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A GENERAL SPECTRAL TRANSFORMATION SIMULTANEOUSLY INCLUDING A
FOURIER TRANSFORMATION AND A LAPLACE TRANSFORMATION

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16. Abstract

A general spectral transformation is proposed and described. Its spectrum can be interpreted as a Fourier spectrum or a Laplace spectrum. Hence it describes the relationship between these two transformations. The laws and functions of the new method are discussed in comparison with the known transformations, and a sample application is shown.

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# A GENERAL SPECTRAL TRANSFORMATION SIMULTANEOUSLY INCLUDING A FOURIER TRANSFORMATION AND A LAPLACE TRANSFORMATION

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A general spectral transformation is proposed and described. Its spectrum can be interpreted as a Fourier spectrum or a Laplace spectrum. Hence it describes the relationship between these two transformations. Using this transformation, general time functions, e.g. functions with exponential growth in both negative and positive time directions, can be represented. This general spectral transformation is characterized by the introduction of two complex frequency variables (p and q) for the positive and the negative time domain, respectively. Thus it is possible to express the spectral distributions of the Fourier transformation (the Dirac function and its derivatives) by rational functions. The laws and operations of the new method are discussed in comparison with the known transformations. With the general spectral transformation it is possible to define the stability criteria for general (i.e. causal and non-causal) systems. As an example, its application is shown for the solution of linear differential equations, taking into account the initial state due to past system excitation.

## 1. Introduction

Spectral transformations have gradually become the most important calculation procedure in information technology. Unfortunately, there are two procedures, the Fourier transformation [1] and the Laplace transformation [2], each valid under different

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* Numbers in the margin indicate pagination in the foreign text.

preconditions. It is also impossible to limit oneself to only one of these methods. Of course, the Laplace transformation is always applicable when causal signals are involved (time functions only for t > 0) and when one wants to describe realizable systems. It then has the advantage of yielding the spectrum by an algebraic expression of the complex frequency p (in many cases a rationally broken function).

With the increased use of the system theory introduced by K. Küpfmüller [3], however, it is also necessary to consider non-causal signals and non-realizable (idealized) systems. The Laplace transformation cannot be used for this, and one must resort to the Fourier transformation. When working with statistical signal theory, one must also used the Fourier transformation. Of course, for stationary or semistationary processes the Fourier spectrum contains distributions (i.e. the Dirac function and its derivatives), which is an undesirable trait that often impedes mathematical handling. There also arises the question of whether there are different spectra for the same time signal, and what these differences may be. This question yields the further question of whether there might be a still more general spectral transformation that would include both the Fourier and the Laplace transformation, and from which both processes would develop as special cases. Such a transformation must on the one hand not be limited to causal signals, and must on the other hand permit representation of all the processes accessible to the Laplace transformation, e.g. functions that develop exponentially.

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The following describes such a general spectral transformation and discusses its laws and peculiarities. In particular, it also establishes the connection between the Fourier and Laplace transformations and permits conversion of their spectra. The present report can contain only a short sketch of the procedure. The reasoning behind the general spectral transformation

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given in detail in [4]; the necessary proofs and the derivation of their laws are also contained there[1].

## 2. Common Grounds and Differences between the Fourier and Laplace Transformations

Both procedures are based on the Fourier integral:

$$U(f) = \int_{-\infty}^{+\infty} u(t)\, e^{-j2\pi ft}\, dt, \qquad (1)$$

$$u(t) = \int_{-\infty}^{+\infty} U(f)\, e^{+j2\pi ft}\, df. \qquad (2)$$

The differences are essentially rooted in the manner in which the convergence of this integral is reached. Namely, with stationary (or unilaterally stationary) time procedures, the Fourier integral does not converge.

The Fourier transformation forces convergence by means of a "convergence factor", e.g. $e^{-\varepsilon|t|}$, yielding

$$U_F(f) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} u(t)\, e^{-\varepsilon|t|}\, e^{-j2\pi ft}\, dt, \qquad (3)$$

$$u(t) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} U(f)\, e^{-\varepsilon|f|}\, e^{+j2\pi ft}\, df. \qquad (4)$$

(In Eq. 4 a convergence factor $e^{-\varepsilon|f|}$ was introduced for reasons of symmetry.) Thus it is possible to represent stationary processes, but one gets distributions by passing the limit $\varepsilon \to 0$. For instance, for u(t) = 1, one gets

$$U_F(f) = \delta(f),$$

where (f) is the Dirac function with the function integral $\int_{-\infty}^{+\infty} \delta(f)\,df = 1$ and $\delta(f) = 0$ for $f \neq 0$.

For the convergence of the Fourier integral the time function must be exponentially limited, i.e. it must satisfy the condition

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[1] The author wishes to thank his colleague Prof. Bordewijk, Delft Technical University, and his assistant W. Wolf, for stimulating discussions on this theme.

$$\int_{-\infty}^{+\infty} |u(t)| e^{-\epsilon|t|} dt < \infty$$

for any $\epsilon > 0$. Thus time functions such as $u(t) = ct^n$ are permitted and are represented in the Fourier spectrum by derivatives of the Dirac function.

The Laplace transform is limited to causal time processes for which $u(t) = 0$ for $t < 0$. By a substitution, $p = j\omega = j2\pi f$, it achieves the convergence of the Fourier integral for complex values of p. In unilaterally stationary processes, $\text{Re}(p) > 0$ is additionally required. The transformation equations are

$$U_L(p) = \int_0^\infty u(t) e^{-pt} dt, \qquad (5)$$

$$u(t) = \frac{1}{2\pi j} \int_{s-j\infty}^{s+j\infty} U_L(p) e^{pt} dp. \qquad (6)$$

For the reverse transformation here should be chosen large anough that the integration path running parallel to the imaginary axis runs within the convergence domain of $U_L(p)$. In this way even unilaterally exponential processes can be represented. The reversal formula (6) can be transformed to a ring integral separately for $t < 0$ and $t > 0$, using Jordan's lemma, and can then be evaluated simply using the residue theorem of function theory. The result of evaluation is called a Heaviside development theorem.



Fig. 1. Pole position of a single pole in the complex frequency plane, and corresponding time function according to Laplace and Fourier trnasformations.
Key: a. p-plane  b. according to
c. real

To compare the two procedures, we now observe a spectrum as in Fig. 1, with a single real pole at point a, i.e. the spectrum

$$U(p) = \frac{1}{p-a}. \qquad (7)$$

Depending on whether a < 0, a = 0 or a > 0, the reverse transformation yields different results via the Laplace and Fourier transformations. For a > 0 (Fig. 1, Case 1), the result is the same with both methods. For a < 0 (Case 3) it is different, because the integration path with the Fourier transformation runs along the imaginary axis of the p plane (for real frequencies) -- i.e. left of the pole -- while with the Laplace transform it is right of the pole. The exponentially developing process appears in the right time domain (t > 0) with the Laplace transformation, while it is reverse-poled and in the left domain (t < 0) with the Fourier transformation. The case of a pole directly on the imaginary axis for a = 0 (Case 2) is especially difficult, for now one must know how the Fourier integral is to be evaluated. If one takes Cauchy's main value, one gets $\frac{1}{2}$sgn(t) with the Fourier transformation. With the Laplace transformation one gets the unit step for t > 0. Thus one and the same spectrum yields different time functions with /365 the two procedures, or in other words, the "spectrum" as defined up to now is ambiguous and one must know which method is being used.

## 3. Definition of a General Spectral Transformation

One can now combine both methods and find a common representation, if one assigns the time domains t > 0 and t < 0 different complex frequency symbols. We choose p for t > 0 and q for t < 0, and write

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$$U(p,q) = \int_0^\infty u(t)\, e^{-pt}\, dt + \int_{-\infty}^0 u(t)\, e^{-qt}\, dt =$$
$$= U_+(p) + U_-(q),\tag{8}$$
$$u(t) = \frac{1}{2\pi j}\int_{(J)} U(\lambda, \lambda)\, e^{\lambda t}\, d\lambda.\tag{9}$$

We get a spectrum $U_{(p,q)}$ with two differentiated complex frequency variables, p and q. Here the partial spectra $U_{+(p)}$ and $U_{-(q)}$ can be calculated by a right- or left-side Laplace transformation, for

$$U_+(p) = \mathcal{L}\{u(t)\},\tag{10}$$
$$U_-(-q) = \mathcal{L}\{u(-t)\}.\tag{11}$$

For the reverse transformation the integration path J from $-j\infty$ to $+j\infty$ is run so that all poles $p_\nu$ of $U_+(p)$ are left of it and all poles $q_\nu$ of $U_-(q)$ are right of it. Under these conditions, $p = q = \lambda$ can again be established for the reverse transformation. Fig. 2 shows the integration path for $U(\lambda,\lambda)$ in the $\lambda$ plane. The p poles $p_\nu$ are shown by the symbol ✕ and the q poles of $q_\nu$ are shown with the symbol ✕. Since the position of the p and q poles is arbitrary, this method can represent time processes that develop exponentially on both sides. With respect to representability, the general spectral transformation thus combines the possibilities of the Fourier and the Laplace transformations.
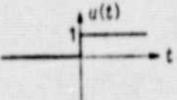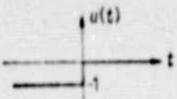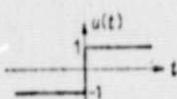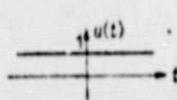


Fig. 2. Integration path in the $\lambda$ plane with the general spectral transformation.
Key: a. $\lambda$ plane
b. real

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| Diagram | Time function $u(t)$ | $U(p,q)$ | Spectrum $U_F(f)$ | Symbol |
|---|---|---|---|---|
|  $u(t)$ | $\gamma(t)$ | $\dfrac{1}{p}$ | $\dfrac{1}{j\omega} + \pi\,\delta(\omega)$ | ⋈ |
|  $u(t)$ | $-\gamma(-t)$ | $\dfrac{1}{q}$ | $\dfrac{1}{j\omega} - \pi\,\delta(\omega)$ | ⋈ |
|  $u(t)$ | $\mathrm{sgn}(t)$ | $\dfrac{1}{p} + \dfrac{1}{q}$ | $\dfrac{1}{j\omega}$ | × |
|  $u(t)$ | $1$ | $\dfrac{1}{p} - \dfrac{1}{q}$ | $2\pi\,\delta(\omega)$ | ↑ |

Fig. 3.  General spectral transformation; time function and spectrum of a single pole at the frequency zero point.

Fig. 3 shows the time function, general spectrum and Fourier spectrum for a single pole in the frequency zero point, and Fig. 4 shows the same for a multiple pole.  One can see that the distributions occurring with the Fourier spectrum (as well as their derivatives with the multiple pole) can be represented by rationally broken functions in p and q.

One gets the Fourier spectrum from the general spectrum by the limit passage

$$U_F(f) = \lim_{\varepsilon \to 0} U(p + \varepsilon, q - \varepsilon). \qquad (12)$$

For the Laplace spectrum

$$U_L(p) = U_+(p), \qquad (13)$$

applies, i.e. the partial spectrum $U_-(q)$ dependent on q is

| Diagram | Time function u(t) | U(p,q) | Spectrum U_F(f) | Symbol |
|---|---|---|---|---|
|  u(t) | $\gamma(t)\dfrac{t^{\mu-1}}{(\mu-1)!}$ | $\dfrac{1}{p^\mu}$ | $\dfrac{1}{(j\omega)^\mu}+\dfrac{\pi j^{\mu-1}}{(\mu-1)!}\delta^{(\mu-1)}(\omega)$ | ⋊ |
|  u(t) | $-\gamma(-t)\dfrac{t^{\mu-1}}{(\mu-1)!}$ | $\dfrac{1}{q^\mu}$ | $\dfrac{1}{(j\omega)^\mu}-\dfrac{\pi j^{\mu-1}}{(\mu-1)!}\delta^{(\mu-1)}(\omega)$ | ⋉ |
|  u(t) | $\mathrm{sgn}(t)\dfrac{t^{\mu-1}}{(\mu-1)!}$ | $\dfrac{1}{p^\mu}+\dfrac{1}{q^\mu}$ | $\dfrac{2}{(j\omega)^\mu}$ | × |
|  u(t) | $\dfrac{t^{\mu-1}}{(\mu-1)!}$ | $\dfrac{1}{p^\mu}-\dfrac{1}{q^\mu}$ | $\dfrac{2\pi j^{\mu-1}}{(\mu-1)!}\delta^{(\mu-1)}(\omega)$ | ↑ |

Fig. 4. General spectral transformation. Time function and spectrum of a μ-multiple pole in the frequency zero point.

simply omitted.

For instance, for the spectrum of u(t) = 1 (direct current),

$$\delta(f) = \lim_{\varepsilon \to 0}\left(\frac{1}{p+\varepsilon}-\frac{1}{q-\varepsilon}\right). \qquad (14)$$

applies. This means that the following correspondence applies for the spectral Dirac function (spectral line):

$$\delta(f) \triangleq \frac{1}{p}-\frac{1}{q}. \qquad (15)$$

For the Dirac function differentiated $\rho$ times, likewise[2]

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[2] The conversion $\delta^{(\rho)}(f) = (2\pi)^{\rho+1}\delta^{(\rho)}(\omega)$ applies.

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$$\delta^{(\varrho)}(f) \triangleq (-2\pi j)^\varrho \varrho! \cdot \left(\frac{1}{p^{\varrho+1}} - \frac{1}{q^{\varrho+1}}\right). \qquad (16)$$

Thus spectral distributions of the $\varrho$th order correspond to the difference between p and q poles of the $\varrho+1^{th}$ order. According to Fig. 4 this includes time functions increasing with the power of $\varrho$. If one only wants the causal portion (for $t > 0$)(Laplace spectrum), one simply omits the q component.

Through the initial complication of having to use two frequency variables p and q, one gains the advantage of always being able to distinguish the positive and negative time domain, even in the spectrum and of obtaining the Fourier spectrum (if it exists) by passing a limit. The advantage of the Laplace transformation is maintained, namely spectral representation by rational functions in p and q instead of distributions, and thus the possibility of applying the rules of function theory (residue theorem). In addition, one can represent time processes that grow exponentially to both sides. Further to be noted is that the procedure is significantly more efficient than the already-known bilateral Laplace transform, which requires the convergence of the spectrum in a strip of the p plane parallel to the imaginary axis.

## 4. The Development Theorem According to Eigenfunctions

If $U_{(p,q)}$ is given by a rational broken function, one can get a development theorem corresponding to the Heaviside theorem of the Laplace transformation. First one must split off an entire function in p or q, setting $p = q = \lambda$. The remaining function component $U_{+(p)}$ is developed into a partial fraction series with the coefficients $A_{\nu\mu}$ of the p poles $p_\nu$ of the order $u_{p\nu}$. One proceeds correspondingly with the function component $U_{-(q)}$ and gets a partial fraction development with the coefficients $B_{\nu\mu}$ of the q poles $q_\nu$ of the order $u_{q\nu}$:

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$$U(p,q) = \sum_{(\mu)} c_\mu \lambda^\mu + \sum_{(\nu)} \sum_{\mu=1}^{u_{p_\nu}} \frac{A_{\nu\mu}}{(p-p_\nu)^\mu} + \\ + \sum_{(\nu)} \sum_{\mu=1}^{u_{q_\nu}} \frac{B_{\nu\mu}}{(q-q_\nu)^\mu}. \qquad (17)$$

By member-by-member transformation into the time domain one gets:

$$u(t) = \sum_{(\mu)} c_\mu \delta^{(\mu)}(t) + \gamma(t) \sum_{(\nu)} \sum_{\mu=1}^{u_{p_\nu}} A_{\nu\mu} \frac{t^{\mu-1}}{(\mu-1)!} e^{p_\nu t} - \\ - \gamma(-t) \sum_{(\nu)} \sum_{\mu=1}^{u_{q_\nu}} B_{\nu\mu} \frac{t^{\mu-1}}{(\mu-1)!} e^{q_\nu t}. \qquad (18)$$

Here $\delta^{(\mu)}(t)$ is the $\mu^{th}$ time derivative of the temporal Dirac function $\delta(t)$. The function

$$\gamma(t) = \begin{cases} 1 \text{ for } t > 0 \\ 0 \text{ for } t < 0 \end{cases}$$

is the unit step. Aside from the distributions in the time zero point one gets a development according to the eigenfunctions at the pole points, which are exponential oscillations of the polar frequencies multiplied by a power of t. The separation of the two time domains (p poles for t > 0 and q poles for t < 0) is very evident here.

## 5. Operations

Table 1 contains the most important operations of spectral transformation, comparing Fournier, Laplace and general spectral transformations. For the derivation and proof of the latter one must refer to [4]. Some peculiarities should be noted.

### 5.1. Causal and Acausal Differentiation and Integration

By distinguishing p from q in the spectrum, not only the time domain but also the time direction becomes distinguishable. For in the usual limit passage that occurs in forming the temporal differential quotients, the causal relation (distinction

## TABLE 1.  SPECTRAL TRANSFORMATION OPERATIONS

| Type | $u(t)$ | $U_F(f)$ | $U_L(p)$ | $U(p, q)$ |
|---|---|---|---|---|
| Linearity | $a_1 u_1(t) + a_2 u_2(t)$ | $a_1 U_{1F}(f) + a_2 U_{2F}(f)$ | $a_1 U_{1L}(p) + a_2 U_{2L}(p)$ | $a_1 U_1(p, q) + a_2 U_2(p, q)$ |
| Similarity | $u(at)$ $(a$ reell$)$ a | $\dfrac{1}{\lvert a \rvert} U_F\left(\dfrac{f}{a}\right)$ | $\dfrac{1}{a} U_L\left(\dfrac{p}{a}\right) \ (a > 0)$ | $\dfrac{1}{a} U\left(\dfrac{p}{a}, \dfrac{q}{a}\right) \quad a > 0$ $\dfrac{(-1)}{a} U\left(\dfrac{q}{a}, \dfrac{p}{a}\right) a < 0$ |
| Time displacement | $u(t - t_0)$ $(t_0$ reell$)$ a | $U_F(f)\, e^{-j2\pi t_0 f}$ | $U_L(p)\, e^{-pt_0}$ $(t_0 > 0)$ | $U(p, q)\, e^{-pt_0} \quad t_0 > 0$ $U(p, q)\, e^{-qt_0} \quad t_0 < 0$ |
| Frequency displacement | $u(t)\, e^{at}$ | $U_F(f - f_0)$ $(a = j2\pi f_0,\ f_0$ reell$)$ a | $U_L(p - a)$ | $U(p - a, q - a)$ |
| Attenuation theorem | $u(t)\, e^{-a\lvert t \rvert}$ $(a$ reell$)$ a | — | $U_L(p + a)$ | $U(p + a, q - a)$ |
| Temporal differentiation | $\dfrac{du}{dt}$ | $U_F(f)\, j2\pi f$ | $U_L(p) - u(+0)$ | $U(p, q)\, p$ kausal b $U(p, q)\, q$ akausal c |
| Temporal integration | $\displaystyle\int_{-\infty}^{t} u(t)\, dt$ $\displaystyle\int_{+\infty}^{t} u(t)\, dt$ | $U_F(f)\left(\dfrac{1}{j2\pi f} + \dfrac{1}{2}\delta(f)\right)$ $U_F(f)\left(\dfrac{1}{j2\pi f} - \dfrac{1}{2}\delta(f)\right)$ | $\dfrac{U_L(p)}{p}$ $\dfrac{U_L(p) - U_L(0)}{p}$ | $\dfrac{U(p, q)}{p}$ kausal b $\dfrac{U(p, q)}{q}$ akausal c |
| Spectral differentiation | $-t\, u(t)$ | $\dfrac{1}{j2\pi}\dfrac{dU_F(f)}{df}$ | $\dfrac{dU_L(p)}{dp}$ | $\dfrac{dU_+(p)}{dp} + \dfrac{dU_-(q)}{dq}$ |
| Spectral integration | $\dfrac{u(t)}{(-t)}$ | $j2\pi\left(\displaystyle\int_{-\infty}^{f} U_F(f)\, df - \tfrac{1}{2}u(0)\right)$ | $\displaystyle\int_{+\infty}^{p} U_L(p)\, dp$ | $\displaystyle\int_{+\infty}^{p} U_+(p)\, dp + \displaystyle\int_{-\infty}^{q} U_-(q)\, dq$ |
| Spectral folding | $u_1(t) \cdot u_2(t)$ | $U_{1F}(f) * U_{2F}(f)$ | $U_{1L}(p) * U_{2L}(p)$ | $U_1(p, q) * U_2(p, q) =$ $= U_{+1}(p) * U_{+2}(p) +$ $+ U_{-1}(q) * U_{-2}(q)$ |
| Temporal folding | $u_1(t) * u_2(t)$ | $U_{1F}(f) \cdot U_{2F}(f)$ | $U_{1L}(p) \cdot U_{2L}(p)$ | $U_1(p, q) \cdot U_2(p, q)$ $\left(\begin{matrix}\mathrm{Re}(p_{1r}) < \mathrm{Re}(q_{2r}) \\ \mathrm{Re}(p_{2r}) < \mathrm{Re}(q_{1r})\end{matrix}\right)$ |
| Causal component | $\gamma(t) \cdot u(t)$ | $\tfrac{1}{2}(U_F(f) - j\hat{U}_F(f))$ | $U_L(p)$ | $U_+(p) = U(p, q) * 1/p$ |
| Acausal component | $\gamma(-t) \cdot u(t)$ | $\tfrac{1}{2}(U_F(f) + j\hat{U}_F(f))$ | $0$ | $U_-(q) = U(p, q) * \dfrac{1}{(-q)}$ |
| Hilbert transformation of the Fourier spectrum | $\mathrm{sgn}(t) \cdot u(t)$ | $-j\hat{U}_F(f)$ | $U_L(p)$ | $U_+(p) - U_-(q)$ |

Key:  a. reell = real
b. kausal = causal
c. akausal = acausal

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between cause and effect) is lost. This has the result that
in differential equations from physics time is symmetrical and
apparently directionless, which has already led to may contra-
dictions in understanding the development of events. (In the
author's opinion it is unnecessary to resort to the Second Law
of Thermodynamics to give time a direction on the grounds of
irreversibility.) For if one forms the differential quotients
while observing the causality requirement, i.e. with the stipu-
lation that only past function values may be used, then ($\Delta t > 0$)

$$\left(\frac{du}{dt}\right)_k = \lim_{\Delta t \to 0} \frac{u(t) - u(t - \Delta t)}{\Delta t}. \qquad (19)$$

Considering the displacement theorem

$$u(t - \Delta t) \circ\!-\!\bullet\, U(p, q)\, e^{-p\,\Delta t} \quad \text{for } \Delta t > 0 \qquad (20)$$

this yields, after the limit has been passed, to the operation

$$\left(\frac{du}{dt}\right)_k \circ\!-\!\bullet\, U(p, q)\, p, \qquad (21)$$

which is designated as a causal differentiation. Correspondingly,
an "acausal" differentiation in which the future function values
are used,

$$\left(\frac{du}{dt}\right)_a = \lim_{\Delta t \to 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} \qquad (22)$$

would lead, with the help of the correspondence of the displace-
ment theorem

$$u(t + \Delta t) \circ\!-\!\bullet\, U(p, q)\, e^{q\,\Delta t} \quad \text{for } \Delta t > 0 \qquad (23)$$

to the operation

$$\left(\frac{du}{dt}\right)_a \circ\!-\!\bullet\, U(p, q)\, q \qquad (24)$$

This means that a "causal" differentiation requires the multi-

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plication of the spectrum by p, and an "acausal" differentiation requires multiplication by q. Thus a temporally causal differential equation yields a spectral power expression only in p, and thus the conversion factor S of a causal system is also only a function of p:

$$S = S_+(p). \tag{25}$$

The impulse response $s(t)\circ\!\!-\!\!\bullet\ S_{+(p)}$ is thus a causal temporal function. By requiring a causal differentiation, the direction of time established, which would otherwise not be the case.

The equivalent situation applies to integration, in which past values of u(t) are used in the causal case and future values in the acausal case. Accordingly, the spectrum must be multiplied by $\frac{1}{p}$ (causal) or $\frac{1}{q}$ (acausal).

## 5.2. Multiplication and Folding

In multiplying two general spectra one should note that there are no restrictions on the products of p-partial spectra and q-partial spectra. However, with a mixed product of p and q partial spectra one must require a parallel to the imaginary axis, so that all p poles of the one factor are left of the axis and all q poles of the other are right of it. The folding of the corresponding time functions corresponds to the product of the spectra.

On the other hand, the folding of the spectra corresponds to the products of the time functions. From this it follows that folding a p-spectrum with a q-spectrum must always yield zero:

$$U_1(p) * U_2(q) = 0. \tag{26}$$

Spectral folding may be defined by a complex integral as follows:

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$$U_1(p,q) * U_2(p,q) = \tag{27}$$

$$= \frac{1}{2\pi j} \int_{(J_1)} U_1(\lambda,\lambda) \, U_2(p-\lambda,q-\lambda) \, d\lambda.$$

Here the integration path $J_1$ of $U_1$ is used, and one should choose $Re(p) > Re(\lambda) + Re(p_{2v})$ and $Re(q) < Re(\lambda) + Re(q_{2v})$ in performing the integration.

Using spectral folding one can calculate partial spectra, if they are not already available:

$$U_+(p) = U(p,q) * \frac{1}{p}, \tag{28}$$

$$U_-(q) = U(p,q) * \left(-\frac{1}{q}\right). \tag{29}$$

This corresponds to a multiplication of the time function by $\gamma(t)$ or $\gamma(-t)$.

## 5.3. Limit Value Theorems

If u(t) at t = 0 has no distribution, one can calculate the left and right-hand initial values for t = +0 or t = -0 from the partial spectrum by the following limit passages:

$$u(+0) = \lim_{Re(p) \to \infty} [p \cdot U_+(p)], \tag{30}$$

$$u(-0) = \lim_{Re(q) \to -\infty} [-q \, U_-(q)]. \tag{31}$$

Likewise, for the asymptotic final values at t = +∞ or t = -∞ (if these exist),

$$u(+\infty) = \lim_{p \to 0} [p \, U_+(p)], \tag{32}$$

$$u(-\infty) = \lim_{q \to 0} [-q \, U_-(q)]. \tag{33}$$

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## 5.4. Hilbert Transformation

The Hilbert transformation of the Fourier spectrum

$$\hat{U}(f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{U(\varphi)}{f - \varphi} \, d\varphi$$

can be expressed especially simply using the general spectrum. Here

$$u(t) \circ \!\!\!-\!\!\!\bullet U_F(f) \triangleq U_+(p) + U_-(q), \qquad (34)$$

$$\text{sgn}(t) \, u(t) \circ \!\!\!-\!\!\!\bullet \frac{1}{j} \hat{U}_F(f) \triangleq U_+(p) - U_-(q). \qquad (35)$$

applies. One thus gets the Hilbert-transformed $\hat{U}_F(f)$ from $U_F(f)$ by reversing the sign of the q partial spectrum and multiplying by j. The correspondence sign $\triangleq$ indicates that the Fourier spectrum is obtained from the general spectrum by limit passage (Eq. 12).

## 6. Stability Criteria for General Systems

The general spectral transformation permits statements on stability for general -- i.e. non-causal -- system functions such as occur in system theory. Since the relationship $S(p,q)$ $\bullet\!\!\!-\!\!\!\circ s(t)$ applies, where $S(p,q)$ is the system function (conversion factor) and $s(t)$ is the impulse response, the following three stability criteria apply:

a) If $s(t)$ is restricted by energy, i.e.

$$\int_{-\infty}^{+\infty} |s(t)|^2 \, dt < \infty,$$

then all p poles of $S(p,q)$ must be in the open left half of the frequency plane and all q poles must be in the open right half. The imaginary axis (including $\omega = \infty$) is poleless.

b) If s(t) is restricted by power, i.e.

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |s(t)|^2 \, dt < \infty,$$

then the p poles of S(p,q) must be in the closed left half of the frequency plane and the q poles must be in the closed right half, i.e. including the imaginary axis. On the imaginary axis (including $\omega = \infty$) the poles are single.

c) If s(t) is exponentially restricted, i.e.

$$\int_{-\infty}^{+\infty} |s(t)| \, e^{-\epsilon|t|} \, dt < \infty$$

for all positive $\epsilon$ , then all p poles of S(p,q) must be in the closed left half of the frequency plane and all q poles must be in the closed right half. Multiple poles may occur on the imaginary axis (including $\omega = \infty$).

(Note: Such systems develop, for instance, in series circuits of ideal integration amplifiers.)

In addition to system theory, which uses idealized but not immediately realizable acausal systems (for which s(t) = 0 does not apply for t < 0), such systems also occur practically, if one replaces time t with place x. This is the case in "homogenous layer system theory"[5], which serves to describe neuron systems (nerve networks in a layer-like structure). Here the same stability criteria apply with respect to the local stimulus transmittance of such systems. The same problems arise, for instance, in realizing digital recursive filters for picture processing.

## 7. Solution of Differential Equations with Initial Values

Here the Laplace transformation is actually particularly

well suited, since it can take account of the charge condition
of the system as expressed in the initial values. On the other
hand it cannot include charging that occurred in the past. Here
the general spectral transformation establishes the relationship.
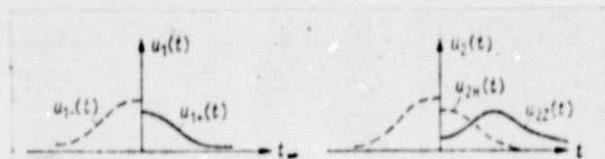


Fig. 5. Time graphs of the compulsive force
$u_1(t)$ and the effect $u_2(t)$. - - - - = charging
of the system by $u_{1-}(t)$ and homogenous solution
$u_{2H}(t)$ for $t > 0$; ——— = compulsive force
$u_{1+}(t)$ and its effect $u_{2Z}(t)$ for $t > 0$.

In Fig. 5 the dotted line represents the charging process
of the system and the solid line represents the compulsive force
$u_{1+}(t)$ and its effect $u_{2+}(t) = u_{2H}(t) + u_{2Z}(t)$. The term $u_{2H}(t)$
is the homogenous solution of the differential equation for $t > 0$
taking into account the intial state. $u_{2Z}(t)$ is the effect of
the compulsive force $u_{1+}(t)$. Both components should be calcu-
lated separately. From the differential equation of the system
(which is assumed to be causal),

$$\sum_{r=0}^{n} a_r u_2^{(r)}(t) = \sum_{r=0}^{m} b_r u_1^{(r)}(t) \qquad (36)$$

one gets by transformation into the spectral domain using the
general spectral transformation:

$$\sum_{r=0}^{n} a_r p^r U_2(p,q) = \sum_{r=0}^{m} b_r p^r U_1(p,q). \qquad (37)$$

This yields

$$U_2(p,q) = S(p) \cdot U_1(p,q). \qquad (38)$$

Here the causal system function

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$$S(p) = \sum_{r=0}^{m} b_r \, p^r \Big/ \sum_{r=0}^{n} a_r \, p^r \qquad (39)$$

is determined by the (real) coefficients of the presumably causal differential equation (36). With the partial spectra of $U_1$, corresponding to both time domains,

$$U_I(\dot{p}, q) = \dot{U}_{1+}(p) + \dot{U}_{1-}(q), \qquad (40)$$

one gets

$$U_2(p, q) = S(p) \cdot U_{1+}(p) + S(p) \cdot U_{1-}(q). \qquad (41)$$

The first term in Eq. (41) describes the effect of the compulsive force $u_{1+}(t)$ commencing at $t = 0$, so that

$$u_{2z}(t) \circ\!\!-\!\!\bullet \, S(p) \cdot U_{1+}(p). \qquad (42)$$

The second term of Eq. (41) describes the influence of the past, i.e. the effect of the cause $u_{1-}(t)$. Of this, only the /370 $t > 0$ component is of interest -- namely the homogenous solution of the differential equation $u_{2H}(t)$. We must thus calculate the p-partial spectrum from the spectral product using Eq.(28), and get

$$u_{2H}(t) \circ\!\!-\!\!\bullet \, [S(p) \cdot U_{1-}(q)] * \frac{1}{p}. \qquad (43)$$

This spectral expression represents a causal system function $H(p)$ describing the homogenous solution (because of the charging):

$$H(p) = [S(p) \, U_{1-}(q)] * \frac{1}{p}. \qquad (44)$$

Thus the replacement circuit in Fig. 6 applies, in which the effect of the compulsive power starting at $t = 0$ and the homogenous solution are represented separately. The latter is pro-

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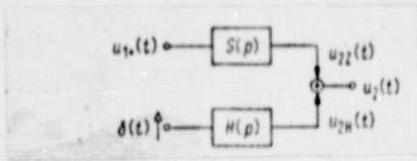duced by a Dirac function $\delta(t)$ over $H(p)$ acting at $t = 0$.



Fig. 6. Replacement circuit diagram of the differential equation for the effect of the compulsive force $u_{2Z}(t)$ (for $t > 0$) and the homogenous solution $u_{2H}(t)$ based on the charging of the system.

Eq.(44) can also be transformed into the time domain, yielding for the homogenous solution

$$u_{2H}(t) = [s(t) * u_{1-}(t)] \cdot \gamma(t). \tag{45}$$

Here an ambiguity is permissible insofar as totally different functions $u_{1-}(t)$ lead to the same function $u_{2H}(t)$. With the determination of $H(p)$, however, the process is established unambiguously. $H(p)$ describes the charge condition of the system at the time $t = 0$. On the other hand, $H(p)$ can also be calculated using the Laplace transformation. If one applies the differentiation theorem of the Laplace transformation, which takes into account the initial condition, one gets, for $t > 0$, $u_{1+}(t) = 0$ as the compulsive force disappears (cf.[4]).

$$H(p) = \sum_{\nu=0}^{n-1} c_\nu p^\nu \Big/ \sum_{\nu=0}^{n} a_\nu p^\nu. \tag{46}$$

Here the denominator polynomial is the same as that of $S(p)$ and the coefficients of the numerator polynomial are given by the initial values of the homogenous solution for $t = +0$ and its derivatives.

$$c_\nu = \sum_{\varrho=0}^{n-1-\nu} a_{1+\nu+\varrho} u_{2H}^{(\varrho)}(+0). \tag{47}$$

Since $u_{2H} \circ\!\!-\!\!\bullet H(p)$ applies, these initial values may also be determined in reverse from $H(p)$ via the limit theorem (30):

$$u_{2u}^{(\rho)}(+0) = \lim_{\text{Re}(p)\to\infty} [p^{\rho+1} H(p)]. \qquad (48)$$

If one knows the coefficient $c_\nu$, whether by knowing H(p) from Eq.(44) in development towards Eq.(46), or from the initial value from Eq.(470, one can get the analog computer realization for the solution of the differential equation as in Fig. 7. (Here the real constant k is an arbitrary time factor and $\tau$ is the time constant of the integration amplifier used, determined by $u_2 = (\frac{1}{\tau}\int u_1 dt$.)
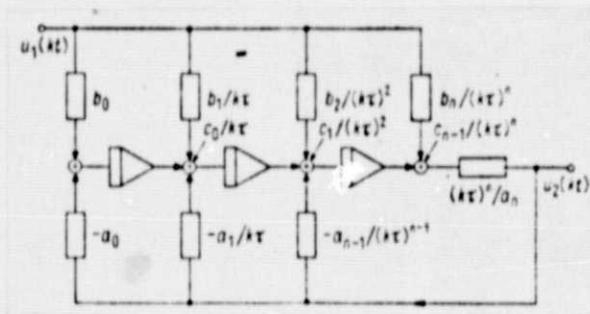


Fig. 7. Realization of the differential equation in analog computer technology with integration amplifiers of the system function $1/p\tau$ and a time scale factor k. The coefficients $a_\nu$ and b are given by Eq.(36) or (39), and the coefficients $c_\nu$ are given by Eq.(46) or (47).

The initial state of the system, which is expressed in the coefficients $c_\nu$ and which must be known for a realization by analog computer, can thus be determined from the system's past using the general spectral transformation according to Eq.(44) and (46).

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