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Final Report on NASA NSG-1367

"Finite Element Analysis of Periodic Transonic Flow Problems"

by

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1. The physical problem.

We considered a flow about an oscillating thin airfoil in a transonic stream. We assume that the flow field can be decomposed into a mean flow plus a periodic perturbation. Letting $\varphi(x,y)$ denote the velocity potential associated with the latter we have

$$(1) \quad [[K - (\gamma-1)u_o]\varphi_x]_x + \varphi_{yy} - (2i\omega/\epsilon)\varphi_x$$
$$+ [\omega^2/\epsilon - i\omega(\gamma-1)u_{ox}]\varphi = 0,$$

where $K = (1-M^2)/M^2\epsilon$, M is the free stream Mach number, γ is the ratio of specific heats, $\epsilon = (\delta/M)^{2/3}$, δ is a measure of the airfoil thickness, $u_0(x,y)$ is the mean flow velocity, x and y have been suitably scaled, and φ is the velocity potential of the flow perturbation. On the surface of the airfoil the usual Neumman conditions are imposed. For the problem considered here, this boundary condition takes the form

$$(2) \quad \varphi_y = (f_\pm)_x + i\omega(f_\pm) \quad \text{for} \quad y = \pm 0, \quad x \quad \text{on the wing}$$

where $f_+(x)$ and $f_-(x)$ described the spatial behavior of the upper and lower surfaces of the wing, respectively.

Since the mean flow is transonic, (1) is a partial differential equation of the mixed type. Furthermore, for large enough values of the frequency, (1) is an indefinite equation. If in the far field the flow is uniform and subsonic, (1) reduces to

$$(3) \quad (1-M^2)\varphi_{xx} - 2i\omega M\varphi_x + \varphi_{yy} + \omega^2\varphi = 0,$$

which is a convected Helmholtz equation.

Of crucial importance is the correct radiation at infinity. M. Gunzburger* has shown that this is given by

$$(4) \qquad (\tfrac{x}{R})\varphi_x + (\tfrac{y}{R})\varphi_y + \frac{i\omega M^2}{(1-M^2)}[1 - \tfrac{x}{R}]\varphi = 0$$

where $R = (x^2 + (1 - M^2)y^2)^{1/2}$ is large.

To complete the specification of the problem we need to impose a condition across the wake. Following standard practice we assume that the wake is a line starting at the trailing edge of the airfoil. Continuity of pressure and normal velocity are required, and assuming the airfoil lies on the line $y = 0$, these become

$$[\varphi_x + i\omega\varphi] = 0$$

$(5) \hspace{4cm} y = 0, \ x \quad \text{in the wake region}$

$$[\varphi_y] = 0$$

where

$$(6) \qquad [\xi] = \xi(x,0+) - \xi(x,0-).$$

The problems of numerical approximation are two fold.** The first and more serious one is that for a range of ω of physical interest, the matrix problem resulting from a discritization of the partial differential equations is indefinite and non-symmetric. This has made the use of standard iterative techniques for solving this sparse matrix system impossible to use. The second difficulty is that different descritizations

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*See Final Report on NASA contract NSG-1366.

**W. H. Weatherill, F. E. Ehlers, and J. D. Sebastian, NASA CR-2599, 1975.

had to be used in the hyperbolic region and the elliptic region.
The new method outlined in the next section obviates both of
these difficulties.

2. The least squares finite element method.

As a first step we reduce the governing system (1), (2), (3)
and (4) into an equivalent first order system

$$[[K - (\gamma + 1)u_0]u]_x + v_y - (2i\omega/\epsilon)u + [\omega^2/\epsilon - i\omega(\gamma-1)u_{0x}]\varphi = 0$$

$$\varphi_x - u = 0$$

$$\varphi_y - v = 0$$

$$v = f_x + i\omega f \quad \text{on the wing}$$

$$\Delta u + i\omega\Delta\varphi = 0$$

$$\text{on the wake}$$

$$\Delta v = 0$$

$$\frac{x}{R} u + \frac{y}{R} v + \frac{i\omega M^2}{1-M^2} (1 - \frac{x}{R})\varphi = 0 \quad \text{in the far field.}$$

Gunzburger's radiation condition (3) will be imposed on a rectangle
enclosing the wing, and whose edges are "far removed" from the
wing. The least squares method consists of minimizing, over a
finite element space of vector valued function, a functional
which is obtained by taking the square of the absolute value of
each of the above terms and integrating over the appropriate .
region.

This yields a linear system of algebraic equations whose coefficient matrix is <u>banded</u>, <u>sparse</u>, <u>Hermitian</u> <u>and</u> <u>positive</u> <u>definite</u>. Therefore iterative methods, which take full advantage of the sparcity, can be used. Furthermore, the method uses the same discritization technique in elliptic and hyperbolic regions.

3. Implementation of method.

Two computer programs have been written, both using linear basis functions over triangles for the finite element space. The first program uses a banded Gaussian elimination solver to solve the matrix problem, while the second uses an iterative technique, namely SOR. Both programs are considered to be working, but certainly not optimized. The only results obtained are for an oscillating flat plate (not a true transonic problem) although the programs have been written to admit general input. The major accomplishment of these programs is their success for values of ω, using SOR, well above the critical value at which other approaches failed.

The following projects would be of immediate interest:
a) Improving the accuracy of the codes by using different grids. Theoretical and numerical experiments on model problems have shown that a triangularization of the type (i) below yields answers on order of magnitude more accurate than that using type (ii) (which is presently being used in the program). It is a relatively trivial matter to convert the program to type (ii) triangles.

(i)   (ii)

b)   Inputing some realistic transonic mean flows. The programs
are built to accept such input, so this is also a relatively easy
matter.

c)   Optimize the codes to achieve maximum efficiency.

d)   Write an extensive report detailing the method and results of
its implementation.

Of longer range interest are the solution of 3-D problems
and of problems with complicated geometries. In both cases the
difficulty of the task is alleviated by the least squares method,
in the first case because of the ability to use iterative methods,
and in the second by the flexibility of finite element methods.