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THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES

PART B: DEVELOPMENT OF CONTINUUM MODEL AND COMPUTER SIMULATION

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THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES

PART B: DEVELOPMENT OF CONTINUUM MODEL AND COMPUTER SIMULATION

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ABSTRACT

The equations of motion of an arbitrary flexible body in orbit are derived. The model includes the effects of gravity with all its higher harmonics. As a specific example, the motion of a long, slender, uniform beam in circular orbit is modelled. The example considers both the inplane and three dimensional motion of the beam in orbit. In the case of planar motion with only flexural vibrations, the pitch motion is not influenced by the elastic motion of the beam. For large values of the square of the ratio of the structural modal frequency to the orbital angular rate the elastic motion is decoupled from the pitch motion. However, for small values of this ratio and small amplitude pitch motion, the elastic motion is giverned by a Hill's 3-term equation. Numerical simulation of this equation indicates the possibilities of instability for very low values of the square of the ratio of the modal frequency to the orbit angular rate. Also numerical simulations of the first order non-linear equations of motion for a long flexible beam in orbit have been performed. The effect of varying the initial conditions and the number of modes has been demonstrated.

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LIST OF SYMBOLS

A_n	Modal amplitude function
B^*, $B^{(0)}$, $B^{(s)}$	Matrices defined in Appendix II
$\bar{c}$	Resultant of external disturbances torques acting on the body
(C_x, C_y, C_z)	Components of $\bar{c}$ in τ_3 frame
$\bar{D}^{(n)}$, $D'_{(n)}$	Terms accounting for the center of mass shift due to elastic motion (Eqns. (30) and (48)
E	External force on the elemental mass dm
E_n	Modal components of external disturbances
$\bar{F}$	Gravity force expressed in $\tau_2(0)$ frame
$\bar{F}_0$	Gravity force at point O, expressed in the intrinsic frame at that point
$(F_\rho, F_\eta, F_\omega)$	Gravity force components in τ_1 frame
$\bar{G}_R$	Gravitational torque due to rigid body motion
$\bar{G}^{(n)}$	Gravity torque due to elastic motion in the n^{th} mode
$H_{\alpha\beta}^{(n)}$	See Eqn. (III - 1) in Appendix - III
$I_\alpha^{(n)}$	See Eqn. (III - 1) in Appendix - III
(J_x, J_y, J_z)	Principal moments of inertia of the body in the undeformed state
K_s	See Eqn. (5)
K_{s0}	Zeroth harmonic amplitude of gravity potential
L	Structural operator which transforms the elastic displacement to elastic force

$L_{\alpha\beta}^{(mn)}$	See Eqn. (III - 1) in Appendix - III
M, $M^{(0)}$, $M^{(s)}$	Matrices defined in Eqn. (6)
M_n	Generalized mass in the n^{th} mode, see Eqn. (12)
$M_{ij}^{(0)}$, $M_{ij}^{(s)}$	Elements of $M^{(0)}$ and $M^{(s)}$ respectively
O'	Origin of τ_0 frame (center of the earth)
O	Origin of the body axes
$P_s^{(m)}(\eta)$	m^{th} Legendre associated function of order s
P	An arbitrary point in the body
$\overline{Q}^{(n)}$	Terms accounting for the inertia torques due to elastic motion (Eqn. (28)).
$(Q_x^{(n)}, Q_y^{(n)}, Q_z^{(n)})$	Components of $\overline{Q}^{(n)}$
$\overline{R}$	See Eqn. (27)
T_1, T_2, T_3	Transformation matrices given in Eqns. (1), (2) and (3).
T	Orbital period
V	Gravitational potential per unit mass (see Eqn. (57)
O'XYZ	Inertial axes system (τ_0 frame)
$O\,X_0\,Y_0\,Z_0$	Orbit fixed reference frame (τ_2 frame)
Oxyz	Principal axes of the body (τ_3 frame)
$Z^{(n)}$	Mode shape of a free-free beam in its n^{th} mode (see Eqn. (IV - 4)
$\overline{a}$	Inertial acceleration of dm
a	Equatorial radius of the earth
$\overline{a}_{cm}$	Acceleration of the center of mass

c	An arbitrary constant in Eqn. (67)
$\bar{e}$	External disturbance force per unit mass
$\bar{f}$	Gravity force per unit mass at an arbitrary point in the body, expressed in the body frame
$\bar{f}_0$	Gravity force at point 0 expressed in the body frame
g_n	Gravitational force acting on n^{th} mode
g_{mn}	See Eqn. (46) and Eqn. (III - 1)
$0i_1\ i_2\ i_3$	Local intrinsic frame (τ_1 frame)
ℓ	Length of the beam
m	Mass of the body
$\bar{q}$	Elastic displacement vector
$\bar{r}$	Instantaneous position vector of the elemental mass dm measured from 0
$\bar{r}_0$	Position vector of dm measured from 0 in the undeformed state
t	Time
z_n	Non-dimensionalized modal amplitude (= A_n/ℓ)
β_n	See Eqn. (IV - 3)
γ	Phase angle
δ_{mn}	Kronecker delta symbol
ζ	See Eqn. (IV - 3)
η	Co-latitude
ϕ, θ, γ	Euler angles

μ	Mass density
ν	Gravitational constant (=9.79m/sec^2 for earth)
(ξ_x, ξ_y, ξ_z)	Co-ordinates of an elemental mass dm in τ_3 frame in the undeformed state
ρ	Distance of dm from the inertial origin O'
τ	Dimensionless time; $\tau = \frac{1}{2}(\sqrt{3}\omega t+\gamma)$
$\overline{\phi}^{(n)}$	Mode shape of nth mode (components $\phi_x^{(n)}$, $\phi_y^{(n)}$, $\phi_z^{(n)}$)
ϕ_{sm}	See Eqn. (5)
ω	Longitude
ω_c	orbital angular velocity
ω_n	nth mode natural frequency
$\Omega(\eta, \omega)$	See Eqn. (5)
Ω_n	See Eqn. (IV - 3)
s( )	sin ( )
c( )	cos ( )
(')	$\frac{d}{d\eta}$ ( )
(^)	$\frac{d}{d\omega}$ ( )
$(\dot{\ })$	$\frac{d}{dt}$ ( )

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1. INTRODUCTION

This report presents the development of the equations of motion of an arbitrary flexible body in orbit and its specialization to a long, slender, uniform beam in circular orbit. In the literature,[1,2,3] many models of a beam satellite in orbit are presented. The model considered by Pringle[3] consists of a massive central rigid body to which a massless elastic beam is rigidly attached. The other end of the beam carries a tip mass. For beam satellites with lengths of the order of 100 meters or more and mass distributed throughout the length this cannot be used. Ashley's model[1,2] of a beam satellite is based on a continuum approach. He arrives at a set of partial differential equations for the beam model through energy methods. However, partial differential equations are not as convenient for numerical simulation. Ashley arrives at the conclusion that it is impossible to excite flexural vibrations of a beam directly through gravity gradient. The same conclusion is arrived at in section 5.1.1 of this report.

The development of the beam model presented in this report follows as a specialization of the equations of motion of an arbitrary flexible body in orbit. Also, the model presented here can be quite easily adopted for numerical simulation and also for inclusion of various control laws.

The development of equations of motion for an arbitrary flexible body presented in this report essentially follows that of Santini.[4]

However, the distinguishing features of the present development from that of Santini are (a) extensive use of the techniques of vector calculus (b) introduction of the orbit fixed reference frame as an intermediate frame between the local intrinsic frame at the origin of the body axes system and the body axes frame. Hence, the Euler angle rotations used in the present report are different from those of Santini.[4]

The equations of motion presented in this report are obtained by the Galerkin integration[5,6] of the equations of motion of a generic point in the body. The motion of the generic point is assumed to be described by the superposition of rigid body motion plus a combination of the structural modes.

Section 2 of this report describes the various co-ordinate reference frames employed in the development of the equations, and the transformation relations between these reference frames. The detailed derivations of the transformation relations are presented in Appendix - I.

Section 3 presents an expression for the gravitational force on a generic element in the body. The expression also includes the higher harmonics of the earth's gravitational field. The details of this development are presented in Appendix - II.

In Section 4, the development of the equations of motion of an arbitrary flexible body in orbit is presented. The expansions of the vector expressions in equations (28), (31), (32), (43), (44), (45), and (46) are presented in Appendix - III.

Specializations of the equations of motion in section 4 to a long, slender, uniform beam are presented in section 5. Sections 5.1 and 5.2 discuss the planar, and the three dimensional motion of the beam in orbit, respectively. In Appendix - IV, the expressions for the natural mode shapes, frequencies and the modal mass for a free-free uniform beam are presented.

In Section 6, the numerical solutions to the response of the planar motion of a beam in circular orbit, undergoing only flexural vibrations, are presented. The responses indicate the possibilities of instability at very low values of $(\omega_n/\omega_c)^2$.

The final section of the report, Section 7, outlines the conclusions based on the numerical results in Section 6.

2. CO-ORDINATE FRAMES

The following co-ordinate frames are used in the development of the equations of motion. (See Fig. 1):

τ_0 : $0'XYZ$ Inertial reference frame.

$0'Z$ along the earth's spin axis.

$0'X$ along the line of the ascending node.

$0'Y$ perpendicular to $0'X$ and $0'Z$.

τ_{1P} : $Pi_1i_2i_3$ Local intrinsic frame at a generic point, P.

Pi_1 along the radius vector from 0' to P.

Pi_2 perpendicular to Pi_1 in the plane of $Z0'P$.

Pi_3 perpendicular to Pi_1 and Pi_2.

τ_{10} : $0i_1i_2i_3$ Local intrinsic frame at 0.

τ_2 : $0X_0Y_0Z_0$ Orbit fixed reference frame.

$0X_0$ along the local vertical

$0Y_0$ along the orbit normal and in the negative direction of the orbit angular momentum vector.

$0Z_0$ perpendicular to $0X_0$ and $0Y_0$.

τ_3 : $0XYZ$ Principal axes of the body.

The above reference frames are related to each other as follows:

$$\begin{bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{bmatrix}_{\tau_{10}} = T_1 \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{\tau_0} ; \quad \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}_{\tau_2} = T_2 \begin{bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{bmatrix}_{\tau_{10}} ; \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{\tau_3} = T_3 \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}_{\tau_2}$$

where,

$$T_1 = \begin{bmatrix} s\eta c\omega & s\eta s\omega & c\eta \\ c\eta c\omega & c\eta s\omega & -s\eta \end{bmatrix} \qquad - (1)$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi & s\chi \\ 0 & -s\chi & c\chi \end{bmatrix} \tag{2}$$

$$T_3 = \begin{bmatrix} c\phi c\theta & (s\phi c\psi + c\phi s\theta s\psi) & (s\phi s\psi - c\phi s\theta c\psi) \\ -s\phi c\theta & (c\phi c\psi - s\phi s\theta s\psi) & (c\phi s\psi + s\phi s\theta c\psi) \\ s\theta & -c\theta s\psi & c\theta c\psi \end{bmatrix} \tag{3}$$

and s,c represent sine and cosine functions respectively. The detailed development of these transformations are presented in Appendix - I.

The body angular velocity components (ω_x, ω_y, ω_z) and the Euler angular rates ($\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$) are related as follows:

$$\begin{aligned} \omega_x &= \dot{\theta} s\phi + \dot{\psi} c\phi c\theta - \omega_c \quad (s\phi c\psi + c\phi s\theta s\psi) \\ \omega_y &= \dot{\theta} c\phi - \dot{\psi} s\phi c\theta - \omega_c \quad (c\phi c\psi - s\phi s\theta s\psi) \\ \omega_z &= \dot{\psi} s\theta + \dot{\phi} + \omega_c c\theta s\psi \end{aligned} \tag{4}$$

3. GRAVITATION

In this section an expression for the gravitational force per unit mass, expressed in the body fixed frame (τ_3) is presented. In deriving this expression, it is assumed that $|\bar{r}|/\rho << 1$, where $\bar{r}$ is the position vector of an arbitrary point in the body with respect to 0 and ρ is the distance of the same point from 0'.

We can write the gravitational[4] potential in the most general form as:

$$V(\rho, \eta, \omega) = \frac{\nu a^2}{\rho} + \nu a \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^{s+1} \Omega_s(\eta,\omega) \qquad (5)$$

where,

$$K_s = K_{so} \cos \phi_{so}$$

$$\Omega(\eta,\omega) = \sum_{m=0}^{s} [P_s^{(m)}(\eta) \cos(m\omega + \phi_{sm})]/K_s$$

$P_s^{(m)}(\eta)$ is the m^{th} associated Legendre function of order s.

K_{so} and ϕ_{sm} are constants to be given experimentally through geodetic satellite techniques.

The gravitational force per unit mass at the origin of the body axes, 0, in the $\tau_1(0)$ frame is $\bar{F}_0 = \nabla V|_0$.

For a point at a distance $\bar{r}$ from 0, neglecting small quantities of the order of $\frac{|\bar{r}|}{\rho}$, the gravity force in the τ_3 frame is given by

$$\bar{f} = \bar{f}_0 + M\,\bar{r} \qquad (10)$$

where

$\bar{f}_0$ = Gravitational force at 0 expressed in the principal body axes frame (τ_3)

$$M = [\, M^{(0)} + \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^{s} M^{(s)}]$$

$$M^{(0)} = \frac{\nu a^2}{\rho^3} \begin{bmatrix} 3c^2\phi c^2\theta - 1 & -3s\phi c\phi c^2\theta & 3c\phi c\theta s\theta \\ -3s\phi c\phi c^2\theta & 3s^2\phi c^2\theta - 1 & -3s\phi c\theta s\theta \\ 3c\phi c\theta s\theta & -3s\phi c\theta s\theta & 3s^2\theta - 1 \end{bmatrix}$$

$$M^{(s)} = \frac{\nu a^2}{\rho^3} T_3\, T_2\, B^{(s)}\, T_2^T\, T_3^T \qquad (6)$$

The matrix $B^{(s)}$ is given by eqn. (II-12) in Appendix - II. It can be observed that $M^{(0)}$ and $M^{(s)}$ are symmetric matrices.

The expression for the gravitational force in the form given in eqn.(6) is used in the development of the equations of motion presented in the next section.

4. EQUATIONS OF MOTION

In this section the equations of rotational motion and elastic motion of an arbitrary flexible body in orbit are presented. The body is assumed to be subjected to small amplitude elastic displacements, $\bar{q}$, which are transformed to elastic forces by the linear operator L.

Consider an elemental mass, dm, whose instantaneous position vector from the center of mass of the body is $\bar{r}$ (Fig. 2).

The equation of motion of dm can be written as:

$$\bar{a}\, dm = L\,(\bar{q}) + \bar{f}\, dm + \bar{E} \tag{7}$$

where

$\bar{a}$ = Inertial acceleration of dm

$L(\bar{q})$ = Elastic forces acting on dm

$\bar{f}$ = Gravitational force per unit mass

$\bar{E}$ = External forces acting on dm

$\bar{q}$ = Elastic displacement

The above vector equaton can be written in the body fixed reference frame (τ_3) as,

$$[\bar{a}_{cm} + \ddot{\bar{r}} + 2\,\bar{\omega} \times \dot{\bar{r}} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})]\, dm$$

$$= L\,(\bar{q}) + \bar{f}\, dm + \bar{E} \tag{8}$$

It is important to note that $\dot{\bar{r}}$ and $\ddot{\bar{r}}$ are the velocity and acceleration of dm respectively as seen from the body fixed reference frame, τ_3. All the vectors in the above equation must be expressed in the body frame τ_3.

We can write the instantaneous position vector $\bar{r}$ of dm as

$$\bar{r} = \bar{r}_0 + \bar{q} \tag{9}$$

where

$\bar{r}_0$ is the position vector of dm with respect to 0 in the undeformed state; $\bar{q}$ is the elastic displacement of dm.

Hence

$$\dot{\bar{r}} = \dot{\bar{q}} \text{ and } \ddot{\bar{r}} = \ddot{\bar{q}} \tag{10}$$

For small amplitude elastic displacements, we can write the elastic displacement, q, as a superposition of the various modal contributions according to

$$\bar{q} = \sum_{n=1}^{\infty} A_n(t)\, \bar{\Phi}^{(n)}\, (\bar{r}_0) \tag{11}$$

where

$$\bar{\Phi}^{(n)}\, (\bar{r}_0) = \phi_x^{(n)}\, \hat{i} + \phi_y^{(n)}\, \hat{j} + \phi_z^{(n)}\, \hat{k}$$

$A_n(t)$ = Modal amplitude

$\bar{\Phi}^{(n)}\, (\bar{r}_0)$ is the mode shape associated with the natural frequency ω_n and satisfies the following orthogonality condition,

$$\int_{vol} \bar{\Phi}^{(m)} \cdot \bar{\Phi}^{(n)}\, dm = \delta_{mn}\, M_n \tag{12}$$

Also

$$L\, (\bar{\Phi}^{(n)}) = -\omega_n^{\,2}\, \bar{\Phi}^{(n)}\, dm \tag{13}$$

Further, if the body is unconstrained, the elastic modes must be orthogonal to the rigid body modes ie,

$$\int_{vol} \bar{\Phi}^{(n)}\, dm = 0 \tag{14}$$

$$\int_{vol} \bar{r}_0 \times \bar{\Phi}^{(n)} dm = 0 \tag{15}$$

If the body is constrained against translation and rotation at the undeformed center of mass, the corresponding modes are called "fixed modes." For fixed modes the orthogonality conditions (14) and (15) do not hold. It should be noted that, for the case of fixed modes, the origin of the body frame, 0, no longer coincides with the center of mass in the deformed state. Hence, only for free modes, $\int_{vol} \bar{r}\, dm = 0$. However for fixed modes $\int_{vol} \bar{r}\, dm = \int_{vol} \bar{q}\, dm \neq 0$.

4.1 Equations of rotational motion

The equations of rotational motion of the body are obtained by the following operation

$$\int_{vol} \bar{r} \times \text{Eqn. (8)}$$

i.e.

$$\int_{vol} \bar{r} \times [\bar{a}_{cm} + \ddot{\bar{r}} + 2\,\bar{\omega} \times \dot{\bar{r}} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})]\, dm$$

$$= \int_{vol} \bar{r} \times [\underline{L(\bar{q})} + \bar{f} + \bar{e}]\, dm \tag{16}$$
$$\qquad\qquad\qquad dm$$

where $\bar{e}$ is the external force per unit mass.

The various terms in Eqn (16) can now be evaluated using the techniques of vector calculus. Assuming $\frac{|\bar{q}|}{|\bar{r}|} \ll 1$, only the first order terms in $\bar{q}$ are retained in the following expansions.

$$\int_{vol} \bar{r} \times \bar{a}_{cm}\, dm = \int_{vol} \bar{q}\, dm \times \bar{a}_{cm} \tag{17}$$

$$\int_{vol} \bar{r} \times \ddot{\bar{r}}\, dm = \int_{V} (\bar{r}_0+\bar{q}) \times \ddot{\bar{q}}\, dm \cong \int_{V} \bar{r}_0 \times \ddot{\bar{q}}\, dm \tag{18}$$

$$\int_{vol} \bar{r} \times 2\, (\bar{\omega} \times \dot{\bar{r}})\, dm = 2 \int_{V} (\bar{r}_0+\bar{q}) \times (\bar{\omega} \times \dot{\bar{q}})\, dm$$

$$\tilde{=}\ 2 \int_{V} \bar{r}_0 \times (\bar{\omega} \times \dot{\bar{q}})\, dm \tag{19}$$

$$\int_{vol} \bar{r} \times (\dot{\bar{\omega}} \times \bar{r})\, dm = \int_{V} (r_0+\bar{q}) \times [\dot{\bar{\omega}} \times (\bar{r}_0+\bar{q})]dm$$

$$\tilde{=}\ \int_{V} \bar{r}_0 \times (\dot{\bar{\omega}} \times \bar{r}_0)dm + \int_{V} r_0 \times (\dot{\bar{\omega}} \times \bar{q})dm$$

$$+ \int_{V} q \times (\dot{\bar{\omega}} \times \bar{r}_0)dm \tag{20}$$

$$\int_{vol} \bar{r} \times (\bar{\omega} \times (\bar{\omega} \times \bar{r}))dm = \int_{V} [(\bar{r} \cdot \bar{\omega} \times \bar{r})\bar{\omega} - (\bar{r} \cdot \bar{\omega})\, (\bar{\omega} \times \bar{r})]dm$$

$$= - \int_{V} (\bar{r} \cdot \bar{\omega})\, (\bar{\omega} \times \bar{r})dm$$

$$= - \int_{V} \{(\bar{r}_0+\bar{q}) \cdot \bar{\omega}\}\{\omega \times (\bar{r}_0+\bar{q})\}dm$$

$$\tilde{=}\ - \int_{V} (\bar{r}_0 \cdot \bar{\omega})\, (\bar{\omega} \times \bar{r}_0)dm$$

$$- \int_{V} [(\bar{r}_0 \cdot \bar{\omega})\, (\bar{\omega} \times \bar{q}) + (\bar{q} \cdot \bar{\omega})\, (\bar{\omega} \times \bar{r}_0)dm \tag{21}$$

$$\int_{vol} \bar{r} \times L\, (\bar{q})dm \;\tilde{=}\; \int_{V} \bar{r}_0 \times L\, (\bar{q})dm$$

$$= - \sum_{n=1}^{\infty} \omega_n{}^2 A_n \int \bar{r}_0 \times \bar{\phi}^{(n)}dm \text{ (using eqn. (13))}$$

$$= \quad 0 \text{ (for free modes. See Eqn. (15)).} \tag{22}$$

$$\int_{vol} \bar{r} \times \bar{f}\, dm = \int_V \bar{r} \times (\bar{f}_0 + M\,\bar{r})dm$$

$$= \int_V \bar{q}\, dm \times \bar{f}_0 + \int_V (\bar{r}_0 + \bar{q}) \times M\, (\bar{r}_0 + \bar{q})dm$$

$$\tilde{=} \quad \int_V \bar{q}\, dm \times \bar{f}_0 + \int_V \bar{r}_0 \times M\, \bar{r}_0\, dm$$

$$+ \int_V [\bar{r}_0 \times M\, \bar{q} + \bar{q} \times M\, \bar{r}_0]dm \tag{23}$$

$$\int_{vol} \bar{r} \times \bar{e}\, dm = \bar{C}$$

It can be easily shown that,

$$\int_V \bar{r}_0 \times (\dot{\bar{\omega}} \times \bar{r}_0)\, dm = J_x\, \dot{\omega}_x\, \hat{i} + J_y\, \dot{\omega}_y\, \hat{j} + J_z\, \dot{\omega}_z\, \hat{k} \tag{24}$$

and $\int (\bar{r}_0 \cdot \bar{\omega})\, (\bar{\omega} \times \bar{r}_0)\, dm = (J_y - J_z)\, \omega_y\, \omega_z\, \hat{i} + (J_z - J_x)\, \omega_z\, \omega_x\, \hat{j}$

$$+ (J_x - J_y)\, \omega_x\, \omega_y\, \hat{k} \tag{25}$$

where, J_x, J_y, J_z are the principal moments of inertia of the body in the undeformed state.

After substitution of the values for the integrals in Eqn. (16) and rearrangement of terms, one can obtain the following form for the equations of rotational motion.

$$\bar{R} + \sum_{n=1}^{\infty} \bar{Q}^{(n)} + \bar{C} + \sum_{n=1}^{\infty} \bar{D}^{(n)} = \bar{G}_R + \sum_{n=1}^{\infty} \bar{G}^{(n)} \tag{26}$$

where,

$$\bar{R} = \{J_x\, \dot{\omega}_x + (J_z - J_y)\, \omega_y\, \omega_z\}\, \hat{i}$$

$$+ \{J_y\, \dot{\omega}_y + (J_x - J_z)\, \omega_z\, \omega_x\}\, \hat{j}$$

$$+ \{J_z\, \dot{\omega}_z + (J_y - J_x)\, \omega_x\, \omega_y\}\, \hat{k} \tag{27}$$

$$\Sigma\, \bar{Q}^{(n)} = \int_{vol} [r_0 \times \ddot{\bar{q}} + 2\bar{r}_0 \times (\omega \times \dot{\bar{q}}) + \bar{r}_0 \times (\dot{\bar{\omega}} \times \bar{q})$$

$$+ \bar{q} \times (\dot{\bar{\omega}} \times \bar{r}_0) - (\bar{r}_0 \cdot \bar{\omega})\, (\bar{\omega} \times \bar{q})$$

$$- (\bar{q} \cdot \bar{\omega})\, (\bar{\omega} \times \bar{r}_0)]\; dm \qquad (28)$$

$$\bar{C} = \int_{vol} \bar{r} \times \bar{e}\; dm \qquad (29)$$

$$\sum_{n=1}^{\infty} \bar{D}^{(n)} = \int_{vol} \bar{q}\; dm \times (\bar{a}_{cm} - \bar{F}_0) + \sum_{n=1}^{\infty} \omega_n^{\,2} A_n \int_v \bar{r}_0 \times \bar{\Phi}^{(n)} dm \qquad (30)$$

$$\bar{G}_R = \int_{vol} \bar{r}_0 \times M\, \bar{r}_0\; dm \qquad (31)$$

$$\sum_{n=1}^{\infty} \bar{G}^{(n)} = \int_{vol} [\bar{r}_0 \times M\, \bar{q} + \bar{q} \times M\, \bar{r}_0]\; dm \qquad (32)$$

The expansions of the above vector expressions are presented in Appendix - III.

The significance of the various terms in Eqn. (26) will now be briefly discussed. The terms, $\bar{Q}^{(n)}$, reflect the inertia torque associated with the elastic deformations. The term, $\bar{G}_R$, corresponds to the gravitational torque on a rigid body. The terms, $\bar{G}^{(n)}$, correspond to the gravitational torque due to the elastic deformations. The terms, $\bar{D}^{(n)}$, account for the difference in position between the actual center of mass and the center of mass of the undeformed body. For the case of free modes, $\bar{D}^{(n)} = 0$.

4.2. Generic mode equations

The generic mode equation is obtained by the following operation:

$$\int_{vol} \bar{\Phi}^{(n)} \cdot [\text{Eqn. } (8)]$$

ie

$$\int_{vol} \bar{\Phi}^{(n)} \cdot [\bar{a}_{cm} + \ddot{\bar{r}} + 2\,\bar{\omega} \times \dot{\bar{r}} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})]\, dm$$

$$= \int_{vol} \bar{\Phi}^{(n)} \cdot [\frac{L(\bar{q})}{dm} + \bar{F} + \bar{e}]\, dm \qquad (33)$$

With the use of equations (6), (9), (10), (11), (12) and (13), the various terms appearing in Eqn. (33) can now be expanded as follows:

$$\int_{vol} \bar{\Phi}^{(n)} \cdot \bar{a}_{cm}\, dm = \int_{v} \bar{\Phi}\, dm \cdot \bar{a}_{cm} \qquad (34)$$

$$\int_{vol} \bar{\Phi}^{(n)} \cdot \ddot{\bar{r}}\, dm = \int_{v} \bar{\Phi}^{(n)} \cdot \ddot{\bar{q}}\, dm$$

$$= \sum_{m} \ddot{A}_m \int \bar{\Phi}^{(n)} \cdot \bar{\Phi}^{(m)}\, dm$$

$$= \ddot{A}_n M_n \qquad (35)$$

$$\int_{vol} \bar{\Phi}^{(n)} \cdot (2\,\bar{\omega} \times \dot{\bar{r}})\, dm = 2 \int_{v} (\bar{\Phi}^{(n)} \cdot \bar{\omega} \times \dot{\bar{q}})\, dm \qquad (36)$$

$$\int_{vol} (\bar{\Phi}^{(n)} \cdot \dot{\bar{\omega}} \times \bar{r})\, dm = \int_{v} (\bar{\Phi}^{(n)} \cdot \dot{\bar{\omega}} \times \bar{r}_0)\, dm + \int_{v} (\bar{\Phi}^{(n)} \cdot \dot{\bar{\omega}} \times \bar{q})\, dm \qquad (37)$$

$$\int_{vol} \bar{\Phi}^{(n)} \cdot \bar{\omega} \times (\bar{\omega} \times \bar{r})\, dm = \int_{v} \bar{\Phi}^{(n)} \cdot \bar{\omega} \times (\bar{\omega} \times \bar{r}_0)\, dm$$

$$+ \int_{v} \bar{\Phi}^{(n)} \cdot \bar{\omega} \times (\bar{\omega} \times \bar{q})\, dm \qquad (38)$$

$$\int_{vol} \bar{\Phi}^{(n)} \cdot L(\bar{q}) = - \sum_{m=1} \omega_m^2 A_m \int_V \bar{\Phi}^{(n)} \cdot \bar{\Phi}^{(m)} \, dm$$

$$= - \omega_n^2 A_n M_n \tag{39}$$

$$\int_{vol} \bar{\Phi}^{(n)} \cdot \bar{f} \, dm = \int_V \bar{\Phi}^{(n)} \, dm \cdot \bar{f}_0 + \int_V \phi^{(n)} \cdot M \bar{r} \, dm$$

$$= \int_V \bar{\Phi}^{(n)} \, dm \cdot \bar{f}_0 + \int_V \bar{\Phi}^{(n)} \cdot M \bar{r}_0 \, dm$$

$$+ \int_V \bar{\Phi}^{(n)} \cdot M \bar{q} \, dm \tag{40}$$

$$\int_{vol} \bar{\Phi}^{(n)} \cdot \bar{e} \, dm = E_n \tag{41}$$

After substitution of the values for the various integrals in Eqn. (33) and rearrangement of the terms, the generic mode equation is obtained in the following form.

$$\ddot{A}_n + \omega_n^2 A_n + \frac{\varphi_n}{M_n} + \frac{1}{M_n} \sum_{m=1}^{\infty} \varphi_{mn} = \frac{1}{M_n} [g_n + \sum_{m=1}^{\infty} g_{mn} + E_n + D_n'] \tag{42}$$

where

$$\varphi_n = \int_V [\bar{\Phi}^{(n)} \cdot \dot{\bar{\omega}} \times \bar{r}_0 + \bar{\Phi}^{(n)} \cdot \bar{\omega} \times (\bar{\omega} \times \bar{r}_0)] \, dm \tag{43}$$

$$\sum_{m=1}^{\infty} \varphi_{mn} = \int_V [2 \, \bar{\Phi}^{(n)} \cdot \bar{\omega} \times \dot{\bar{q}} + \bar{\Phi}^{(n)} \cdot \dot{\bar{\omega}} \times \bar{q} + \bar{\Phi}^{(n)} \cdot \bar{\omega} \times (\bar{\omega} \times \bar{q})] \, dm \tag{44}$$

$$g_n = \int_V \bar{\Phi}^{(n)} \cdot M \bar{r}_0 \, dm \tag{45}$$

$$\sum_{m=1}^{\infty} g_{mn} = \int_V \bar{\Phi}^{(n)} \cdot M \bar{q} \, dm \tag{46}$$

$$E_n = \int_V \bar{\Phi}^{(n)} \cdot \bar{e} \, dm \tag{47}$$

$$D_n' = \int_V \bar{\Phi}^{(n)} \, dm \cdot (\bar{a}_{cm} - \bar{f}_0) \tag{48}$$

$$M_n = \int \bar{\Phi}^{(n)} \cdot \bar{\Phi}^{(m)} \, dm \tag{49}$$

The expanded forms of the above expressions are presented in Appendix III.

The significance of the various terms in Eqn. (42) will now be briefly discussed. The term φ_n corresponds to the forcing term due to rigid body motion. φ_{mn} is the forcing term due to the elastic motion in the m^{th} mode. The term, g_n, represents the gravitational force acting on the n^{th} mode due to the rigid body motion. The terms g_{mn} correspond to the gravitational force acting on the n^{th} mode, due to the elastic motion in the m^{th} mode. E_n is the component of the external force acting on the n^{th} mode. D'_n is the term corresponding to the displacement of the center of mass from the point 0.

In the next section, an application of the equations of motion developed in this section is presented.

5. APPLICATIONS

In this section, we consider the application of the equations of motion presented in the previous section, to the specific example of a beam in circular orbit shown in Fig. 3.

In section 5.1 it is assumed that the motion of the beam is restricted to the orbit plane. In section 5.2, the general case of the three dimensional beam is considered.

5.1. Planar motion of a long, slender, uniform beam in circular orbit

Since the beam motion is restricted to the orbit plane, the yaw and the roll angles vanish, i.e.

$$\psi\ (t) = \phi\ (t) = 0 \tag{50}$$

Also, the elastic deformation out of the orbit plane is assumed to be zero, ie, $\phi_y^{(n)} = 0$.

For unconstrained structures,

$$\overline{D}^{(n)} = 0;\ D'_n = 0 \text{ and } H_{\alpha\beta}^{(n)} = H_{\beta\alpha}^{(n)} \tag{51}$$

where

$$H_{\alpha\beta}^{(n)} \equiv \int_{vol} \xi_\alpha\ \phi_\beta^{(n)}\ dm \quad (\alpha,\ \beta = x,y,z)$$

and (ξ_x, ξ_y, ξ_z) are coordinates of dm in the undeformed state measured in the body frame.

In the absence of out-of-plane deformations, i.e. $\phi_y^{(n)} = 0$, we can deduce that,

$$H_{yy}^{(n)} = H_{xy}^{(n)} = H_{yx}^{(n)} = H_{zy}^{(n)} = H_{yz}^{(n)} = 0 \tag{52}$$

Noting that, for a beam (see Fig. 3) with uniform cross section,

$$\iint \xi_z \; d\xi_y \; d\xi_z = \iint \xi_y \, d\xi_y \, d\xi_z = 0 \tag{53}$$

we can further deduce that,

$$H_{zx}^{(n)} = \int_{vol} \mu \xi_z \, \phi_x^{(n)}(\xi_x) \, d\xi_x \; d\xi_y \; d\xi_z = 0 = H_{xz}^{(n)} \tag{54}$$

$$H_{zz}^{(n)} = \int_{vol} \mu \xi_z \, \phi_z^{(n)} \, (\xi_x) d\xi_x \; d\xi_y \; d\xi_z = 0 \tag{55}$$

where, μ is the density of the beam material.

Since, by definition,

$$L_{\alpha\beta}^{(mn)} \equiv \int_{vol} \phi_\alpha^{(m)} \, \phi_\beta^{(n)} \, dm \quad (\alpha, \beta = x, y, z)$$

in the absence of out-of-plane elastic deformation, i.e., $\phi_y^{(n)} = 0$, it can be easily shown that

$$L_{xy}^{(mn)} = L_{yx}^{(mn)} = L_{yz}^{(mn)} = L_{zy}^{(mn)} = L_{yy}^{(mn)} = 0 \tag{56}$$

With the use of the above results and the results presented in Appendix - III, the pitch and the generic mode equations for the present case are obtained as:

$$J_y \dot{\omega}_y + \sum_{n=1}^{\infty} Q_y^{(n)} + C_y = G_{Ry} + \sum_{n=1}^{\infty} G_y^{(n)} \quad \text{(pitch equation)} \tag{57}$$

where,

$$\dot{\omega}_y = \ddot{\theta}$$

$$Q_y^{(n)} = 2 \, [\dot{A}_n(\dot{\theta} - \omega_c) + A_n \, \ddot{\theta}] \, H_{xx}^{(n)}$$

$$G_{Ry} = (J_x - J_z) \, M_{31} \cong - J_z \, M_{31} \; (\text{for } J_x << J_z)$$

$$G_y^{(n)} = - 2 \, M_{31} \, H_{xx}^{(n)}$$

$$M_{31} = \frac{\mu a^2}{\rho^3} \, [s\theta \, c\theta + \sum_{s=1}^{\infty} K_s \, (\frac{a}{\rho})^s \, M_{31}{}^{(s)}]$$

The generic mode equations:

$$\ddot{A}_n + \omega_n^2 A_n + \frac{\varphi_n}{M_n} + \frac{1}{M_n} \sum_{m=1}^{\infty} \varphi_{mn} = \frac{1}{M_n} (g_n + \sum_{m=1}^{\infty} g_{mn} + E_n) \tag{58}$$

where,

$$\varphi_n = - (\dot{\theta} - \omega_c)^2 H_{xx}^{(n)}$$

$$\varphi_{mn} = [2 \dot{A}_m (\dot{\theta} - \omega_c) + A_m \ddot{\theta}] (L_{xz}^{(mn)} - L_{zx}^{(mn)})$$

$$- A_m (\dot{\theta} - \omega_c)^2 (L_{xx}^{(mn)} + L_{zz}^{(mn)})$$

$$g_n = M_{11} H_{xx}^{(n)}$$

$$g_{mn} = A_m [M_{11} L_{xx}^{(mn)} + M_{13} (L_{xz}^{(mn)} + L_{zx}^{(mn)}) + M_{33} L_{zz}^{(mn)}]$$

We will now consider several specific cases of the long slender beam assumed to be subjected to rotations and deformations only within the orbital plane.

5.1.1. The case of no longitudinal vibrations: i.e., $\phi_x^{(n)} = 0$.

For this case it can be seen that,

$$H_{xx}^{(n)} = L_{xz}^{(mn)} = L_{zx}^{(mn)} = L_{xx}^{(mn)} = 0 \tag{59}$$

By choosing $\phi_z^{(n)}$ to represent the eigen-modes of bending vibrations of a free-free beam, we obtain,

$$L_{zz}^{(mn)} = \delta_{mn} M_n \tag{60}$$

where, δ_{mn} is the kronecker delta and M_n is the generalized mass of the beam in its n^{th} mode. An expression for the value of M_n for free-free beams is presented in Appendix - IV.

Substituting the above results in equations (57) and (58), one obtains the pitch equation and generic mode equation in the following form.

$$\ddot{\theta} + M_{31} + \frac{C_y}{J_y} = 0 \tag{61}$$

Generic mode equation:

$$\ddot{A}_n + [\omega_n{}^2 - M_{33} - (\dot{\theta} - \omega_c)^2]A_n = \frac{E_n}{M_n} \tag{62}$$

where,

$$M_{31} = \frac{\nu a^2}{\rho^3} \quad [s\theta \; c\theta \; + \sum_{s=1}^{\infty} \quad K_s \; (\frac{a}{\rho})^s \; M_{31}^{(s)}]$$

$$M_{33} = \frac{\nu a^2}{\rho^3} \; [3 \; s^2 \; \theta - 1 + \sum_{s=1}^{\infty} K_s \; (\frac{a}{\rho})^s \; M_{33}^{(s)}]$$

Assumptions of a spherically symmetric gravitational field and circular orbit result in the following simplification of M_{31} and M_{33},

$$\frac{\nu a^2}{\rho^3} = \omega_c{}^2$$

$$M_{31} = \frac{3}{2}\,\omega_c{}^2 \sin 2\theta$$

$$M_{33} = \omega_c{}^2 \; (\sin^2 \theta - 1)$$

Hence, equations (61) and (62) simplify to,

$$\ddot{\theta} + \frac{3}{2}\,\omega_c{}^2 \sin 2\theta + \frac{C_y}{2} = 0 \tag{63}$$

$$\ddot{A}_n + [\omega_n{}^2 - \omega_c{}^2 \; (3 \sin^2 \theta - 1) - (\dot{\theta} - \omega_c)^2] \; A_n = \frac{E_n}{M_n} \tag{64}$$

Equation (63) is just the rigid body pitching equation. Thus from Eqn. (63), one can conclude that there is no influence of the elastic motion on the rigid body motion, under the assumptions of the present case. The same conclusions were arrived at by Ashley[1] for the case of a thin beam undergoing flexural deformations in the orbit plane.

If the rigid body pitch oscillations are small ie. $\theta \ll 1$, equations (63) and (64) further simplify to (assuming no external disturbances),

$$\ddot{\theta} + 3\,\omega_c^2\,\theta = 0 \tag{65}$$

$$\ddot{A}_n + [\omega_n^2 - \dot{\theta}^2 + 2\omega_c\dot{\theta}]\,A_n \cong 0 \tag{66}$$

Without loss of generality, one can assume the solution of equation (65) as,

$$\theta = c\,\sin\,(\sqrt{3}\,\omega_c t + \gamma) \tag{67}$$

where, c is the amplitude of the pitch motion.

Substitution of Eqn. (67) into Eqn. (66) yields,

$$\ddot{A}_n + [\omega_n^2 - 3\,\omega_c^2\,c^2\cos^2\,(\sqrt{3}\,\omega_c t + \gamma) + 2\sqrt{3}\,\omega_c^2\,c\,\cos(\sqrt{3}\,\omega_c t + \gamma)]\,A_n = 0 \tag{68}$$

With the introduction of dimensionless variables, $\tau = \frac{1}{2}(\sqrt{3}\,\omega_c t + \gamma)$ and $z_n = A_n/\ell$ where ℓ is the length of the beam, one can write,

$$\dot{A}_n = \frac{\sqrt{3}}{2}\,\omega_c\,\ell\,\frac{dz_n}{d\tau} \quad \text{and} \quad \ddot{A}_n = \frac{3}{4}\,\omega_c^2\,\ell\,\frac{d^2z_n}{d\tau^2} \tag{69}$$

After substituting Eqn. (69) into Eqn. (68), one obtains the generic mode equation in the following non-dimensional form,

$$\frac{d^2z_n}{d\tau^2} + \frac{4}{3}\;[(\frac{\omega_n^2}{\omega_c^2} - 3\,c^2\cos^2 2\tau + 2\,\sqrt{3}\,c\,\cos\,2\tau]\,z_n = 0 \tag{70}$$

Using the trigonometric identity, $\cos^2 2\tau = \frac{1}{2}(\cos\,4\tau + 1)$, Eqn. (70) can be re-written as,

$$\frac{d^2z_n}{d\tau^2} + \frac{4}{3}\,[(\frac{\omega_n^2}{\omega_\tau^2} - \frac{3c^2}{2}) + 2\sqrt{3}\,c\,\cos\,2\tau - \frac{3}{2}\,c^2\,\cos\,4\tau]\,z_n = 0 \tag{71}$$

Equation (71) is in the form of "Hill's 3 - term equation" or "Whittaker's equation."[7] For small pitch amplitudes ie., $c \ll 1$, the above equation can be further approximated by the Mathieu equation

$$\frac{d^2 z_n}{d\tau^2} + (\delta + \varepsilon \cos 2\tau)\, z_n = 0 \tag{72}$$

where,

$$\delta = \frac{4}{3}\left(\frac{\omega_n^2}{\omega_c^2} - \frac{3}{2} c^2\right)$$

$$\varepsilon = \frac{8\sqrt{3}}{3} c = 4.62\, c$$

Figure 3 shows the Mathieu stability diagram[8] with ε and δ as parameters. It can be seen from the Mathieu chart that for the values of δ around unity i.e., $\frac{\omega_n^2}{\omega_c^2} = \mathcal{O}(1)$ the system may enter the region of instability. However for large values of $(\omega_n/\omega_c)^2$ the co-efficient of z_n in Eqn. (72) is dominated by $(\omega_n/\omega_c)^2$. Hence, in the high frequency range, the elastic modes are essentially governed by the following simple equation:

$$\frac{d^2 z_n}{d\tau^2} + \frac{4}{3}\left(\frac{\omega_n}{\omega_c}\right)^2 z_n \cong 0 \quad \left(\frac{\omega_n}{\omega_c} \gg 1\right)$$

Thus, it can be concluded that for beams with $(\omega_n/\omega_c)^2 \gg 1$ ($n = 1, 2, \ldots \infty$), the elastic motion and the rigid body pitching motion are completely decoupled from each other. However, if $(\omega_n/\omega_c)^2 = \mathcal{O}(1)$, (highly flexible beams) one has to consider Eqn.(71) to study the elastic motion, and thus the elastic motion is coupled with the rigid body pitching motion.

In Section 6, numerical solutions of Eqn. (71) are presented, for some typical values of ω_n/ω_c.

5.1.2. The case of no flexural vibrations: $(\phi_z^{(n)} = 0)$

In this Section, we specialize equations (57) and (58) by restricting the beam to have no bending or flexural vibrations, but allow for the possibility of longitudinal vibrations. With the above assumption and the assumption of circular orbit in a spherically symmetric gravitational field, the following simplifications result,

$$L_{zz}^{(mn)} = L_{zx}^{(mn)} = L_{xz}^{(mn)} = 0 \tag{73}$$

$$M_{31} = \omega_c^{\ 2}\ s\theta\ c\theta \tag{74}$$

$$M_{11} = \omega_c^{\ 2}\ (3\ c^2\theta - 1) \tag{75}$$

Substitution of the above results in equations (57) and (58) leads to, (assuming $C_y = 0$),

$$J_y\ \ddot{\theta} + 2\sum_{n=1}^{\infty} [\dot{A}_n\ (\dot{\theta} - \omega_c) + A_n\ \ddot{\theta}]\ H_{xx}^{(n)} + [J_y + \sum_{n=1}^{\infty} H_{xx}^{(n)}]\frac{\omega_c^2}{2}\ \sin 2\theta = 0 \tag{76}$$

$$\ddot{A}_n + \omega_n^{\ 2}\ A_n - [(\dot{\theta} - \omega_c)^2 + \omega_c^{\ 2}\ (3\ c^2\theta - 1)](\frac{H_{xx}^{(n)}}{M_n} + A_n) = \frac{E_n}{M_n} \tag{77}$$

Equations (76) and (77) are consistent with Ashley's equations (A-8) and (A-9) in Ref. 2, which were derived from an energy approach.

Ashley has further shown in Ref. 1, that if the beam is spinning at a high enough spin rate, ie. $\omega_c/\Omega << 1$, where Ω here represents the spin angular velocity, the response of Eqn. (77) for only the first mode, consists of steady stretching on which can be superimposed an oscillation with the physical frequency of $(\omega_{10}^{\ 2} + 2.93\Omega^2)^{1/2}$ where, ω_{10} is the fundamental frequency of the beam in the longitudinal vibrations assuming no spin.

Ashley[1] has also investigated another special case of equation (77) by assuming $\theta(t)$ = constant. Physically this implies that the beam is forced to maintain a constant orientation with respect to the local vertical. In this case, it can be easily seen that Eqn. (77) reduces to that of an harmonic oscillator with the physical frequency of $[\omega_n{}^2 - \omega_c{}^2 + \omega_c{}^2 (1 - 3c^2\theta)]^{\frac{1}{2}}$. The third term in this frequency expression is due to the gravity-gradient effect. Thus, the gravity - gradient contribution to the frequency changes sign when $c^2\theta = \frac{1}{3}$. For beams with $\omega_n = \mathcal{O}(\omega_c)$, the possibility of buckling instability at $\theta = 0$ is also evident from the frequency expression.

5.2. Three dimensional motion of a long, slender, uniform beam in circular orbit: (assuming yaw to be zero, ie. $\psi = 0$).

This section presents the equations of three dimensional motion of a beam in circular orbit. Some of the following results which were obtained in Section 5.1 also hold for the present case, ie.

$$\bar{\bar{D}}^{(n)} = 0;\ D'_n = 0 \text{ and } H^{(n)}_{\alpha\beta} = H^{(n)}_{\beta\alpha} \qquad (51)$$

$$H^{(n)}_{xz} = H^{(n)}_{zx} = H^{(n)}_{xy} = H^{(n)}_{yx} = H^{(n)}_{yy} = H^{(n)}_{zz} = H^{(n)}_{yz} = H^{(n)}_{zy} = 0 \qquad (78)$$

Also, $J_y = J_z$ and $J_y - J_x \tilde{=} J_x$ for long, slender beams.

Hence, the pitch equation is,

$$J_y\,\dot{\omega}_y - J_y\,\omega_z\,\omega_x + \sum_{n=1}^{\infty} Q_y^{(n)} + C_y \tilde{=} G_{R_y} + \sum_{n=1}^{\infty} G_y^{(n)} \qquad (79)$$

where,

$$\omega_x = (\dot{\theta} - \omega_c)\, s\phi \;;\; \omega_y = (\dot{\theta} - \omega_c)\, c\phi \;;\; \omega_z = \dot{\phi}$$

$$Q_y^{(n)} = 2\,[\dot{A}_n\, \omega_y + A_n\, (\dot{\omega}_y - 2\,\omega_z\,\omega_x)]H_{xx}^{(n)}$$

$$G_{Ry} \,\tilde{=}\, - J_z\, M_{31}$$

$$G_y^{(n)} = - 2M_{31}\, A_n\, H_{xx}^{(n)}$$

The roll equation is,

$$J_z\, \dot{\omega}_z + J_z\, \omega_x\, \omega_y + \sum_{n=1}^{\infty} Q_z^{(n)} + C_z = G_{Rz} + \sum_{n=1} G_z^{(n)} \tag{80}$$

where,

$$Q_z^{(n)} = 2\,[\dot{A}_n\, \omega_z + A_n\, (\dot{\omega}_z + \omega_y\, \omega_z)]H_{xx}^{(n)}$$

$$G_{Rz}^{(n)} \,\tilde{=}\, J_z\, M_{12}$$

$$G_z = 2\, M_{12}\, A_n\, H_{xx}^{(n)}$$

$$M_{12} = - 3\,\frac{\nu a^2}{\rho^3}\, s\phi c\phi c^2\theta + \frac{\nu a^2}{\rho^3} \sum_{s=1}^{\infty} K_s\, \left(\frac{a}{\rho}\right)^s M_{12}^{(s)}$$

The generic mode equation is,

$$\ddot{A}_n + \omega_n^{\,2}\, A_n + \frac{\varphi_n}{M_n} + \frac{1}{M_n} \sum_{m=1}^{\infty} \varphi_{mn} = \frac{1}{M_n}\, (g_n + \sum_{m=1}^{\infty} g_{mn} + E_n) \tag{81}$$

where,

$$\varphi_n = - (\omega_y^{\,2} + \omega_z^{\,2})\, H_{xx}^{(n)}$$

$$\begin{aligned}\varphi_{mn} = {} & 2\,\dot{A}_m\, (\omega_y - \omega_z)\, (L_{zx}^{(mn)} - L_{xz}^{(mn)}) \\ & + A_m\, [(\dot{\omega}_y - \dot{\omega}_z)\, (L_{zx}^{(mn)} - L_{xz}^{(mn)}) + \omega_x(\omega_y + \omega_z)\, (L_{zx}^{(mn)} + L_{xz}^{(mn)}) \\ & + (2\,\omega_y\,\omega_z - 2\,\omega_x^{\,2} - \omega_y^{\,2} - \omega_z^{\,2})\, L_{zz}^{(mn)} - (\omega_y^{\,2} + \omega_z^{\,2})\, L_{xx}^{(mn)}]\end{aligned}$$

$$g_n = M_{11}\, H_{xx}^{(n)}$$

$$\begin{aligned}g_{mn} = {} & A_m\, [L_{xx}^{(mn)}\, M_{11} + (M_{22} + M_{33} + 2M_{23})\, L_{zz}^{(mn)} \\ & + (L_{zx}^{(mn)} + L_{xz}^{(mn)})\, (M_{12} + M_{13})]\end{aligned}$$

5.2.1 The case of no longitudinal vibrations: ($\phi_x^{(n)} = 0$).

In this case, the following further simplifications result, i.e.

$$H_{xx}^{(n)} = L_{zx}^{(mn)} = L_{xz}^{(mn)} = L_{xx}^{(mn)} = 0 \tag{82}$$

Furthermore, if we choose the eigen modes of bending vibrations of a free-free beam to be $\phi_z^{(n)}$, then,

$$L_{zz}^{(mn)} = \delta_{mn} M_n. \tag{83}$$

Assumption of a spherically symmetric gravity field results in,

$$\begin{aligned} M_{12} &= -3\,\omega_c^2\ s\phi\ c\phi\ \ c^2\theta;\ M_{22} = (3\ s^2\phi\ c^2\theta - 1)\omega_c^2 \\ M_{23} &= -3\,\omega_c^2\ s\phi\ c\theta\ \ s\theta\ \ ;\ M_{33} = (3\ s^2\theta - 1)\omega_c^2 \\ M_{31} &= 3\,\omega_c^2\ \ c\phi\ c\theta\ s\theta \end{aligned} \tag{84}$$

With the above results substituted in equations (79), (80) and (81), one obtains the following equations of:

Pitch:

$$\dot{\omega}_y - \omega_z\,\omega_x + M_{31} + C_y/J_y = 0 \tag{85}$$

Roll:

$$\dot{\omega}_z + \omega_x\,\omega_y - M_{12} + C_z/J_z = 0 \tag{86}$$

Generic mode:

$$\ddot{A}_n + (\omega_n^2 + 2\,\omega_y\,\omega_z - 2\,\omega_x^2 - \omega_y^2 - \omega_z^2 - M_{22} - M_{33} - 2\,M_{23})\,A_n = \frac{E_n}{M_n} \tag{87}$$

where,

$$\omega_x = (\dot{\theta} - \omega_c)\ s\phi$$

$$\omega_y = (\dot{\theta} - \omega_c)\ c\phi$$

$$\omega_z = \dot{\phi}$$

The assumption of small amplitude pitch and roll, i.e. $\theta \ll 1$ and $\phi \ll 1$, yields the following set of linearized equations for pitch and roll, after neglecting the terms containing the products and powers of ϕ, $\dot{\phi}$, θ and $\dot{\theta}$.

Pitch:
$$\ddot{\theta} + 3\,\omega_c^{\,2}\,\theta + \frac{C_y}{J_y} = 0 \tag{88}$$

Roll:
$$\ddot{\phi} + 3\,\omega_c^{\,2}\,\phi + \frac{C_z}{J_z} = 0 \tag{89}$$

The generic mode equation is,

$$\ddot{A}_n + [\omega_n^{\,2} + 2\,\dot{\phi}\,(\dot{\theta} - \omega_c) - (\dot{\theta} - \omega_c)^2 - \dot{\phi}^2 + 2\,\omega_c^{\,2}]A_n = \frac{E_n}{M_n} \tag{90}$$

5.2.2 The case of no flexural vibrations ($\phi_y^{(n)} = \phi_y^{(n)} = 0$):

In this case, the assumption of no flexural vibration results in,

$$L_{zz}^{(mn)} = L_{zx}^{(mn)} = L_{xz}^{(mn)} = 0 \tag{91}$$

Also, the assumption of $\phi_x^{(n)}$ to be the eigen-modes of longitudinal vibration of the beam leads to the result,

$$L_{xx}^{(mn)} = \delta_{mn}\,M_n \quad (m = 1,2,\,\ldots\,\infty) \tag{92}$$

As in Section 5.2.1, the assumption of a spherically symmetric gravity field, yeilds

$$\begin{aligned}
M_{11} &= (3\,c^2\phi\;c^2\theta - 1)\omega_c^{\,2}\;;\; M_{12} = -\,3\,\omega_c^{\,2}\;s\phi\;c\phi\;\;c^2\theta\\
M_{22} &= (3\,s^2\phi\;c^2\phi - 1)\omega_c^{\,2}\;:\; M_{23} = -\,3\,\omega_c^{\,2}\;s\phi\;c\theta\;s\theta\\
M_{23} &= 3\,\omega_c^{\,2}\;c\phi\;c\theta\;s\theta\quad ;M_{33} = (3\,s^2\theta - 1)\omega_c^{\,2}
\end{aligned} \tag{93}$$

For this case, the pitch and roll equations remain the same as equations (79) and (80). The generic mode Eqn., (81), simplifies to the following form:

$$\ddot{A}_n + \omega_n^2 A_n + \frac{\varphi_n}{M_n} + \frac{1}{M_n} \sum_{n=1}^{\infty} \varphi_{mn} = \frac{1}{M_n} (g_n + \sum_{m=1}^{\infty} g_{mn} + E_n) \tag{94}$$

where,

$$\varphi_n = - (\omega_y^2 + \omega_z^2) H_{xx}^{(n)}$$

$$\varphi_{mn} = - A_m [\omega_y^2 + \omega_z^2] M_n \delta_{mn}$$

$$g_n = M_{11} H_{xx}^{(n)}$$

$$g_{mn} = A_m M_{11} M_n \delta_{mn}.$$

6. NUMERICAL RESULTS

In this section, the numerical solutions of equation (71), for a few typical values of $(\omega_n/\omega_c)^2$ shown in Table -1, are presented. Cases 1-5 correspond to very flexible beams. Eqn. (71) was integrated, using a Runge-Kutta fourth order method with variable step size, on a Nova 840 digital computer. As an initial guess a step size of approximately $^1/500$th of the orbital period was chosen for all the computer runs. A pitch amplitude of 0.2 radians was assumed, which is the upper limit for θ for which the approximation $\sin\theta \cong \theta$ is still valid.

The responses shown in Figs. 5 and 6 for the value of $(\omega_n/\omega_c)^2 = 1.0$ indicate the instability of the system at very low natural frequencies. Point 1 corresponding to this case on the Mathieu stability diagram in Fig. 4 also indicates this instability. The other points on the Mathieu stability diagram correspond to other cases shown in Table 1 and are referred to by the appropriate case number. For the other values of $(\omega_n/\omega_c)^2$ considered, the beam response is sinusoidal to a very close approximation, see Figs. 7, 8, 9 and 10. Phase plane plots of cases 1, 2, 3, 4 are shown in Figs. 11, 12, 13, and 14, respectively.

It is to be noted however, that when the pitch amplitude exceeds 0.2 radians, one has to use equations (63) and (64) for simulation. The development of a general computer program which treats the combined pitch and generic modal equations will be discussed in Section 6.1.

Case No.	Assumed data				Comments
	$(\omega_n/\omega_c)^2$	c**	$z_n(0)$	$dz_n(0)/d\tau$	
1.	1.0	0.2	0.5×10^{-4}	0.0	Range of τ: 0 to 11
2.	1.0	0.05	0.5×10^{-4}	0.0	Range of τ: 0 to 11
3.	2.0	0.2	0.5×10^{-4}	0.0	Range of τ: 0 to 11
4.	5.0	0.2	0.5×10^{-4}	0.0	Range of τ: 0 to 5.5
5.	10.0	0.2	0.5×10^{-4}	0.0	Range of τ: 0 to 5.5
6.*	3200	0.2	0.5×10^{-4}	0.0	Range of τ: 0 to 1.1

* This case corresponds to a beam with a fundamental natural frequency of $^1/_{100}$ cps and moving in a circular orbit of 250 n. miles altitude.

** c - Pitch amplitude in radians.

Table 1: Data assumed for numerical simulation

6.1 The Flexbeam Computer Program

A. Development of Computer Program

The first order non-linear equations of motion for the long-flexible beam in orbit as developed in section 5, Eqs. (57) and (58), were coded for numerical simulation using the Nova 840 digital computer system. The program, Flexbeam, consists of nine subprograms whose names and functions are:

FBSET - (a) set up initial state, (b) an input quantity, Precision, dependent upon the desired precision (range 1.0 to 10^{-5}); the size of this input parameter is inversely proportional to the desired accuracy (and also the number of iterations required per computational time step).

FBHST - set $H_{ij}^{(n)}$ where $n = 1, 2, \ldots, M$, $M_{max} = 20$ modes

FBLST - set $L_{ij}^{(r,n)}$ where $r = 1, 2, \ldots, M$, and $n = 1, 2, \ldots, M$.

FBDIF - set differential equations which include primary and secondary functions

RKSCL - (a) set full scale values of the initial conditions: ϕ_0, $\dot{\phi}_0$, An_0, and $\dot{An}_0$, (b) set the number of ordinary differential equations to be evaluated, (c) set the bounds on the maximum number of iterations (11) that the Runge-Kutta subroutine is allowed per time step in order to fulfill the desired precision specified in FBSET. If more than this number of iterations would be required the Runge-Kutta subroutine is automatically terminated.

In this case, the precision parameter or full-scale values would have to be adjusted.

RKGS - Uses the Runge-Kutta method to obtain an approximate solution of the system of first order differential equations, given initial conditions.

FBPIF - calculation of primary intermediate functions, i.e. M_{11}, M_{13}, R_2, $\sin\phi$ and $\cos\phi$, etc.

FBSIF - calculation of secondary intermediate functions, i.e. ϕ_r, g_r, ϕ_{rn} and g_{rn}

FBOUT - sets the scale magnitudes of the output variables ϕ, $\dot{\phi}$, A_n, and $\dot{A}_n$, equal to the maximum expected amplitudes for use in the plot routine.

The flow diagram, Fig. 15, depicts the various subprograms and how input information, as well as the resulting output of each subprogram, is passed through the Flexbeam main program.

B. Flexbeam Usage

Flexbeam is a computer simulation of the dynamics of a flexible beam in a circular orbit about the earth. The programming language used in the main program is Fortran Five. This language allows the results to be obtained faster than the same results obtained, using Fortran Four. (The Flexbeam out-put program is coded in Fortran Four.) Flexbeam, contains three principal components: Flexbeam Main Program, (refer to Fig. 15), Flexbeam Input Data and Flexbeam Output Program.

Inside the main program the subroutine FBINP is associated with Flexbeam Input Data. The purpose of this subroutine is to specify the order and format with which the input data is read into the computer. Block data, also inside the main program, sets the constant parameters, i.e. gravitational constant, NU, orbit frequency, WD, external forces, QCP, C2, and terms involving the higher harmonics of the earth's gravitational field, J2 and J3, to their full-scale values; also constant parameters associated with each mode are set, i.e. the modal frequency, DW. The number of modes (M) and the number of differential equations being considered, NORD, are also specified here.

Flexbeam Input Data is the component program that allows the main program to be executed. The first input of the first data card denotes the total time interval over which the numerical integration is to be performed, e.g. 0.0 to 300.0 sec. The second input on this card is the computational time step, 0.5 sec. The third input is the precision, 1.0. The second data card contains the initial conditions: ϕ_0, $\dot{\phi}_0$ and their scale values. The remaining data cards contain An_0, $\dot{A}n_0$, their scale values and the modal frequency, in that order for each mode being considered. Sample data cards are shown in Fig. 16 and input data, for a specific case, is shown on page 12 of the program listing (Appendix V - 12).

The Flexbeam Output Program prints and/or plots any and/or all of the following: pitch angle (and pitch rate), modal amplitudes (and rates), and deflections taken at particular points along the undeformed beam, all as functions of time.

(The designated points at which the deflections are to be calculated are specified according to the position of the particular point from the left end of the beam, non-dimensionalized by the total beam length.) The desired outputs can be implemented by placing the "amplitude vs. time" deck of cards, (page V - 17 of the program listing), "print out-put" deck of cards, (page V - 16) of the program listing), or the "call to deflection plot (DFPLO)" subroutine, (page V -13 of the program listing), into the main part of this program which is also on page V - 13.

6.2 Numerical results of Flexbeam Program

The results obtained from the computer simulation of the dynamics of a flexible beam in orbit using the Flexbeam Program will be considered. The beam is assumed to be a long slender hollow tube made of wrought aluminum (2014T6). The length of the beam is taken as 100 meters, its outside diameter and thickness are: 0.05 meters and 0.005 meters, respectively. The structural rigidity (EI) of the beam is 7.707×10^3 nt - m^2 and the mass per unit length is 9.906×10^4 kg/m. It is further assumed that the c.m. of the system follows a 250 n.mi (463.31 km) altitude circular orbit.

The initial conditions which remained constant throughout all simulation runs are: $\phi(0) = 0.2$ rad., $\dot{\phi}(0) = 0.0$ rad./sec., $A_n(0) =$ 0.5 meters (maximum value) and $\dot{A}_n(0) = 0.0$ meters/sec. Other parameters, i.e. the time interval of the numerical simulation and the generic modal frequencies, were varied. Variations in these parameters and their effect upon the deflection of the beam throughout its entire length will be discussed.

The first two cases of the simulation were constructed to test the program. The first, depicted in Fig. 17, is a simulation of pitch motion with a superposition of the first and second generic modes, (refer to Table 2). Each mode is assumed to have an initial amplitude coefficient of 0.5 meters. The deflection is calculated using Eq. (11), where $\bar{\Phi}^{(n)}(\bar{r}_0)$ has only one component, $\phi_z^{(n)}\hat{k}$.

The deflected beam was examined at three locations, the left and right nodal points of the first mode and the central nodal point of the second mode, (Table 2). In Figs. 17a and c, we observe the predominant effect of the frequency response of the second mode, whereas in Fig. 17b, that of the first mode. Discrepancies between the responses shown in Fig. 17 and purely simple harmonic motion at a particular modal frequency, are attributed to: (1) small numerical errors in calculating the exact location of the nodal points and (2) nonlinearities associated with pitch-rate coupling in the generic modal equations. In the second case, Fig. 18, the frequency of the first mode is set equal to the orbital frequency. The frequencies of modes two and three are calculated based on criterion consistent with free-free beams, (Table 2). Figs. 18a and b. depict the responses of the first generic mode and then all three generic modes, respectively. In each figure, we note the growth of the amplitude of the first mode due to orbital resonance as simulated. Fig. 19 is associated with Fig. 18b. This figure illustrates the deflection of the entire beam for a one hundred second time interval. This interval was chosen to show the dominating effect of the first mode during this part of the response. It should be noted that the case shown in Fig. 18 for the response of the first generic mode, duplicates the earlier result described in Fig. 5, after accounting for the nondimensionalization of A_1 and noting that the initial conditions in ϕ and $\dot{\phi}$ shown in Fig. 18a, will result in a pitch amplitude of 0.2 radians.

The third and remaining cases of the simulation shown in Figs. 20 - 29, all depict the following plots; modal amplitude and deflection vs. non-dimensional beam length as functions of time. The modal amplitude responses are used to study the interaction of the modes superimposed upon one another; first with initial values of the mode amplitudes equally weighted, i.e. An(0) = 0.5 meters (Fig. 20), then unequally weighted, i.e. Figs. 22, 25a, and 27. Figs. 22 and 25a can be compared since the initial conditions are similar. It can be observed that with the frequencies chosen for the first generic mode, $\omega_1 = 0.0628$ in Fig. 22 and $\omega_1 = 0.628$ in Fig. 25a. which represents $^1/100$ and $^1/10$ cycles per second, respectively, the responses shown, illustrate the effect of the greater rigidity in the latter case. The deflection plots are used to study the deflection of the entire beam during specific time intervals. As a particular example, we consider Figs. 20 and 21 jointly, to study the effect of the superimposition of the different modes at different intervals. We note that at the beginning of the simulation the signs of all three modal amplitudes ($A_1 - A_3$) are positive. Figures 21a. and b. show the deflection of the beam throughout the length of the entire beam during the first 21 seconds. From Table 2, it is observed that the deflections at the left end of the beam due to the first three (or more) generic modes, will be additive providing the modal amplitude factors have the same sign. This phenomena is apparent in Fig. 21a, where the initial larger deflection at the left end of the beam should be noted.

The first and second modal amplitudes have negative values between 42.5 and 48 secs. whereas the third modal amplitude has a positive value, (Fig. 20). It should also be noted that the first modal amplitude reaches a maximum negative value at 45 secs. During this time period, the dominating influence of the first mode in the deflection response (Fig. 21c), is apparent. The contributions of the second and third mode tend to compensate each other.

The remaining amplitude and deflection responses showing the effect of different initial conditions and the numbers of modes in the model can be examined in a similar manner and will be useful in the forthcoming simulation of the free-free beam under the action of various control devices.

Figure 30 shows a typical response of the pitch motion of the beam for a simulation which involved only one generic mode. Since the pitch motion is not coupled to first order with the generic modal motion [Eqns. (88) and (90)], this type of response is representative of all pitch motions simulated for small pitch amplitudes.

For all numerical cases reported here, an average of 10-12 minutes computational time was required to simulate the dynamics over an 80 minute interval of real time using the NOVA 840 computer system.

7. CONCLUSIONS

This report presents the development of the equations of motion of an arbitrary flexible body in orbit. In the case of planar motion of a long, slender beam in a circular orbit, undergoing small pitch oscillations and flexural vibrations, the pitch motion completely decouples from the elastic motion and the elastic motion is governed by a Hill's 3 - term equation. For large values of $(\omega_n/\omega_c)^2$ the elastic motion completely decouples from the pitch motion and the elastic motion closely approximates that of an harmonic oscillator. However, the numerical results indicate the possibilities of instabilities at very low values of $(\omega_n/\omega_c)^2$. A general computer program which treats both the pitching motion and the first-order flexural vibrations of a thin beam in orbit is developed; this program can be modified to simulate the effects of both external environmental torques and control torques that may be provided by actuators located at various positions along the beam.

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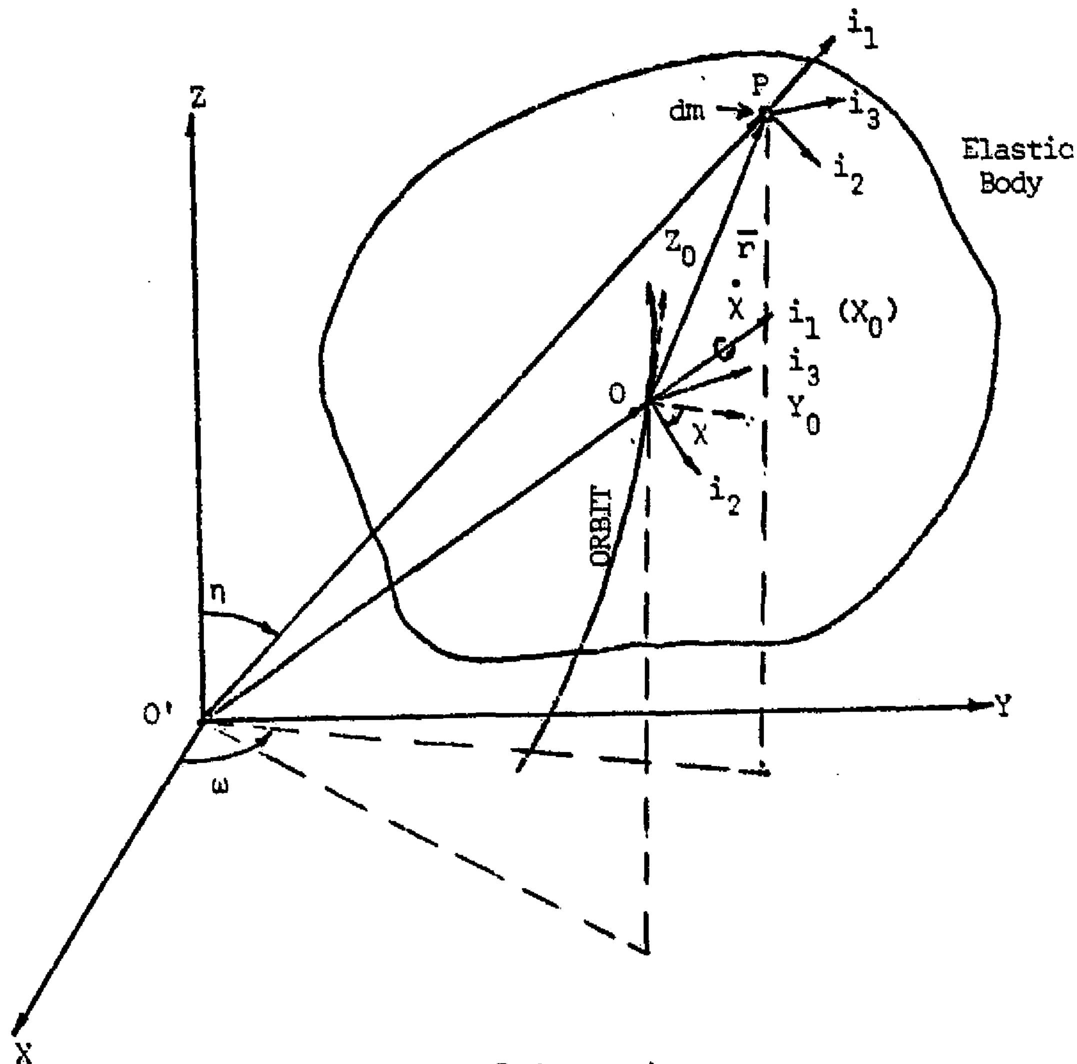
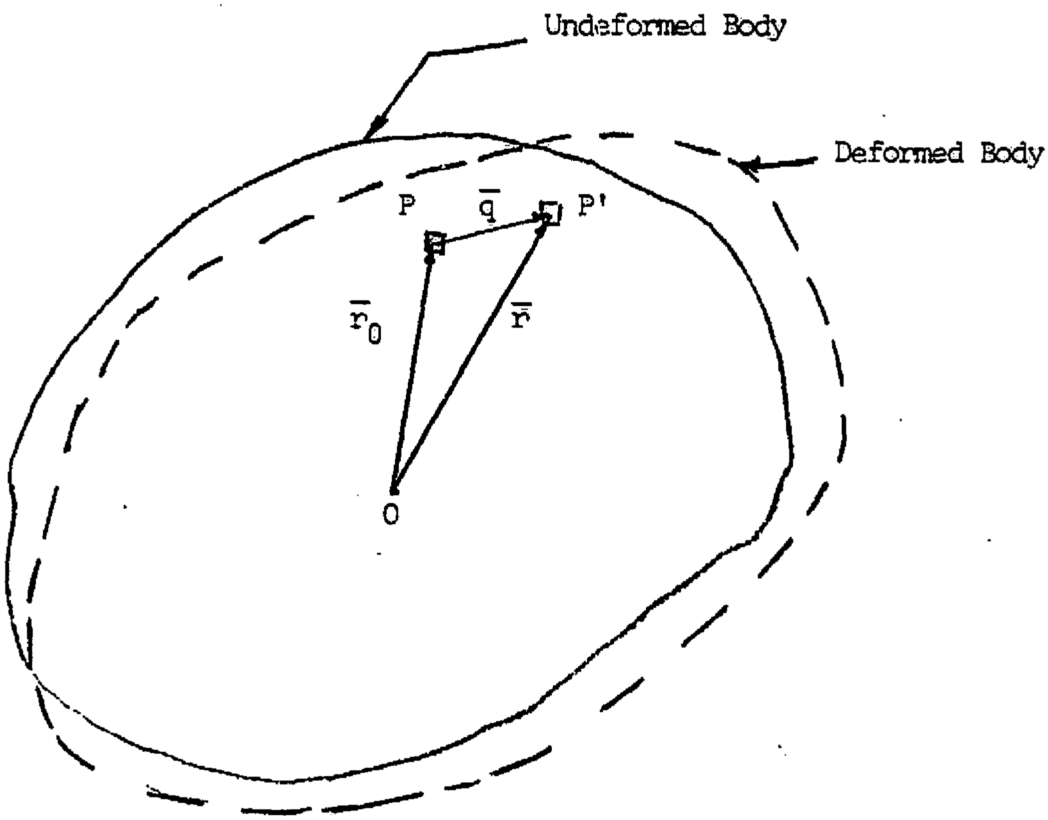


O' - Center of the earth

O - Center of mass of the body

P - An arbitrary point in the body

Fig. 1: Co-ordinate Frames



O - Center of mass

$\bar{r}_0$ - Position vector of P before deformation

$\bar{r}$ - Position vector of P after deformation

$\bar{q}$ - Elastic displacement

Fig. 2: Deformed and Undeformed Configurations of the Body.

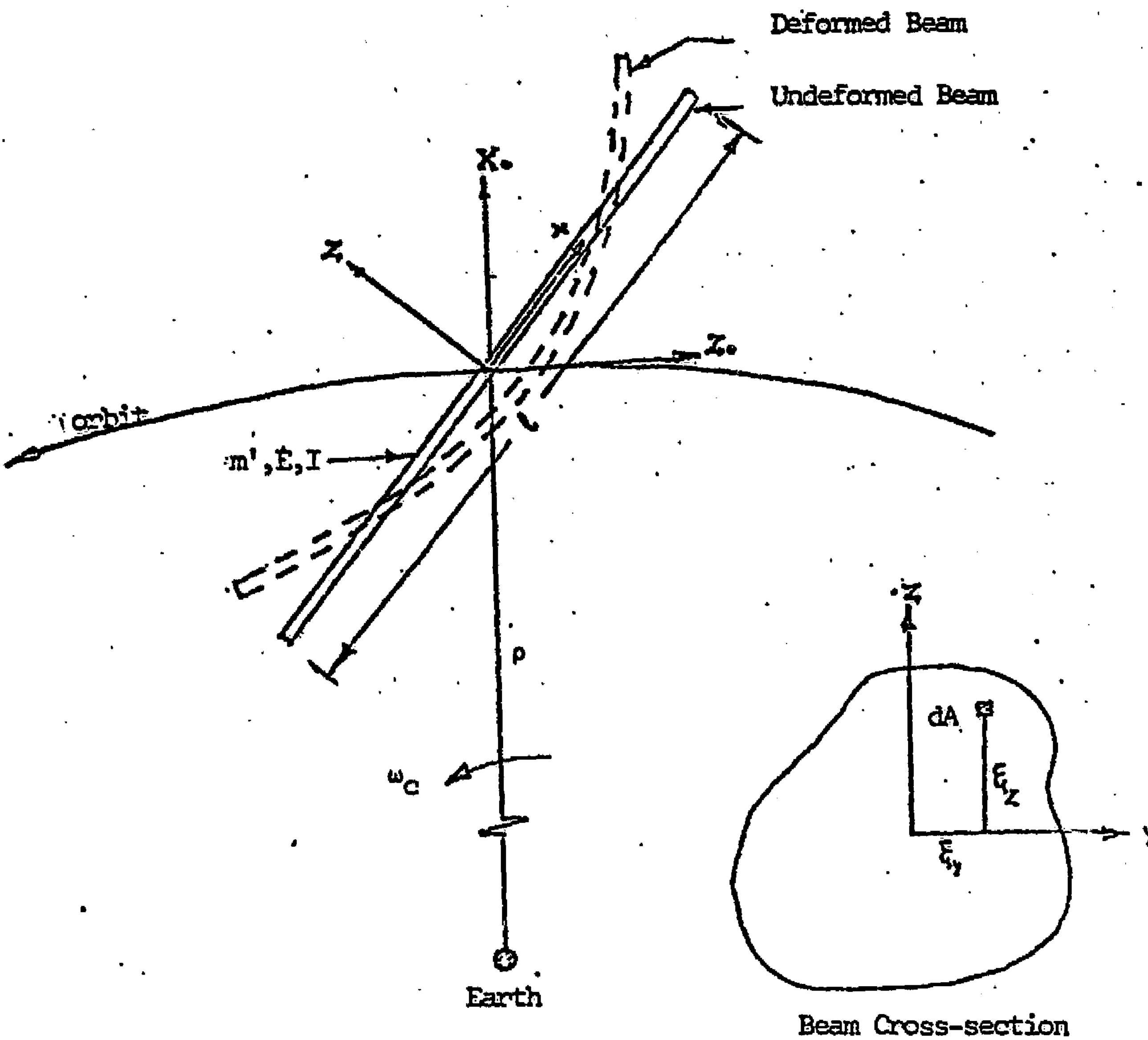


Fig. 3 : Beam Satellite

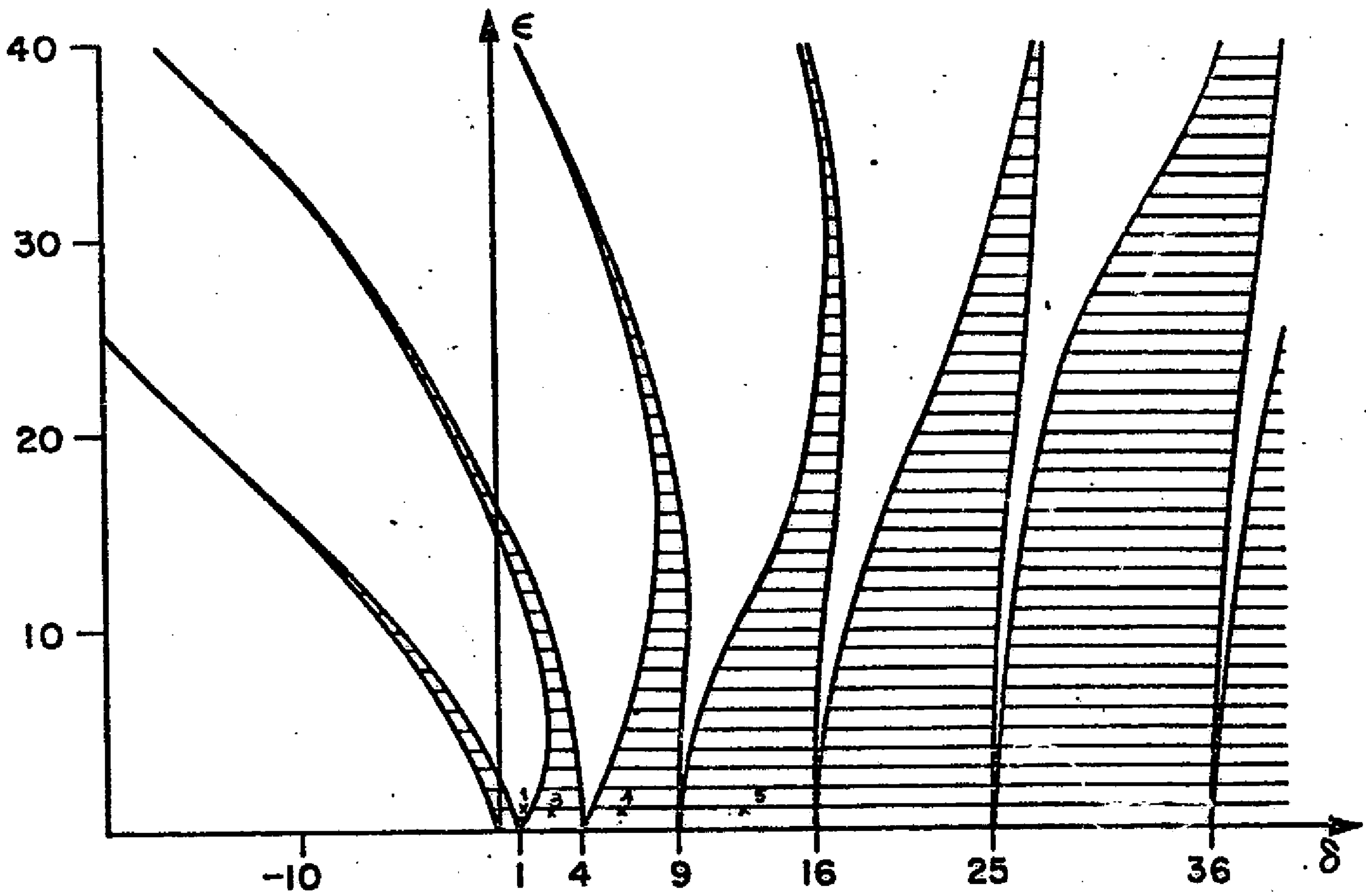


Fig. 4: Mathieu Stability Diagram
(Shaded regions indicate stability)

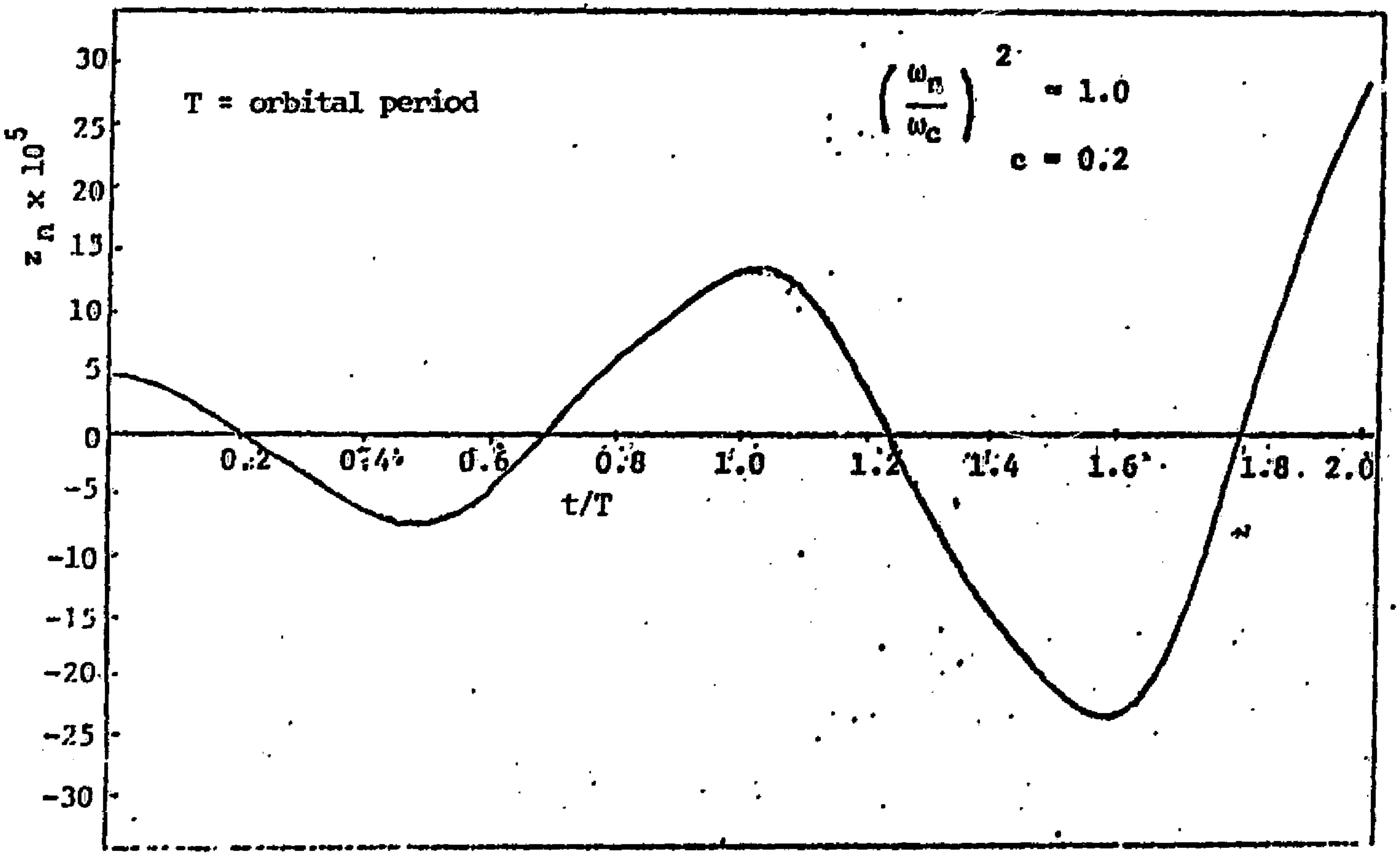


Fig. 5 : Modal Amplitude Response - Very Flexible Beam

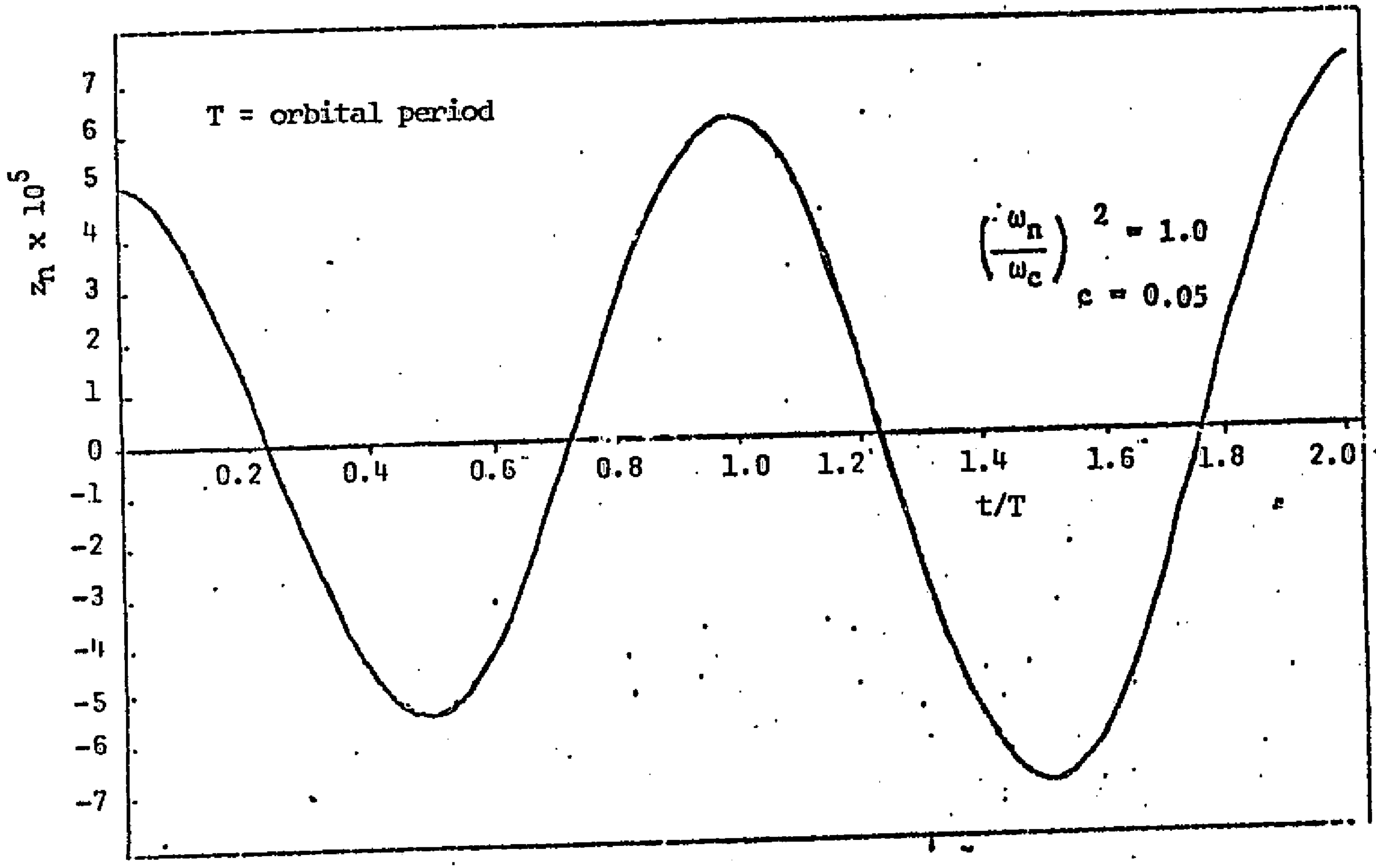


Fig. 6: Modal Amplitude Response - Very Flexible Beam

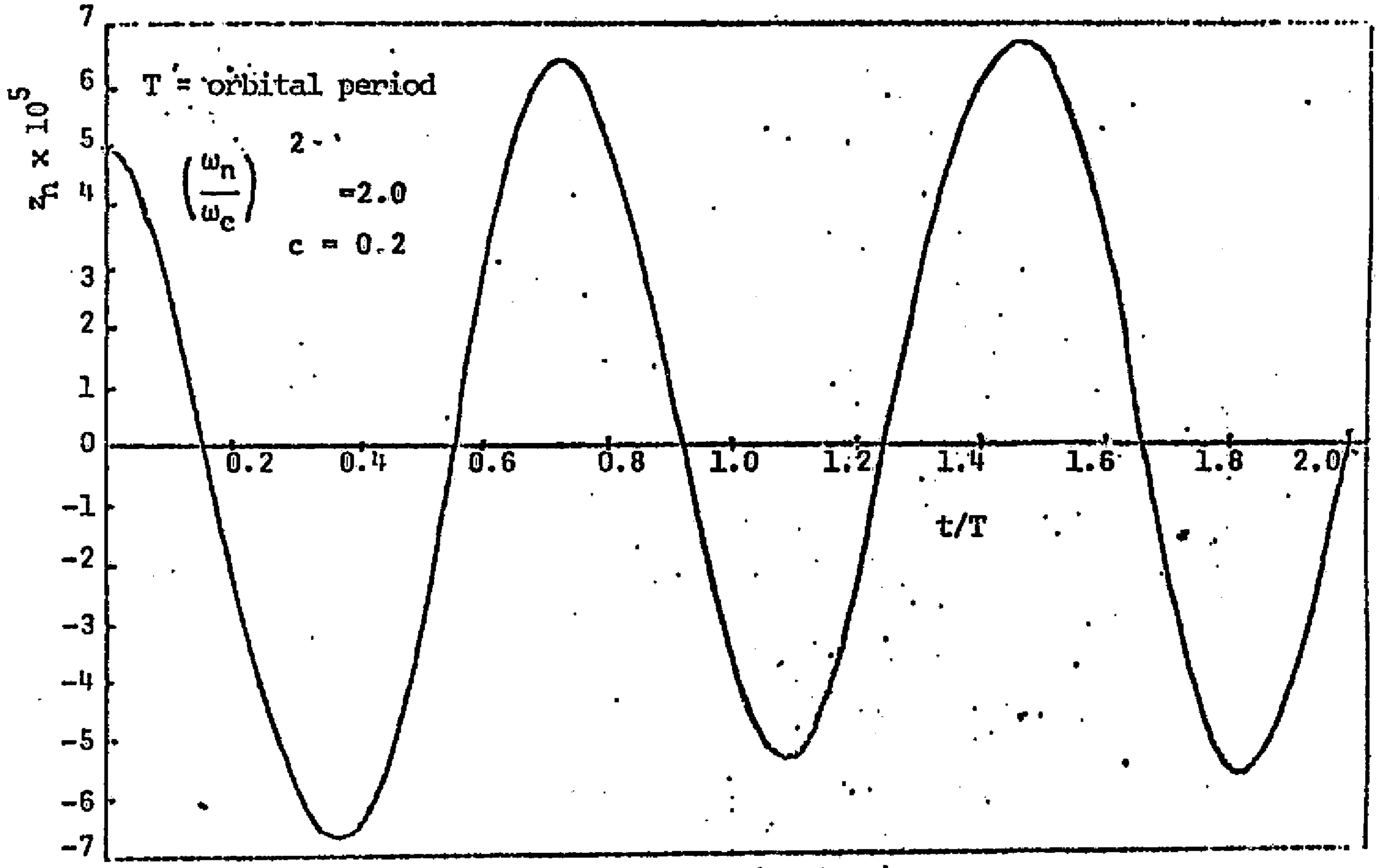


Fig. 7: Modal Amplitude Response - Effect of Increased Beam Stiffness

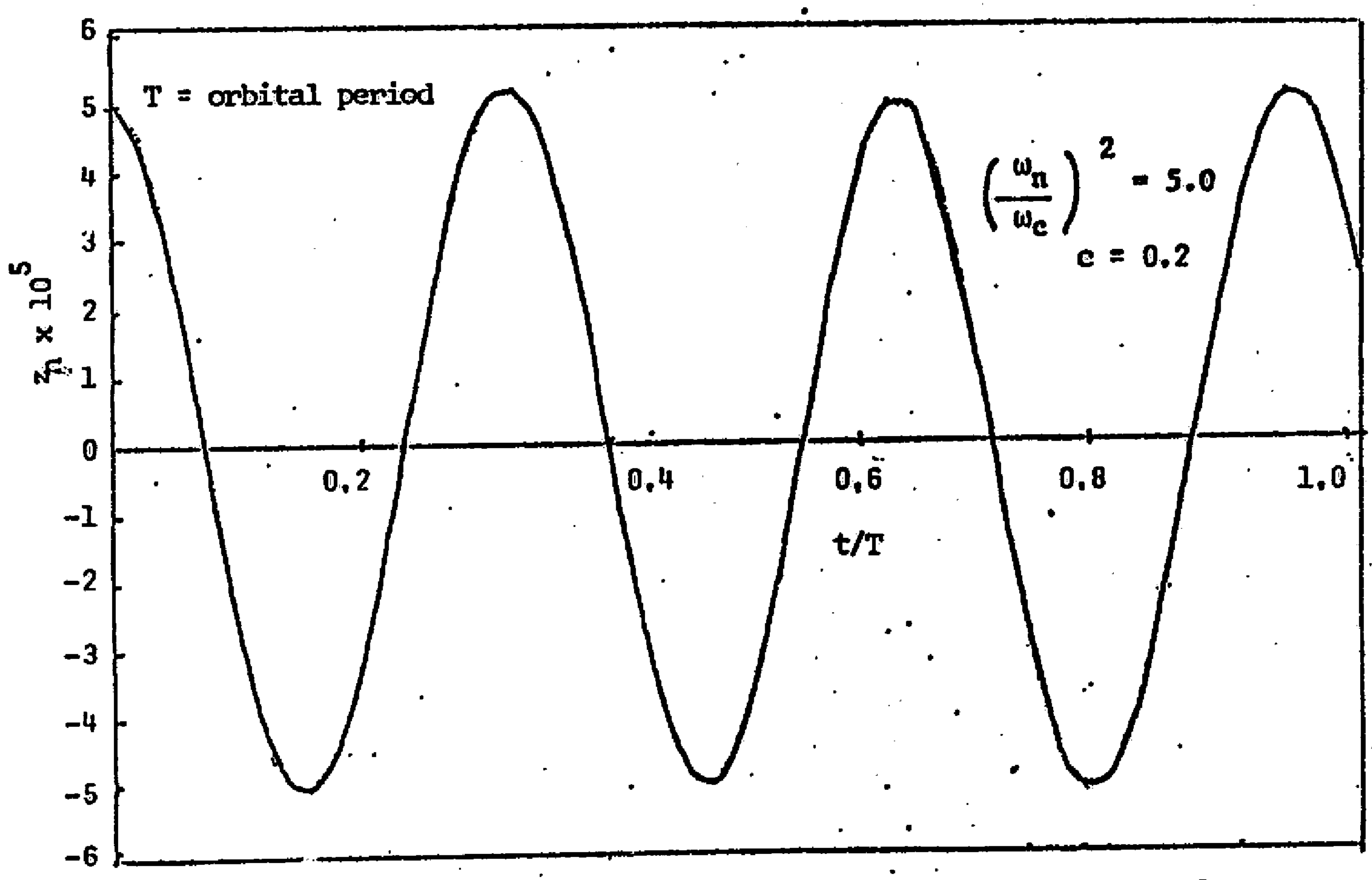


Fig. 8: Modal Amplitude Response - Effect of Increased Beam Stiffness

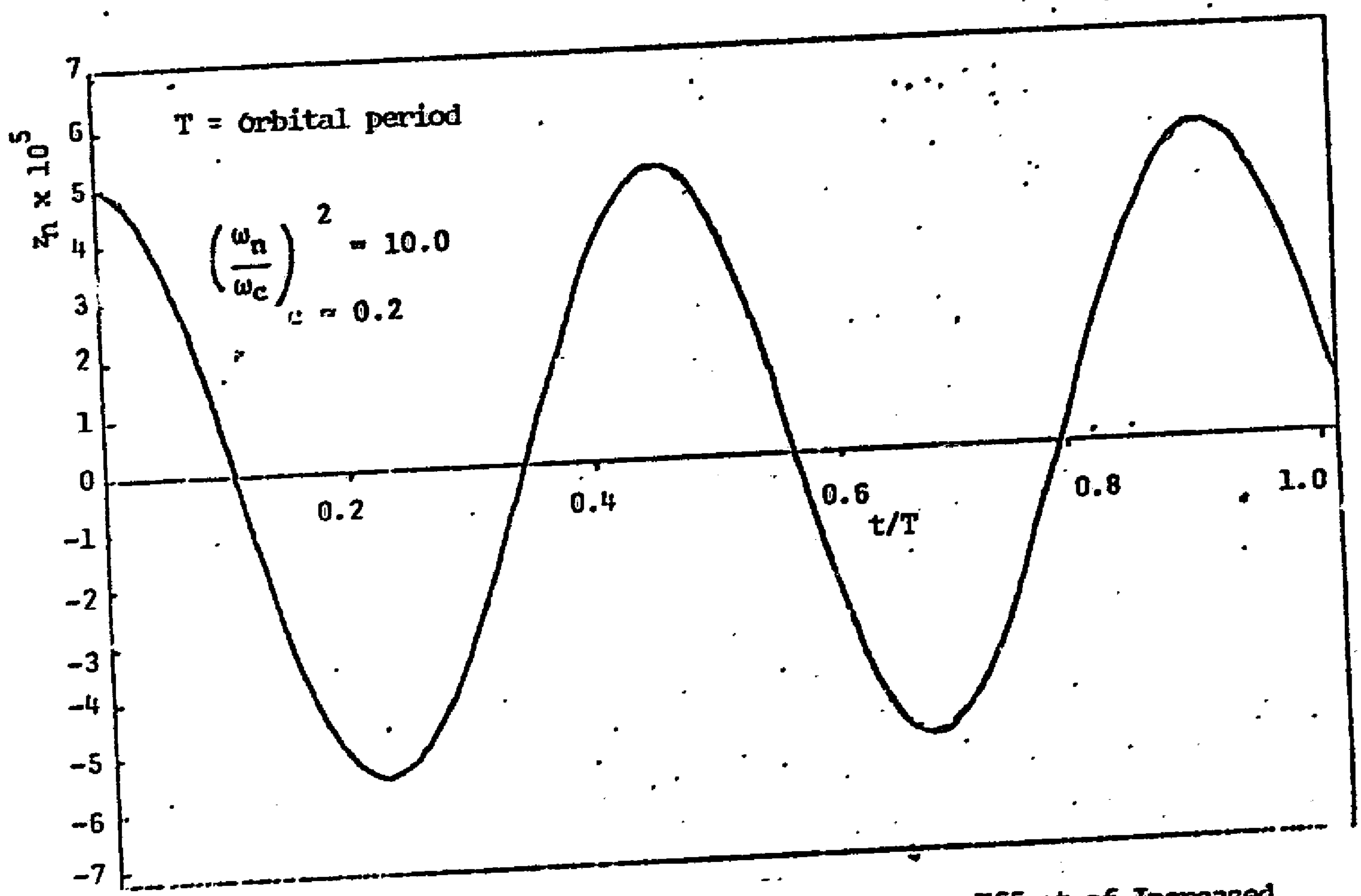


Fig. 9: Modal Amplitude Response - Effect of Increased Beam Stiffness

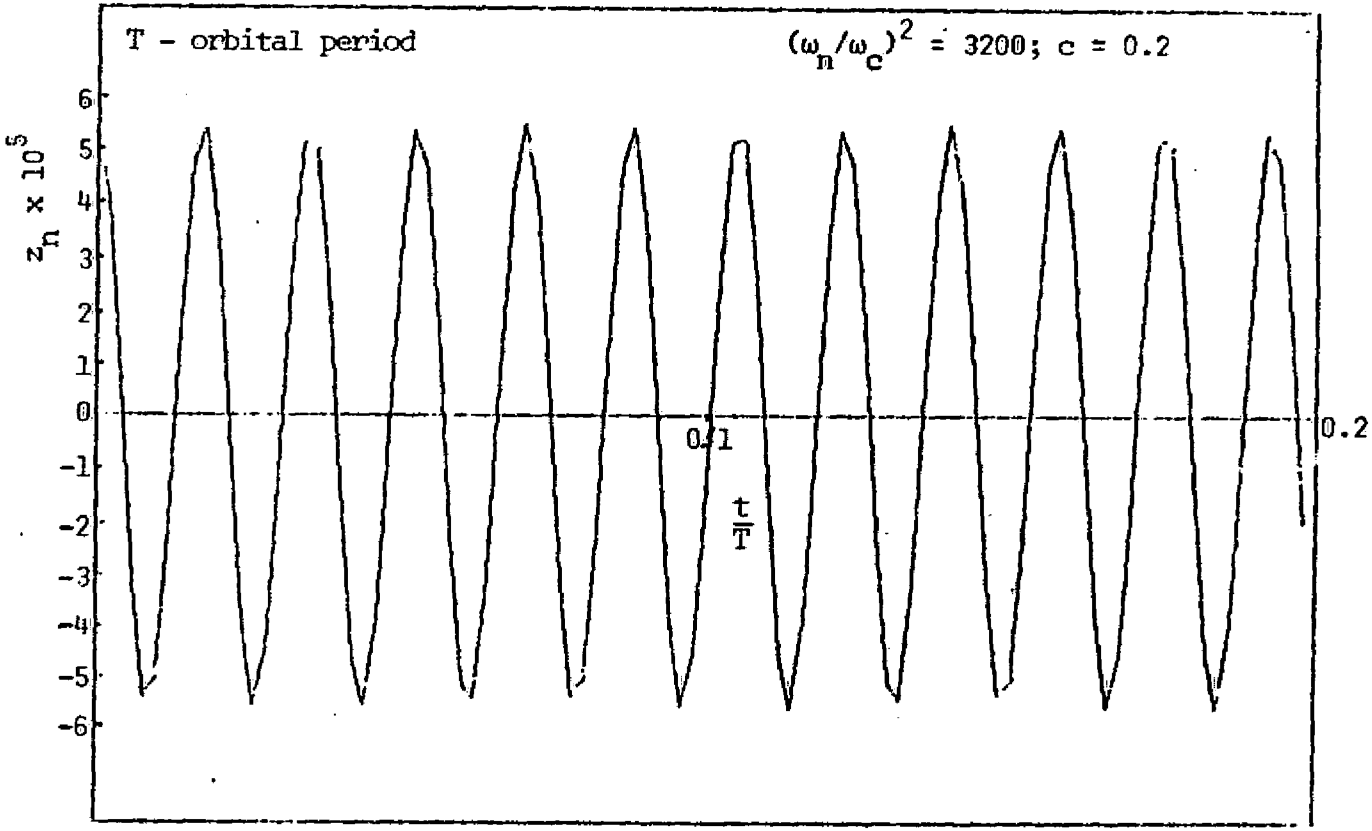


Fig. 10: Modal Amplitude Response - Effect of Increased Beam Stiffness.

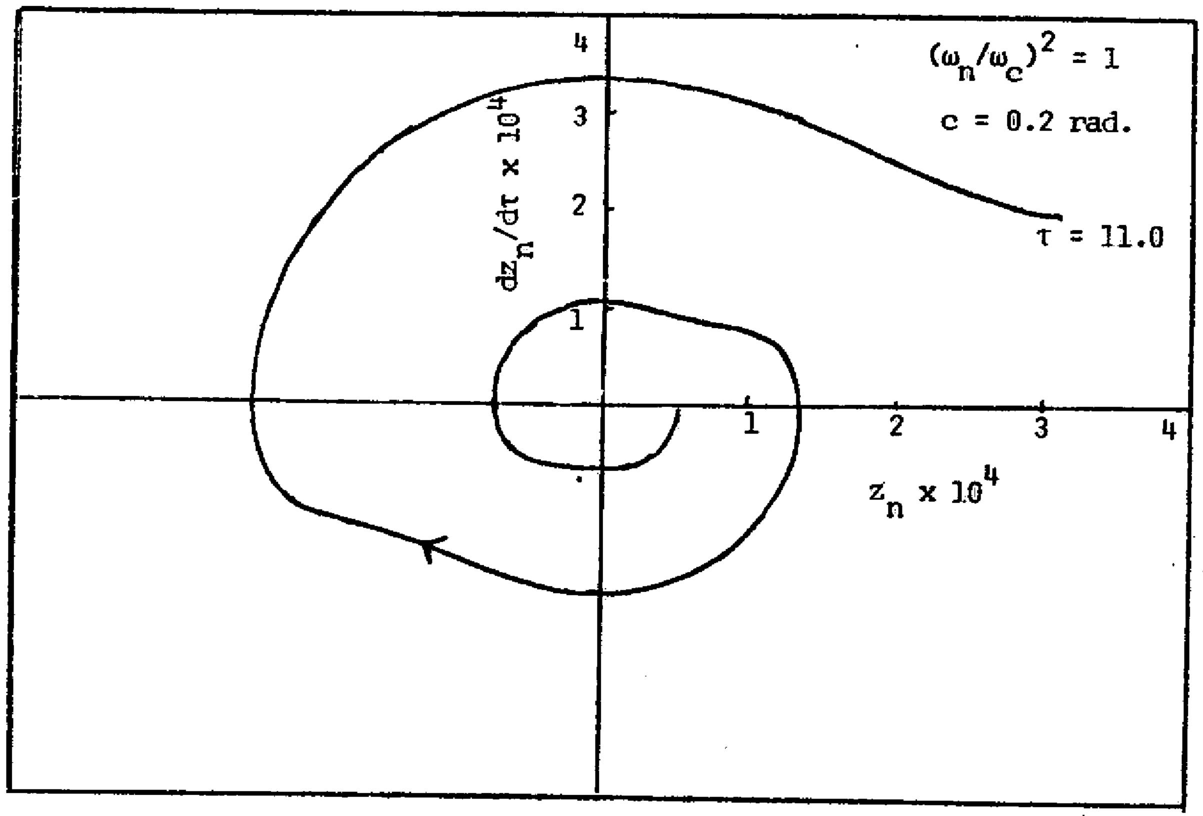


Fig. 11: Phase Plane Plot (case 1)

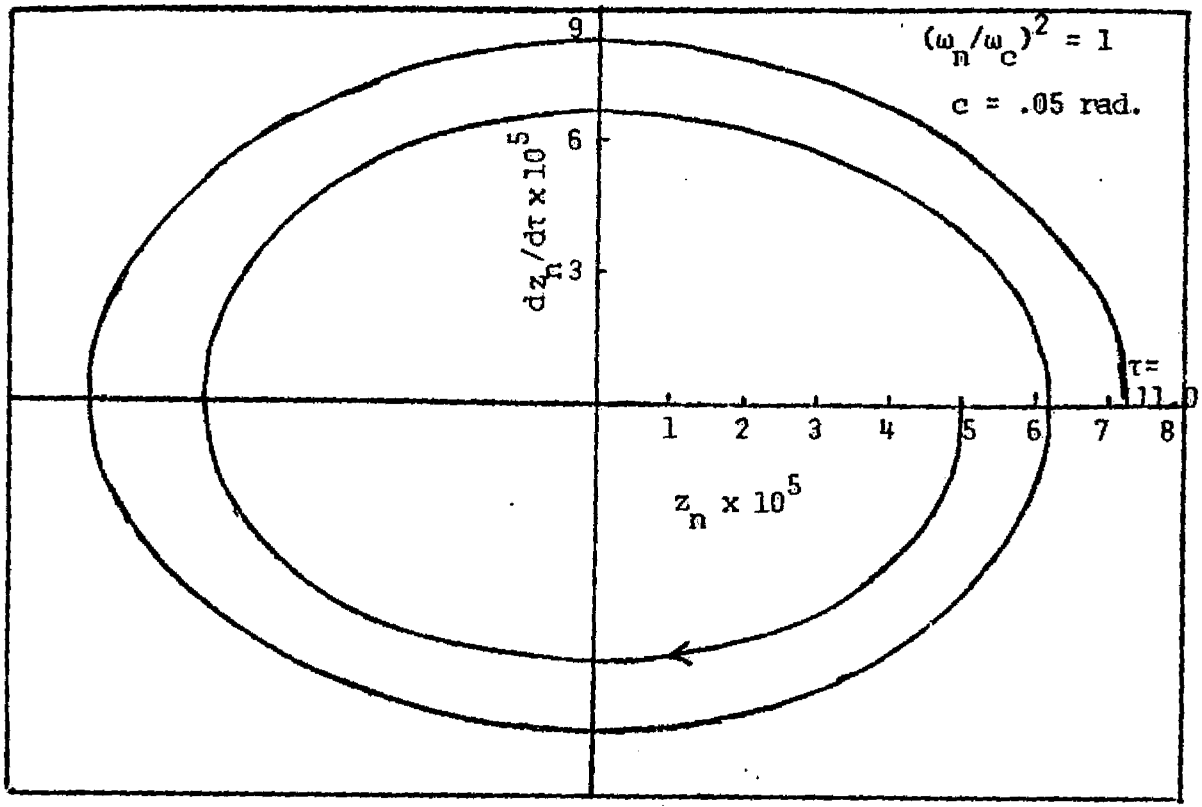


Fig. 12: Phase Plane Plot (case 2)

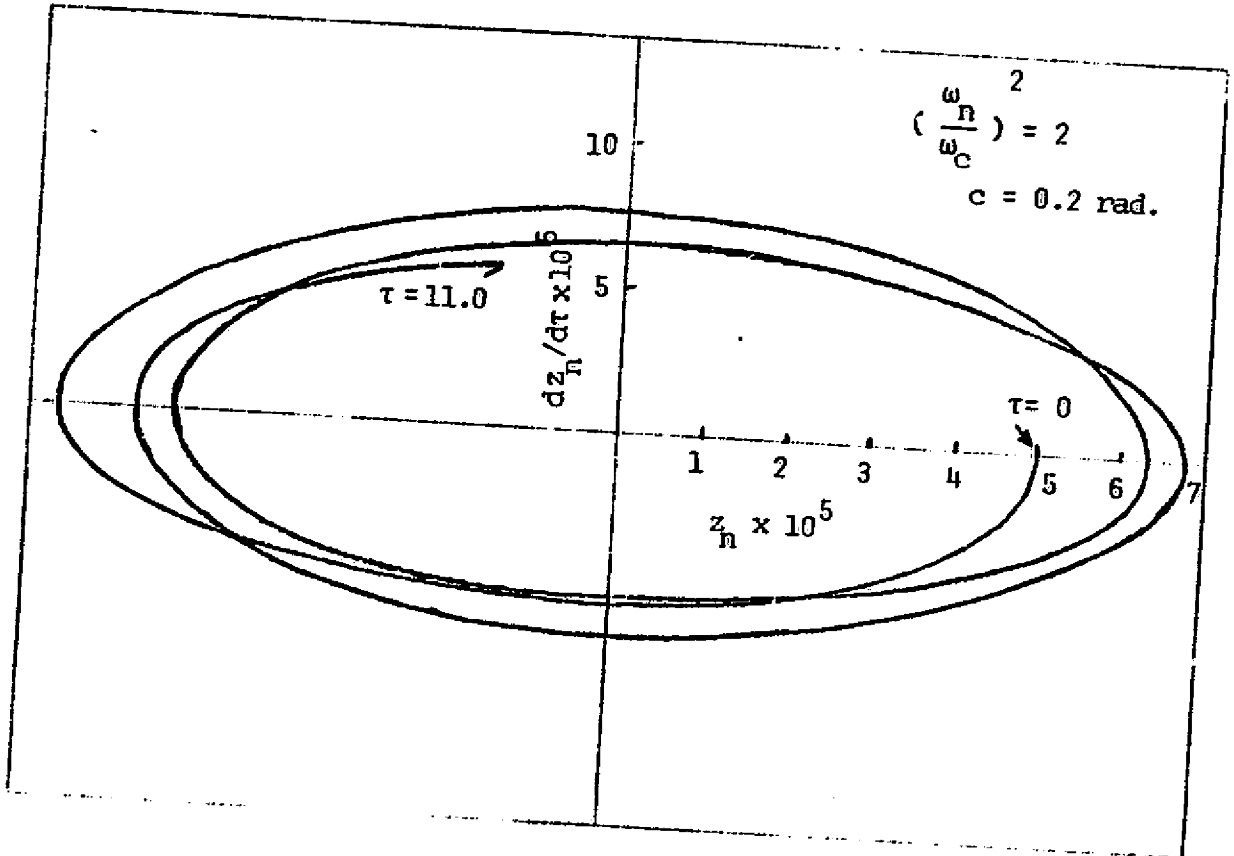


Fig. 13: Phase Plane Plot (case 3)

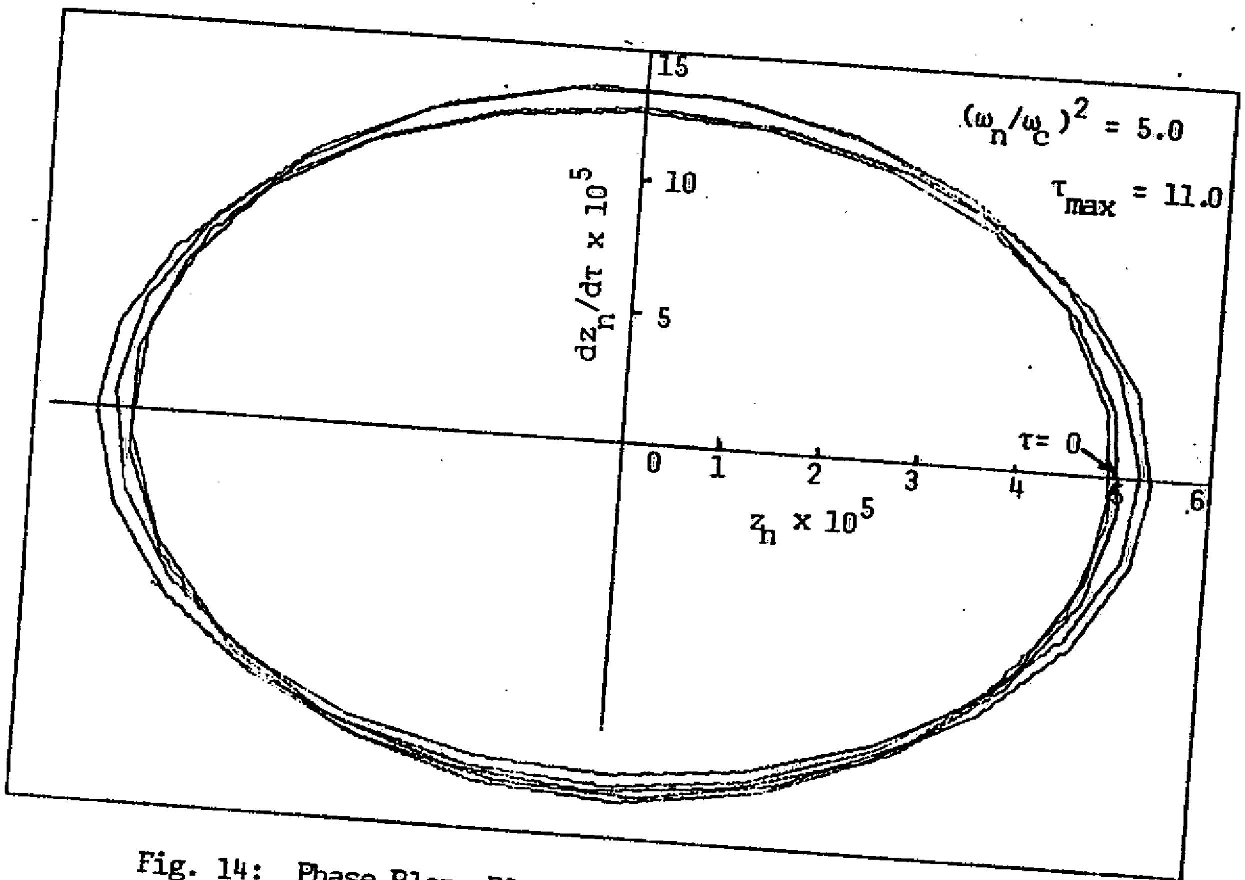


Fig. 14: Phase Plane Plot (case 4)

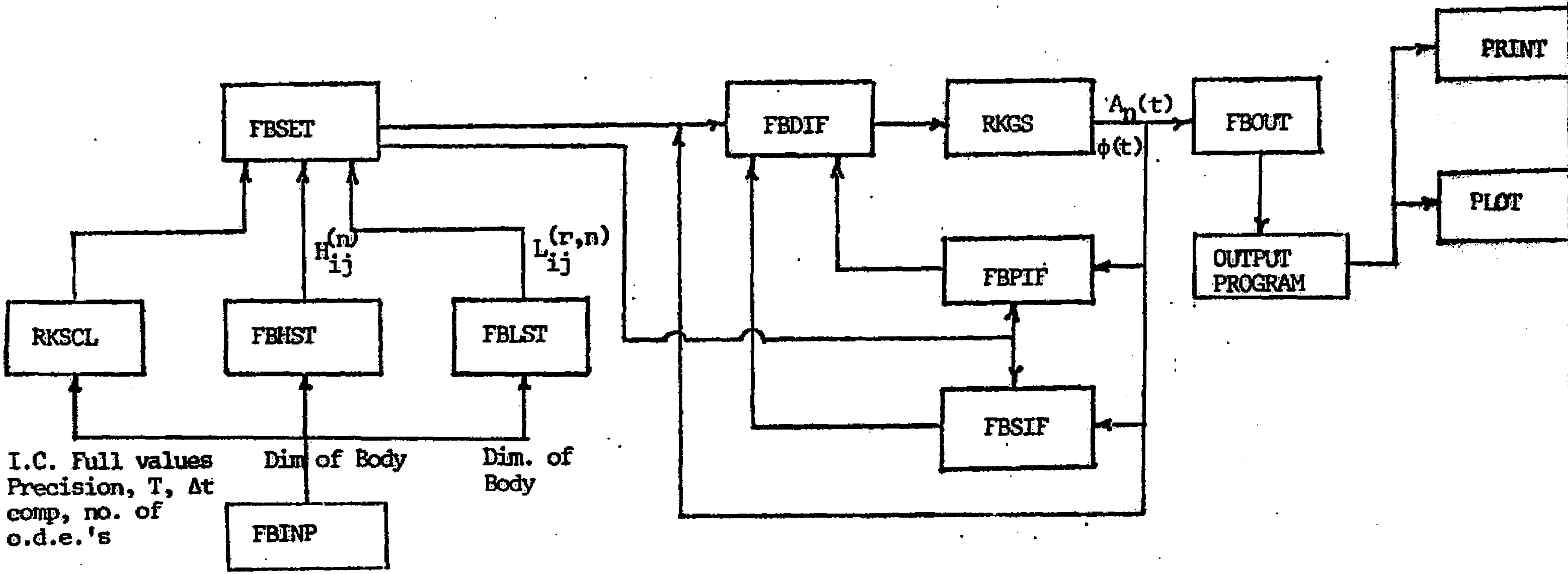


Fig. 15: Flow Diagram of Flexbeam Program
Howard University NOVA - 840 Digital Computer

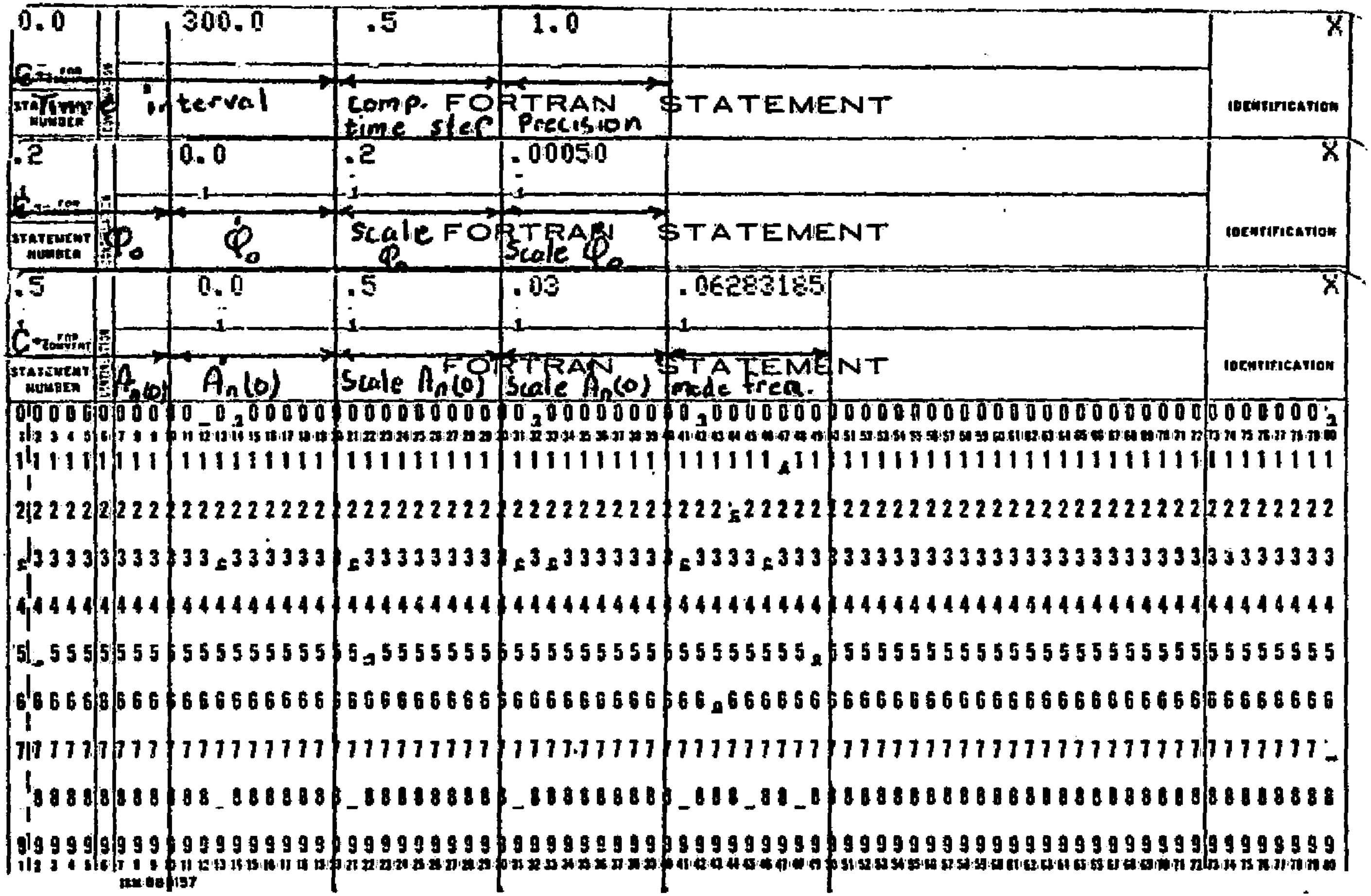


Fig. 16. Sample Data Input Cards

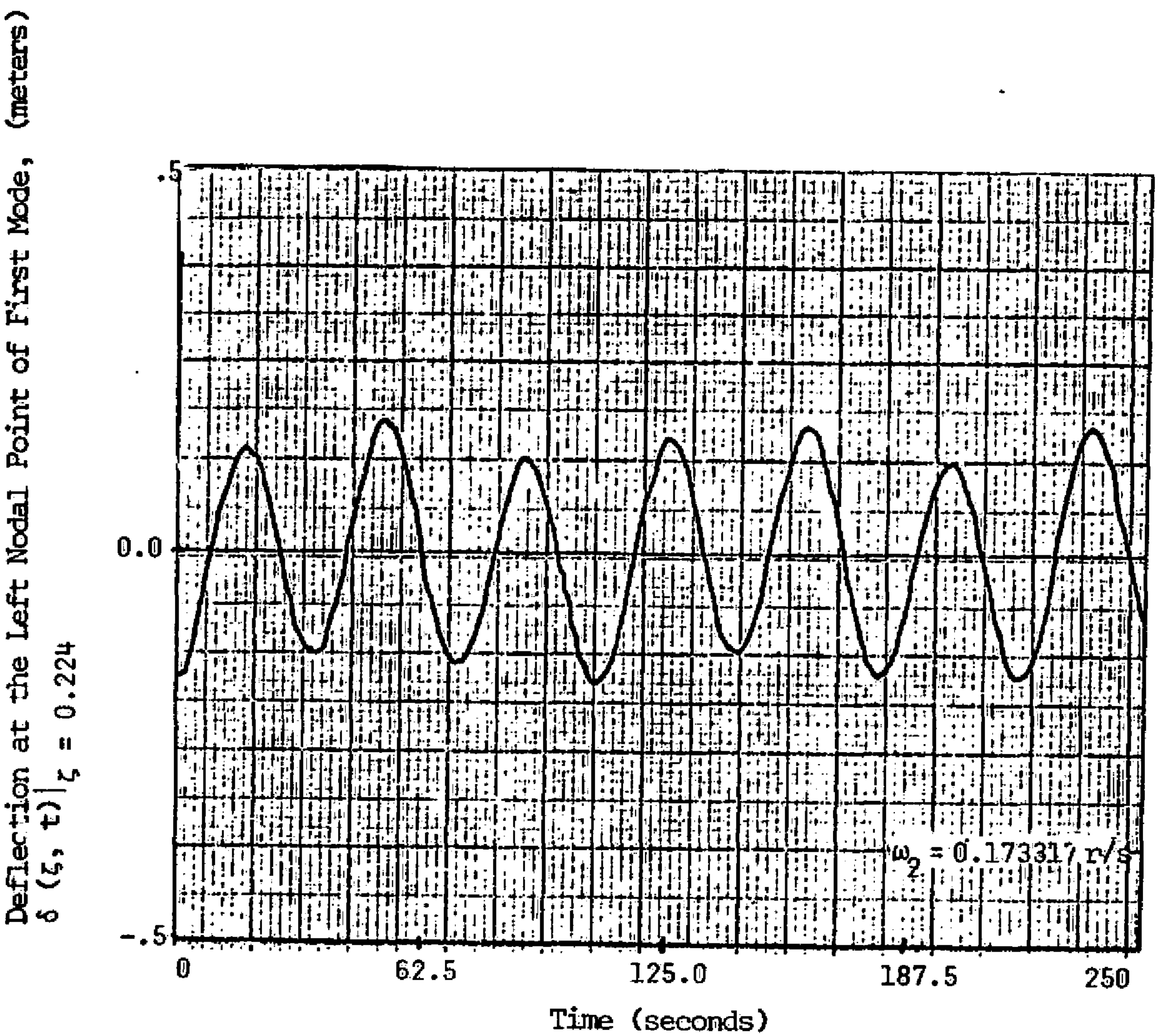


Fig. 17a: Beam Deflection Time Response

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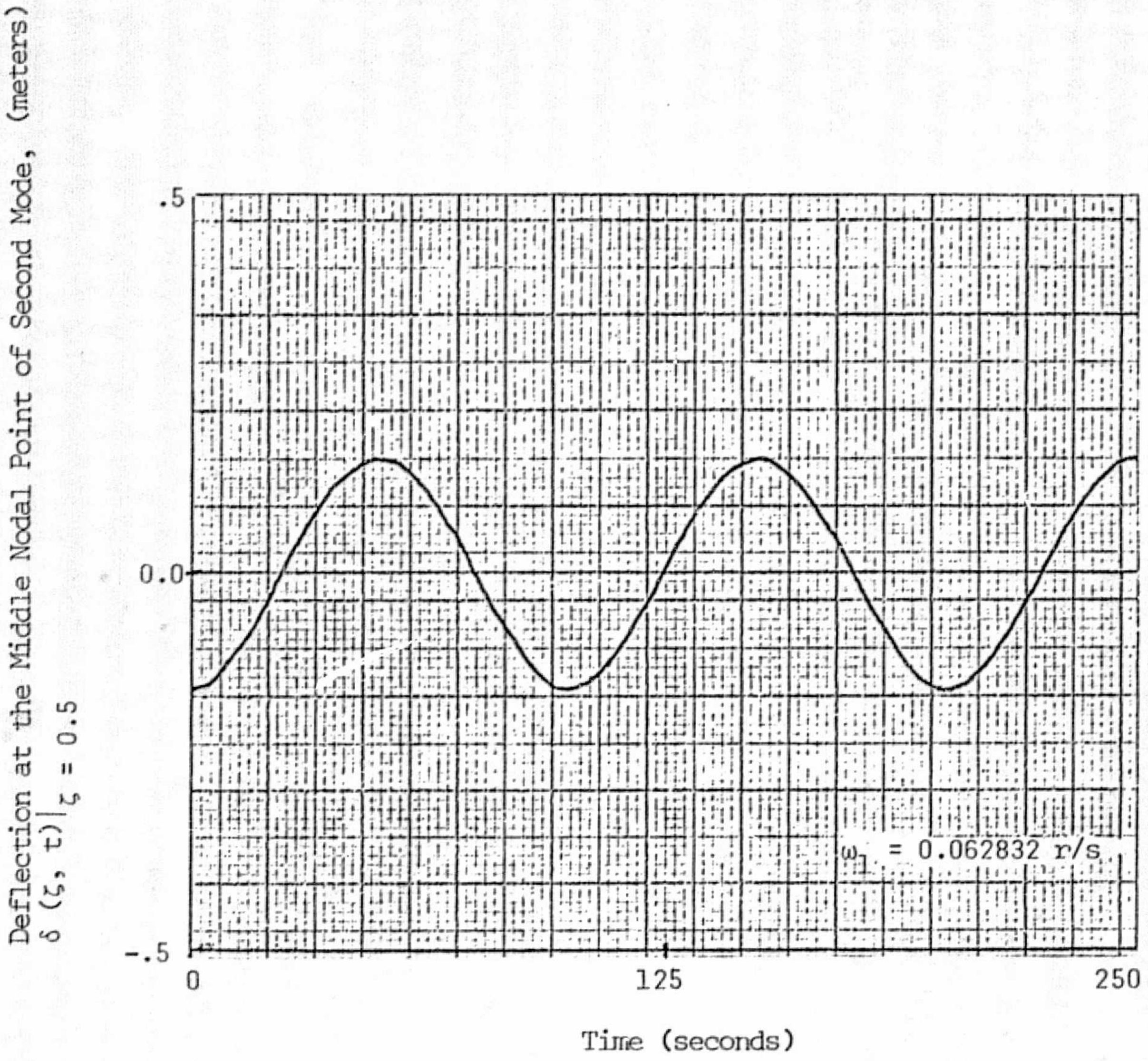

Fig. 17b: Beam Deflection Time Response

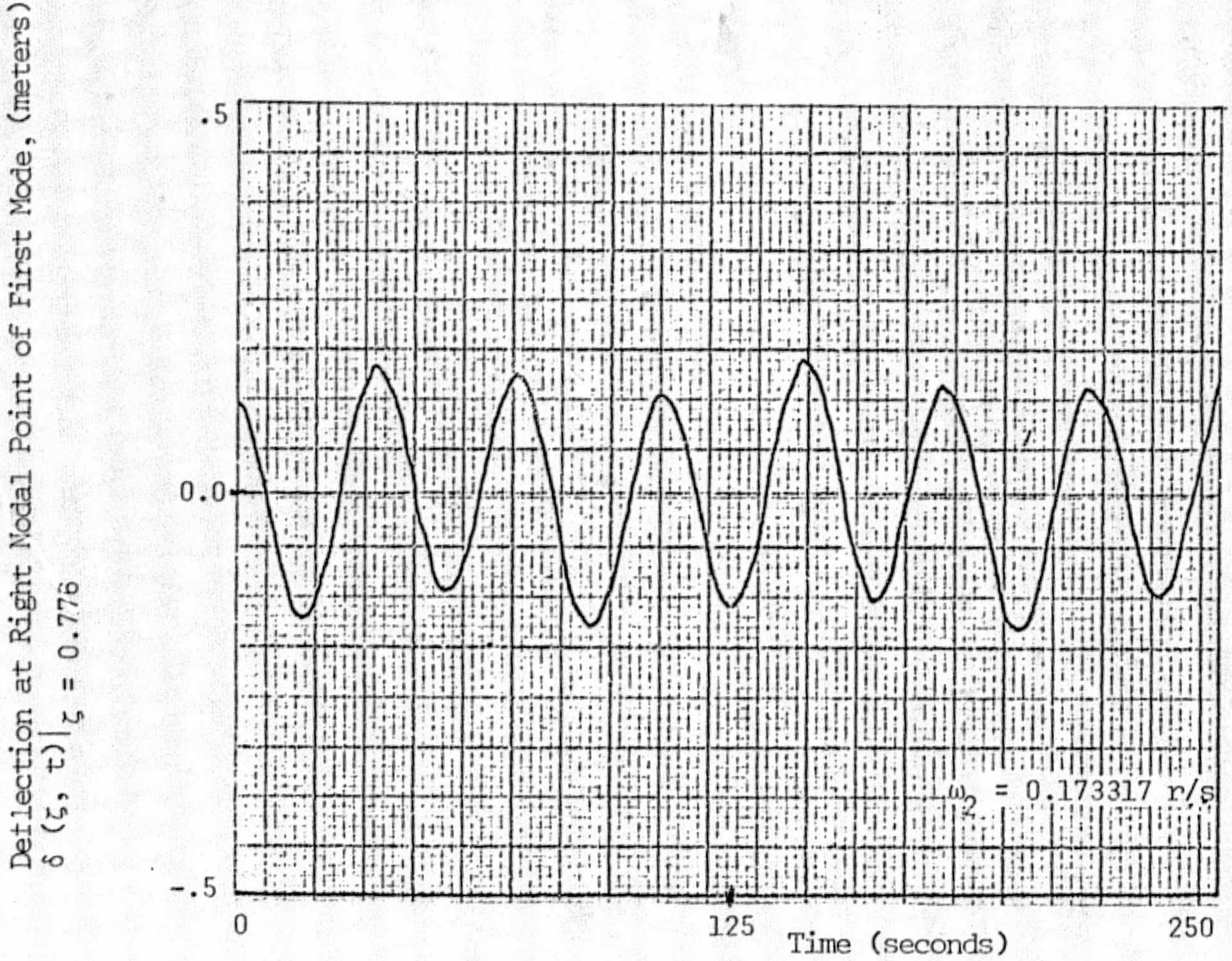

Fig. 17c: Beam Deflection Time Response

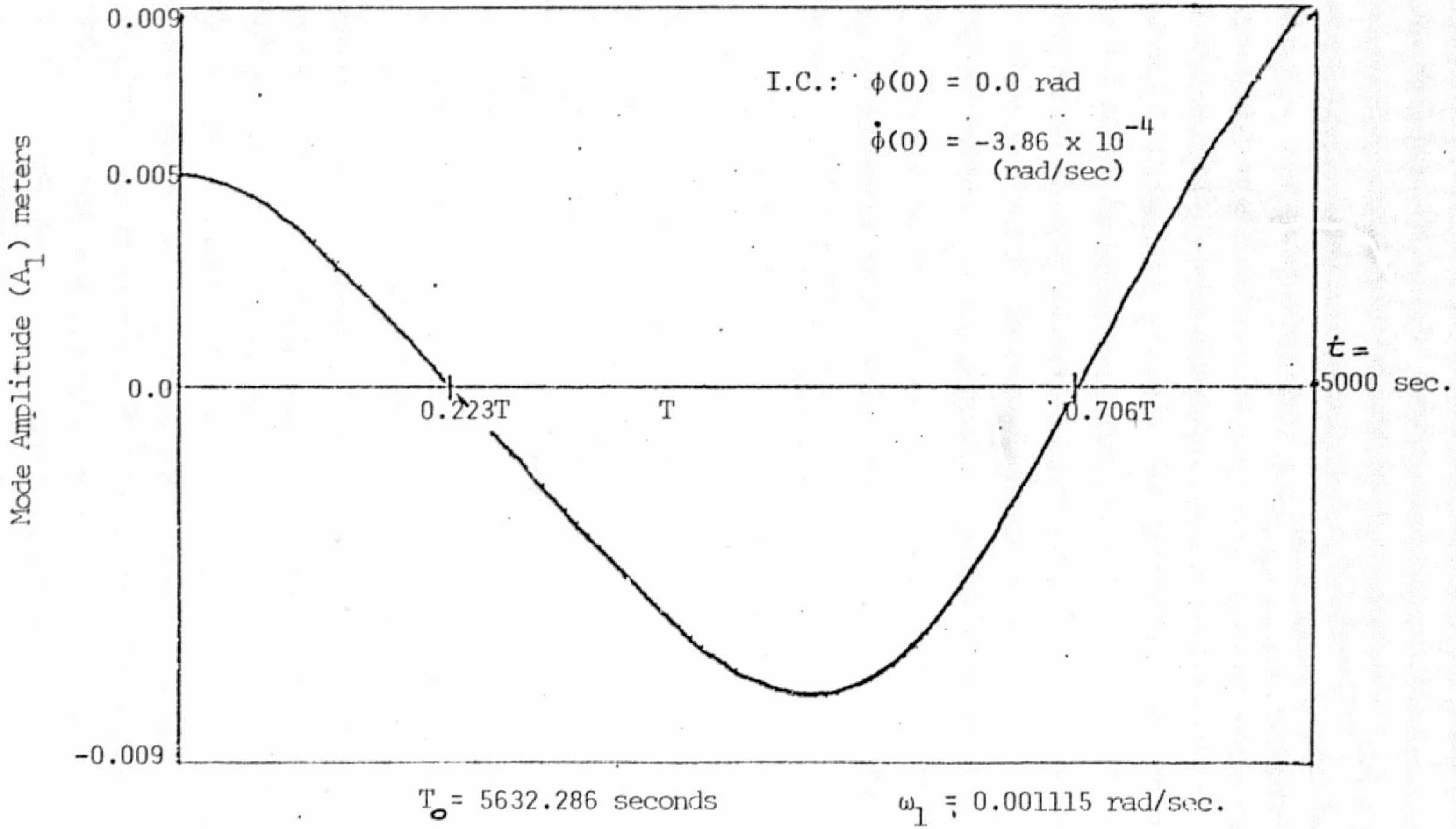

Fig. 18a: Time Response of First Generic Mode
(Orbital Frequency Equals Mode Frequency)

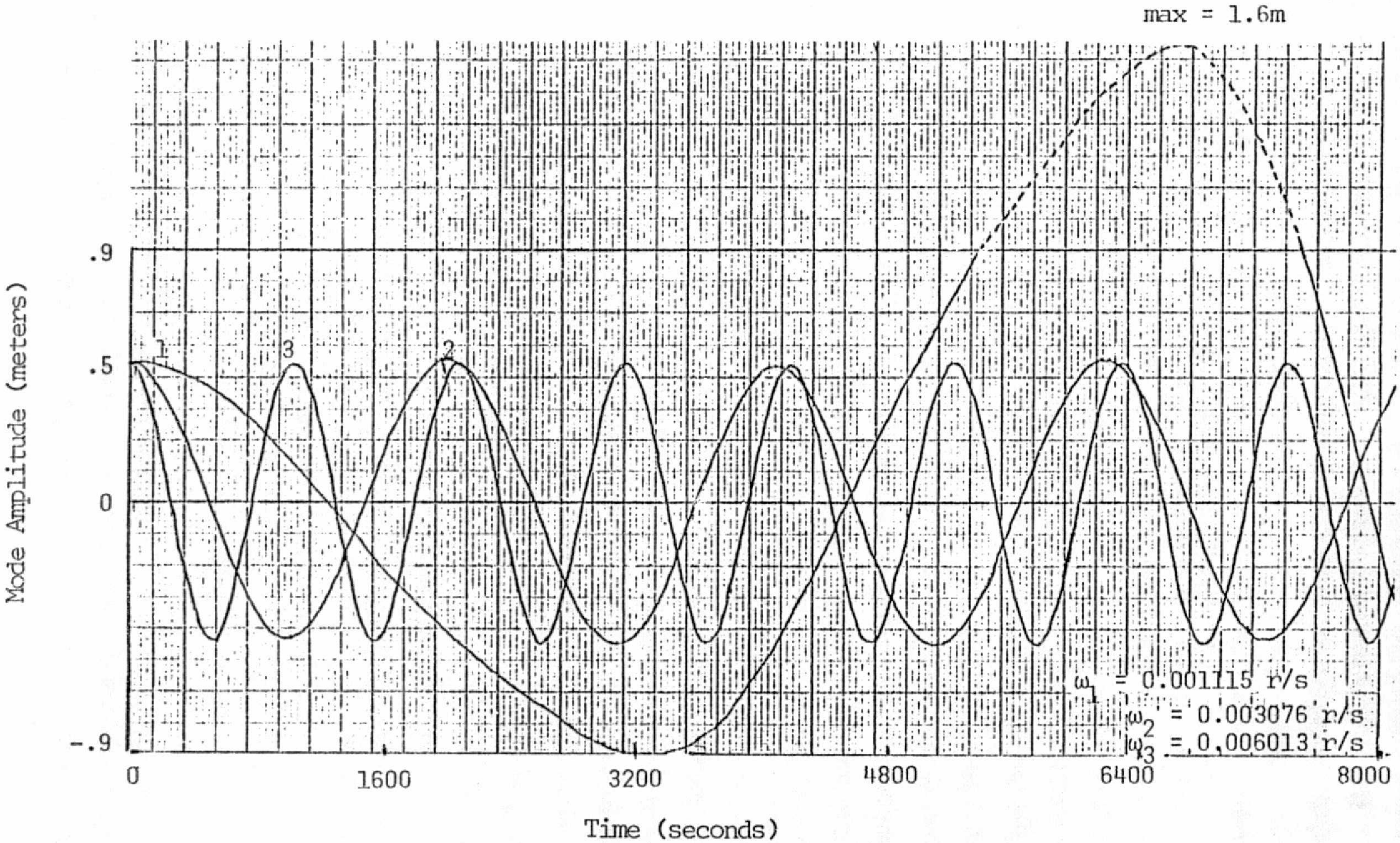

Fig. 18b: Time Response of Three Modes Equally Weighted in $A_n(0)$

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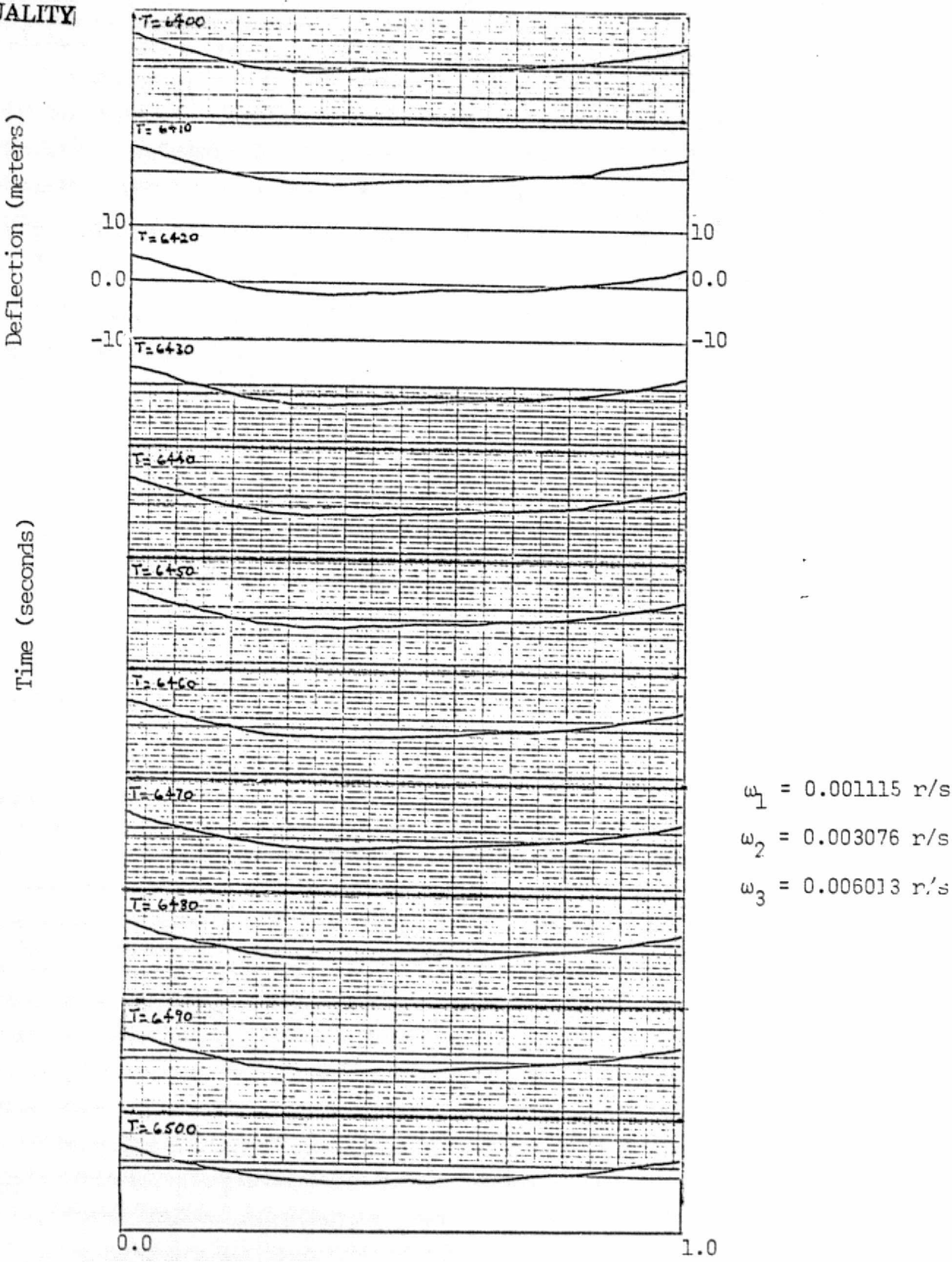

Fig. 19: Deflection vs. Non-dimensional Beam Length for $6400 \leq T \leq 6500$ (secs)

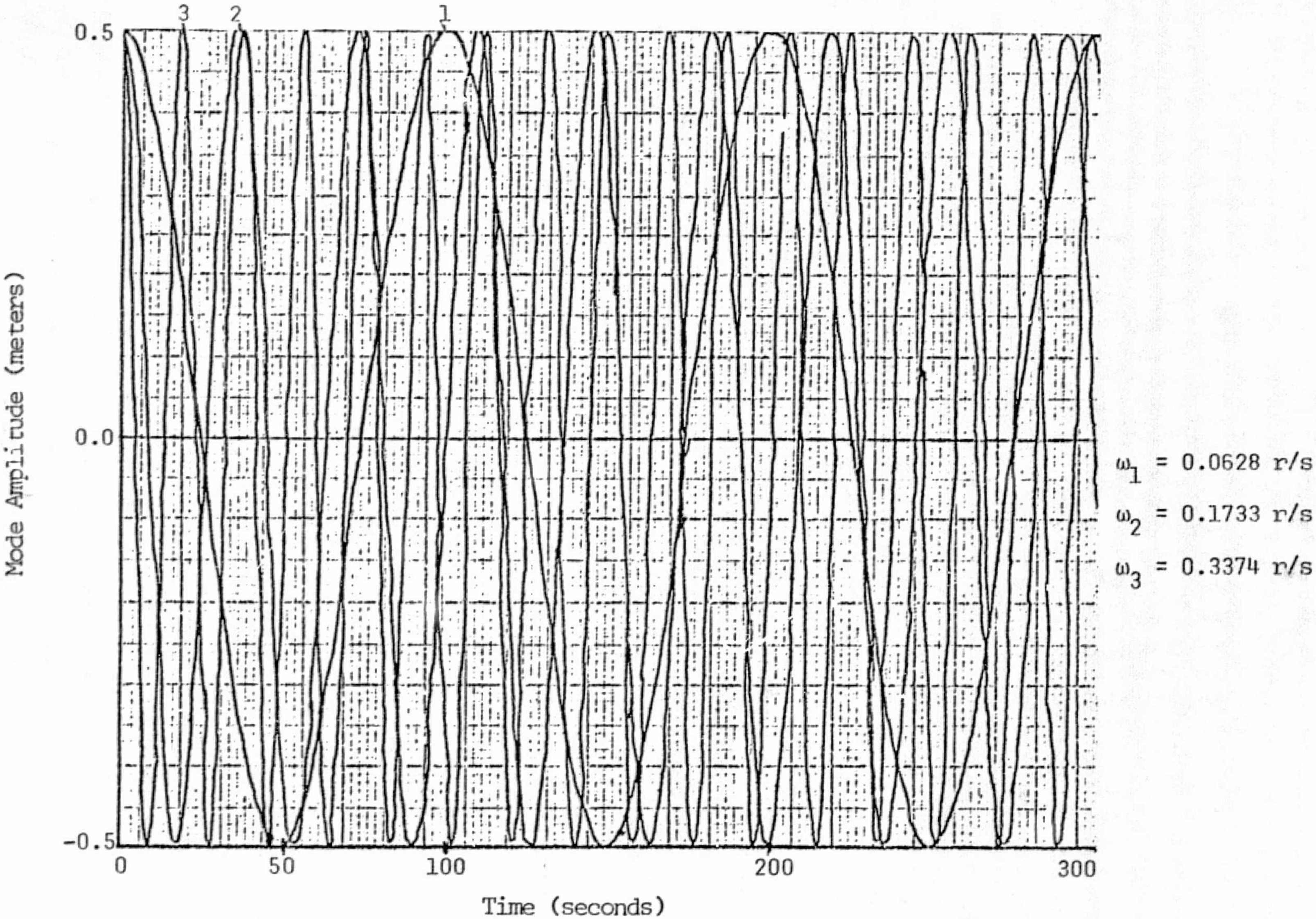

Fig. 20: Time Response of Three Modes Equally Weighted in $A_n(0)$

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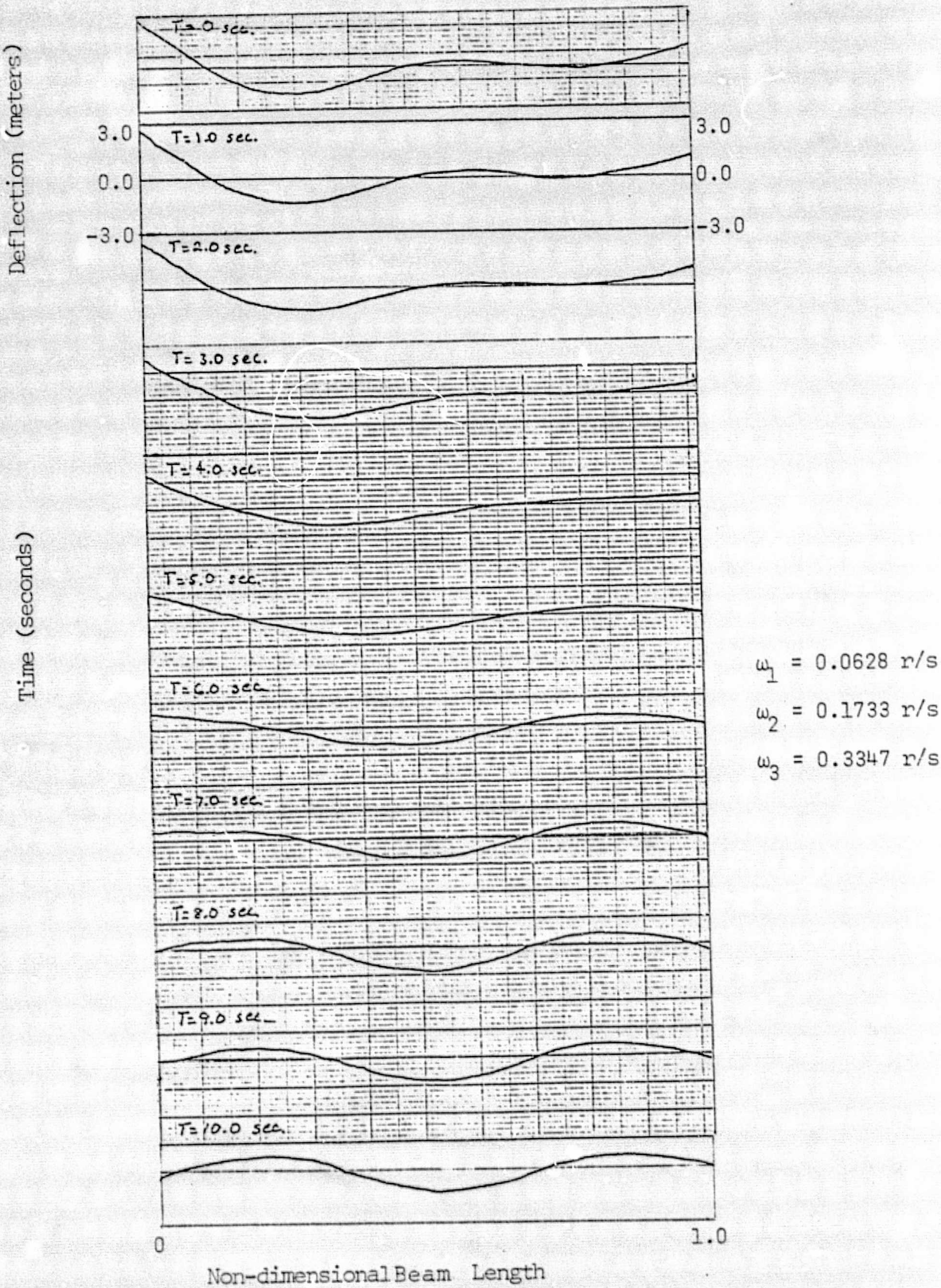


Fig. 21a: Deflection vs. Non-dimensional Beam Length for $0 \leq T \leq 10$ (secs)

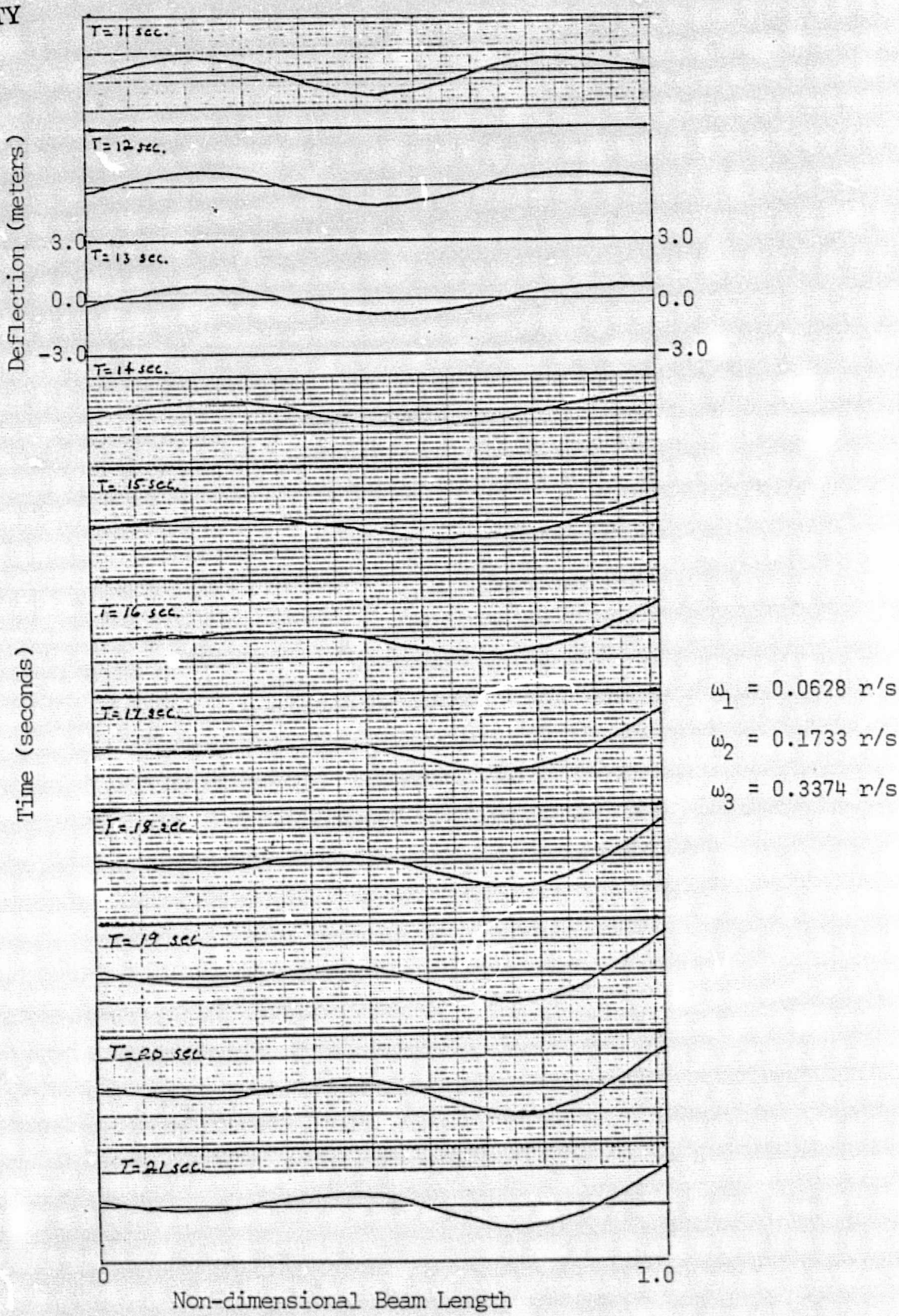


Fig. 21b: Deflection vs. Non-dimensional Beam Length for $11 \leq T \leq 21$ (secs)

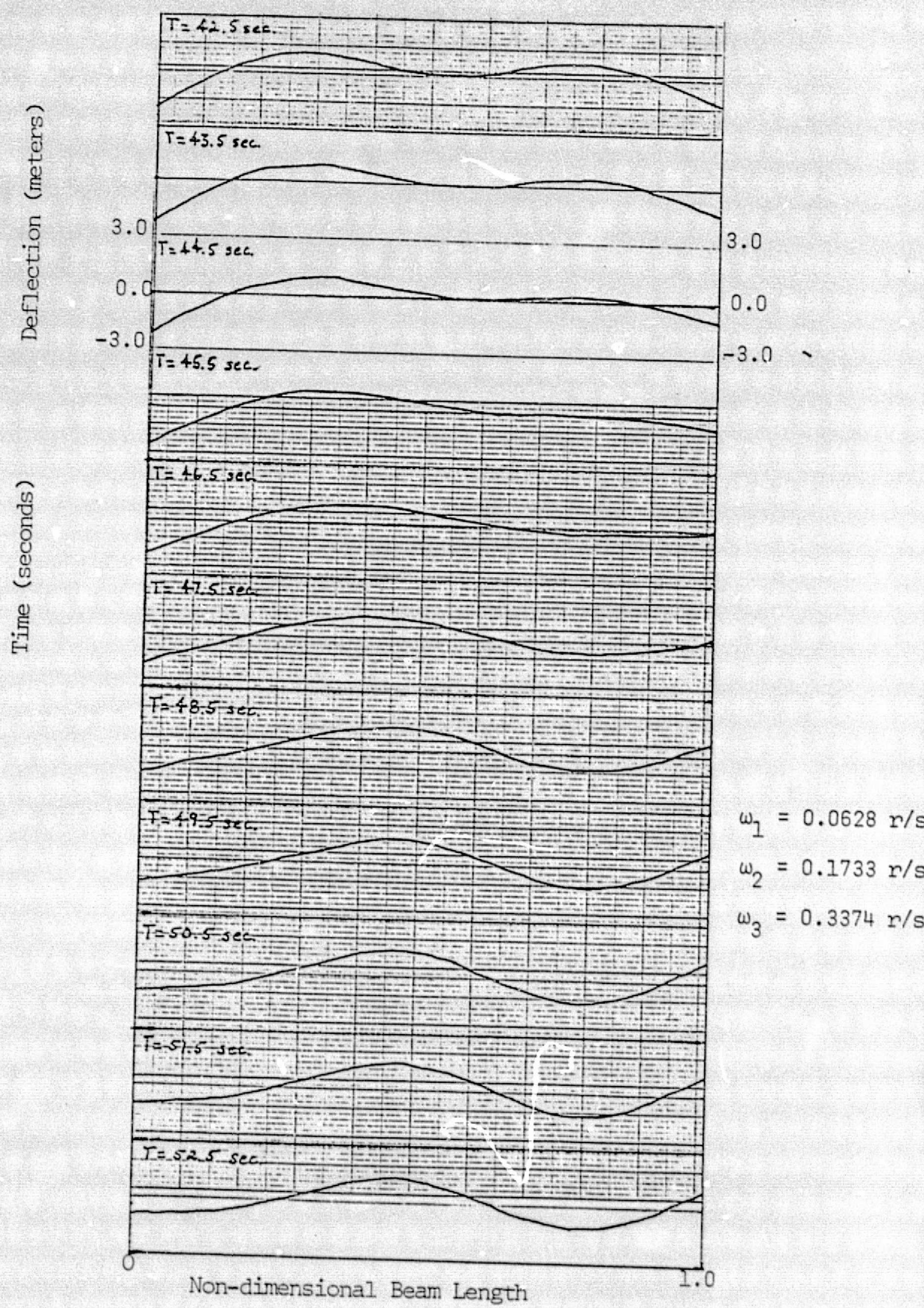


Fig. 21c: Deflection vs. Non-dimensional Beam Length for $42.5 \leq T \leq 52.5$ (secs)

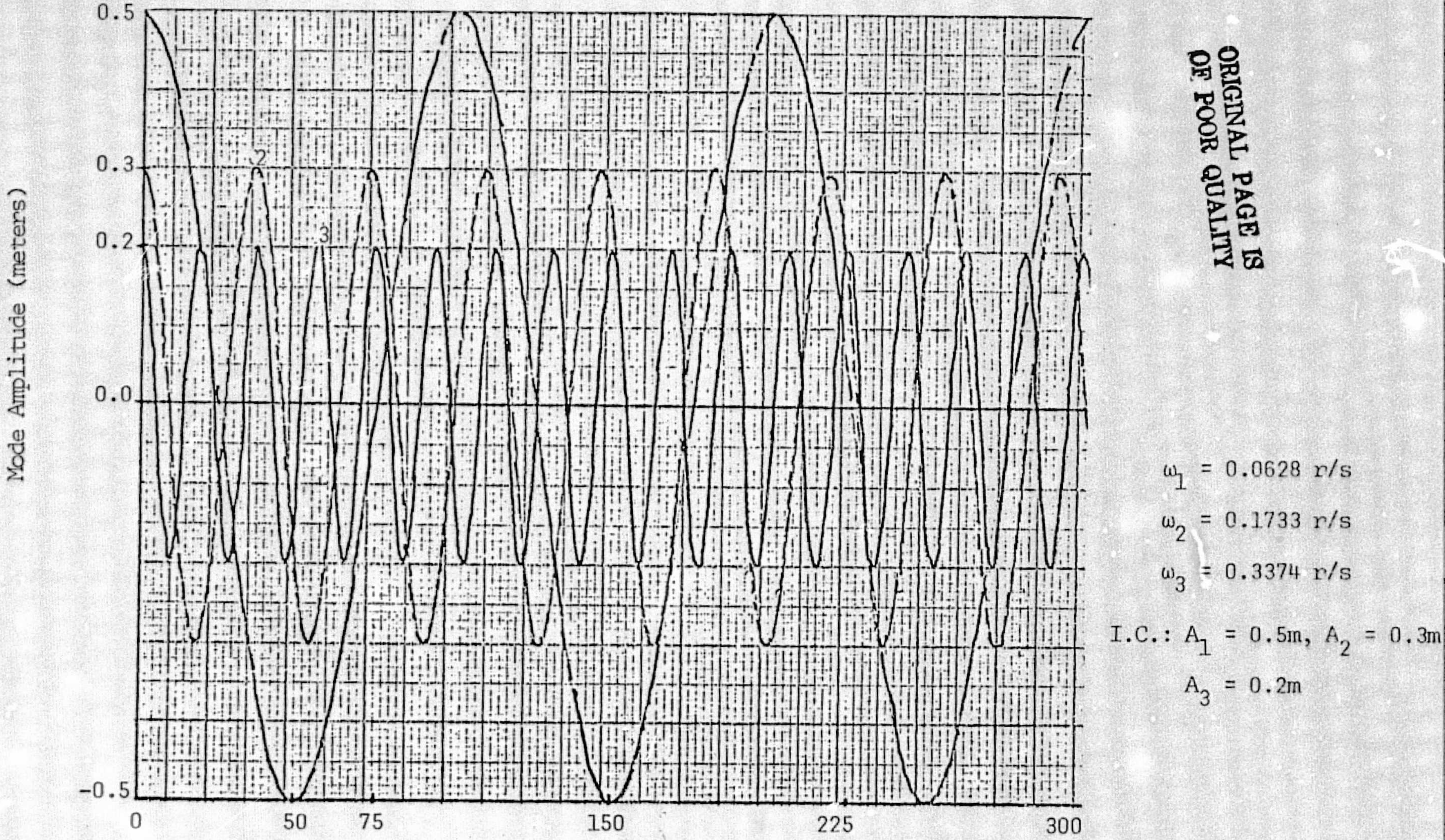


Fig. 22: Time Response of Three Modes Unequally Weighted In $A_n(0)$

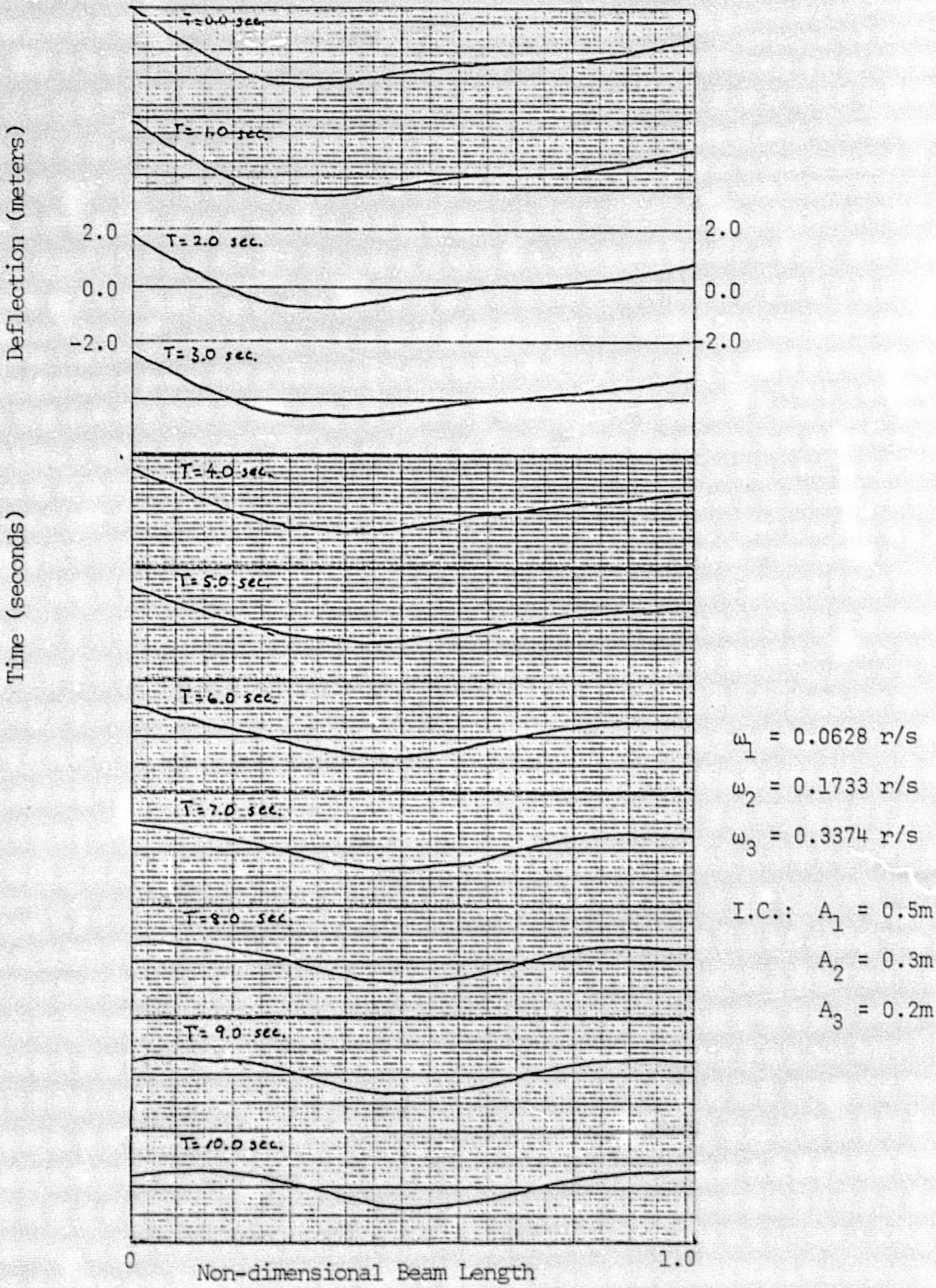


Fig. 23: Deflection vs. Non-dimensional Beam Length for $0 \leq T \leq 10$ (sec.)

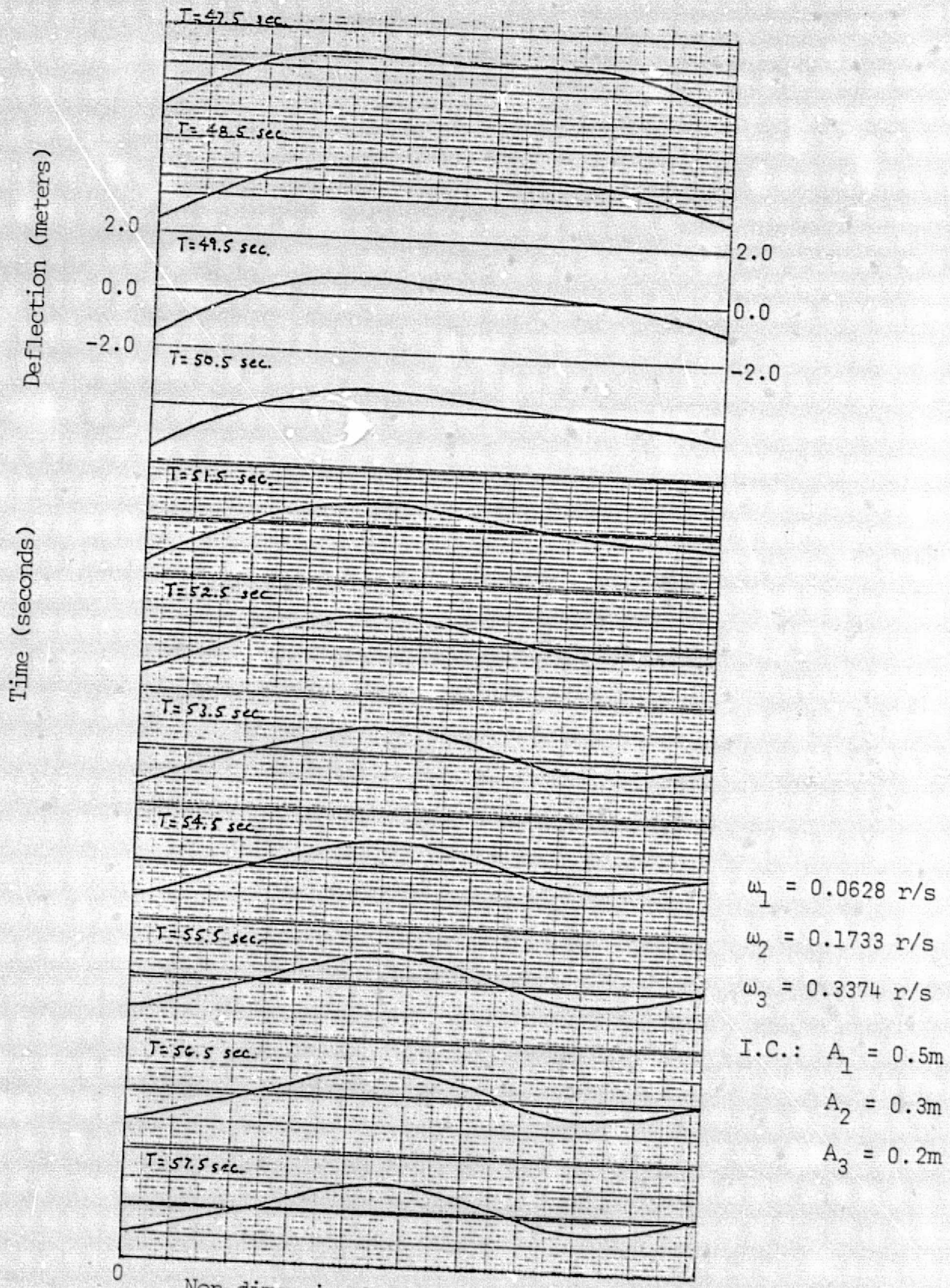
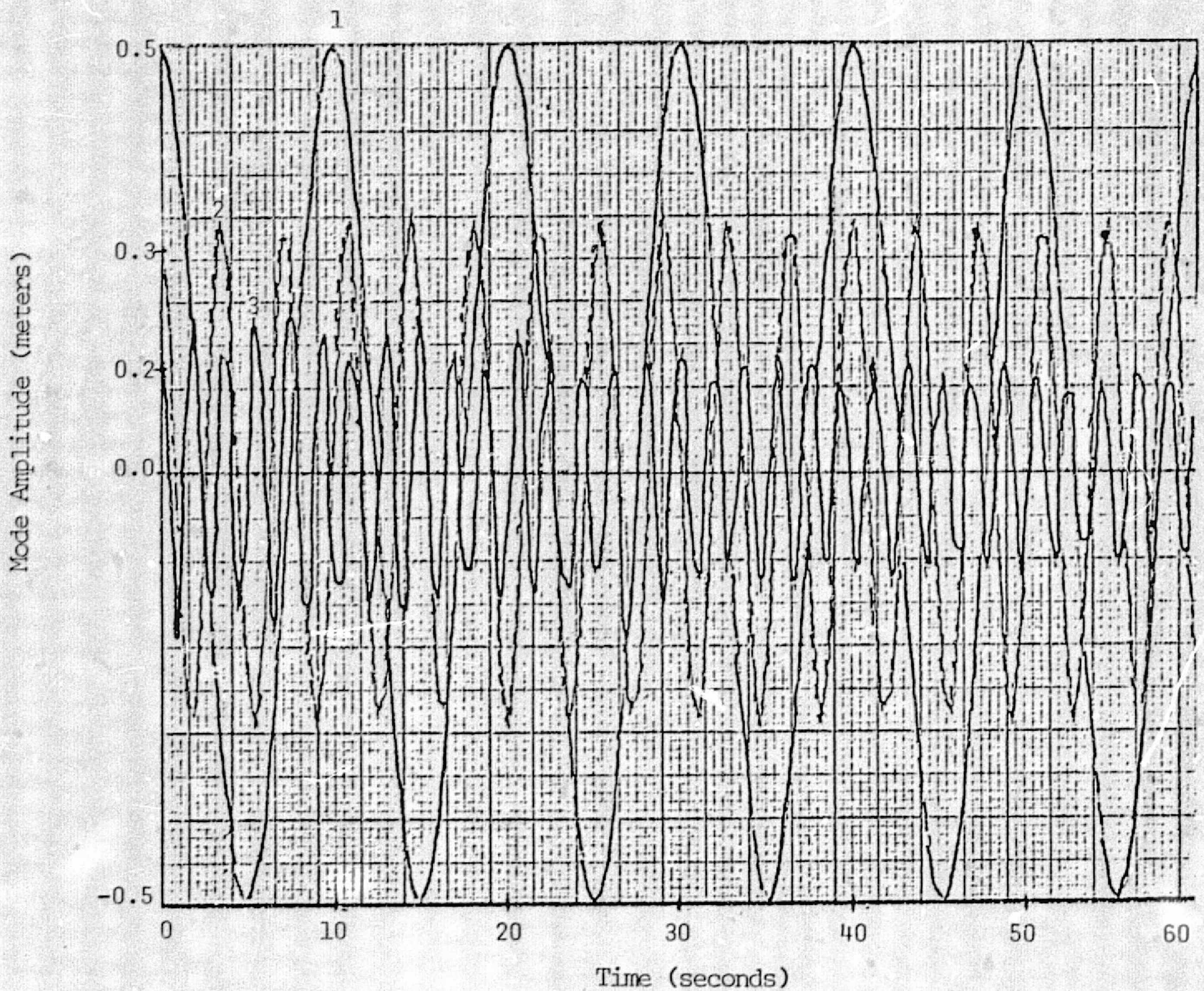


Fig. 24: Deflection vs. Non-dimensional Beam Length for $47.5 \leq T \leq 57.5$ (secs)



$\omega_1 = 0.6283$ r/s

$\omega_2 = 1.7332$ r/s

$\omega_3 = 3.3741$ r/s

I.C.: $A_1 = 0.5$m

$A_2 = 0.3$m

$A_3 = 0.2$m

Fig. 25: Time Response of Three Modes Unequally Weighted in $A_n(0)$

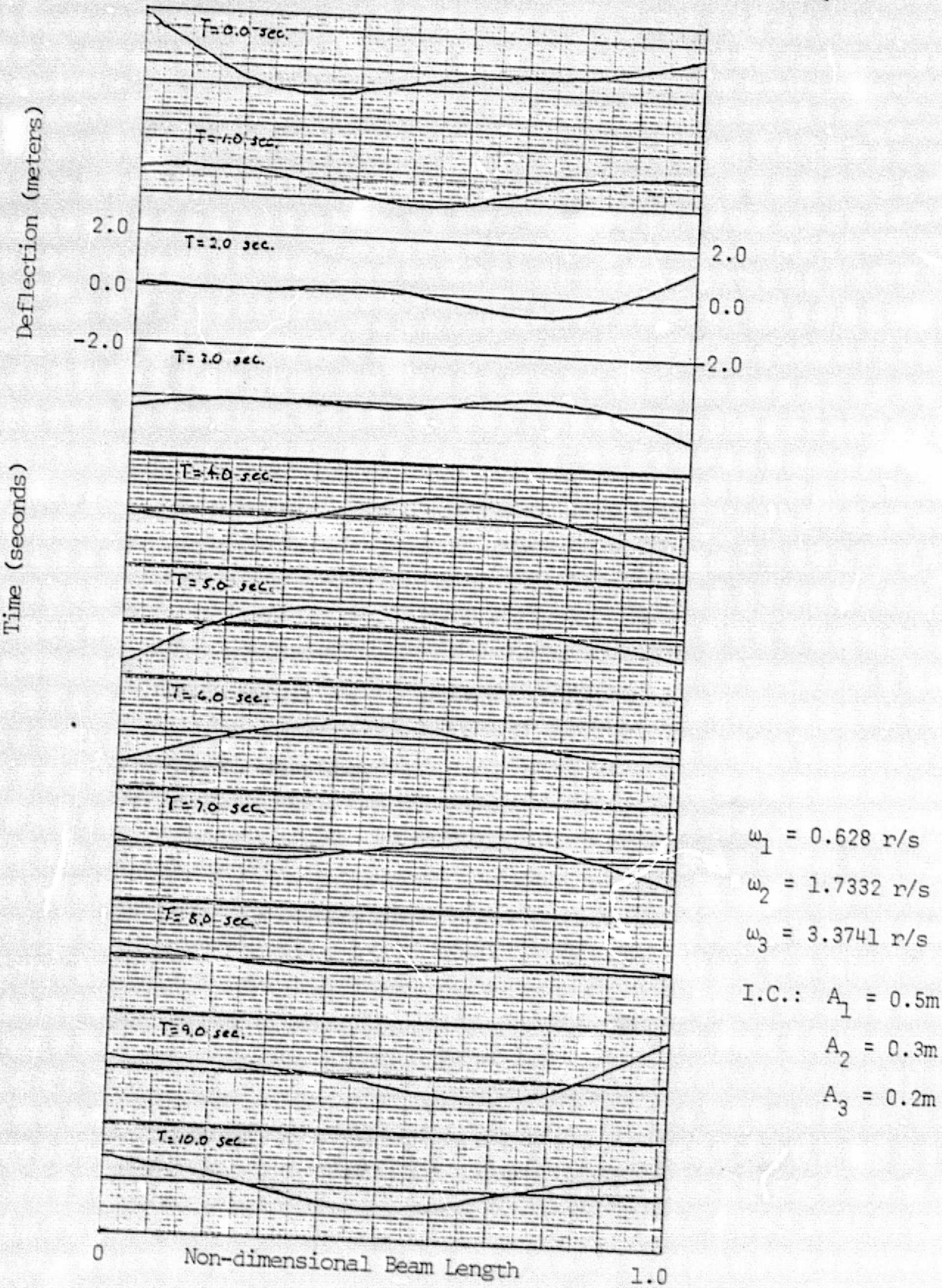


Fig. 26: Deflection vs. Non-dimensional Beam Length for $0 \leq T \leq 10$ (secs)

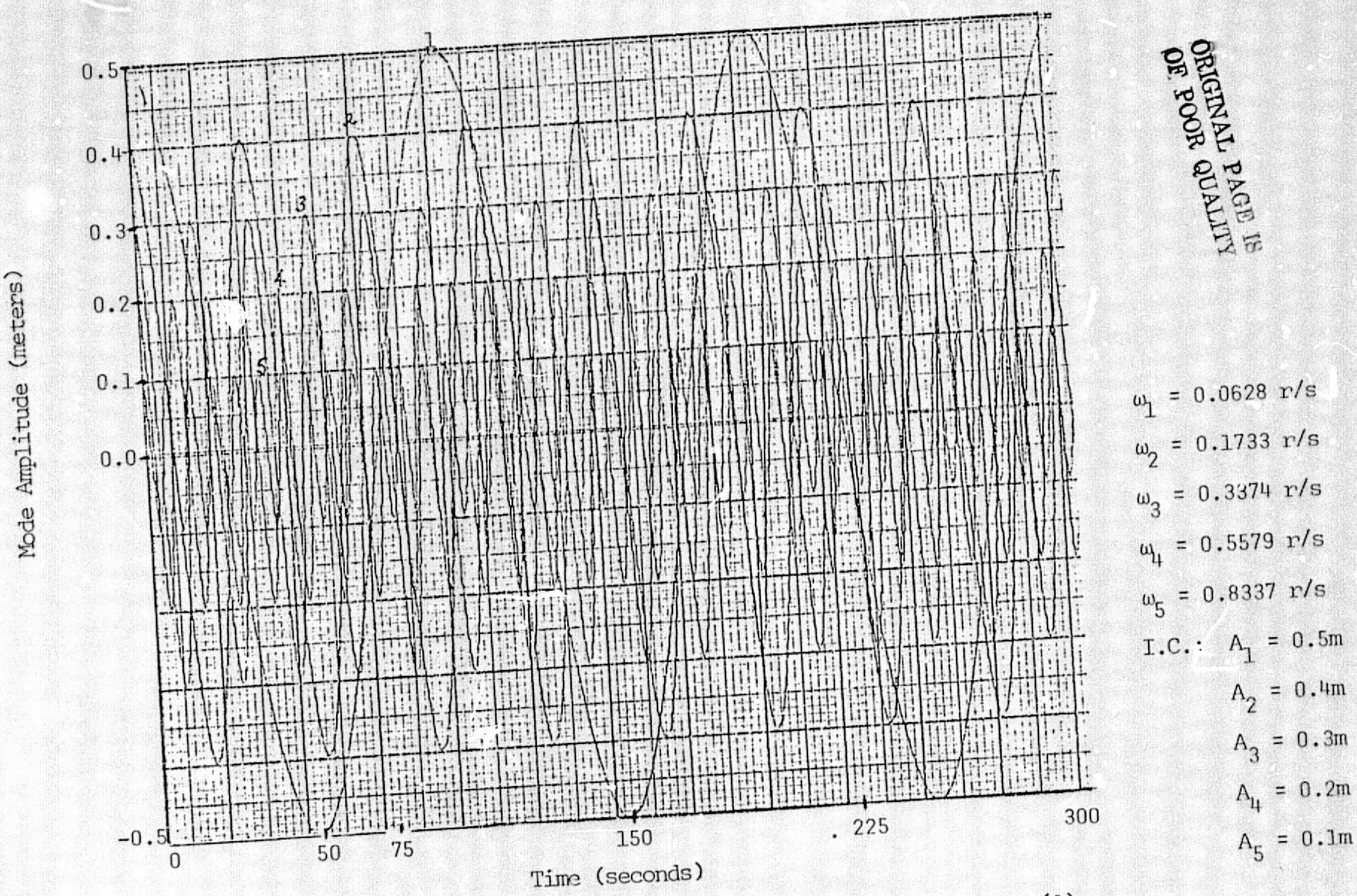


Fig. 27: Time Response of Five Modes Unequally Weighted in $A_n(0)$

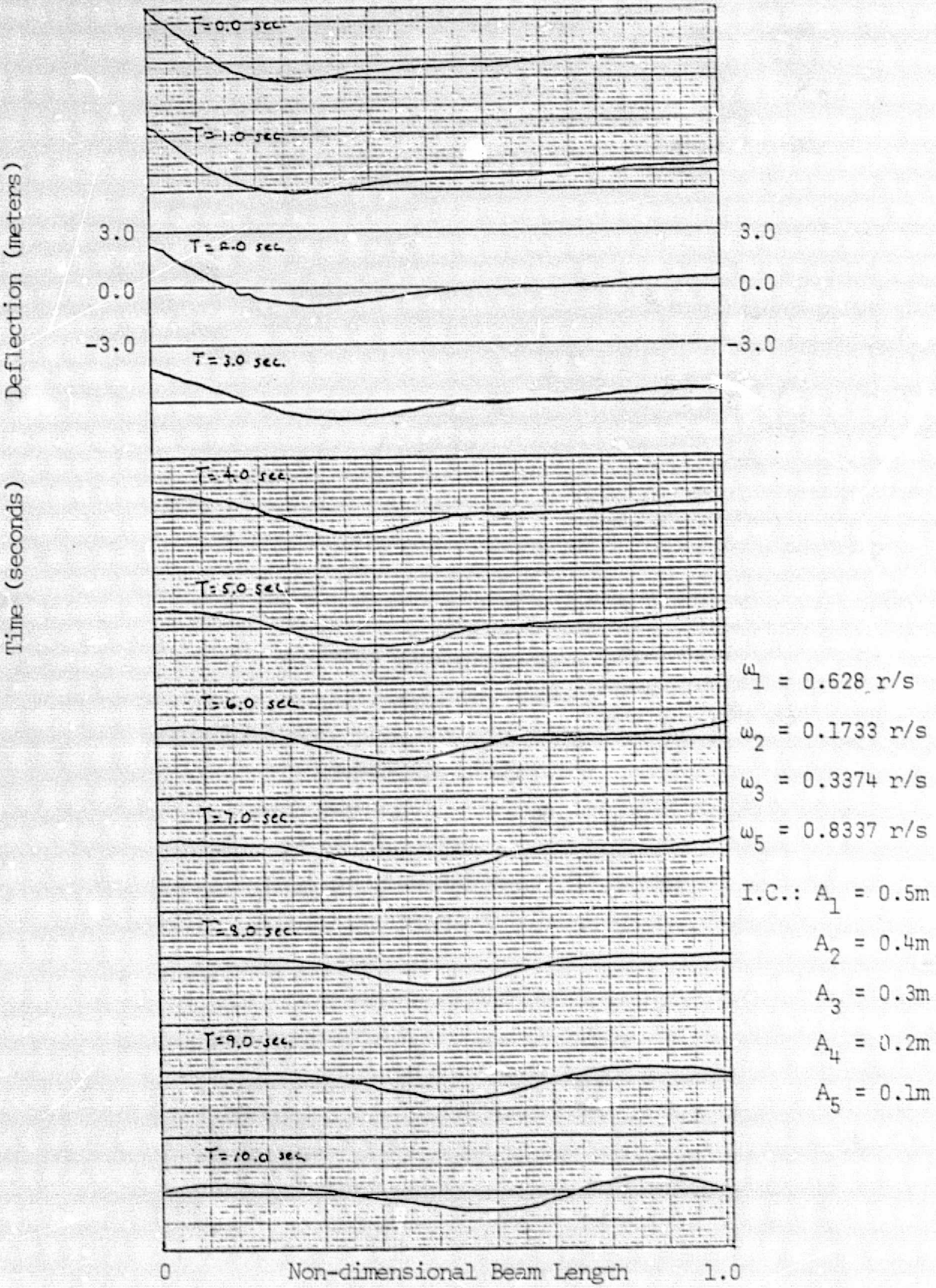


Fig. 28: Deflection vs. Non-dimensional Beam Length for $0 \leq T \leq 10$ (sec)

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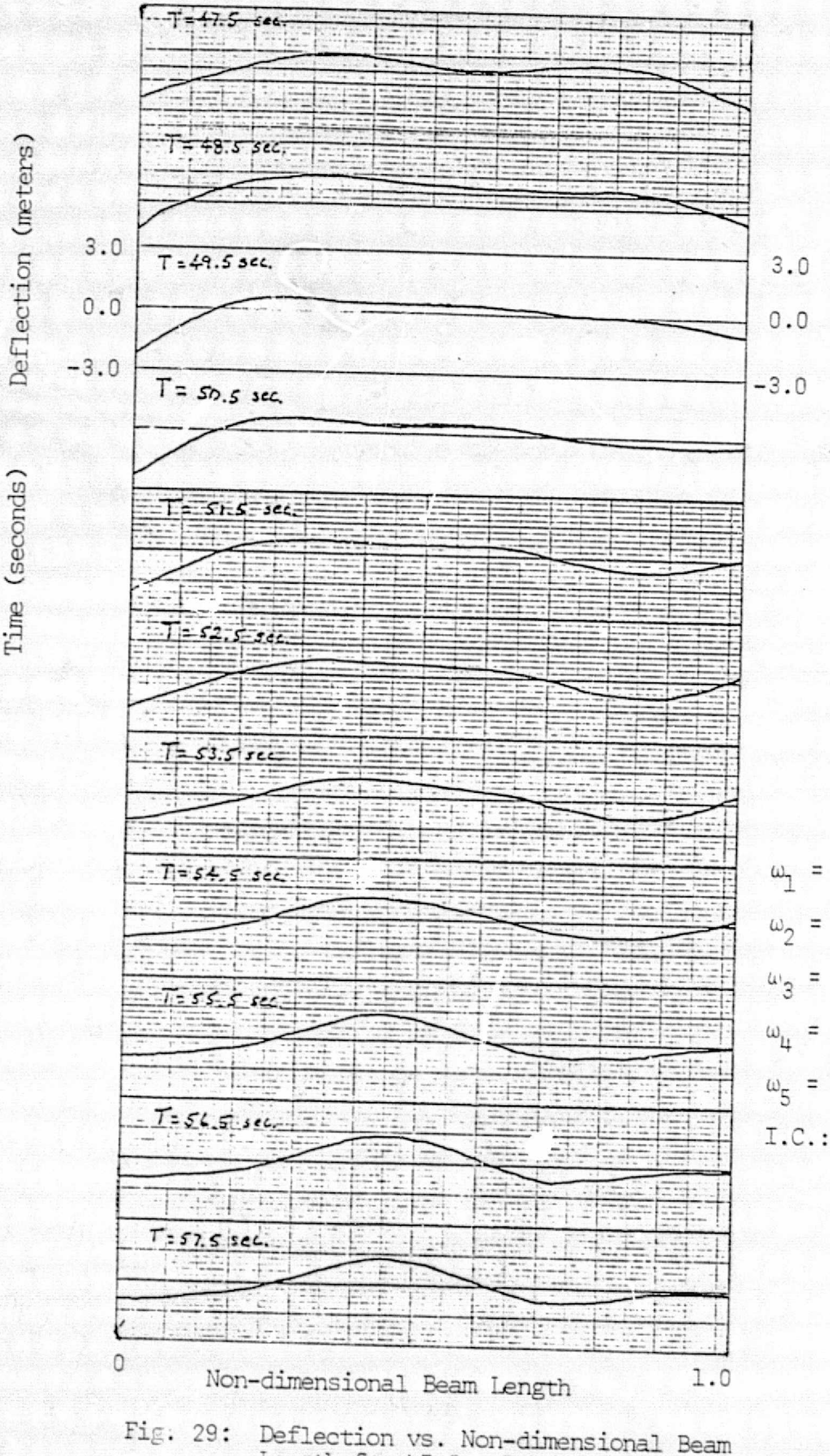


$\omega_1 = 0.0628$ r/s

$\omega_2 = 0.1733$ r/s

$\omega_3 = 0.3374$ r/s

$\omega_4 = 0.5579$ r/s

$\omega_5 = 0.8337$ r/s

I.C.: $A_1 = 0.5$m

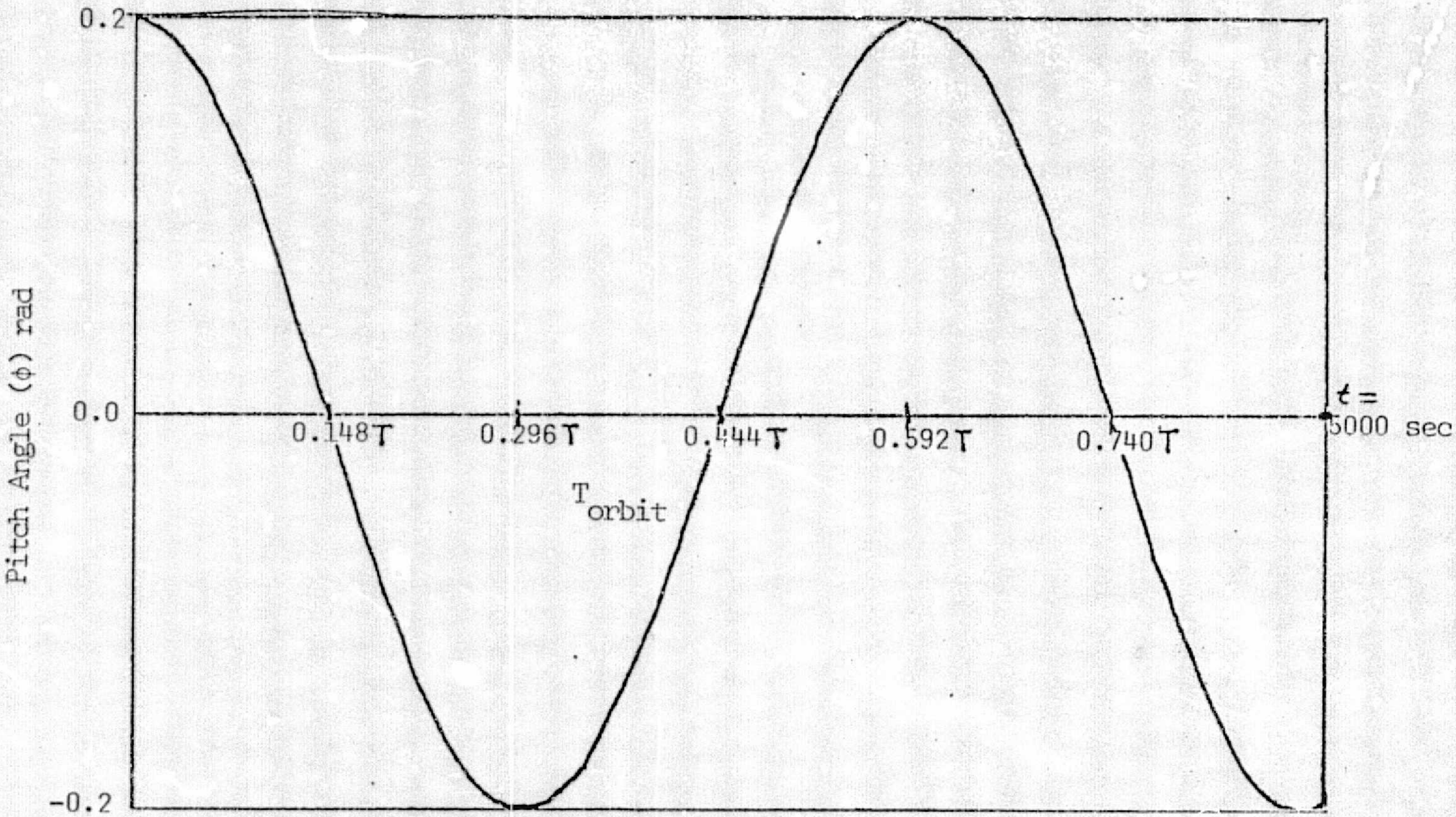
$A_2 = 0.4$m

$A_3 = 0.3$m

$A_4 = 0.2$m

$A_5 = 0.1$m

Fig. 29: Deflection vs. Non-dimensional Beam Length for $47.5 \leq T \leq 57.5$ (sec)



T_{orbit} = 5632.28 sec ω_1 = 0.0051 rad/sec   I.C.: $\phi(0)$ = 0.2 rad

$\dot{\phi}(0)$ = 0.0 rad/sec

Fig. 30: Time Response of the Pitch Motion

Appendix I

Transformation Relations:

1. Transformation from τ_0 to τ_1

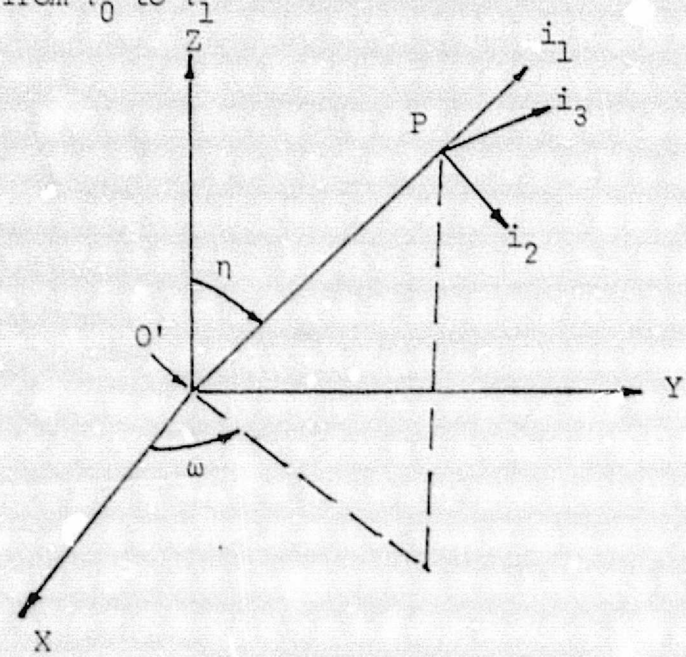


Fig. (A-1): Inertial and intrinsic frames

From Fig. (A-1) it is evident that,

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}_{\tau_1(P)} = \underbrace{\begin{bmatrix} s\,\eta\, c\,\omega & s\,\eta\, s\,\omega & c\,\eta \\ c\,\eta\, c\,\omega & c\,\eta\, s\,\omega & s\,\eta \\ -s\,\omega & c\,\omega & 0 \end{bmatrix}}_{T_1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{\tau_0} \qquad \text{(I-1)}$$

2. Transformation from the intrinsic frame (τ_1) to the orbit fixed frame (τ_2):

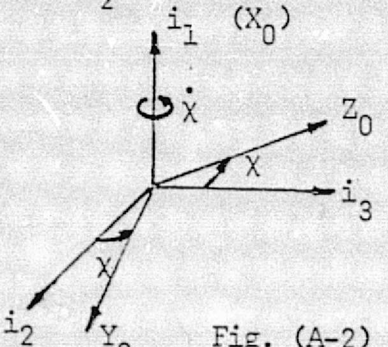


Fig. (A-2)

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi & s\chi \\ 0 & -s\chi & c\chi \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \qquad \text{(I-2)}$$

3. Transformation from orbit frame (τ_2) to the body frame (τ_3):

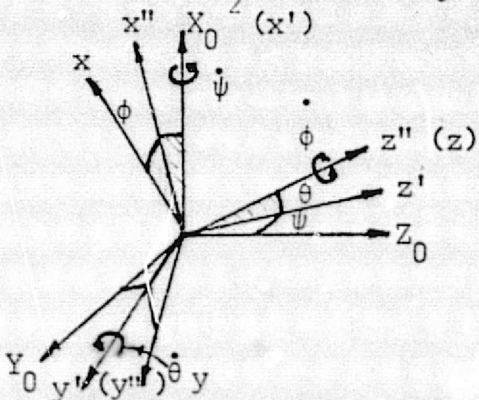


Fig (A-3): Euler angle rotations

$$X_0\ Y_0\ Z_0 \xrightarrow[\text{yaw}]{\psi} x'\ y'\ z' \xrightarrow[\text{pitch}]{\theta} x'\ y''\ z'' \xrightarrow[\text{roll}]{\phi} x\ y\ z$$

From Fig. (A-3),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & s\psi \\ 0 & -s\psi & c\psi \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

$$\text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} c\phi c\theta & s\phi c\psi + c\phi s\theta s\psi & s\phi s\psi - c\phi s\theta c\psi \\ -s\phi c\theta & c\phi c\psi - s\phi s\theta s\psi & c\phi s\psi + s\phi s\theta c\psi \\ s\theta & -c\theta s\psi & c\theta c\psi \end{bmatrix}}_{T_3\ (\phi,\ \theta,\ \psi)} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \qquad \text{(I-3)}$$

Appendix - II

The transformation matrix connecting τ_1 (P) and τ_3, where P is a general point in the body, is developed as below,

(i) Transform a vector in τ_1 (P) to a vector in τ_0 using the transformation T_1^{-1} (P).

(ii) Transform the vector in τ_0 to a vector in $\tau_1(0)$ by the transformation T_1 (0)

we can write,

$$T_1^{-1}(P) = T_1^{-1}(0) + \Delta T_1^{-1}$$

$$= T_1^{-1}(0) + \frac{\partial T_1}{\partial \eta}\Delta\eta + \frac{\partial T_1}{\partial \omega}\Delta\omega + \text{H.O.T.}$$

Hence,

$$T_1(0)\,T_1^{-1}(P) = I + \underline{\Delta} \qquad \text{(II - 1)}$$

where,
$$\underline{\Delta} = T_1(0)\left[\frac{\partial T_1}{\partial \eta}\Delta\eta + \frac{\partial T_1}{\partial \omega}\Delta\omega\right]$$

and from Eqn. (I - 1)

$$\underline{\Delta} = \Delta\eta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta\omega \begin{bmatrix} 0 & 0 & -s\eta \\ 0 & 0 & -c\eta \\ s\eta & c\eta & 0 \end{bmatrix} \qquad \text{(II - 2)}$$

Thus, the transformation ($I+\underline{\Delta}$) transforms a vector in τ_1 (P) to a vector in τ_1 (0), i.e.

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}_{\tau_1(0)} = (I + \underline{\Delta}) \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}_{\tau_1(P)} \qquad \text{(II - 3)}$$

The gravity force per unit mass at O, expressed in the intrinsic frame at that point, τ_1 (O) is given by,

$$\overline{F}_0 (\rho, \eta, \omega) = \nabla V|_0 = \begin{bmatrix} \frac{\partial v}{\partial \rho} \\ \frac{1}{\rho}\frac{\partial v}{\partial \eta} \\ \frac{1}{\rho s\eta}\frac{\partial v}{\partial \omega} \end{bmatrix}_0 \qquad \text{(II - 4)}$$

Where, V is the gravitational potential given by Eqn (5).

The gravity force at P referred to the intrinsic frame at P can be written as:

$$\overline{F}_p = \overline{F}_0 (\rho, \eta, \omega) + \frac{\partial \overline{F}}{\partial(\rho,\eta,\omega)} \begin{bmatrix} \Delta\rho \\ \Delta\eta \\ \Delta\omega \end{bmatrix} + \text{H.O.T} \qquad \text{(II - 5)}$$

Hence, the gravity force at P can be referred to the intrinsic frame at O, τ_1 (O), through the use of transformation Eqn. (II-3), by:

$$\overline{F} = (I + \underline{\Delta})\, \overline{F}_p$$

i.e.,

$$\overline{F} = (I + \underline{\Delta}) \quad [\overline{F}_0 + \frac{\partial \overline{F}}{\partial(\rho,\eta,\omega)} \begin{bmatrix} \Delta\rho \\ \Delta\eta \\ \Delta\omega \end{bmatrix}] \qquad \text{(II -6)}$$

To a first order approximation in $\Delta\rho$, $\Delta\eta$, $\Delta\omega$,

$$\overline{F} \cong \overline{F}_0 + \underline{\Delta}\, \overline{F}_0 + \frac{\partial F}{\partial(\rho,\eta,\omega)} \begin{bmatrix} \Delta\rho \\ \Delta\eta \\ \Delta\omega \end{bmatrix} \qquad \text{(II - 7)}$$

From Eqns. (II - 2) & (II - 3), it is easy to show that,

$$\underline{\Delta}\, \overline{F}_0 = \begin{bmatrix} 0 & -F_\eta & -F_\omega s\eta \\ 0 & F_\rho & -F_\omega c\eta \\ 0 & 0 & F_\rho s\eta + F_\eta c\eta \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta\eta \\ \Delta\omega \end{bmatrix} \qquad \text{(II - 8)}$$

where,

$$F_\rho = \frac{\partial V}{\partial \rho} = -\frac{\nu a^2}{\rho^2} + \nu a \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^{s+1} \Omega_s$$

$$F_\eta = \frac{1}{\rho}\frac{\partial V}{\partial \eta} = \frac{\nu a}{\rho} \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^{s+1} \Omega'_s$$

$$F_\omega = \frac{1}{\rho s\eta}\frac{\partial V}{\partial \omega} = \frac{\nu a}{\rho s\eta} \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^{s+1} \hat{\Omega}_s$$

(II - 9)

By some matrix manipulations, it can be shown that,

$$\underline{\Delta} F_0 + \frac{\partial F}{\partial(\rho,\eta,\omega)} \begin{bmatrix} \Delta\rho \\ \Delta\eta \\ \Delta\omega \end{bmatrix} = B^* \begin{bmatrix} \Delta\rho \\ \rho\Delta\eta \\ \rho s\eta\Delta\omega \end{bmatrix} \tag{II -10}$$

where,

$$B^* = \left[\begin{array}{c|c|c} \frac{\partial F_\rho}{\partial \rho} & \frac{1}{\rho}\left(\frac{\partial F_\rho}{\partial \eta} - F_\eta\right) & \frac{1}{\rho s\eta}\left(\frac{\partial F_\rho}{\partial \omega} - F_\omega s\eta\right) \\ \hline \frac{\partial F_\eta}{\partial \rho} & \frac{1}{\rho}\left(\frac{\partial F_\eta}{\partial \eta} + F_\rho\right) & \frac{1}{\rho s\eta}\left(\frac{\partial F_\eta}{\partial \omega} - F_\omega c\eta\right) \\ \hline \frac{\partial F_\omega}{\partial \rho} & \frac{1}{\rho}\left(\frac{\partial F_\omega}{\partial \eta}\right) & \frac{1}{\rho s\eta}\left(\frac{\partial F}{\partial \omega} + F_\rho s\eta + F_\eta c\eta\right) \end{array}\right] \tag{II - 11}$$

Evaluating the partial derivatives in B*, using Eqn. (II-9), we arrive at the following ex ression.

$$B^* = \frac{\nu a^2}{\rho^3} \left[B^{(0)} + \sum_{s=1}^{\infty} K_s \left(\frac{a}{\rho}\right)^s B^{(s)}\right] \tag{II - 12}$$

where,

$$B^{(0)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$B^{(s)} = \begin{bmatrix} (s+1)\ (s+2)\Omega_s & -(s+2)\Omega_s' & -s(s+2)\dfrac{\hat{\Omega}_s}{s\eta} \\ -(s+2)\Omega_s' & [\Omega_s'' - (s+1)\ \Omega_s] & \left(\dfrac{\hat{\Omega}_s}{s\eta}\right)' \\ -(s+2)\dfrac{\hat{\Omega}}{s\eta} & \left(\dfrac{\hat{\Omega}_s}{s\eta}\right)' & [\Omega_s' \cot\eta - \Omega_s(s+1+\dfrac{m^2}{s^2\eta})] \end{bmatrix} \tag{II - 13}$$

Also, to a first order approximation,

$$\begin{bmatrix} \Delta\rho \\ \rho\Delta\eta \\ \rho s\eta\Delta\omega \end{bmatrix} = (T_3T_2)^{-1}\,\bar{r} \tag{II - 14}$$

where $\bar{r}$ is the postion vector expressed in the body axes frame, τ_3.

After substitution of Eqns. (II - 10) and (II - 14) into Eqn. (II - 7), there results,

$$\bar{F} \cong \bar{F}_0 + B^* (T_3\ T_2)^{-1}\,\bar{r} \tag{II - 15}$$

Reprojecting on the body fixed frame, we get

$$\bar{\mathcal{F}} \cong \bar{\mathcal{F}}_0 + M\,\bar{r} \tag{II - 16}$$

where,

$$\bar{\mathcal{F}}_0 = T_3\ T_2\ \bar{F}_0$$

$$M = T_3\ T_2\ B^* (T_3\ T_2)^{-1} \tag{II - 17}$$

If we write, $M = M^{(0)} + \Sigma\, K_s\, (\frac{a}{\rho})^s\, M^{(s)}$

then,

$$M^{(0)} = T_3\ T_2\ B^{(0)}(T_3\ T_2)^{-1}\ \frac{\nu a^2}{\rho^3} \tag{II - 18}$$

$$M^{(s)} = T_3\ T_2\ B^{(s)}(T_3\ T_2)^{-1}\ \frac{\nu a^2}{\rho^3}$$

The matrix $M^{(0)}$ is given in Eqn. (6).

Appendix - III

This appendix presents the expanded forms of the vector expressions in equations (28), (31), (32), (43), (44), (45), and (46).

Let us introduce the following notations,

$$H_{\alpha\beta}^{(n)} = \int_{vol} \xi_\alpha \, \phi_\beta^{(n)} \, dm$$

$$L_{\alpha\beta}^{(mn)} = \int_{vol} \phi_\alpha^{(m)} \, \phi_\beta^{(n)} \, dm \qquad \text{(III - 1)}$$

$$I_\alpha^{(n)} = \frac{1}{m} \int_{vol} \phi_\alpha^{(n)} \, dm$$

By the assumption $\bar{q} = \sum_{n=1}^{\infty} A_n(t) \, \bar{\Phi}^{(n)} (\bar{r}_0)$, we have

$$\dot{\bar{q}} = \sum_{n=1}^{\infty} \dot{A}_n(t) \, \bar{\Phi}^{(n)} (\bar{r}_0)$$

$$\text{and} \quad \ddot{\bar{q}} = \sum_{n=1}^{\infty} \ddot{A}_n(t) \, \bar{\Phi}^{(n)} (\bar{r}_0) \qquad \text{(III - 2)}$$

Consider Eqn. (28),

$$\sum_{n=1}^{\infty} \bar{Q}^{(n)} = \int_{vol} [\bar{r}_0 \times \ddot{\bar{q}} + 2 \, \bar{r}_0 \times (\bar{\omega} \times \dot{\bar{q}}) + \bar{r}_0 \times (\dot{\bar{\omega}} \times \bar{q})$$

$$+ \bar{q} \times (\dot{\bar{\omega}} \times \bar{r}_0) - (\bar{r}_0 \cdot \bar{\omega}) (\bar{\omega} \times \bar{q}) - (\bar{q} \cdot \bar{\omega}) (\bar{\omega} \times \bar{r}_0)] dm$$

$$\sum^{\infty} \bar{Q}^{(n)} = \sum_{n=1}^{\infty} \int_{vol} [\ddot{A}_n (\bar{r}_0 \times \bar{\Phi}^{(n)}) + 2 \, \dot{A}_n \, (\bar{r}_0 \times (\bar{\omega} \times \bar{\Phi}^{(n)}))$$

$$+ A_n \, \{\bar{r}_0 \times (\dot{\bar{\omega}} \times \bar{\Phi}^{(n)}) + (\bar{\Phi}^{(n)} \times (\dot{\bar{\omega}} \times \bar{r}_0)$$

$$- (\bar{r}_0 \cdot \bar{\omega}) (\bar{\omega} \times \bar{\Phi}^{(n)}) - (\bar{\Phi}^{(n)} \cdot \bar{\omega}) (\bar{\omega} \times \bar{r}_0)\}] dm \qquad \text{(III - 3)}$$

Using the methods of vector algebra, the above expression can be expanded in the component form to obtain

$$Q_x^{(n)} = \ddot{A}_n (H_{yz}^{(n)} - H_{zy}^{(n)}) + 2\,\dot{A}_n\,[(H_{yy}^{(n)} + H_{zz}^{(n)})\omega_x - H_{yx}^{(n)}\omega_y - H_{zx}^{(n)}\omega_z]$$

$$+ A_n[2(H_{yy}^{(n)} + H_{zz}^{(n)})\,\dot{\omega}_x - (H_{xy}^{(n)} + H_{yx}^{(n)})\dot{\omega}_y - (H_{zx}^{(n)} + H_{xz}^{(n)})\dot{\omega}_z$$

$$- 2\,\omega_y\omega_z\,(H_{zz}^{(n)} - H_{yy}^{(n)}) - \omega_x\omega_y\,(H_{xz}^{(n)} + H_{zx}^{(n)})$$

$$+ \omega_x\omega_z\,(H_{xy}^{(n)} + H_{yx}^{(n)}) + (\omega_z^{\ 2} - \omega_y^{\ 2})\,(H_{yz}^{(n)} + H_{zy}^{(n)})] \qquad \text{(III - 4)}$$

$Q_y^{(n)}$ and $Q_z^{(n)}$ are obtained by the cyclic permutation of x, y, z in the expression for $Q_x^{(n)}$.

Now consider equation (31), which is given by,

$$\bar{G}_R = \int_{vol} \bar{r}_0 \times M\,\bar{r}_0\,dm$$

On expanding in the component form,

$$\bar{G}_R = (J_z - J_y)\,M_{23}\,\hat{i} + (J_x - J_z)\,M_{31}\,\hat{j} + (J_y - J_x)\,M_{12}\,\hat{k} \qquad \text{(III - 5)}$$

where, M_{ij} is an element of M matrix

J_x, J_y, J_z are the principal moments of inertia of the body in the undeformed state.

Considering Eqn. (32) for $\bar{G}^{(n)}$, we have,

$$\sum_{n=1}^{\infty} G^{(n)} = \int_{vol} (\bar{r}_0 \times M\bar{q} + \bar{q} \times M\bar{r}_0)dm$$

$$= \sum_{n=1}^{\infty} A_n \int_{vol} (\bar{r}_0 \times M\,\bar{\Phi}^{(n)} + \bar{\Phi}^{(n)} \times M\,\bar{r}_0)dm \qquad \text{(III - 6)}$$

Hence, the components of (III - 6) are,

$$G_x^{(n)} = A_n[(M_{33} - M_{22})(H_{yz}^{(n)} + H_{zy}^{(n)}) - M_{21}(H_{xz}^{(n)} + H_{zx}^{(n)})$$

$$+ M_{31}(H_{xy}^{(n)} + H_{yx}^{(n)}) + 2M_{23}(H_{yy}^{(n)} - H_{zz}^{(n)})]$$

$G_y^{(n)}$ and $G_z^{(n)}$ components are obtained by the cyclic permutation of x, y, z in (III - 7).

Now consider the following scalar quantities.

Eqn. (43):

$$\phi_n = \int_{vol} [\overline{\Phi}^{(n)} \cdot \dot{\overline{\omega}} \times \overline{r}_0 + \overline{\Phi}^{(n)} \cdot \overline{\omega} \times (\overline{\omega} \times \overline{r}_0)]dm$$

$$= \dot{\omega}_x (H_{yz}^{(n)} - H_{zy}^{(n)}) + \dot{\omega}_y (H_{zx}^{(n)} - H_{xz}^{(n)}) + \dot{\omega}_z (H_{xy}^{(n)} - H_{yx}^{(n)})$$

$$+ \omega_x\omega_y (H_{xy}^{(n)} + H_{yx}^{(n)}) + \omega_y\omega_z (H_{yz}^{(n)} + H_{zy}^{(n)}) + \omega_z\omega_x (H_{zx}^{(n)} + H_{xz}^{(n)})$$

$$- \omega_x^2 (H_{yy}^{(n)} + H_{zz}^{(n)}) - \omega_y^2 (H_{zz}^{(n)} + H_{xx}^{(n)}) - \omega_z^2 (H_{xx}^{(n)} + H_{yy}^{(n)}) \qquad \text{(III - 8)}$$

Eqn. (44):

$$\sum_{m=1}^{\infty} \phi_{mn} = \int_{vol} [2\,\overline{\Phi}^{(n)} \cdot \overline{\omega} \times \dot{\overline{q}} + \overline{\Phi}^{(n)} \cdot \dot{\overline{\omega}} \times \overline{q} + \overline{\Phi}^{(n)} \cdot \overline{\omega} \times (\overline{\omega} \times \overline{q})]dm$$

$$= \sum_{m=1}^{\infty} \int_{vol} [2\,\dot{A}_m (\overline{\Phi}^{(n)} \cdot \overline{\omega} \times \overline{\Phi}^{(m)})$$

$$+ A_m(\overline{\Phi}^{(n)} \cdot \dot{\overline{\omega}} \times \overline{\Phi}^{(m)} + \overline{\Phi}^{(n)} \cdot \overline{\omega} \times (\overline{\omega} \times \overline{\Phi}^{(m)})]dm$$

Hence,

$$\phi_{mn} = 2\dot{A}_m[\omega_x (L_{yz}^{(mn)} - L_{zy}^{(mn)}) + \omega_y (L_{zx}^{(mn)} - L_{xz}^{(mn)}) + \omega_z (L_{xy}^{(mn)} - L_{yx}^{(mn)})]$$

$$+ A_m [\dot{\omega}_x (L_{yz}^{(mn)} - L_{zy}^{(mn)}) + \dot{\omega}_y (L_{zx}^{(mn)} - L_{xz}^{(mn)}) + \dot{\omega}_z (L_{xy}^{(mn)} - L_{yx}^{(mn)})$$

$$+ \omega_x\omega_y (L_{xy}^{(mn)} + L_{yx}^{(mn)}) + \omega_y\omega_z (L_{yz}^{(mn)} + L_{zy}^{(mn)}) + \omega_z\omega_x (L_{zx}^{(mn)} + L_{xz}^{(mn)})$$

$$- \omega_x^2 (L_{yy}^{(mn)} + L_{zz}^{(mn)}) - \omega_y^2 (L_{zz}^{(mn)} + L_{xx}^{(mn)}) - \omega_z^2 (L_{xx}^{(mn)} + L_{yy}^{(mn)})]$$

(III - 9)

Eqn. (45): $g_n = \int_{vol} \overline{\Phi}^{(n)} \cdot M\, \overline{r}_0 \, dm = \sum_{\alpha} \sum_{\beta} H^{(n)}_{\alpha\beta} M_{\alpha\beta}$ (III - 10)

Eqn. (46): $\sum_{m=1}^{\infty} g_{mn} = \int_{vol} \overline{\Phi}^{(n)} \cdot M\, \overline{q} \, dm = \sum_{m=1}^{\infty} A_m \int_{vol} \overline{\Phi}^{(n)} \cdot M\, \overline{\Phi}^{(m)} dm$

Hence,

$$g_{mn} = A_m \sum_{\alpha} \sum_{\beta} L^{(mn)}_{\alpha\beta} M_{\alpha\beta} \qquad \text{(III - 11)}$$

where, $\alpha\ \beta$ = x, y, z or 1, 2, 3. For example, when α is x in $H^{(n)}_{\alpha\beta}$, the corresponding value of α in $M_{\alpha\beta}$ is 1. In a similar way when α is y in $H^{(n)}_{\alpha\beta}$, α is 2 in $M_{\alpha\beta}$ and when α is z in $H^{(n)}_{\alpha\beta}$, α is 3 in $M_{\alpha\beta}$. Same reasoning holds for β also.

Appendix IV

<u>Natural frequencies, mode shapes and modal mass for free-free uniform beams:</u>

The natural frequencies of a free-free beam can be obtained by solving the following frequency equation.[9]

$$\cos \beta\ell \ \cosh \beta\ell = 1 \qquad \text{(IV - 1)}$$

where,

ℓ - length of the beam

$\beta^4 = \dfrac{\omega^2 m'}{EI}$

m' - mass per unit length of the beam

EI - bending modulus

ω - natural frequency

The mode shapes corresponding to the frequencies obtained by Eqn. (IV - 1) are given by,

$$Z'^{(n)}(x') = D_n\left[\left(\frac{\cos\beta_n\ell - \cosh\beta_n\ell}{\sinh\beta_n\ell - \sin\beta_n\ell}\right)(\sin\beta_n x' + \sinh\beta_n x') + (\cos\beta_n x' + \cosh\beta_n x')\right] \qquad \text{(IV - 2)}$$

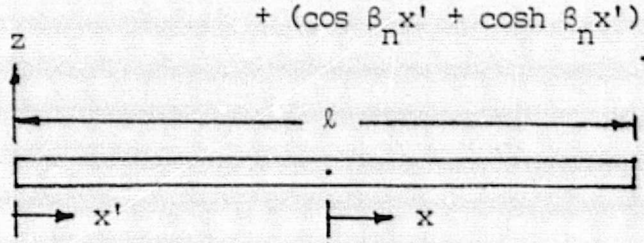


Fig. (D-1): Free-free beam

where, β_n ($n = 1,2, - - -$) are roots of Eqn. (IV - 1).

It should be noted that x' is measured from one end of the beam, as shown in the Fig (D-1). Hence, $x' = x + {}^{\ell}/_{2}$

$$\text{Define,} \quad \frac{x'}{\ell} = \zeta \qquad\qquad \text{(IV - 3)}$$

$$\beta_n \ell = \Omega_n$$

Hence, the non-dimensionalized form of the mode shape in Eq. (IV -2) is,

$$Z^{(n)}(\zeta) = \left[\left(\frac{\cos\Omega_n - \cosh\Omega_n}{\sinh\Omega_n - \sin\Omega_n}\right)(\sin\Omega_n \zeta + \sinh\Omega_n \zeta) + (\cos\Omega_n \zeta + \cosh\Omega_n \zeta)\right] \qquad \text{(IV - 4)}$$

The following Table - 1 gives the first five natural modes and the approximate values of the corresponding natural frequencies of a free-free beam.[10]

The modal mass (generalized mass) in the n^{th} mode is given by,

$$M_n = \int_V \phi_z^{(n)^2}(x)\ \mu\ dv \qquad \text{(IV - 5)}$$

Since the beam is assumed to be uniform, μ = constant,

Hence,

$$M_n = m' \int_{x=-\ell/2}^{+\ell/2} \phi_z^{(n)^2}(x)\ dx \qquad \text{(IV - 6)}$$

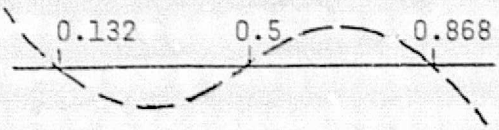
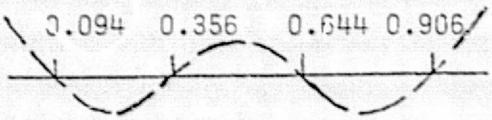
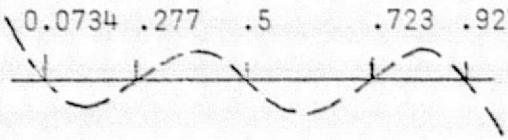
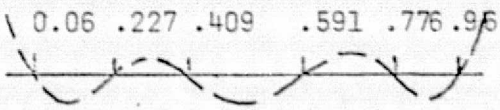
Using the change of variable $\zeta = \frac{x}{\ell} + \frac{1}{2}$, we get

$$M_n = m'\ell \int_0^1 [Z^{(n)}(\zeta)]^2 d\zeta \qquad \text{(IV - 7)}$$

Substituting (IV - 4) in (IV - 7),

$$M_n = D_n^2\, m'\ell \int_0^1 [K_n(\sin\Omega_n\zeta + \sinh\Omega_n\zeta) + (\cos\Omega_n\zeta + \cosh\Omega_n\zeta)]^2 d\zeta \qquad \text{(IV - 8)}$$

Table - 2: Mode shapes and frequencies of free-free beams

Mode shape	Frequency
$\vdash\!\!\rightarrow \zeta$ $\qquad\qquad$ $\zeta = 1$	
	$\omega_1 = 3.56\sqrt{\frac{EI}{m'\ell^4}}$
	$\omega_2 = 9.82\sqrt{\frac{EI}{m'\ell^4}}$
	$\omega_3 = 19.2\sqrt{\frac{EI}{m'\ell^4}}$
	$\omega_4 = 31.8\sqrt{\frac{EI}{m'\ell^4}}$
	$\omega_5 = 47.5\sqrt{\frac{EI}{m'\ell^4}}$

Note: The frequency value of zero corresponds to the rigid body modes.

where $K_n = (\cos \Omega_n - \cosh \Omega_n)/(\sinh \Omega_n - \sin \Omega_n)$

The various definite integrals appearing in Eqn. (II - 8) are evaluated in Table - 3.

Table - 3: Useful definite integrals

$$\int_0^1 \sin^2 \Omega_n \zeta \, d\zeta = \frac{1}{2} - \frac{\sin 2\Omega_n}{4\Omega_n}$$

$$\int_0^1 \sin \Omega_n \zeta \cos \Omega_n \zeta \, d\zeta = \frac{\sin^2 \Omega_n}{2\Omega_n}$$

$$\int_0^1 \sin \Omega_n \zeta \sinh \Omega_n \zeta \, d\zeta = \frac{\cosh \Omega_n \sin \Omega_n - \sinh \Omega_n \cos \Omega_n}{2\Omega_n}$$

$$\int_0^1 \sin \Omega_n \zeta \cosh \Omega_n \zeta \, d\zeta = \frac{\sinh \Omega_n \sin \Omega_n - \cosh \Omega_n \cos \Omega_n}{2\Omega_n} + \frac{1}{2\Omega_n}$$

$$\int_0^1 \cos^2 \Omega_n \zeta \, d\zeta = \frac{1}{2} + \frac{\sin 2\Omega_n}{4}$$

$$\int_0^1 \cos \Omega_n \zeta \sinh \Omega_n \zeta \, d\zeta = \frac{\cosh \Omega_n \cos \Omega_n + \sinh \Omega_n \sin \Omega_n - 1}{2\Omega_n}$$

$$\int_0^1 \cos \Omega_n \zeta \cosh \Omega_n \zeta \, d\zeta = \frac{\sinh \Omega_n \cos \Omega_n + \cosh \Omega_n \sin \Omega_n}{2\Omega_n}$$

$$\int_0^1 \sinh^2 \Omega_n \zeta \, d\zeta = \frac{\sinh 2\Omega_n}{4\Omega_n} - \frac{1}{2}$$

$$\int_0^1 \sinh \Omega_n\zeta \cosh \Omega_n\zeta \, d\zeta = \sinh^2\Omega_n/2\Omega_n$$

$$\int_0^1 \cosh^2\Omega_n\zeta \, d\zeta = \frac{\sinh 2\Omega_n}{4\Omega_n} + \frac{1}{2} .$$

Substituting the values of various definite integrals in Eqn. (IV - 8), after simplification we get

$$M_n = \frac{D_n^2 m' \ell}{\Omega_n} \Big[(K_n^2 + 1) \left(\frac{\sinh 2\Omega_n}{4} + \cosh \Omega_n \sin \Omega_n\right)$$

$$- (K_n^2 - 1) \left(\frac{\sin 2\Omega_n}{4} + \sinh \Omega_n \cos \Omega_n\right)$$

$$+ K_n (\sin \Omega_n + \sinh \Omega_n)^2 + \Omega_n \Big] \qquad \text{(IV - 9)}$$

```
*******************************************************************************
*                                                                             *
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*                                                                             *
*******************************************************************************

FORTRAN/B/E/P TMP T01.RB/B TMP.LS/L
      COMPILER DOUBLE PRECISION
C         FLEX BEAM MAIN PROGRAM
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7  QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
      COMMON /LOG/ SIZNOM(42), SIZMAX(42), LINES, LSTEP
      DIMENSION AUX(12, 42)

      OPEN 5,"SYSOUT",ATT="AP"
      DELETE "FLEXBEAM"
      OPEN 3, "FLEXBEAM"
      OPEN 2,"TMP",ATT="I"
      CALL FBINP(PRCSN)
      CALL FBSET(PRCSN)
      LINES = 0
      LSTEP = 0.20*( (PARM(2) - PARM(1))/PARM(3) ) + 0.5
      CALL RKGS(PARM,Y,DY,NORD,IHLF,AUX)
      DO 7 I = 1, NORD
          RATIO = (SIZMAX(I)/SIZNOM(I))*100.0
          WRITE (5, 91) I, RATIO
    7 CONTINUE
      WRITE(5,9) IHLF
      CALL EXIT

    9 FORMAT('0FINAL VALUE OF IHLF:',I3)
   91 FORMAT ('0Y(', I2, ') PEAK:', F6.1)
      END
TPLIST
PRINT 1
```

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```
DELETE TMP.LS
FORTRAN/R/E/P TMP TMP.LS/L FCT.RB/R
      COMPILER DOUBLE PRECISION
      SUBROUTINE  FCT (T,AY,ADY)
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7  QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD

              FBDIF--DIFF EQN'S

      CALL FBPIF
      CALL FBSIF
      DY(1)=ALPHA
      DY(2)=Y(1)
      DO 2 I=1,M
              DY(2*I+1)=FBA2R(I)
              DY(2*I+2)=Y(2*I+1)
    2 CONTINUE
      RETURN

      END
```

```
MPLIST
RINT 1
PRINT 1
ORTRAN/B/E/B TMP TMP,LS/L FBSET,RB/B
      COMPILER DOUBLE PRECISION
      SUBROUTINE FBSET(PRECSN)
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7  QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD

               FBSET -SET UP INITAL STATE
               (PRECSN=PRECISION, RELATIVE TO 1.0)

      CALL FBHST
      CALL FBLST
      Z=1.2448E-6
      CALL RKSCL(NORD,DY,DY,PRECSN,PARM)
      RETURN

      END
```

```
LIST
NT 1
RTRAN/R/F/P TMP TMP.LS/L FBHST.RB/R
      COMPILER DOUBLE PRECISION
      SUBROUTINE FBHST
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,RD,J2,J3,
     7  QC(20),QCP,ND,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD

C             FBHST--SET 'HTJ'

      DO 10 I=1,M
      H11(I)=0.0
      H13(I)=0.0
      H33(I)=0.0
      HS(I)=0.0
      HD(I)=0.0
   10 CONTINUE
      RETURN

      END
```

V2-1412

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```
MPLIST
PINT 1
!FORTRAN/B/E/P TMP TMP.LS/L FBLST.RB/B
      COMPILER DOUBLE PRECISION
      SUBROUTINE FBLST
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARY(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7  QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
C
C     FBLST--SET 'LTJ'
C
      DO 20 I=1,M
      DO 20 J=1,M
   20 L11(I,J)=0.0
      DO 30 I=1,M
      DO 30 J=1,M
   30 L13(I,J)=0.0
      DO 40 I=1,M
      DO 40 J=1,M
   40 L31(I,J)=0.0
      DO 50 I=1,M
      DO 60 J=1,M
   60 L33(I,J)=0.0
   50 L33(I,I)=1.0
      DO 61 I=1,M
      DO 61 J=1,M
      LD(I,J)=L31(I,J)-L13(I,J)
      LS(I,J)=L11(I,J)+L33(I,J)
   61 CONTINUE
      RETURN
C
C
      END
```

```
LIST
NT 1
RTRAN/R/E/P TMP TMP,LS/L FBPIF,RB/R
      COMPILER DOUBLE PRECISION
      SUBROUTINE FBPIF
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2 H11(20),H33(20),H13(20),
     3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4 M11,M33,M13,M31,MS,ALPHA,
     5 P(20),PP(20,20),G(20),GG(20,20),
     6 DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7 QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD

              FBPIF--PRIMARY INTERMEDIATE FUNCTIONS

      SPH=SIN(Y(2))
      CPH=COS(Y(2))
      M11=3*CPH**2-1
      M13=-3*SPH
      M31=-3*SPH*CPH
      M33=3*SPH**2-1
      MS=M13+M31
      R2=Y(1)+WD
      RETURN

      END
```

VR-1412

```
 MPLIST
 PINT 1
IFORTRAN/B/E/P TYP TMP,LS/L FBSIF,RB/B
      COMPILER DOUBLE PRECISION
      SUBROUTINE FBSIF
      INTEGER R
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2 H11(20),H33(20),H13(20),
     3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4 M11,M33,M13,M31,MS,ALPHA,
     5 P(20),PP(20,20),G(20),GG(20,20),
     6 DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7 QC(20),QCP,ND,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
C
C             FBSIF--SECONDARY INTERMEDIATE FUNCTIONS
C
      Y2=Y(2)
      TY2=2.0*Y2
      CTY2=COS(TY2)
      STY2=SIN(TY2)
      SUM1 = 0.0
      SUM2 = 0.0
      DO 3  R = 1, M
         SUM1 = SUM1 + 3.0*(H13(R)*CTY2 - 0.5*HD(R)*STY2)*Y(2*R+2)
     *                - HS(R)*R2*Y(2*R+1)
         SUM2 = SUM2 + HS(R)*Y(2*R+2)
    3 CONTINUE
      ALPHA = Z*(-1.5*J3*STY2 + 2.0*SUM1 - C2 - QCP) / (J2 + 2.0*SUM2)
      DO 70 R=1,M
      P(R)=-R2**2*HS(R)
      G(R)=Z*(H11(R)*M11+H33(R)*M33+2*H13(R)*MS)
      DO 80 N=1,M
      PP(R,N)=(2*Y(2*N+1)*R2+Y(2*N+2)*ALPHA)*LD(R,N)
     2-Y(2*N+2)*R2**2*LS(R,N)
      GG(R,N)=Z*Y(2*N+2)*(L11(R,N)*M11+L13(R,N)*M13+L31(R,N)*M31
     2+L33(R,N)*M33)
   80 CONTINUE
   70 CONTINUE
      RETURN
C
C
      END
```

```
!IMPLTST
!PRINT 1
!FORTRAN/B/E/P TMP TMP.LS/L FBA2R.RB/B
      COMPILER DOUBLE PRECISION
      FUNCTION FBA2R(R)
      INTEGER R
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2 H11(20),H33(20),H13(20),
     3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4 M11,M33,M13,M31,MS,ALPHA,
     5 P(20),PP(20,20),G(20),GG(20,20),
     6 DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
     7 QC(20,,QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
C
C             FBA2R--EVALUATE 2-ND DER OF A(R)
C
      SUM=0.0
      DO 90 N=1,M
      SUM=SUM-PP(R,N)+GG(R,N)
   90 CONTINUE
      FBA2R=-DW(R)**2*Y(2*R+2)-P(R)+SUM+G(R)+E(R)+QC(R)
      RETURN
C
C
      END
```

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```
!TMPLIST
!PRINT 1
!LFE N DP0:FLEXBEAM5.LB/0
#  OUTP.RB FBSET.RB FBHST.RB FBLST.RB FBPTF.RB FBSIF.RB FBA2R.RB
!DELETE
#  OUTP.RB FBSET.RB FBHST.RB FBLST.RB FBPTF.RB FBSIF.RB FBA2R.RB
!GTOD
!DELETE TMP.LS
!FORTRAN/B/E/P TMP TMP.LS/L OUTP.RB/B
      COMPILER DOUBLE PRECISION
      SUBROUTINE  OUTP(T,AY,ADY,IHLF,N,APARM)
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,R0,J2,J3,
     7  QC(20),QCP,W0,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
      COMMON /LOG/ SIZNOM(42), SIZMAX(42), LINES, LSTEP
      LOGICAL  RKNXT
C
C                FBOUT --TAKE OUTPUT VALUES
C
      DO 2 I = 1, NORD
           YMAG = ABS(Y(I))
           IF(YMAG .GT. SIZMAX(I)) SIZMAX(I) = YMAG
    2 CONTINUE
      IF (.NOT. RKNXT(IHLF))  GO TO 8
           WRITE BINARY(3) T, IHLF,(Y(I),I=1,NORD)
           IT = ((T - PARM(1))/PARM(2))*100.0 + 0.5
           IF (MOD(LINES,LSTEP) .EQ. 0 )TYPE IT,IHLF
           LINES = LINES + 1
    8 CONTINUE
      RETURN
C
C
      END
```

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```
!FORTRAN/R/F/P TMP T03.RB/B TMP.LS/L
      COMPILER DOUBLE PRECISION
       SUBROUTINE FBINP (PRCSN)
      REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
      COMMON /BEAM/  PARM(5),Y(42),DY(42),
     2  H11(20),H33(20),H13(20),
     3  L11(20,20),L33(20,20),L13(20,20),L31(20,20),
     4  M11,M33,M13,M31,MS,ALPHA,
     5  P(20),PP(20,20),G(20),GG(20,20),
     6  DW(20),HS(20),HD(20),Z,NU,A,RD,J2,J3,
     7  QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
      COMMON /LOG/ SIZNOM(42), SIZMAX(42), LINES, LSTEP
C
C                 FLEXBEAM -- INPUT
C
      MMP1 = 2*M + 1
      MMP2 = 2*M + 2
      READ (2, 81) (PARM(I), I = 1, 3), PRCSN
      WRITE (5, 91) (PARM(I), I = 1, 3), PRCSN
      READ (2, 82) Y(2), Y(1), DY(2), DY(1)
      WRITE (5, 92) Y(2), Y(1), DY(2), DY(1)
      DO 2 L = 1, M
          I1 = 2*L + 2
          I2 = 2*L  + 1
          READ (2, 82) Y(I1), Y(I2), DY(I1), DY(I2), DW(L)
          WRITE (5,93) L, Y(I1), Y(I2), DY(I1), DY(I2), DW(L)
    2 CONTINUE
      DO 4 I = 1, NORD
          SIZNOM(I) = DY(I)
          SIZMAX(I) = 0.0
    4 CONTINUE
      RETURN
C
C
   81 FORMAT (4F10.0)
   82 FORMAT (7F10.0, F9.0, 1X)
   91 FORMAT ('0T FROM', F7.1, '  TO', F7.1,  '  BY', F7.1, 8X,
     2   ' PRECISION =', F9.6)
   92 FORMAT (10X, 'INIT VAL   INIT DER      SCALE VAL   SCALE DER',
     2   6X, 'OMEGA' / ' THETA ', 2F11.6, F13.6, F12.6)
   93 FORMAT (' MODE', I2, 2F11.6, F13.6, F12.6, F13.6)
       END
!TMPLIST
!PRINT 1
```

```
!FORTRAN/B/E/P TMP TO2.RB/B TMP.LS/L
     COMPILER DOUBLE PRECISION
          BLOCK DATA
     REAL L11,L13,L31,L33,M11,M13,M31,M33,MS,J2,J3,LS,LD,NU
     COMMON /BEAM/  PARM(5),Y(42),DY(42),
    2 H11(20),H33(20),H13(20),
    3 L11(20,20),L33(20,20),L13(20,20),L31(20,20),
    4 M11,M33,M13,M31,MS,ALPHA,
    5 P(20),PP(20,20),G(20),GG(20,20),
    6 DW(20),HS(20),HD(20),Z,NU,A,RO,J2,J3,
    7 QC(20),QCP,WD,E(20),R2,LS(20,20),LD(20,20),C2,M,NORD
        COMMON  /RKNXO/  IHOLD, NEXT
C
C
     DATA M/5/, NORD/12/
     DATA NU/9.797/,A/6378388./,RO/6881388./,J2/825500./
     DATA J3/825500./,QCP/0.0/,WD/.001115/,C2/0.0/
     DATA Y/42*0.0/
     DATA DY/42*0.0/
     DATA DW/20*0.0/
     DATA QC/0.0,19*0.0/
     DATA E/0.0,19*0.0/
     DATA PARM/5*0.0/
        DATA  IHOLD/0/, NEXT/1/
C
C
     END
!MPLIST
!PRINT I
```

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```
**************************************************************************************************

        FLEXBEAM INPUT DATA
DP0:FLEXBEAM
.       300.0       .5          1.0
2       0.0         .2          .00050
5        0.0        .5          .03          .06283185
4       0.0         .4          .003          .1733170807
3       0.0         .3          .003         .33740705
2       0.0         .2          .003         .5579468
1       0.0         .1          .003         .8337786
GOOD
DELETE DP0:FLEXBEAM
XFER FLEXBEAM DP0:FLEXBEAM
```

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```
*******************************************************************************
*
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*
*******************************************************************************
C
C         FLEXBEAM OUTPUT PROGRAM
// FORTRAN
      PARAMETER NORD = 12, POINTS = 601, MODES = 5
      DOUBLE PRECISION  TT, Y(NORD)
      DIMENSION T(POINTS)
      DIMENSION PHI(POINTS)
      DIMENSION A(POINTS, MODES)
C
C
      CALL INOUT (2, 5)
      CALL FOPEN (3, "FLEXBEAM")
      DO 2 I = 1, POINTS
          READ BINARY (3) TT, IHLF, Y
          T(I) = SNGL(TT)
          PHI(I) = SNGL(Y(2))
          DO 1 J = 1, MODES
              A(I, J) = SNGL(Y(2*J+2))
    1 CONTINUE
    2 CONTINUE
      CALL DFPLO(POINTS, MODES, A)
      CALL EXIT
C
      END
```

```
// FORTRAN
      PARAMETER MAXM = 5, NX = 101
      PARAMETER IT1 = 95, IT2 = 116, DT = 2
      PARAMETER DMAX = 3.0
      SUBROUTINE DEPLO(NPTS, MODES, A)
          DIMENSION A(NPTS, MODES), SHAPE(NX , MAXM),DEF(NX)
          DOUBLE PRECISION OMEGA
          COMMON /DEPLD/ OMEGA(MAXM)
          DATA OMEGA /4.7300408D0, 7.8532046D0, 10.9956078D0,
     2                 14.137165D0, 17.2787596D0/
C
C
      CALL DEPLS (OMEGA, MODES, NX, SHAPE)
      CALL PSIZE (5.0, 11.0)
      CALL PGRID (1, 11)
      CALL PLOGO (0.0, 11.0)
      CALL PSIZE (5.0, 1.0)
      DO 8 IT = IT1, IT2, DT
          CALL PLOGO (0.0, - 1.0)
          DO 5 IX = 1, NX
              Q=0.0
              DO 4 IM = 1, MODES
                  Q = Q + A(IT, IM)*SHAPE(IX,IM)
    4         CONTINUE
              DEF(IX) = Q
    5     CONTINUE
          CALL PLOX (-DMAX, DEF, DMAX, NX)
    8 CONTINUE
      RETURN
C
C
      END
```

```
// FORTRAN
      SUBROUTINE DFPLS(OMEGA, MODES, NX, SHAPE)
           DOUBLE PRECISION OMEGA (MODES)
           DIMENSION SHAPE (NX, MODES)
           DOUBLE PRECISION DZ, W, EW, SHW, CHW, COEF
           DOUBLE PRECISION ZETA, WZ, EWZ, SHWZ, CHWZ, DSHAPE
C
C
      DZ = 1.0D0/DFLOAT(NX - 1)
      DO 8 IM = 1, MODES
           W = OMEGA(IM)
           EW = DEXP(W)
           SHW = (EW - 1.0D0/EW)/2.0D0
           CHW = (EW + 1.0D0/EW)/2.0D0
           COEF =(DCOS(W) - CHW)/(SHW - DSIN(W))
           DO 4 IX = 1, NX
               ZETA = DFLOAT(IX - 1)*DZ
               WZ = W*ZETA
               EWZ = DEXP(WZ)
               SHWZ = (EWZ - 1.0D0/EWZ)/2.0D0
               CHWZ = (EWZ + 1.0D0/EWZ)/2.0D0
               DSHAPE = COEF*(DSIN(WZ) + SHWZ) + (DCOS(WZ) + CHWZ)
               SHAPE(IX, IM) = SNGL(DSHAPE)
    4      CONTINUE
    8 CONTINUE
      RETURN
C
C
      END
// LOAD PLOT/L
!XFER  DP0:FLEXBEAM  FLEXBEAM
// XEQ
```

```
      DO 4 L = 1,41
          IPHI = 10*L - 9
          IA1A = L
          IA1B = L + 960
    4 CONTINUE
      DO 6 L = 42, 101
          IPHI = 10*L - 9
    6 CONTINUE
          WRITE (5,91) T(IPHI), PHI(IPHI), T(IA1A), A(IA1A,1), A(IA1A,2)
     I                   ,T(IA1B), A(IA1B,1), A(IA1B,2)
          WRITE (5,91) T(IPHI), PHI(IPHI)
   91 FORMAT (1X, F8.1, F14.9, 2(F13.1,2F14.9))
```

ORIGINAL PAGE IS
OF POOR QUALITY

```
      PLOT AMPLITUDE VS. TIME
   CALL PSIZE (6.0, 5.0)
   CALL PBOX
   CALL PAXES
   CALL PLOX (-0.5, A(1,1), 0.5, POINTS)
   CALL PLOX (-0.5, A(1,2), 0.5, POINTS)
   CALL PLOX (-0.5, A(1,3), 0.5, POINTS)
   CALL PLOX (-0.5, A(1,4), 0.5, POINTS)
   CALL PLOX (-0.5, A(1,5), 0.5, POINTS)
```