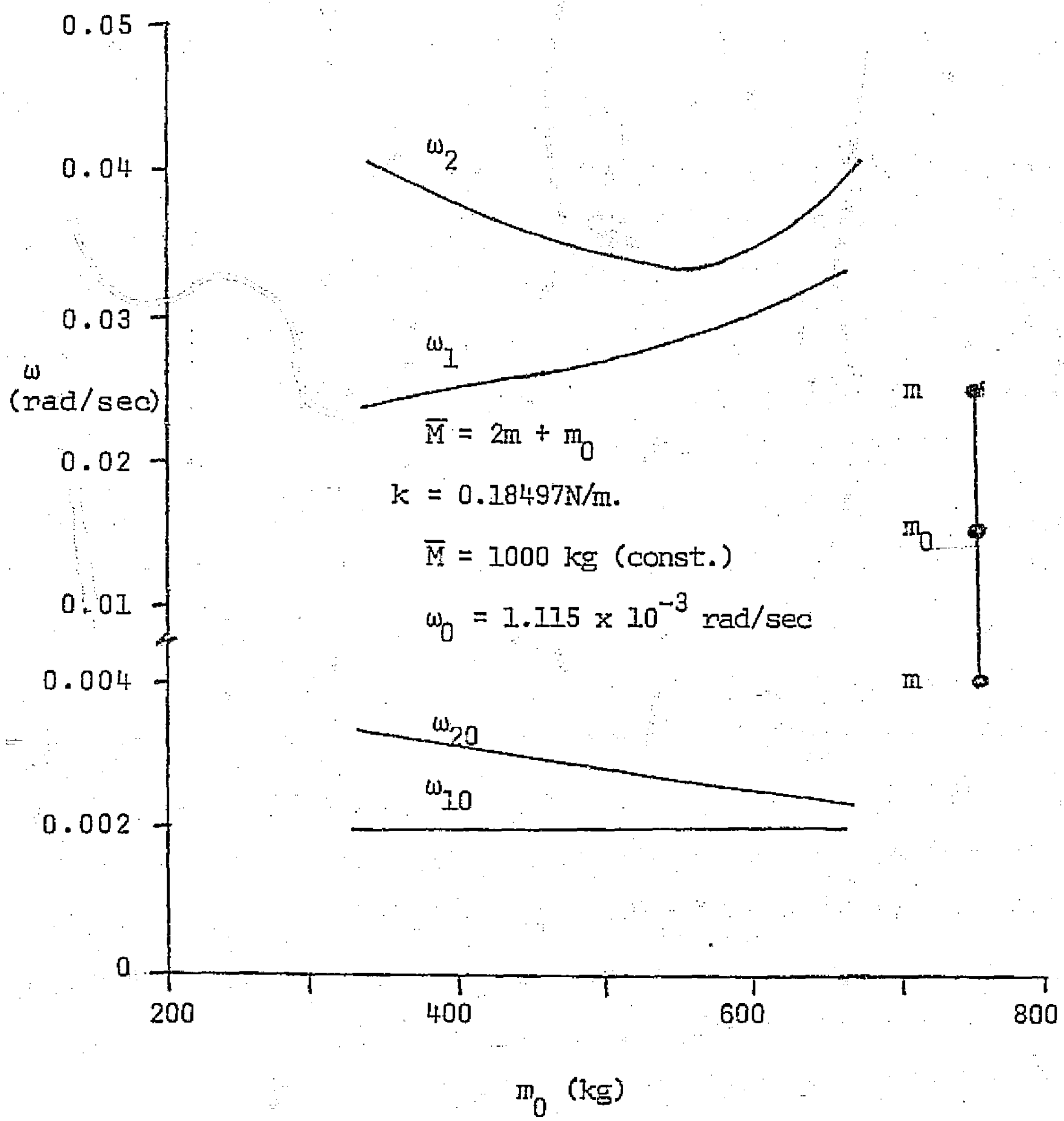


Fig. 3.3. Variation of ω_1 and ω_2 with m_0 (local vertical system).

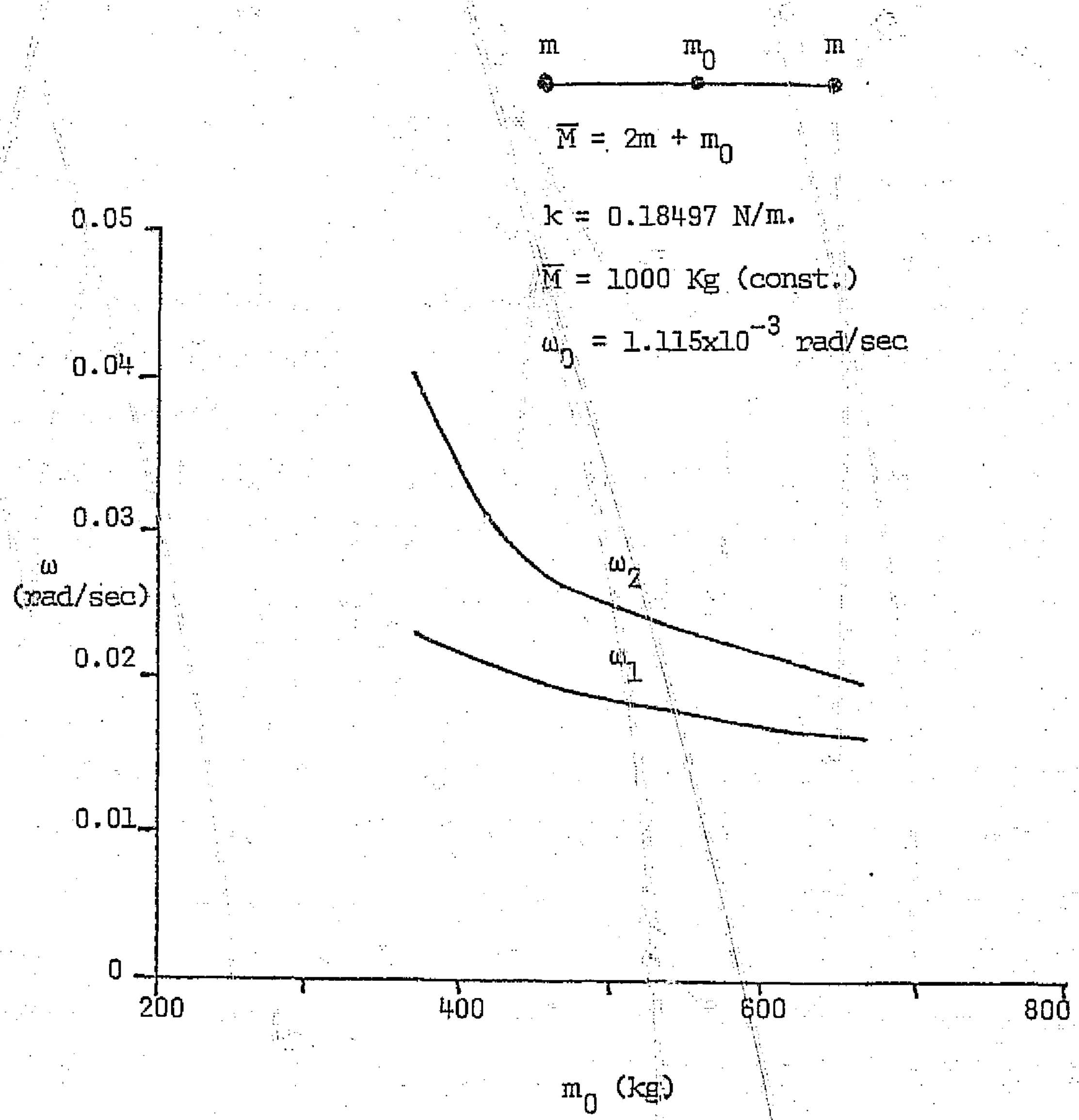


Fig. 3.4. Variation of ω_1 and ω_2 with m_0 (local horizontal system).

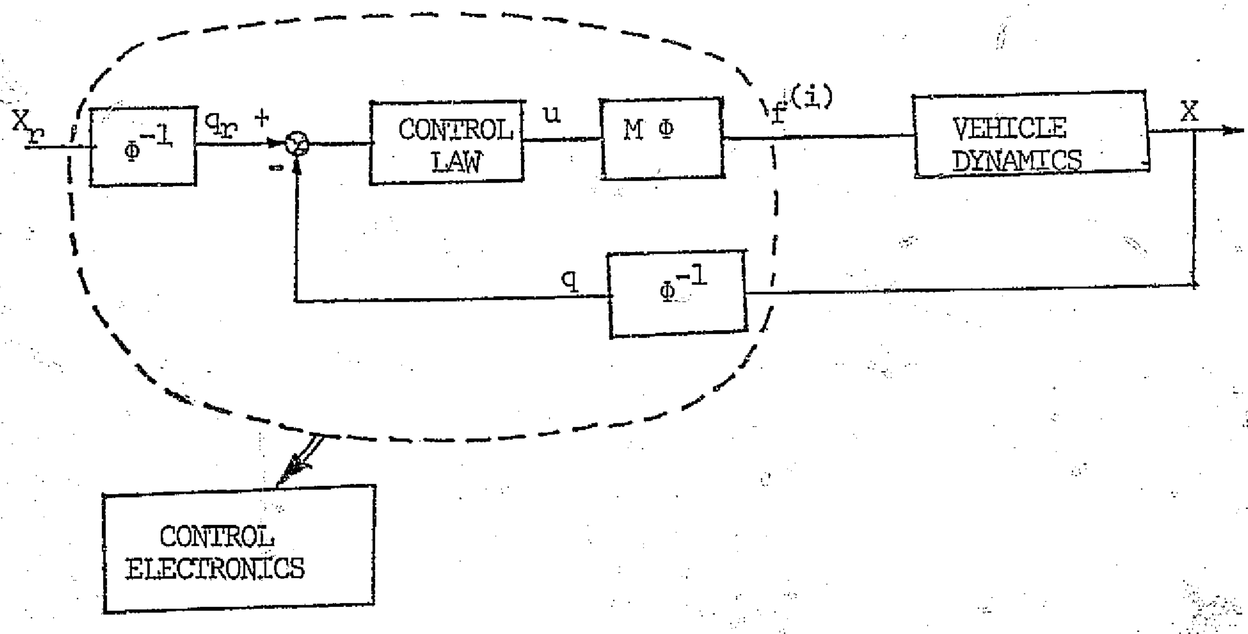


Fig. 4.1. Mode control implementation.

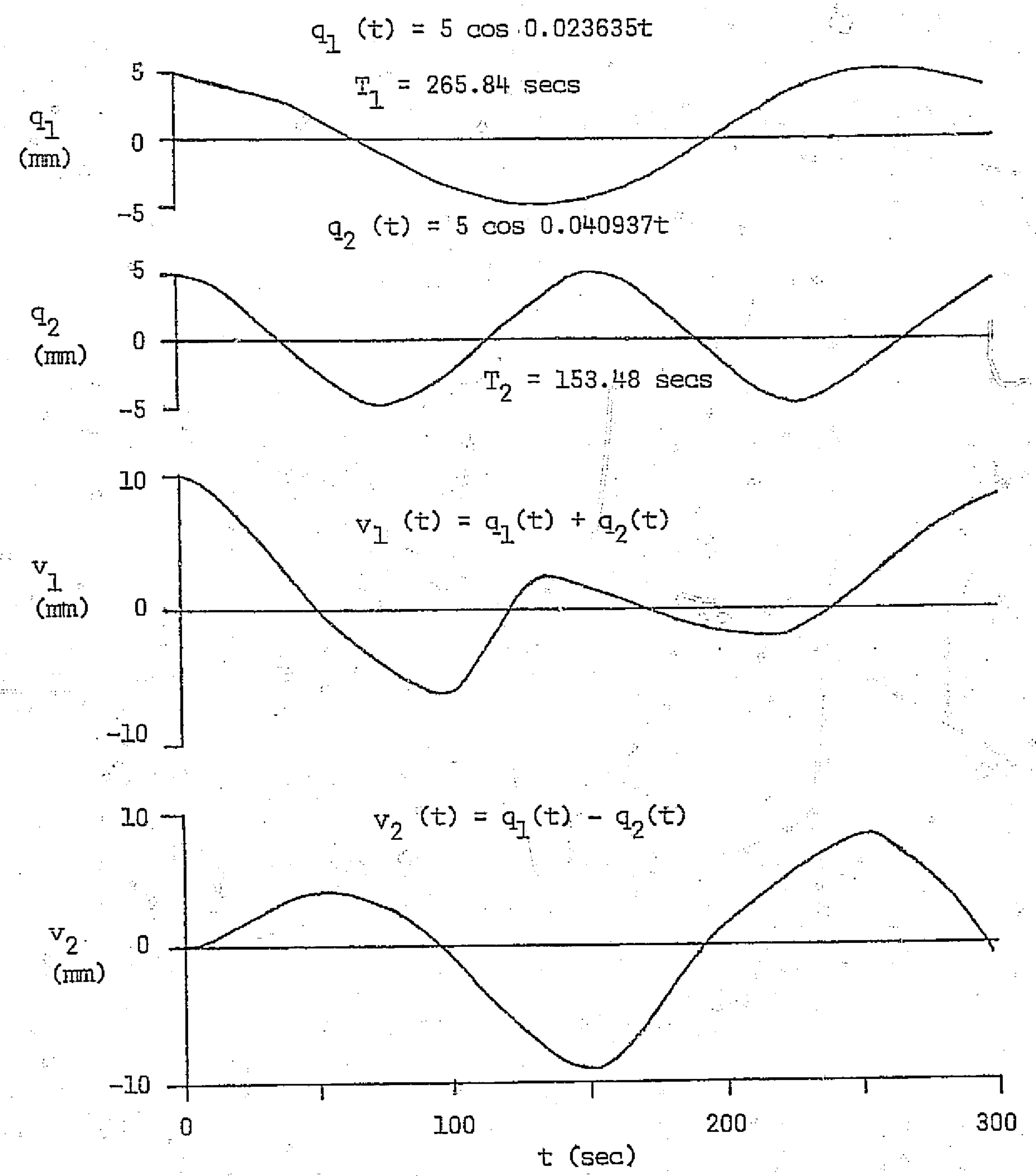


Fig. 5.1. Uncontrolled motion of the system ($f_i = 0$).

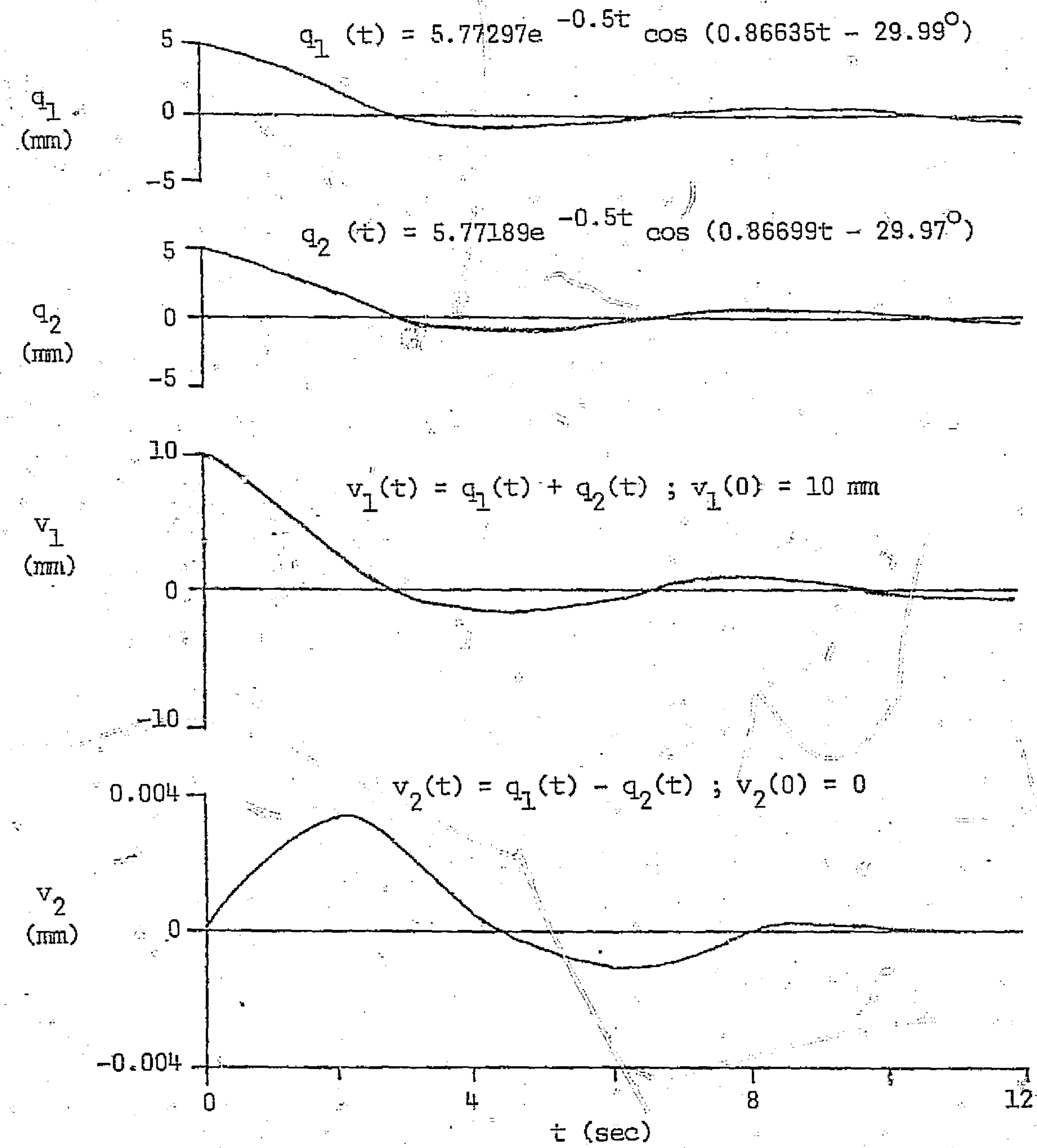


Fig. 5.2. Dynamic response of the system with the number of actuators equal to the number of modes (f_i =1)

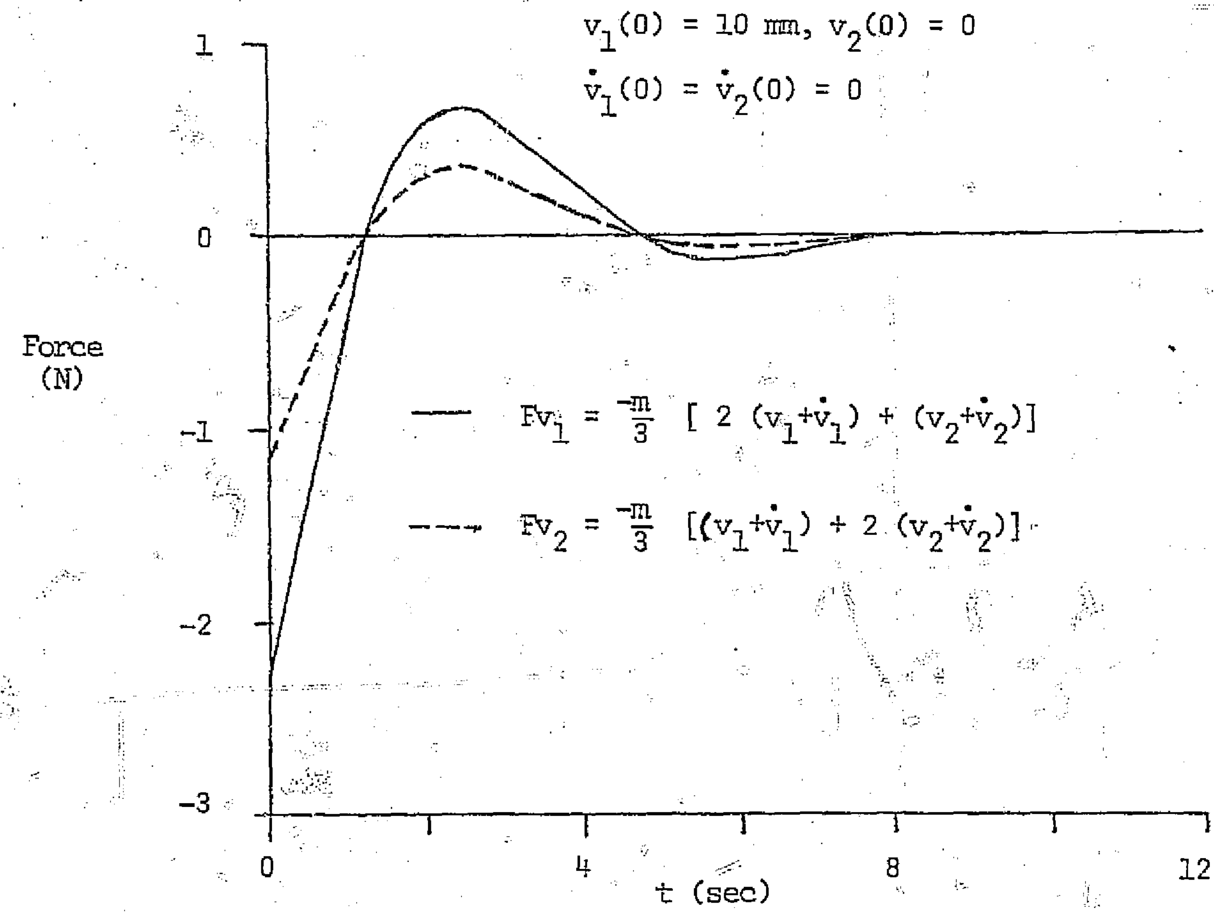


Fig. 5.3. Time history of the control forces with the number of actuators equal to the number of modes ($f_i = 1$)

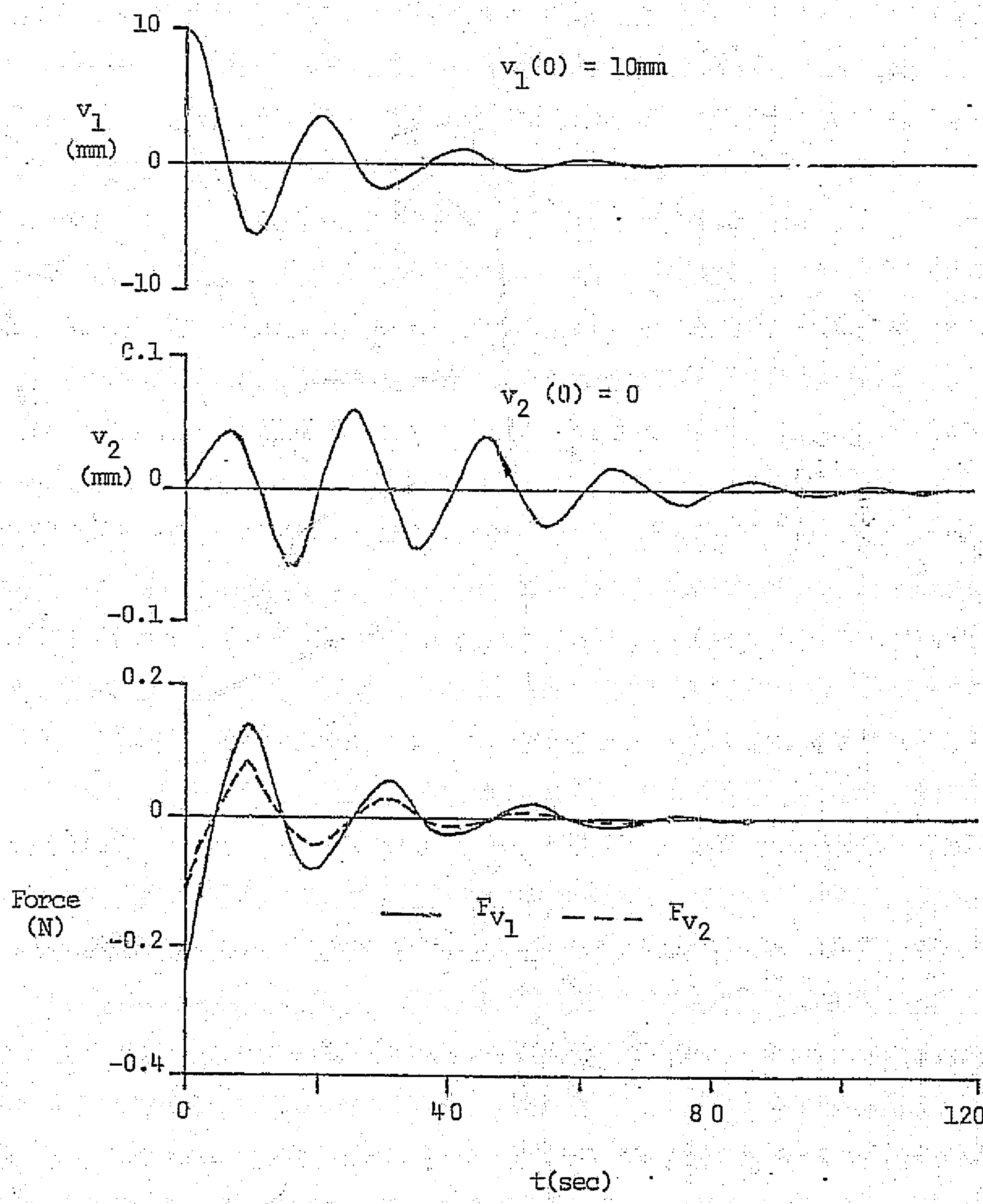


Fig. 5.4. Dynamic response of the system with the number of actuators equal to the number of modes (f_i = 0.1).

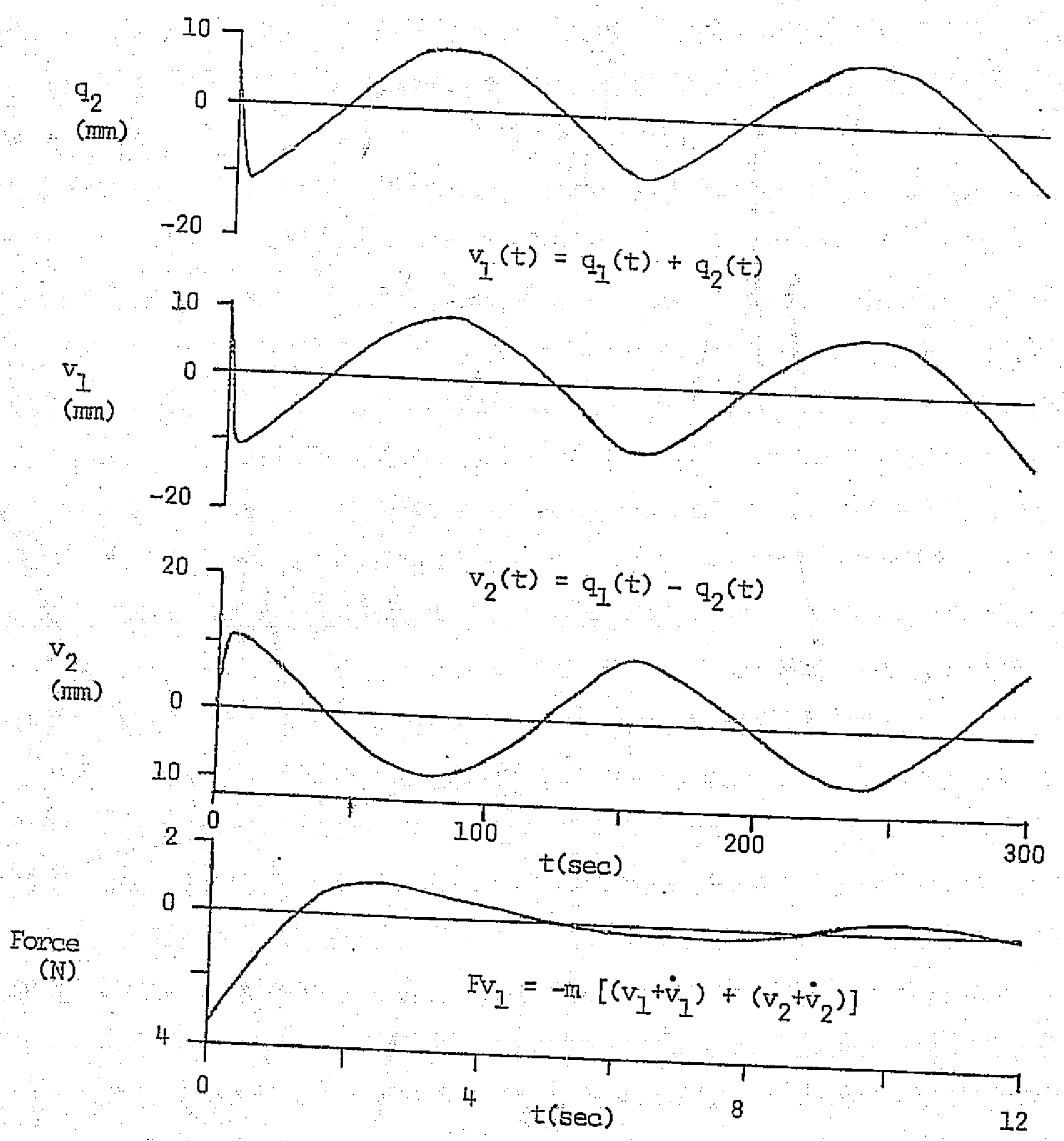


Fig. 5.5. Time response of the system with the number of actuators less than the number of modes ($f_i = 1$)

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COMPUTER PROGRAM

```
1/28/78    10:50:5
!JOB [READ IN AT 10:49:55]      SELLAPPAN
!FORT/A/B/E/P/S FORT.LS/L
!LISTING

C.....+<--------------------    FORTRAN  STATEMENT   -----------

C       MODAL CONTROL
        EXTERNAL  RGS01,RGS02
        DIMENSION PARM(5),X(4),DX(4),SIZE(4),WORK(8,4)
        COMMON F1,F2,F3,F4,G
        CALL INOUT(2,5)
        CALL OPEN (1, 'SELLAPPAN' ,3,IER)
        IF(IER.NE.1) STOP UNABLE TO OPEN FILE
        READ(2,91) TMAX,STEP,TOL
        READ(2,91) X
        READ(2,91) SIZE
        READ(2,91) F1,F2,F3,F4,G
     91 FORMAT( 8F10.0 )
        PARM(1)=0.0
        PARM(2)=TMAX
        PARM(3)=STEP
        N=4
        WRITE(5,92) TMAX,STEP,TOL
     92 FORMAT( '1TMAX=',F8.2,10X,'STEP=',F8.4,10X,'TOL=',F8.6)
        CALL RKSCL(N,SIZE,DX,TOL,PARM)
        CALL RKGS(PARM,X,DX,N,IHLF,RGS01,RGS02,WORK)
        WRITE(5,99)IHLF
     99 FORMAT('0IHLF=',I3)
        CALL EXIT
        END

COMPILATION SUCCESSFUL -- OBJECT CODE IN FILE NAMED 001.RB

!FORT/A/B/E/P/S FORT.LS/L
!LISTING
```

```
C....+<--------------------  FORTRAN  STATEMENT  -----------

      SUBROUTINE  RGS01(T,X,DX)
      DIMENSION X(4), DX(4)
      COMMON F1,F2,F3,F4,G
      DX(1)=X(2)
      WO=0.001115
      AK=0.0005549
      AL1=3.0*WO*WO+AK
      DX(2)=-F2*X(2)-(AL1+F1)*X(1)
      DX(3)=X(4)
      AL2=3.0*AL1
      DX(4)=-F4*X(4)-(AL2+F3)*X(3)-G*F2*X(2)-G*F1*X(1)
      RETURN
      END

COMPILATION SUCCESSFUL -- OBJECT CODE IN FILE NAMED 002.RB

!FORT/A/B/E/P/S FORT.LS/L
!LISTING

C....+<--------------------  FORTRAN  STATEMENT  -----------

      SUBROUTINE  RGS02(T,X,DX,IHLF,N,P)
        LOGICAL RKNXT
      DIMENSION X(4),DX(4),DUMMY(4)
      COMMON F1,F2,F3,F4,G
      CALL  RGS01(T,X,DUMMY)
      U1=-F1*X(1)-F2*X(2)
      U2=-F3*X(3)-F4*X(4)
      AM=1000.0/3.0
      FV1=AM*(U1+(U2/3.0))
      FV2=AM*(U1-(U2/3.0))
      V1=X(1)+X(3)
      V2=X(1)-X(3)
      V3=X(2)+X(4)
      V4=X(2)-X(4)
      VCM=-(2.0/3.0)*X(3)
      IF(.NOT.RKNXT(IHLF)) GO TO 8
      WRITE(5,1) T,X(1),X(3),V1,V2,VCM,FV1,FV2,V3,V4
    1 FORMAT (1X,F9.1,9F13.7)
      WRITE BINARY(1) T,FV2,FV1,VCM,V2,V1,X(3),X(1)
    8 CONTINUE
      RETURN
      END

COMPILATION SUCCESSFUL -- OBJECT CODE IN FILE NAMED 003.RB

!RLDR TMP/S SYS/E 001 002 003 DP0:SSP.LB FORT.LB
!LOADED
!DELETE/V SELLAPPAN
DELETED SELLAPPAN
!CREATE SELLAPPAN
!EXEC
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