NASA Contractor Report 3016

# Lectures on Algebraic System Theory: Linear Systems Over Rings

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Prepared for
Ames Research Center
under Contract A-43119-B

NASA

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CONTRACT A-43119-B
JULY 1978

NASA
National Aeronautics
and Space Administration

Scientific and Technical
Information Office

1978

# PREFACE

This report provides an introduction to the theory of linear systems defined over rings. The theory is presented with a minimum of mathematical details, so that anyone having some knowledge of linear system theory should find the work easy to read. The presentation centers on four classes of systems that can be treated as linear systems over a ring. These are (i) discrete-time systems over a ring of scalars such as the integers; (ii) continuous-time systems containing time delays; (iii) large-scale discrete-time systems; (iv) time-varying discrete-time systems.

The material given here is an expanded version of a series of lectures given at NASA Ames Research Center, Moffett Field, California during the week of November 7, 1977. I am very grateful to Professor R. E. Kalman and Brian Doolin for the opportunity of giving these lectures. My gratitude also extends to Eduardo Sontag, Yves Rouchaleau, and Bostwick Wyman for countless hours of discussions on linear systems over rings. Part of the research work presented here was supported by the U. S. Army Research Office, Research Triangle Park, N. C., under Grant DAAG29-77-G-0085.

# 1. INTRODUCTION

## 1.1 Linear systems over fields.

Since the late 1950's, a great deal of effort has been devoted to the study of linear finite-dimensional continuous-time and discrete-time systems specified by state equations defined over a field K. More precisely, given a triple (F,G,H) of $n\times n$, $n\times m$, $p\times n$ matrices over a field K, a m-input p-output n-dimensional time-invariant discrete-time system is defined by the state equations

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned} \tag{1.1}$$

where $t\in Z$ = set of integers. In the continuous-time case, with K = R = field of real numbers, a time-invariant system is given by the state equations

$$\begin{aligned} \frac{dx(t)}{dt} &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned} \tag{1.2}$$

where $t\in R$. In both (1.1) and (1.2), the state x(t), input u(t), and output y(t) are column vectors over the field K.

Initial efforts in studying linear systems defined over a field K usually assumed that K was a particular infinite field, such as the field R of real numbers. Infinite fields are fields that contain a subset which can be put into a one-to-one correspondence with the set of integers. In addition to the field R, the field of rational numbers and the field of complex numbers are examples of infinite fields. There are several textbooks (e.g., [1,2,3]) on the theory of time-invariant and time-varying linear systems defined over the field R.

In a sequence of publications beginning in 1965, R. E. Kalman [4,5,6] developed an algebraic theory for discrete-time systems of the form (1.1) defined over an arbitrary (finite or infinite) field K. Hence, in addition to being applicable to systems over the real or complex numbers, Kalman's theory can be applied to systems over finite fields, which includes linear sequential circuits [7].

1.2 Discrete-time systems over a ring of scalars.

After the completion of his work on discrete-time systems over arbitrary fields, Kalman initiated the study of discrete-time systems over rings. The notion of a ring is a generalization of the notion of a field: A ring is a set with two operations, called addition and multiplication; however, unlike a field, a ring can contain nonzero elements that do not have a multiplicative inverse (see[8, Chapter II]).

An example of a ring is the set of integers Z with the usual addition and multiplication operations. Since $n \in Z$ has a multiplicative inverse belonging to Z if and only if $n = 1$ or $-1$, Z is not a field. Another example of a ring is the set K[z] of polynomials in a symbol z with coefficients in a field K, with the usual operations of polynomial addition and multiplication. The only elements in K[z] that are invertible are the nonzero polynomials of degree zero.

The first detailed results dealing with discrete-time systems over arbitrary commutative rings were derived in Rouchaleau's Ph.D. thesis [9] in 1972 under the supervision of R. E. Kalman. This work was concerned with the class of linear time-invariant discrete-time systems over a commutative ring A given by the state equations

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned} \tag{1.3}$$

where $t \in Z$, F, G, H are $n\times n$, $n\times m$, $p\times n$ matrices over the ring A, and $x(t)$, $u(t)$, $y(t)$ are column vectors over A.

An interesting example of a system over a ring is a system with the ring of scalars equal to Z. By definition, such a system accepts vector sequences over Z, processes these sequences using integer operations, and then outputs vector sequences over Z. These systems are of interest from a computational standpoint, since integer operations can be implemented "exactly" on a digital computer (assuming that magnitudes are less than $10^{12}$ for 12-digit precision). Further, systems over Z appear in various applications, such as coding theory [10] and scheduling or sequencing problems [11].

## 1.3 Continuous-time systems over rings of operators.

In 1973, Kamen [12] showed that a large class of linear infinite-dimensional continuous-time systems can be represented by first-order vector differential equations defined over a ring of operators (due to publication delays, [12] did not appear in print until 1975). The ring approach developed in [12] and [13,14] applies to time-invariant and time-varying continuous-time systems containing pure and distributed time delays. Independently of this work, Williams and Zakian [15,16] constructed the same type of operator setting for a class of time-invariant continuous-time systems with pure time delays.

In this section we shall define the operator framework for the class of systems with commensurate delays given by the dynamical equations

$$\begin{aligned} \frac{dx(t)}{dt} &= \sum_{i=0}^{q} F_i x(t - ia) + \sum_{i=0}^{r} G_i u(t - ia) \\ y(t) &= \sum_{i=0}^{s} H_i x(t - ia) \end{aligned} \tag{1.4}$$

where a is a fixed positive number, the $F_i$, $G_i$, $H_i$ are $n\times n$, $n\times m$, $p\times n$ matrices over the field of real numbers R, and $x(t)$, $u(t)$, $y(t)$ are column vectors over R.

The n-vector $x(t)$ in (1.4) is usually referred to as the state at time t; however, as a result of the delay terms, the <u>actual state</u> at time t is the function segment $x(\tau)$, $t - qa \le \tau \le t$. To solve (1.4) for $t > 0$, we need to know the actual state at time $t = 0$, which consists of the function segment $x(\tau)$, $-qa \le \tau \le 0$.

In the mathematics literature, equations of the form (1.4), or generalizations of (1.4), are usually treated as ordinary differential equations in a Banach or Hilbert space consisting of function segments (see [17,18,19]). In contrast, we shall view (1.4) as a vector differential equation with coefficients belonging to a ring of delay operators. The constructions are as follows.

Let V denote the linear space consisting of all R-valued functions defined on R with support bounded on the left (i.e., for each $v \in V$, there is a $t_v \in R$ such that $v(t) = 0$ for all $t < t_v$). Let $d: V \to V$ denote the a-second delay operator on V defined by $(dv)(t) = v(t-a)$, $v \in V$. We can extend d to column vectors $v = (v_1, \ldots, v_n)'$ over V by defining

$$(dv)(t) = (v_1(t-a), \ldots, v_n(t-a))'$$

where the prime denotes the transpose operation.

Let R[d] denote the set of all finite sums $\sum_i a_i d^i$ where $a_i \in R$ and $(d^i v)(t) = v(t-ia)$, $v \in V$. With the usual operations of polynomial addition and multiplication, R[d] is a ring of delay operators.

Now viewing the state trajectory x and the input function u as column vectors over V (or some subspace of V), we can write (1.4) in the form

$$\frac{dx(t)}{dt} = (F(d)x)(t) + (G(d)u)(t)$$
$$y(t) = (H(d)x)(t) \tag{1.5}$$

where $F(d)$, $G(d)$, $H(d)$ are $n\times n$, $n\times m$, $p\times n$ matrices over the operator ring $R[d]$ given by

$$F(d) = \sum_{i=0}^{q} F_i d^i, \quad G(d) = \sum_{i=0}^{r} G_i d^i, \quad H(d) = \sum_{i=0}^{s} H_i d^i$$

Thus the class of time-delay systems given by (1.4) can be studied in terms of the vector differential equation (1.5) defined over the ring $R[d]$. We shall refer to the system (1.5) as a linear time-invariant continuous-time system over the ring of operators $R[d]$.

EXAMPLE 1.1 Consider the time-delay system given by the dynamical equations

$$\frac{dx_1(t)}{dt} = -x_1(t-a) + x_2(t) - x_2(t-2a) + u(t)$$

$$\frac{dx_2(t)}{dt} = x_1(t) + x_1(t-a) - x_2(t) + u(t-a)$$

$$y(t) = x_1(t-a) - x_1(t-2a) + x_2(t)$$

This system of equations can be written in the form (1.5) with

$$F(d) = \begin{bmatrix} -d & 1-d^2 \\ 1+d & -1 \end{bmatrix} \qquad G(d) = \begin{bmatrix} 1 \\ d \end{bmatrix} \qquad H(d) = \begin{bmatrix} d-d^2 & 1 \end{bmatrix}$$

By exploiting the structure of the operator-ring representation (1.5), we can develop an algebraic theory for the class of time-delay systems given by (1.4). In particular, we can obtain constructive results given in terms of operations on matrices and vectors defined over the ring R[d], or ring extensions of R[d]. Examples will be given in the next three chapters of these notes.

As noted above, systems with noncommensurate delays and distributed delays can be studied in terms of differential equations over rings of operators. Also, as shown in [14], initial data for equations of the form (1.5), or generalizations of (1.5), can be incorporated into the operator framework.

Note that in the representation (1.5), the elements of the ring R[d] act on functions; whereas in (1.3), the elements of the ring A act on values of functions. This is the reason for referring to (1.5) as a system over a ring of operators and (1.3) as a system over a ring of scalars. We shall show that both types of systems can be studied using the same techniques.

1.4 Discrete-time systems over a ring of operators.

In this section we shall show that there are discrete-time systems that can be treated as systems over a ring of operators. Consider the class of discrete-time systems given by the following dynamical equations

$$\begin{aligned} x(t+1) &= \sum_{i=0}^{q} F_i x(t-i) + \sum_{i=0}^{r} G_i u(t-i) \\ y(t) &= \sum_{i=0}^{s} H_i x(t-i) \end{aligned} \tag{1.6}$$

where $t \in Z$, the $F_i, G_i, H_i$ are $n\times n$, $n\times m$, $p\times n$ matrices over a field K, and $x(t)$, $u(t)$, $y(t)$ are column vectors over K.

Note the similarity between (1.4) and (1.6). In fact, if we proceed as we did above for continuous-time systems with time delays, it is clear that we can write (1.6) as a vector difference equation over the operator ring $K[\sigma]$ consisting of all polynomials in the right-shift operator $\sigma: x(t) \to x(t-1)$ with coefficients belonging to the field K. More precisely, we have that

$$\begin{aligned} x(t+1) &= (F(\sigma)x)(t) + (G(\sigma)u)(t) \\ y(t) &= (H(\sigma)x)(t) \end{aligned} \tag{1.7}$$

where $F(\sigma)$, $G(\sigma)$, $H(\sigma)$ are matrices over $K[\sigma]$ given by

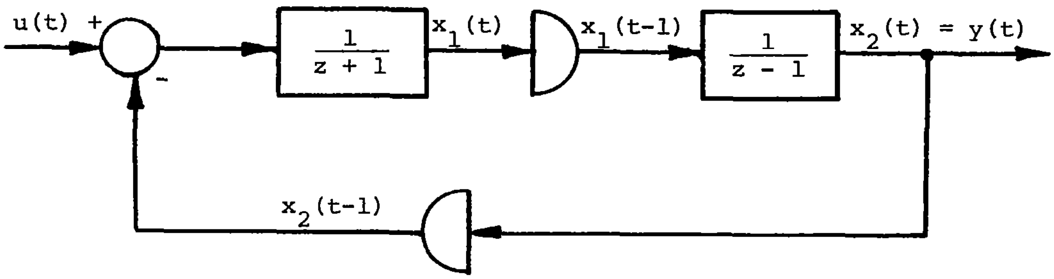
$$F(\sigma) = \sum_i F_i \sigma^i, \quad G(\sigma) = \sum_i G_i \sigma^i, \quad H(\sigma) = \sum_i H_i \sigma^i$$

We shall refer to the system (1.7) as a linear time-invariant discrete-time system over the ring of operators $K[\sigma]$.

Via the representation (1.7), we can study large (high-dimensional) discrete-time systems in terms of some subset of the set of all possible state variables. This is made possible by lumping together components with memory in the coefficient matrices $F(\sigma)$, $G(\sigma)$, $H(\sigma)$. In other words, the representation (1.7) can be viewed as an aggregated model for large discrete-time systems.

The representation (1.7) is a natural model for systems consisting of an interconnection of subsystems separated by time lags with the components of $x(t)$ equal to the states of the subsystems. This is illustrated by the following example.

EXAMPLE 1.2 Suppose that $K = R$. Consider the discrete-time system given by the following block diagram.




Here $x_1(t)$ (resp. $x_2(t)$) is the state of the subsystem having transfer function $1/(z+1)$ (resp. $1/z-1$)). We have that

$$x_1(t+1) = -x_1(t) - x_2(t-1) + u(t)$$

$$x_2(t+1) = x_2(t) + x_1(t-1)$$

$$y(t) = x_2(t)$$

These equations can be written in the form (1.7) with

$$F(\sigma) = \begin{bmatrix} -1 & -\sigma \\ \sigma & 1 \end{bmatrix} \quad G(\sigma) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H(\sigma) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Thus we have a representation of order two over $R[\sigma]$. Defining $x_3(t) = x_1(t-1)$ and $x_4(t) = x_2(t-1)$, we also have a four-dimensional representation over R given by

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{1.8}$$

$$y(t) = x_2(t)$$

The representation over $R[\sigma]$ can be very useful, because in some problems it is more efficient from a computational standpoint to work with the two-dimensional representation over $R[\sigma]$, rather than the four-dimensional representation over R. An example will be given in Chapter 4.

1.5 Additional examples.

In addition to the examples given above, there are many other examples of systems that can be treated as linear systems over rings. These include systems given by discretized partial differential equations [20,21] and linear two-dimensional digital filters viewed as linear systems defined over a ring of proper rational functions in one variable [22].

Linear time-varying systems can also be viewed as systems over rings, in this case a ring of time functions. However, the algebraic theory of time-varying systems differs from the theory of time-invariant systems. The primary reason for the difference stems from the fact that time-varying systems, viewed as systems over a ring of time functions, must be studied in terms of a special type of nonlinear transformation, called a pseudolinear transformation [23]. In the last chapter of these notes, we develop this approach for the class of linear time-varying discrete-time systems. The theory is based on the concept of a semilinear transformation, which is an example of a pseudolinear transformation.

# 2. REPRESENTATION AND REALIZATION THEORY

## 2.1 Abstract systems over rings.

For the three classes of systems defined in Sections 1.2 - 1.4, it turns out that many structural and dynamical properties depend only on the triple (F,G,H) of matrices appearing in the state equations. In other words, the particular form of the dynamical equations (e.g., discrete time or continuous time) is not always a primary consideration. Thus it is possible to develop a general theory of systems over rings specified in terms of a triple of matrices over a ring. This was the approach taken by Sontag in his survey paper [24] on systems over rings. In this section, we define the major components of the theory, beginning with the notion of an abstract system over a ring.

DEFINITION 2.1 Let A be a commutative ring and let n, m, p be fixed positive integers. An abstract (free) linear time-invariant system over A is a triple (F,G,H) of $n \times n$, $n \times m$, $p \times n$ matrices over A. The integer n is called the dimension of the system (F,G,H).

Although the concept of an abstract system does not include any explicit reference to dynamical behavior, an abstract system (F,G,H) can be interpreted as a dynamical system by associating a set of state equations specified in terms of F,G,H. For instance, (F,G,H) can be interpreted as a discrete-time system over the ring of scalars A, given by the state equations (1.3).

Let (F,G,H) be an abstract system over the ring A. Suppose that C is a ring extension of A, i.e., A is a subset of C with the property that addition and multiplication in C, when restricted to elements in A, are identical to addition and multiplication in A. In other words, the ring A is a subring of C. Then since the elements comprising F,G,H can be viewed as elements of C, (F,G,H) can be viewed as an abstract system over C.

It is of particular interest to note that when A is an integral domain, there is a ring extension of A which is a field. Integral domains are commutative rings that do not contain divisors of zero. That is, if $ab = 0$ for some $a,b \in A$, then either a or b must be zero. Examples of integral domains are the ring of integers Z and the operator rings R[d] and K[σ] constructed in the preceding chapter.

An integral domain A can be viewed as a subring of a field Q(A), called the quotient field of A. The field Q(A) is equal to the set of all formal ratios a/b where $a,b \in A$, $b \neq 0$, with addition and multiplication defined by

$$a_1/b_1 + a_2/b_2 = (a_1 b_2 + a_2 b_1)/(b_1 b_2)$$

$$(a_1/b_1)(a_2/b_2) = (a_1 a_2)/(b_1/b_2)$$

For example, the quotient field of Z is the field of rational numbers and the quotient field of R[d] is the field of rational functions in d.

Now given an abstract system (F,G,H) over an integral domain A, we can view (F,G,H) as a system over the quotient field Q(A). Thus there exists the possibility of utilizing results from the theory of systems over fields in the study of systems over rings. Although this approach is useful (e.g., in the realization problem), the quotient-field framework seldom gives a complete solution to a given problem, since in general the results are specified in terms of elements of $Q(A)$, rather than elements of $A$.

Next, we have the concept of an input/output sequence.

DEFINITION 2.2 Let $m,p$ be fixed positive integers. An input/output (i/o) sequence $f$ over a commutative ring $A$ is a sequence $f = (J_1, J_2, \ldots)$ consisting of $p \times m$ matrices over $A$. The i/o sequence of an abstract system $(F,G,H)$ or a dynamical system specified in terms of $(F,G,H)$ is the sequence $f = (J_1, J_2, \ldots)$ where $J_i = H(F^{i-1})G$ for $i = 1,2,\ldots$

In the remainder of this section, we shall show that the i/o sequence associated with the systems defined in Sections 1.2 - 1.4 completely characterizes the input/output behavior of these systems.

(a) Consider the discrete-time system over the ring of scalars $A$ with the dynamical equations (1.3). Solving (1.3) by iteration, we have that the state $x(t)$ at time $t$ resulting from initial state $x(t_o)$ at time $t_o < t$ and input $u(t)$, $t \geq t_o$ is given by

$$x(t) = F^{t-t_o}x(t_o) + \sum_{i=t_o}^{t-1} F^{t-i-1}Gu(i), \quad t > t_o \tag{2.1}$$

If $x(t_o) = 0$, the output response is

$$y(t) = \sum_{i=t_o}^{t-1} HF^{t-i-1}Gu(i), \quad t > t_o \tag{2.2}$$

From (2.2) it is seen that the input/output behavior of the system is completely determined by the sequence $(J_1, J_2, \ldots)$ where $J_i = HF^{i-1}G$ for $i = 1,2,\ldots$

(b) Consider a continuous-time system over the operator ring R[d] with the dynamical equations (1.5). Assuming that $x(t) = 0$ for $t \le 0$ and taking the (one-sided) Laplace transform of (1.5), we get

$$Y(s) = H(e^{-as})(sI - F(e^{-as}))^{-1}G(e^{-as})U(s)$$

where $F(e^{-as})$, $G(e^{-as})$, $H(e^{-as})$ are computed from $F(d)$, $G(d)$, $H(d)$ by setting $d = e^{-as}$, and where $U(s)$ (resp. $Y(s)$) is the Laplace transform of $u(t)$ $(y(t))$. Thus the transfer function matrix $T(s,e^{-as})$ is given by

$$T(s,e^{-as}) = H(e^{-as})(sI - F(e^{-as}))^{-1}G(e^{-as})$$

Expanding $(sI - F(e^{-as}))^{-1}$ into a power series in $s^{-1}$, we have that

$$T(s,e^{-as}) = \sum_{i=1}^{\infty} H(e^{-as})F^{i-1}(e^{-as})G(e^{-as})s^{-i}$$

Hence $T(s,e^{-as})$ can be determined from the sequence $(J_1(d), J_2(d), \ldots)$, where $J_i(d) = H(d)F^{i-1}(d)G(d)$.

(c) Consider a discrete-time system over the operator ring $K[\sigma]$ with the dynamical equations (1.7). We shall specify the input/output behavior using the formal z-transform defined as follows.

Let z be a symbol. Given a function $f:Z \to K$ with $f(t)$ equal to zero for $t<0$, we define the (formal) z-transform of f to be the formal power series $F(z^{-1}) = \sum_{i=0}^{\infty} f(i)z^{-i}$.

Now if we assume that $x(t) = 0$ for $t \leq 0$ and take the z-transform of (1.7), we get

$$Y(z^{-1}) = T(z^{-1})U(z^{-1})$$

where $T(z^{-1}) = H(z^{-1})(zI - F(z^{-1}))^{-1}G(z^{-1})$ is the transfer function matrix. Expanding $(zI - F(z^{-1}))^{-1}$ into a formal power series in $z^{-1}$, we have that

$$T(z^{-1}) = \sum_{i=1}^{\infty} H(z^{-1})F^{i-1}(z^{-1})G(z^{-1})z^{-i} \tag{2.3}$$

Thus the sequence $(J_1(\sigma), J_2(\sigma), \ldots)$, where $J_i(\sigma) = H(\sigma)F^{i-1}(\sigma)G(\sigma)$, determines $T(z^{-1})$. However, it should be noted that the expansion (2.3) is not unique. Therefore, the input/output behavior of a discrete-time system over $K[\sigma]$ cannot be characterized uniquely by a sequence $(J_1(\sigma), J_2(\sigma), \ldots)$ over $K[\sigma]$.

## 2.2. Problem of realization

In this section we present a general approach to realizability based on the concept of abstract systems. The results given here can be applied directly to discrete-time systems over a ring of scalars and continuous-time systems over a ring of delay operators.

DEFINITION 2.3 An abstract system $(F,G,H)$ over A is a realization of an i/o sequence $(J_1,J_2,...)$ over A if and only if $J_i = H(F^{i-1})G$ for $i = 1,2,...$ A realization $(F,G,H)$ is minimal if its dimension is minimal among all possible realizations.

Let us first consider the realization of a scalar sequence $f = (a_1,a_2,...)$, $a_i \in A$. To the sequence f, we associate the formal power series $w(z^{-1}) = \sum_{i=1}^{\infty} a_i z^{-i}$ in the symbol $z^{-1}$ with coefficients in the ring A. The power series $w(z^{-1})$ is said to be rational if it can be written as a ratio of the form

$$w(z^{-1}) = \frac{c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_1 z + c_0}{z^n + e_{n-1}z^{n-1} + \ldots + e_1 z + e_0} \tag{2.4}$$

where the $c_i, e_i \in A$.

We then have the following results on realization. The proofs of these results are omitted, as they are easy generalizations of the field case.

PROPOSITION 2.4 A scalar sequence is realizable if and only if its associated power series $w(z^{-1})$ is rational.

PROPOSITION 2.5 Let f be a scalar sequence with associated power series given by (2.4). If the polynomials comprising $w(z^{-1})$ contain no common factors of degree $\geq 1$, then $(F,g,h)$ is a minimal realization of f where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ -e_0 & -e_1 & -e_2 & \cdots & \cdots & -e_{n-1} \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad h = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix}$$

The result in Proposition 2.4 can be extended to matrix sequences $f = (J_1, J_2, \ldots)$ as follows. Let $W(z^{-1})$ denote the formal matrix power series in $z^{-1}$ associated with f; i.e., $W(z^{-1}) = \sum_{i=1}^{\infty} J_i z^{-i}$. By definition, $W(z^{-1})$ can be written as a matrix whose elements are scalar power series in $z^{-1}$. The matrix $W(z^{-1})$ is rational if each element of $W(z^{-1})$ is rational.

We then have the following result [24].

PROPOSITION 2.6 The matrix sequence f is realizable if and only if its associated power series is rational.

The realization of matrix sequences can be approached by first realizing the elements of $W(z^{-1})$, but the resulting realizations are seldom minimal. As will be seen, for a special class of rings, minimal realizations of a matrix sequence $f = (J_1, J_2, \ldots)$ can be constructed from the Hankel matrix $B(f)$ given by

$$B(f) = \begin{bmatrix} J_1 & J_2 & J_3 & \cdots \\ J_2 & J_3 & J_4 & \cdots \\ J_3 & J_4 & J_5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

It is well known [6] that a matrix sequence f over a field K is realizable if and only if the rank of $B(f)$ is finite. The rank of $B(f)$ is the smallest integer q such that all minors of $B(f)$ of order greater than q are zero. If the rank of $B(f)$ is finite, f has a minimal realization of dimension equal to the rank of B(f).

Realizability results for matrix sequences over fields can be applied to matrix sequences over an integral domain A: Given such an f, we can view f as a sequence over the quotient field Q(A). Then since Q(A) is a field, f has a realization over Q(A) if and only if the Hankel matrix B(f) has finite rank as a matrix over Q(A). But since we are seeking realizations over A, we would like to know when realizability over Q(A) implies realizability over A. As proved in [9], this is the case for the class of rings referred to as principal ideal domains (for more general results see [25,26,27]).

A principal ideal domain (p.i.d.) is an integral domain A with the property that, for any subset S of A such that $a + b \in S$ whenever $a,b \in S$ and $ab \in S$ whenever $a \in A$, $b \in S$, there is an element $a \in S$ such that $S = \{ab : b \in A\}$ (i.e., every ideal S of A is generated by a single element $a \in S$). The ring of integers and the operator rings R[d] and K[$\sigma$] are examples of p.i.d.'s.

Since realizability over Q(A) implies realizability over A when A is a p.i.d., we see that if a matrix sequence over Z has a realization over the field of rational numbers, there is a realization over Z; and if a matrix sequence over R[d] has a realization over the field of rational functions in d, there is a realization over R[d]. These are very interesting results.

It is proved in [9] that a realizable matrix sequence f over a p.i.d. has a minimal realization of dimension equal to the rank of the Hankel matrix B(f) as a matrix over Q(A). In the p.i.d. case, minimal realizations can be computed from B(f) using Rouchaleau's algorithm [9]. The algorithm yields a minimal realization over $Q(A)$ using Silverman's formulas, from which a minimal realization over A is generated via a similarity transformation. The steps of a condensed version of this algorithm are given below.

Let $f = (J_1, J_2 \ldots)$ be a matrix sequence consisting of $p \times m$ matrices defined over a p.i.d. A. Suppose that $B(f)$ has finite rank equal to n. Then a minimal realization of f over A can be constructed by carrying out the following steps.

(1) Let C be a $n \times n$ submatrix of $B(f)$ having rank n as a matrix over $Q(A)$.

(2) Let D be the $n \times nm$ submatrix of $B(f)$ containing the same rows as C and the first n block columns of $B(f)$.

(3) From D construct a $n \times n$ matrix V as follows. Let $b_1$ be the greatest common divisor of the elements in the first row of D. There is an A- linear combination $v_1$ of the columns of D having $b_1$ as first element; and for each column $d_i$ of D, there is a $r_i \in A$ such that the first element of $d_i - r_i v_1$ is zero. Define $D_1 = [d_1 - r_1 v_1 \;\; d_2 - r_2 v_1 \; . \; . \; . \; d_{nm} - r_{nm} v_1]$. Apply the same procedure to $D_1$ working with the second row, which yields a column vector $v_2$ and a matrix $D_2$. Continue until $[v_1, v_2 \ldots v_n] = V$ is constructed.

(4) Let N be the $n \times m$ submatrix of the first block column of $B(f)$ containing the same rows as C.

(5) Let E be the $p \times n$ submatrix of the first block row of $B(f)$ containing the same columns as C.

(6) Let M be the $n \times n$ submatrix of $B(f)$ sitting to the right of C.

Then $(F,G,H)$ is a minimal realization of f where

$$F = (V^{-1})M(C^{-1})V, \; G = (V^{-1})N, \; H = E(C^{-1})V \tag{2.5}$$

## 2.3 Example

Consider the continuous-time system with time delays given by the transfer function matrix

$$T(s,e^{-s}) = \frac{1}{s^2+(e^{-s}-1)s-e^{-s}} \begin{bmatrix} s-1 & e^{-s}s+1+e^{-2s} \\ 1-s & -1 \end{bmatrix}$$

We want to compute a minimal realization of $T(s,e^{-s})$ given by a triple $(F(d), G(d), H(d))$ over the operator ring $R[d]$ with

$$T(s,e^{-s}) = H(e^{-s})(sI - F(e^{-s}))^{-1}G(e^{-s})$$

Since the least common denominator (as a polynomial in s) of $T(s,e^{-s})$ has degree equal to two, it follows that $T(s,e^{-s})$ has a realization over $R[d]$ of dimension 4. We shall compute a minimal realization using the above algorithm. First, expanding $T(s,e^{-s})$ into a power series in $s^{-1}$, we get

$$T(s,e^{-s}) = \begin{bmatrix} 1 & e^{-s} \\ -1 & 0 \end{bmatrix} s^{-1} + \begin{bmatrix} -e^{-s} & 1+e^{-s} \\ e^{-s} & -1 \end{bmatrix} s^{-2} + \begin{bmatrix} e^{-2s} & 1 \\ -e^{-2s} & e^{-s}+1 \end{bmatrix} s^{-3} + \ldots$$

Then

$$
B(f) = \left[\begin{array}{cc|cc|c}
1 & e^{-s} & -e^{-s} & 1+e^{-s} & \cdots \\
-1 & 0 & e^{-s} & -1 & \cdots \\
\hline
-e^{-s} & 1+e^{-s} & e^{-2s} & 1 & \cdots \\
e^{-s} & -1 & -e^{-2s} & e^{-s}+1 & \cdots \\
\hline
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot &
\end{array}\right]
$$

Since there is a realization of dimension 4, the rank of B(f) must be less than or equal to 4, so it is not necessary to consider any more blocks in B(f) than those given above. Checking the minors of the $4\times 4$ submatrix of B(f) given above, we find that the rank of B(f) is 2. Thus there is a realization of dimension 2. We then choose

$$
C = \begin{bmatrix} 1 & e^{-s} \\ -1 & 0 \end{bmatrix}
$$

so that

$$
D = \begin{bmatrix} 1 & e^{-s} & -e^{-s} & 1+e^{-s} \\ -1 & 0 & e^{-s} & -1 \end{bmatrix}
$$

Applying the procedure given in Step (3) to D, we get

$$V = \begin{bmatrix} 1 & 0 \\ -1 & e^{-s} \end{bmatrix}$$

We have that N = E = C and $M = \begin{bmatrix} -e^{-s} & 1+e^{-s} \\ e^{-s} & -1 \end{bmatrix}$

Then from (2.5), we have the following minimal realization of $T(s,e^{-s})$:

$$F(d) = \begin{bmatrix} -d & 1+d+d^2 \\ 0 & 1 \end{bmatrix}, \quad G(d) = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, \quad H(d) = \begin{bmatrix} 1 & 0 \\ -1 & d \end{bmatrix}$$

In component form, the realization is given by

$$\frac{dx_1(t)}{dt} = -x_1(t-1) + x_2(t) + x_2(t-1) + x_2(t-2) + u_1(t) + u_2(t-1)$$

$$\frac{dx_2(t)}{dt} = x_2(t) + u_2(t)$$

$$y_1(t) = x_1(t)$$

$$y_2(t) = -x_1(t) + x_2(t-1)$$

## 3. REACHABILITY, OBSERVABILITY, AND DUALITY

We shall now study the concepts of reachability and observability for discrete-time systems over a ring of scalars and continuous-time systems over the operator ring R[d]. The results obtained here lead to a definition of reachability and observability for abstract systems over rings. In the last part of Section 3.3, we consider a concept of duality for abstract systems.

### 3.1 Discrete-time systems over a ring of scalars.

Consider the discrete-time system given by the state equations

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned} \tag{3.1}$$

where F,G,H are $n\times n$, $n\times m$, $p\times n$ matrices over the ring A. Let $A^n$ denote the set of all n-element column vectors over A.

DEFINITION 3.1 The system (3.1) is reachable if for any $x \in A^n$, there is an integer $N > 0$ and inputs $u(0)$, $u(1)$, $\ldots, u(N-1)$ that drive the system from the zero state at time $t = 0$ to the state x at time $t = N$. The system is reachable in n steps if for any $x \in A^n$, the integer N can be taken to be n.

In Definition 3.1, we have taken the initial time to be zero. This does not imply any loss of generality, since the system is time invariant.

Let $v_1, v_2, \ldots, v_q$ be fixed elements belonging to $A^n$. We say that $x \in A^n$ is an A-linear combination of $v_1, v_2, \ldots, v_q$ if x can be written in the form

$$x = \sum_{i=1}^{q} e_i v_i \quad \text{for some} \quad e_i \in A$$

In terms of this concept, we have the following criterion for reachability.

PROPOSITION 3.2 The following conditions are equivalent.

(1) The system (3.1) is reachable.

(2) The system (3.1) is reachable in n steps.

(3) Every element of $A^n$ can be written as an A-linear combination of the columns of $G, FG, \ldots, F^{n-1}G$.

Proof. From (2.1), the state x(n) at time t = n starting from x(0) = 0 is given by

$$x(n) = \sum_{i=0}^{n-1} F^{n-i-1}Gu(i) \tag{3.2}$$

The equivalence of conditions (2) and (3) follows from (3.2). The equivalence of conditions (1) and (2) follows from the Cayley-Hamilton theorem [8, page 400].

COROLLARY 3.3 Let m = 1 (the single-input case). Then (3.1) is reachable if and only if the $n \times n$ matrix $[G, FG, \ldots, F^{n-1}G]$ is invertible over A, which is the case if and only if the determinant of $[G, FG, \ldots, F^{n-1}G]$ has an inverse in A.

EXAMPLE 3.4 Suppose that (3.1) is a single-input system over the ring of integers Z. Since the only invertible elements of Z are +1 and -1, by Corollary 3.3 we have that the system is reachable if and only if the determinant of $[G, FG, \ldots, F^{n-1}G]$ is equal to +1 or -1. Thus, given a system selected "at random", it is very unlikely that every element in $Z^n$ can be reached using integer-valued controls.

Again consider the system (3.1) defined over the ring A. Let U denote the $n \times mn$ matrix $[G, FG, \ldots, F^{n-1}G]$, and let $u \in A^{mn}$ denote the control vector $u = [u(n-1)\ u(n-2)\ \ldots\ u(1)\ u(0)]'$. It follows from (3.2) that u drives the system to the state $x \in A^n$ at time $t = n$ if and only if u satisfies the equation $x = Uu$. If the system is reachable, in the single-input case there is a unique solution given by $u = U^{-1}x$, where $U^{-1}$ is the inverse of U. In the multi-input case ($m > 1$), the computation of a control u (if one exists) is in general a difficult problem. For a class of fields, we have the following result.

THEOREM 3.5 Let $A = K$, where K is a field with the property that $\sum_i a_i^2 \neq -1$ for any $a_i \in K$. Then the system (3.1) is reachable if and only if the $n \times n$ matrix $U(U')$ is invertible over K, in which case a solution $u \in K^{mn}$ of $x = Uu$ is $u = (U')[U(U')]^{-1}x$.

The above result is well known for the case in which $K = R$ = field of real numbers. The usual proof for the $K = R$ case extends to the class of fields K with the property that $\sum_i a_i^2 \neq -1$ for any $a_i \in K$. It is also well known that when $K = R$, the control $u = (U')[U(U')]^{-1}x$ minimizes $\|u\|$, where $\|\ \|$ denotes the Euclidean norm, and where u ranges over all solutions of $x = Uu$.

In the ring case, invertibility of $U(U')$ over A implies that the system is reachable, but the converse is not true:

EXAMPLE 3.6 Consider the discrete-time system over Z given by $x(t + 1) = x(t) + u_1(t) + u_2(t)$, $y(t) = Hx(t)$. Here $G = [1 \quad 1]$, and $U = G$. Since the columns of U generate Z, the system is reachable. But $U(U') = 2$, which does not have an inverse in Z.

Now assume that A is an integral domain with quotient field $Q(A)$ having the property that $\sum_i a_i^2 \neq -1$ for any $a_i \in Q(A)$. Suppose that the system (3.1), viewed as a system over $Q(A)$, is reachable. Then by Theorem 3.5, any $x \in A^n$ can be reached by applying the control $u = (U')[U(U')]^{-1}x$, which in general is defined over $Q(A)$. This is an acceptable solution if the control over $Q(A)$ can be generated.

EXAMPLE 3.7 Again let (3.1) be a system over Z. Viewing (3.1) as a system over Q(Z), the field of rational numbers, we have that it is reachable if and only if the determinant of $U(U')$ is nonzero. In this case, the control $u = (U')[U(U')]^{-1}x$ would be a vector of rational numbers in general.

For systems over Z, a very interesting problem is the computation of integer-valued controls u that minimize $\|u\|$, where u ranges over all integer-valued solutions of $x = Uu$. When the determinant of $U(U')$ is equal to +1 or -1, an integer-valued solution minimizing $\|u\|$ is $u = (U')[U(U')]^{-1}x$. Unfortunately, invertibility of $U(U')$ is too severe of a condition to be of much value.

For systems over a p.i.d. (such as Z), controls belonging to $A^{mn}$ can be computed by first putting U into diagonal form (the Smith form) using row and column operations (see [28, page 109]). The details of this procedure will not be considered here.

We now consider the concept of observability.

DEFINITION 3.8 The system (3.1) is observable if for any nonzero initial state $x(0) \epsilon A^n$, there is an integer $N > 0$ such that the output response $y(t)$ resulting from $x(0)$ is nonzero for at least one value of $t \epsilon \{0,1,\ldots,N-1\}$. The system is observable in n steps if for any nonzero $x(0) \epsilon A^n$, the integer N can be taken to be n.

PROPOSITION 3.9 The following conditions are equivalent.

(1) The system (3.1) is observable.

(2) The system (3.1) is observable in n steps.

(3) There is no nonzero $x \in A^n$ such that

$$x'[H',(F')(H'),\ldots,(F')^{n-1}(H')] = 0.$$

Proof. From (2.1), the output response $y(t)$ for $0 \leq t \leq n-1$ resulting from initial state $x(0) \epsilon A^n$ can be expressed in the form

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ H(F^{n-1}) \end{bmatrix} x(0) \tag{3.3}$$

The equivalence of conditions (2) and (3) follows directly from (3.3). The equivalence of conditions (1) and (2) follows from the Cayley-Hamilton theorem.

Let V denote the $np \times n$ matrix

$$V = \begin{bmatrix} H \\ HF \\ \vdots \\ H(F^{n-1}) \end{bmatrix}$$

and let $y \epsilon A^{np}$ denote the response vector $[y(0)\ y(1)\ \ldots\ y(n-1)]'$. By (3.3), the response vector y resulting from initial state $x \epsilon A^n$ at time $t = 0$ is given by $y = Vx$. We would like to be able to compute the initial state x from knowledge of y. In the case of systems over fields, this problem is dual to the problem of computing controls in the generation of states. In particular, we have the dual of Theorem 3.5.

THEOREM 3.10 Let $A = K$, where K is a field with $\sum_i a_i^2 \neq -1$ for any $a_i \epsilon K$. Then the system (3.1) is observable if and only if the $n \times n$ matrix $(V')V$ is invertible over K, in which case, given $y = Vx$ for some $x \epsilon K^n$, $x = (V'V)^{-1}(V')y$.

Proof. Recall that a system over a field is observable if and only if its dual, specified in terms of the matrices F',H',G', is reachable [5]. Then apply Theorem 3.5.

Although the duality between reachability and observability does not extend to systems over rings (see Section 3.3), Theorem 3.10 can be extended to systems over an integral domain as follows.

COROLLARY 3.11 Suppose that A is an integral domain with $\sum_i a_i^2 \neq -1$ for any $a_i \in Q(A)$. Then the system (3.1) over A is observable if and only if $(V')V$ is invertible over $Q(A)$, in which case, given $y = Vx$ for some $x \in A^n$, $x = (V'V)^{-1}(V')y$.

Proof. It is easily verified that the system (3.1) is observable if and only if it is observable as a system over the field $Q(A)$. Then apply Theorem 3.10.

3.2 Continuous-time systems over a ring of delay operators

Consider the continuous-time system given by

$$\begin{aligned} \frac{dx(t)}{dt} &= (F(d)x)(t) + (G(d)u)(t) \\ y(t) &= (H(d)x)(t) \end{aligned} \tag{3.4}$$

where $F(d)$, $G(d)$, $H(d)$ are $n\times n$, $n\times m$, $p\times n$ matrices over the operator ring $R[d]$.

Given a fixed positive number $t_1$, let $L^2([0,t_1];R^n)$ denote the Hilbert space of square-integrable functions on $[0,t_1]$ with values in $R^n$. The norm $||f||$ of a function belonging to $L^2([0,t_1];R^n)$ is defined by

$$||f||^2 = \int_0^{t_1} ||f(t)||^2 dt$$

where the norm in the integrand is the Euclidean norm.

We can now define the notion of Euclidean reachability.

DEFINITION 3.12 The system (3.4) is $R^n$-reachable in time $t_1 > 0$ if for any $x \in R^n$, there is an input $u \in L^2([0,t_1];R^m)$ that drives the system to the state x at time $t = t_1$ with initial state function $x(t) = 0$ for $t \leq 0$.

We shall give a necessary and sufficient condition for $R^n$-reachability in terms of the following constructions.

Let $M(t)$ denote the inverse Laplace transform of the n×m matrix $(sI - F(e^{-as}))^{-1}G(e^{-as})$. It can be shown (see [14]) that the state $x(t_1)$ at time $t_1$ resulting from the input $u \in L^2([0,t_1];R^m)$ with $x(t) = 0$ for $t \leq 0$ is given by

$$x(t_1) = \int_0^{t_1} M(t_1 - s)u(s)ds \tag{3.5}$$

Define the map $\lambda(t_1):L^2([0,t_1];R^m) \to R^n : u \to \int_0^{t_1} M(t_1 - s)u(s)ds$.
We then have the following result which follows directly from (3.5).

PROPOSITION 3.13 The system (3.4) is $R^n$-reachable in time $t_1$ if and only if the map $\lambda(t_1)$ is onto.

We can get an explicit condition for reachability in terms of the adjoint $\lambda^*(t_1)$ of $\lambda(t_1)$ defined by

$$\lambda^*(t_1):R^n \to L^2([0,t_1];R^m):x \to M'(t_1 - s)x, \quad 0 \leq s \leq t_1$$

Here we are using the result that the elements of $M(t)$ are square-integrable on any finite interval $0 \leq t \leq t_1$.

The composition $\lambda(t_1)\lambda^*(t_1)$ is a map from $R^n$ into $R^n$ given by

$$\lambda(t_1)\lambda^*(t_1):x \to B(t_1)x$$

where

$$B(t_1) = \int_0^{t_1} M(t_1 - s)M'(t_1 - s)ds$$

The matrix $B(t_1)$ is the controllability gramian for time-delay systems.

THEOREM 3.14 The system (3.4) is $R^n$-reachable in time $t_1$ if and only if $B(t_1)$ is invertible, in which case the control $u = \lambda^*(t_1)B^{-1}(t_1)x$ sets up the state $x \in R^n$ at time $t = t_1$. Further, this control minimizes $\|u\|$, where $u$ ranges over all solutions of $x = \lambda(t_1)u$.

Proof. Follows from standard results in the theory of linear transformations and their adjoints (refer to Luenberger's book [29]).

Although $B(t)$ is constructed from the matrices $F(d)$ and $G(d)$, we would like to have a criterion for $R^n$-reachability specified directly in terms of $F(d)$, $G(d)$. This can be accomplished by first noting that the system (3.4) is $R^n$-reachable in time $t_1$ if and only if $\lambda^*(t_1)$ is one-to-one, which is the case if and only if there is no nonzero $x \in R^n$ such that $x'M(t) = 0$ for $0 \le t \le t_1$. We then have the following result.

THEOREM 3.15 Given the system (3.4), suppose that $F(d) = \sum_{i=0}^{q} F_i d^i$ and $G(d) = \sum_{i=0}^{r} G_i d^i$, where the $F_i$ and $G_i$ are matrices over R. Then the following are equivalent.

(1) The system is $R^n$-reachable in time $t_1$ for some $t_1 > 0$.

(2) The system is $R^n$-reachable in time $t_1$ for any $t_1 > q(n-1)a + ra$.

(3) There is no nonzero $x \in R^n$ such that

$$x'[G(d), F(d)G(d), \ldots, F(d)^{n-1}G(d)] = 0.$$

Proof. Clearly, (2)$\Rightarrow$(1). We shall show that (1) implies (3): Suppose that there is a nonzero $x \in R^n$ such that $x'[G(d), F(d)G(d), \ldots, F(d)^{n-1}G(d)] = 0$. Then by the Cayley-Hamilton theorem, $x'F(d)^{i-1}G(d) = 0$ for all $i \geq 1$. Now by definition of $M(t)$, it can be expanded into the series

$$M(t) = \sum_{i=1}^{\infty} (F(d)^{i-1}G(d))t^{i-1}, \quad t \geq 0$$

Thus, $x'M(t) = 0$ for all $t \geq 0$, so the system is not $R^n$-reachable.

(3)$\Rightarrow$(2): Suppose that the system is not $R^n$-reachable for $t_1 > q(n-1)a + ra$. Then there exists a nonzero $x \in R^n$ such that $x'M(t) = 0$ for $0 \leq t \leq q(n-1)a + ra$. Now let $A(t)$ denote the inverse Laplace transform of $(sI - F(e^{-as}))^{-1}$. Then $M(t) = (AG(d))(t) = \sum_{i=0}^{r} A(t - ia)G_i$. Evaluating $M(t)$ at $t = ia$ and noting that $A(0) = I = n \times n$ identity matrix, we have that $x'G_i = 0$ for $i = 0,1,2,\ldots,r$. Hence $x'G(d) = 0$. Taking the first derivative of $x'M(t)$ for $t > 0$, we have that $x'(F(d)AG(d))(t) = x'\left[\sum_{i=0}^{q} \sum_{j=0}^{r} F_i A(t - ia - ja)G_j\right] = 0$ for $0 \leq t \leq q(n-1)a + ra$. Evaluating this at $t = ia + ja$, we get $x'F_iG_j = 0$ for $i = 0,1,\ldots,q$ and $j = 0,.,\ldots,r$, which implies that $x'F(d)G(d) = 0$. Continuing in this manner, we can show that $x'F(d)^{i-1}G(d) = 0$ for $i = 1,2,\ldots,n$.

Now let $U(d)$ denote the $n \times mn$ matrix $[G(d), F(d)G(d), \ldots, F(d)^{n-1}G(d)]$, and let $R(d)$ denote the quotient field of $R[d]$. We then have the following sufficient condition for $R^n$-reachability.

COROLLARY 3.16 Suppose that the rank of $U(d)$, viewed as a matrix over $R(d)$, is equal to $n$. Then the system (3.4) is $R^n$-reachable.

Proof. The rank of $U(d)$ is equal to $n$ if and only if there is no nonzero $x \in R(d)^n$ such that $x'U(d) = 0$. Then apply Theorem 3.15.

A generalized version of condition (3) in Theorem 3.15 was given by Sontag [24, page 28]. Results similar to those given in Theorem 3.15 were derived by Williams and Zakian [16]. For a geometric version of these results, see Byrnes [30].

Unfortunately, the notions of $R^n$-reachability and $R^n$-controllability (driving $x(t)$ to zero at some time $t_1 > 0$) are of limited use in the design of control systems with time delays, since the value of $x(t)$ at time $t$ is not the complete state of the system. For example, the existence of an open-loop or closed-loop control which drives $x(t)$ to zero at some time $t_1 > 0$ does not imply that $x(t)$ will remain zero for all $t > t_1$. This of course is due to the storage of signals within the delay lines.

In the literature on systems with time delays, conditions based on the concepts of exact or approximate function space reachability are usually considered [31]. These conditions are much stronger than $R^n$-reachability since they deal with the problem of reaching (from zero) or controlling (to zero) $R^n$-valued function segments, rather than points in $R^n$.

It is interesting to note that the rank condition in Corollary 3.16 is stronger than $R^n$-reachability:

EXAMPLE 3.17: Consider the system(3.4) with

$$F(d) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } G(d) = \begin{bmatrix} 1 \\ d \end{bmatrix}, \text{ so that } U(d) = \begin{bmatrix} 1 & 1 \\ d & d \end{bmatrix}$$

There is no nonzero $x \epsilon R^2$ such that $x'U(d) = 0$, so the system is $R^2$-reachable. But the rank of $U(d)$ is equal to one. Thus $R^n$-reachability does not necessarily imply that the rank of $U(d)$ is equal to n.

A very interesting open question is whether or not the rank condition in Corollary 3.16 is equivalent to one of the various notions of function space reachability specified in terms of some topological structure.

There is a much stronger condition than the rank condition given in Corollary 3.16. Namely, we can require that any $x \epsilon R[d]^n$ can be written as an R[d]-linear combination of the columns of U(d). In the single-input case, we can do this if and only if the determinant of U(d) is equal to a nonzero element of R (which is seldom the case). As will be seen in the next chapter, this condition is necessary and sufficient for "pole assignability" using state feedback of the form $u = -B(d)x$, where the feedback matrix B(d) is a $1 \times n$ matrix over R[d].

The last topic of this section is $R^n$-observability.

DEFINITION 3.18 The system (3.4) is $R^n$-observable in time $t_1 > 0$ if for any nonzero initial state $x(0) \epsilon R^n$ with $x(t) = 0$ for all $t < 0$, the output response $y(t)$ resulting from $x(0)$ is nonzero for some range of values of t belonging to $[0, t_1]$.

The next result states that $R^n$-observability is dual to $R^n$-reachability.

PROPOSITION 3.19 The system (3.4) is $R^n$-observable in time $t_1$ if and only if the dual system, specified by the matrices $F(d)'$, $H(d)'$, $G(d)'$, is $R^n$-reachable in time $t_1$.

Proof. Follows from the theory of adjoints [29].

COROLLARY 3.20 Given the system (3.4), suppose that $F(d) = \sum_{i=0}^{q} F_i d^i$ and $H(d) = \sum_{i=0}^{s} H_i d^i$. Then the following are equivalent.

(1) The system is $R^n$-observable in time $t_1$ for some $t_1 > 0$.

(2) The system is $R^n$-observable in time $t_1$ for any $t_1 > q(n-1)a + sa$.

(3) There is no nonzero $x \in R^n$ such that

$$x'[H(d)', F(d)'H(d)', \ldots, (F(d)')^{n-1}H(d)'] = 0$$

Dualizing Corollary 3.16, we have that the system (3.4) is $R^n$-observable if the rank of $[H(d)', F(d)'H(d)', \ldots, (F(d)')^{n-1}H(d)']$, viewed as a matrix over the quotient field $R(d)$, is equal to n. The relationship (if there is one) between this rank condition and the various notions of function space observability is not known at present.

Suppose that (3.4) is $R^n$-observable in time $t_1$. Given the output response $y(t)$ on $[0,t_1]$ resulting from initial state $x(0) \in R^n$, by dualizing Theorem 3.14 we can derive an expression for $x(0)$ in terms of $y(t)$ on $[0,t_1]$. The straight-forward details are omitted.

3.3 Abstract systems.

The results given in Sections 3.1 and 3.2 suggest the following notions of reachability and observability for abstract systems. Here we also define canonical systems and minimal systems.

DEFINITION 3.21 Let $(F,G,H)$ be an n-dimensional abstract system over the commutative ring A. Then $(F,G,H)$ is reachable if any $x \in A^n$ can be written as an A-linear combination of the columns of $G, FG, \ldots, F^{n-1}G$. The system is observable if there is no nonzero $x \in A^n$ such that $x'[H', F'H', \ldots (F')^{n-1}H'] = 0$. The system is canonical if it is reachable and observable. The system is minimal if its input/output sequence $f = (HG, HFG, \ldots, H(F^{i-1})G, \ldots)$

cannot be realized by a system over A with dimension strictly less than n.

When A is a field, it is known [6] that a system is canonical if and only if it is minimal. When A is not a field, it is still true that a canonical system is minimal, but the converse is not necessarily true:

EXAMPLE 3.22 Consider the one-dimensional system $(1,d,1)$ over $R[d]$. The system is obviously minimal. But since $R[d] \neq \{d\pi(d):\pi(d)\in R[d]\}$, it is not reachable, so it is not canonical.

We define the dual of $(F,G,H)$ to be the system $(F',H',G')$. If $(F',H',G')$ is reachable, it can be shown that $(F,G,H)$ is observable. But it is not necessarily true that observability of $(F,G,H)$ implies reachability of $(F',H',G')$:

EXAMPLE 3.23 Consider the system $(F,G,H)$ over $Z$ with $F = 1$ and $H = 2$. There is no nonzero $x\in Z$ such that $Hx = 0$, so the system is observable. But not every $x\in Z$ can be written in the form $x = (H')a = 2a$ for some $a\in Z$, so the dual is not reachable.

Thus reachability of the dual is a stronger condition than observability of the given system. In the next chapter, we shall see that the stronger condition is useful in the construction of state observers.

A reachable system whose dual is also reachable is said to be a split system [24]. For split systems, it is possible to approach the design of regulators by feeding back an estimate of the state obtained from a state observer. An interesting question is whether or not a given input/output sequence can be realized by a split system. For results on this, see Sontag [24] and Byrnes [30].

# 4. CONTROLLERS AND OBSERVERS

## 4.1 Introduction.

In the first part of this chapter, we consider the construction of feedback controllers for abstract systems over rings. The general theory is illustrated by examples involving the three classes of systems defined in Sections 1.2 - 1.4. In the last section of the chapter, we consider the construction of observers by dualizing the results on feedback controllers.

## 4.2 Assignability.

Let $\Omega$ be a ring extension of the ring A. Given a fixed element $\theta$ belonging to $\Omega$, let $A[\theta]$ denote the subring of $\Omega$ consisting of all finite sums of the form $\sum_{i=0}^{q} a_i\theta^i$, where $a_i \in A$ and $\theta^0 = 1$. Given a symbol z, let A[z] denote the ring of polynomials in z with coefficients in A. Finally, let $\rho$ denote the map

$$\rho : A[z] \to A[\theta] : \sum_{i=0}^{q} a_i z^i \to \sum_{i=0}^{q} a_i \theta^i$$

The element $\theta$ is said to be transcendental over A if the map $\rho$ is one-to-one. That is, there does not exist a nonzero polynomial $\pi(z) \in A[z]$ such that $\rho(\pi(z)) = \pi(\theta) = 0$. If $\theta$ is transcendental over A, the rings A[z] and $A[\theta]$ are isomorphic (i.e., $\rho$ is onto and one-to-one, and $\rho(\pi_1\pi_2 + \pi_3) = \rho(\pi_1)\rho(\pi_2) + \rho(\pi_3)$ for any $\pi_1, \pi_2, \pi_3 \in A[z]$).

If $\theta$ is not transcendental over A, it is said to be algebraic over A. If $\theta$ is algebraic over A, an element of $A[\theta]$ does not have a unique expression as a polynomial in $\theta$ with coefficients in A. Nevertheless, we can still say that $b \in A$ is a zero of $\pi(\theta) \in A[\theta]$ if there exists a $\beta(\theta) \in A[\theta]$ such that $\pi(\theta) = (\theta - b)\beta(\theta)$.

Let (F,G,H) be an abstract system over A. As before, we assume that

$F$ is $n \times n$, $G$ is $n \times m$, and $H$ is $p \times n$. Given a $m \times n$ matrix $B$ over $A$, the system $(F-GB,G,H)$ will be referred to as a closed-loop system formed from $(F,G,H)$. The matrix $B$ is called the feedback matrix (or controller). Now given $\theta \in \Omega \supset A$, let $\det(\theta I - F + GB)$ denote the determinant of $\theta I - F + GB$, where $I$ is the $n \times n$ identity matrix. It can be shown that $\det(\theta I - F + GB)$ is an element of $A[\theta]$ of the form $\theta^n + \sum_{i=0}^{n-1} a_i \theta^i$, $a_i \in A$. In terms of these constructions, we have the following concepts.

DEFINITION 4.1 The triple $(F,G,\theta)$ is coefficient assignable if for any $b_0, b_1, \ldots, b_{n-1}$ belonging to $A$, there is a $B$ over $A$ such that $\det(\theta I - F + GB) = \theta^n + \sum_{i=0}^{n-1} b_i \theta^i$. The triple $(F,G,\theta)$ is zero assignable if for any $c_1, c_2, \ldots, c_n$ belonging to $A$, there is a $B$ over $A$ such that $\det(\theta I - F + GB) = (\theta - c_1)(\theta - c_2)\ldots(\theta - c_n)$.

As indicated below, for the three classes of systems defined in Sections 1.2 - 1.4, zero (or coefficient) assignability implies that by employing state feedback, we can specify the "asymptotic" behavior of the state response resulting from arbitrary initial states with zero input.

(a) Consider a discrete-time system over a ring of scalars $A$ given by the dynamical equations

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned} \tag{4.1}$$

We can feed back the state $x(t)$ by setting

$$u(t) = -Bx(t) + r(t) \tag{4.2}$$

where $B$ is the feedback matrix defined over $A$ and $r(t)$ is an external input or disturbance. Combining (4.1) and (4.2), we have that the closed-loop system is given by

$$x(t + 1) = (F - GB)x(t) + Gr(t) \tag{4.3a}$$

$$y(t) = Hx(t) \tag{4.3b}$$

Let $\sigma^{-1}$ denote the left-shift operator defined by $(\sigma^{-1}f)(t) = f(t + 1)$, where f is a function on Z with values in A. Then viewing A as a subring of $A[\sigma^{-1}]$, we can write (4.3a) in the form

$$(\sigma^{-1}I - F + GB)x = Gr \tag{4.4}$$

From a well-known result in matrix theory [8, page 334], we have that

$$[\det(\sigma^{-1}I - F + GB)][\sigma^{-1}I - F + GB]^{-1} = \mathrm{Adj}(\sigma^{-1}I - F + GB) \tag{4.5}$$

where $\mathrm{Adj}(\sigma^{-1}I - F + GB)$ is the transpose of the matrix of cofactors of $\sigma^{-1}I - F + GB$. It follows from (4.4) and (4.5) that if the triple $(F,G,\sigma^{-1})$ is zero or coefficient assignable, by selecting the feedback matrix B, we can specify the behavior of the free response x(t) (i.e., r(t) = 0 for $t \geq 0$) as $t \to \infty$. In particular, by choosing B so that $\det(\sigma^{-1}I - F + GB) = \sigma^{-n}$, we have that the free response x(t) is zero for $t \geq n$, starting from any initial state $x(0) \epsilon A^n$. This is often referred to as "dead-beat" control.

(b) Consider the discrete-time system over the operator ring $K[\sigma]$ given by

$$\begin{aligned} x(t + 1) &= (F(\sigma)x)(t) + (G(\sigma)u)(t) \\ y(t) &= (H(\sigma)x)(t) \end{aligned} \tag{4.6}$$

Setting $u(t) = -(B(\sigma)x)(t) + r(t)$, we have that the closed-loop system is given by

$$x(t+1) = (F(\sigma) - G(\sigma)B(\sigma)x)(t) + (G(\sigma)r)(t) \tag{4.7a}$$

$$y(t) = (H(\sigma)x)(t) \tag{4.7b}$$

where $B(\sigma)$ is a matrix over $K[\sigma]$. In this case, the feedback signal $-(B(\sigma)x)(t)$ consists of delayed versions of $x(t)$.

Let $\sigma^{-1}$ denote the inverse of $\sigma$, so that $\sigma^{-1}$ is the left-shift operator. Now we can view $K[\sigma]$ as a subring of the ring $K[\sigma, \sigma^{-1}]$ consisting of all finite sums $\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{ij}\sigma^i\sigma^{-j}$ with the $a_{ij} \in K$. It is important to note that $\sigma^{-1}$ is not transcendental over $K[\sigma]$. To see this, consider the polynomial $\pi(z) = \sigma z - 1 \in (K[\sigma])[z]$. We have that $\pi(\sigma^{-1}) = \sigma(\sigma^{-1}) - 1 = 0$, so the map $\rho$ defined above is not one-to-one. It follows that the ring $K[\sigma,\sigma^{-1}]$ is not isomorphic to $(K[\sigma])[z]$.

Viewing (4.7a) as an operator equation over $K[\sigma,\sigma^{-1}]$, we have that

$$(\sigma^{-1}I - F(\sigma) + G(\sigma)B(\sigma))x = G(\sigma)r \tag{4.8}$$

As in the case of systems over A, it follows from (4.8) that if the triple $(F(\sigma), G(\sigma), \sigma^{-1})$ is zero or coefficient assignable, we can specify the behavior of the free response $x(t)$ as $t \to \infty$. In fact, if $B(\sigma)$ is chosen so that the determinant of $\sigma^{-1}I - F(\sigma) + G(\sigma)B(\sigma)$ is equal to $\sigma^{-n}$, the free response $x(t)$ will be zero for $t \geq t_1$, where $t_1$ is some positive integer.

(c) Consider the closed-loop continuous-time system over R[d] given by

$$\frac{dx(t)}{dt} = (F(d) - G(d)B(d)x)(t) + (G(d)r)(t) \tag{4.9a}$$

$$y(t) = (H(d)x)(t) \tag{4.9b}$$

As in the previous example, the feedback signal $-(B(d)x)(t)$ consists of delayed versions of $x(t)$.

Letting D denote the derivative operator, we can view R[d] as a subring of R[d,D] which consists of delay differential operators of the form $\sum_{ij}\sum a_{ij}d^iD^j$ acting on some (unspecified) space of functions (see[14] for details). As shown in [12], D is transcendental over R[d], so R[d,D] is isomorphic to the ring (R[d])[z] of polynomials in z with coefficients in R[d].

Viewing (4.9a) as an operator equation over R[d,D], we have that

$$(DI - F(d) + G(d)B(d))x = G(d)r \tag{4.10}$$

In this case, if the triple (F(d),G(d),D) is zero or coefficient assignable, by choosing B(d) we can get the free response x(t) to converge to zero at an exponential rate $e^{-at}$ for any $a > 0$ (see [14]).

4.3 Single-input case.

Let (F,G,H) be an abstract system over A, where now G is an n-element column vector over A (i.e., m = 1, which is the single-input case). Given $\theta \in \Omega$, where $\Omega$ is some ring extension of A, in this section we shall present necessary and sufficient conditions for the triple $(F,G,\theta)$ to be zero or coefficient assignable. Our first approach to this problem is based on the concept of cyclicity which we now define.

DEFINITION 4.2 A $n\times n$ matrix F over a commutative ring A is said to be cyclic with generator $g\in A^n$ if every element of $A^n$ can be expressed as a finite A-linear combination of $g,Fg,\ldots,F^ig,\ldots$

It follows from the Cayley-Hamilton theorem that a $n\times n$ F is cyclic with generator g if and only if the elements $g,Fg,\ldots,F^{n-1}g$ generate $A^n$; i.e., every element of $A^n$ can be written as an A-linear combination of $g,Fg,\ldots,F^{n-1}g$. Hence F is cyclic with generator g if and only if the determinant of the matrix $[g,Fg,\ldots,F^{n-1}g]$ has an inverse in A. We also have the following result.

PROPOSITION 4.3 Let (F,G,H) be a single-input system, so that $G = g\in A^n$. Then (F,g,H) is reachable (Definition 3.21) if and only if F is cyclic with generator g.

As we now show, cyclicity is equivalent to the existence of a particular canonical form. As before, let z be a symbol. Given a $n\times n$ matrix F over A, we have that $\det(zI - F) = z^n + \sum_{i=0}^{n-1} a_i z^i$ for some $a_i\in A$. Define

$$\bar{F} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{4.11}$$

PROPOSITION 4.4 The matrix F is cyclic with generator g if and only if there exists a n×n invertible matrix P over A such that $\bar{F} = P^{-1}FP$ and $\bar{g} = P^{-1}g$, where F and g are given by (4.11). If F is cyclic with generator g, P can be taken to be the matrix $[g,Fg,\ldots,F^{n-1}g][\bar{g},\bar{F}\bar{g},\ldots,(\bar{F})^{n-1}\bar{g}]^{-1}$.

This result is an easy generalization of the field case, so the proof is omitted. Using Proposition 4.4, we can prove the following result relating cyclicity and assignability.

THEOREM 4.5 Let (F,g,H) be a single-input system and let $\theta$ be a fixed element of $\Omega\supset A$, with $\theta$ transcendental over A. Then the following are equivalent.

(1) F is cyclic with generator g.

(2) The triple $(F,g,\theta)$ is coefficient assignable.

(3) The triple $(F,g,\theta)$ is zero assignable.

Proof. Clearly, (2)$\Rightarrow$(3). It follows from Sontag [24, Proposition 4.3] that (3) implies that (F,g,H) is reachable. Hence F is cyclic with generator g. So the proof is completed if it is shown that (1)$\Rightarrow$(2): Given $b_0,b_1,\ldots,b_{n-1}$ belonging to A, let $\bar{B}$ denote the row vector $[b_0-a_0 \;\; b_1-a_1 \ldots b_{n-1} - a_{n-1}]$, where $\det(zI - F) = z^n + \sum_{i=0}^{n-1} a_i z^i$. Then $\det(\theta I - \bar{F} + \bar{g}\bar{B}) = \theta^n + \sum_{i=0}^{n-1} b_i \theta^i$, where $\bar{F},\bar{g}$ are given by (4.11). If F is cyclic with generator g, it follows from Proposition 4.4 that $\det(\theta I - \bar{F} + \bar{g}\bar{B}) = \det(\theta I - F + gB)$ where $B = \bar{B}(P^{-1})$. Thus the triple $(F,g,\theta)$ is coefficient assignable.

By Theorem 4.5, if $\theta$ is transcendental over A, the triple $(F,g,\theta)$ is zero or coefficient assignable if and only if the determinant of $[g,Fg,\ldots,F^{n-1}g]$ has an inverse in A. Unfortunately, this condition can be very severe. For example, in the case of a single-input continuous-time system over R[d], the triple (F(d),g(d),D) is zero or coefficient assignable if and only if the determinant of $[g(d),F(d)g(d),\ldots,F(d)^{n-1}g(d)]$ is a nonzero element of R (which is seldom the case). This limitation can be overcome by considering feedback matrices defined over a ring extension C of R[d]. If C contains a "large" subset of elements that are invertible in C, but not in R[d], assignability with respect to C (i.e., B is over C) will be a much weaker condition than assignability with respect to R[d].

An interesting example of a ring extension of R[d] is the ring consisting of all proper rational functions in $d^{-1}$ which satisfy a stability criterion (see Sontag [24] for details). It may seem reasonable to take C to be the quotient field R(d), since every nonzero element of R(d) is invertible. However, in general it is not possible to implement feedback matrices defined over R(d) since they may contain noncausal or unstable elements. For instance, it is not possible to implement the ideal predictor given by $d^{-1}$.

When $\theta$ is algebraic over A, cyclicity of F is no longer necessary for assignability (see Example 4.8). In this case, a necessary and sufficient condition for coefficient assignability can be derived using the identity

$$\det(\theta I - F + gB) = \det(\theta I - F) + B[\mathrm{Adj}(\theta I - F)]g \tag{4.12}$$

where $\mathrm{Adj}(\theta I - F)$ is the transpose of the matrix of cofactors of $\theta I - F$ (see [14]).

PROPOSITION 4.6 The triple $(F,g,\theta)$ is coefficient assignable if and only if there is a $n\times n$ matrix W over A such that

$$W[\mathrm{Adj}(\theta I - F)]g = \begin{bmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{bmatrix} \tag{4.13}$$

Proof. Suppose that there is a W satisfying (4.13). Given $b_0, b_1, \ldots, b_{n-1}$ belonging to A, let $B = [b_0 - a_0 \;\; b_1 - a_1 \; \ldots \; b_{n-1} - a_{n-1}]W$, where $\det(\theta I - F) = \theta^n + \sum_{i=0}^{n-1} a_i \theta^i$. It follows from (4.12) that $\det(\theta I - F + gB) = \theta^n + \sum_{i=0}^{n-1} b_i \theta^i$. Conversely, suppose that $(F, g, \theta)$ is coefficient assignable. Then for $i = 1,2,\ldots,n$, there is a row vector $B_i$ such that $\det(\theta I - F + gB_i) - \det(\theta I - F) = \theta^{i-1}$. Let W denote the $n\times n$ matrix with $i^{th}$ row equal to $B_i$. Then W satisfies (4.13).

COROLLARY 4.7 Suppose that F is cyclic with generator g. Then (4.13) is satisfied wtih $W = P^{-1}$, where P is the matrix defined in Proposition 4.4.

Proof. It is easily verified that $\text{Adj}(\theta I - \bar{F})\bar{g} = [1 \;\theta \ldots \; \theta^{n-1}]'$, where $\bar{F}, \bar{g}$ are given by (4.11). Using $\bar{F} = P^{-1}FP$, $\bar{g} = P^{-1}g$, we have that $W = P^{-1}$ satisfies (4.13).

EXAMPLE 4.8 Consider the discrete-time system over $R[\sigma]$ with

$$F(\sigma) = \begin{bmatrix} -1 & -\sigma \\ \sigma & 1 \end{bmatrix}, \quad g(\sigma) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is the system given by the block diagram in Example 1.2. For this example, the determinant of $[g(\sigma), F(\sigma)g(\sigma)]$ is equal to $\sigma$, which does not have an inverse in $R[\sigma]$. Thus $F(\sigma)$ is not cyclic with generator $g(\sigma)$. But since $\sigma^{-1}$ is algebraic over $R[\sigma]$, cyclicity is not necessary for assignability. By Proposition 4.6, $(F(\sigma), g(\sigma), \sigma^{-1})$ is coefficient assignable if and only if there is a W over $R[\sigma]$ such that

$$W[\text{Adj}(\sigma^{-1}I - F(\sigma))]g(\sigma) = W \begin{bmatrix} \sigma^{-1} & -1 \\ \sigma & \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma^{-1} \end{bmatrix} \tag{4.14}$$

Multiplying both sides of (4.14) by $\sigma$, we have that $(F(\sigma), g(\sigma), \sigma^{-1})$ is coefficient assignable if and only if there is a W over $R[\sigma]$ such that

$$W \begin{bmatrix} 1-\sigma \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} \sigma \\ 1 \end{bmatrix} \tag{4.15}$$

Since $1-\sigma$ and $\sigma^2$ are relatively prime elements of $R[\sigma]$, there is a W satisfying (4.15). Further, a solution W can be computed using the Euclidean algorithm. This yields

$$W = \begin{bmatrix} \sigma & 1 \\ \sigma+1 & 1 \end{bmatrix}$$

Hence $(F(\sigma), g(\sigma), \sigma^{-1})$ is coefficient assignable. Let's compute $B(\sigma)$ so that $\det(\sigma^{-1}I - F(\sigma) + g(\sigma)B(\sigma)) = \sigma^{-2}$. We have that $\det(\sigma^{-1}I - F(\sigma)) = \sigma^{-2} - 1 + \sigma^2$. Thus

$$B(\sigma) = [b_0-a_0 \quad b_1-a_1]W$$

$$B(\sigma) = [0+1-\sigma^2 \quad 0-(0)] \begin{bmatrix} \sigma & 1 \\ \sigma+1 & 1 \end{bmatrix}$$

$$B(\sigma) = [-\sigma^3+\sigma \qquad 1-\sigma^2]$$

In this case, the feedback signal $-(B(\sigma)x)(t)$ is equal to $x_1(t-3) - x_1(t-1) + x_2(t-2) - x_2(t)$.

As noted in Example 1.2, this system also has a four-dimensional representation over R given by (1.8). Letting $\hat{F}, \hat{g}$ denote the coefficients of (1.8), we have that the rank of $[\hat{g},\hat{F}\hat{g},(\hat{F})^2\hat{g},(\hat{F})^3\hat{g}]$ is equal to four, so $(\hat{F},\hat{g},\sigma^{-1})$ is coefficient assignable. It is interesting to note that in this framework, the computation of feedback matrices using the canonical form (4.11) requires that we work with 4×4 matrices over R; in particular, it is necessary to invert $[\hat{g},\hat{F}\hat{g},(\hat{F})^2\hat{g},(\hat{F})^3\hat{g}]$. In general this procedure appears to be less efficient from a computational standpoint, than the one given above which utilizes the Euclidean algorithm to compute a W satisfying (4.13). This point is currently under investigation.

### 4.4 Multi-input case.

Given an abstract system (F,G,H) over A, in this section we assume that G is n×m with m > 1 (the multi-input case). As before, let $\theta$ be a fixed element of $\Omega \supset A$. We then have the following two results on zero assignability.

THEOREM 4.9 If $\theta$ is transcendental over A, the triple $(F,G,\theta)$ is zero assignable only if (F,G,H) is reachable.

THEOREM 4.10 If the ring A is a p.i.d., reachability of (F,G,H) implies that $(F,G,\theta)$ is zero assignable.

Theorem 4.9 (resp. Theorem 4.10) follows from the work of Sontag [24] (Morse [32]). It should be noted that there is a constructive proof of Theorem 4.10 based on the diagonalization of matrices defined over a p.i.d. (see [32]). Instead of pursuing this, we shall consider the problem of coefficient assignability using the concept of cyclicity.

PROPOSITION 4.11 Given (F,G,H) and $\theta \in \Omega$, suppose that there is a feedback matrix L over A such that F-GL is cyclic with generator g = Gu for some $u \in A^m$. Then $(F,G,\theta)$ is coefficient assignable. Further, there exists an n×n invertible matrix P over A such that, for any $b_0,b_1,\ldots,b_{n-1}$

belonging to A, there is a feedback matrix B over A such that

$$P^{-1}(F-GB)P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & \cdots & -b_{n-1} \end{bmatrix} \tag{4.16}$$

Proof. Suppose that F-GL is cyclic with generator $g = Gu$. Write $\det(\theta I - F + GL) = \theta^n + \sum_{i=0}^{n-1} a_i\theta^i$, and let $\bar{F},\bar{g}$ denote the matrices given by (4.11). Then by Proposition 4.4, $\bar{F} = P^{-1}(F-GL)P, \bar{g} = P^{-1}g$, where

$$P = [g,(F-GL)g,\ldots,(F-GL)^{n-1}g][\bar{g},\bar{F}\bar{g},\ldots,(\bar{F})^{n-1}\bar{g}]^{-1} \tag{4.17}$$

Now given $b_0,b_1,\ldots,b_{n-1}$ belonging to A, let

$$B = L - uq(P^{-1}) \tag{4.18}$$

where $q = [a_0-b_0 \quad a_1-b_1 \quad \cdots \quad a_{n-1}-b_{n-1}]$.
It follows that $\det(\theta I - F + GB) = \theta^n + \sum_{i=0}^{n-1} b_i\theta^i$, so $(F,G,\theta)$ is coefficient assignable. Again applying Proposition 4.4, we have that $P^{-1}(F - GB)P$ is equal to the right side of (4.16).

The result given in Proposition 4.11 has a very interesting interpretation for continuous-time systems over R[d] (or discrete-time systems over $K[\sigma]$): By choosing the $b_i$ to be elements of R (or K), we can construct a closed-loop system that is equivalent to a system $(\hat{F},\hat{G},\hat{H})$ with $\hat{F}$ over R (or K) and with any desired eigenvalues of $\hat{F}$. Here equivalent means that the state $\hat{x}(t)$ of $(\hat{F},\hat{G},\hat{H})$ is related to the state $x(t)$ of the closed-loop system by the transformation $\hat{x}(t) = (P^{-1}x)(t)$, where P is given by (4.17). This construction will be illustrated by an example later.

To utilize the above construction on coefficient assignability, we need to consider conditions for the existence of a feedback matrix L such that F-GL is cyclic with generator $g = Gu$ for some $u \in A^m$. When A is a field, it is known [33] that reachability is necessary and sufficient. When A is a ring, reachability is necessary, but it is not sufficient. In particular, Sontag [24] has shown that when A = Z or R[d], reachability is not sufficient.

A necessary and sufficient condition for the existence of a cyclic F-GL is given in the following theorem (the proof is omitted).

THEOREM 4.12 There exists a matrix L over A such that F-GL is cyclic with generator $g = Gu$ for some $u \in A^m$ if and only if there are elements $u_0, u_1, \ldots, u_{n-1} \in A^m$ such that the elements $g_0, g_1, \ldots, g_{n-1}$ generate $A^n$, where $g_0 = Gu_0$ and $g_i = Fg_{i-1} + Gu_i$ for $i = 1,2,\ldots,n-1$. Given $u_0, u_1, \ldots, u_{n-1}$ and $g_0, g_1, \ldots, g_{n-1}$ satisfying these condtions, L can be taken to be the matrix $-[u_1, u_2, \ldots, u_{n-1}, 0][g_0, g_1, \ldots, g_{n-1}]^{-1}$.

The result in Theorem 4.12 specializes to a result obtained by Hautus [34] in the case when A is a field. When A is a field, there is a constructive procedure (due to Heymann [33]) for finding elements $u_0, u_1, \ldots, u_{n-1}$ and $g_0, g_1, \ldots, g_{n-1}$ satisfying the conditions in Theorem 4.12. When A is a ring, a procedure that yields a solution whenever one exists has not been found as yet. However, there is a procedure involving trial-and-error, which seems to work well in examples for which the dimension is not large. The steps are as follows.

Step 1. If the condition in Theorem 4.12 holds, there exists a basis $\{\alpha_1,\alpha_2, \ldots,\alpha_n\}$ for $A^n$ with $\alpha_1 = Gu_0$ for some $u_0 \epsilon A^m$. When A is a p.i.d., a basis $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ with $\alpha_1 = Gu_0$ can be computed (assuming one exists) by putting G into Smith form. Let $g_0 = Gu_0$.

Step 2. Define the map

$$P_1:A^n \to A^{n-1}: \sum_{i=1}^{n} a_i\alpha_i \quad \to \quad \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Now compute a basis $\{\beta_1,\beta_2,\ldots,\beta_{n-1}\}$ for $A^{n-1}$ with $\beta_1 = P_1(Fg_0 + Gu_1)$ for some $u_1 \epsilon A^m$. Again, when A is a p.i.d., there is an algorithm for computing such a basis (assuming that one exists). If no such basis exists, it is necessary to repeat Step 1 in order to compute another $g_0 = Gu_0$. Given such a basis with $\beta_1 = P_1(Fg_0 + Gu_1)$, let $g_1 = Fg_0 + Gu_1$.

Step 3. Define the map

$$P_2:A^{n-1} \to A^{n-2}: \sum_{i=1}^{n-1} b_i\beta_i \quad \to \quad \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

Compute a basis $\{\gamma_1,\gamma_2,\ldots,\gamma_{n-2}\}$ for $A^{n-2}$ such that $\gamma_1 = P_2P_1(Fg_1 + Gu_2)$ for some $u_2 \epsilon A^m$, and so on. If possible, continue until $g_{n-1}$ is constructed.

4.5 Example.

Consider the continuous-time system over R[d] given by

$$\frac{dx_1(t)}{dt} = -x_1(t) + x_2(t) + u_1(t) + u_2(t)$$

$$\frac{dx_2(t)}{dt} = x_1(t) + x_2(t-a) + u_1(t) + u_2(t-a)$$

In the operator notation, we have that

$$F(d) = \begin{bmatrix} -1 & 1 \\ 1 & d \end{bmatrix} \quad \text{and } G(d) = \begin{bmatrix} 1 & 1 \\ 1 & d \end{bmatrix}$$

Using the procedure given in Section 4.4, we shall first attempt to construct a matrix $L(d)$ such that $F(d)-G(d)L(d)$ is cyclic with generator $g = G(d)u$ for some $u \in R[d]^2$.

We have that $\{\alpha_1, \alpha_2\}$, where $\alpha_1 = [1 \quad 1]'$ and $\alpha_2 = [0 \quad 1]'$, is a basis of $R[d]^2$, with $\alpha_1 = G(d)[1 \quad 0]' = g_0$. Then $P_1(F(d)g_0) = 1 + d$. Now we must find $u_1 \in R[d]^2$, such that $1 + d + P_1(G(d)u_1)$ has an inverse in $R[d]$. This is satisfied with $u_1 = [0 \quad -1]'$. Then $g_1 = F(d)g_0 + G(d)u_1 = [-1 \quad 1]'$. Thus,

$$L(d) = -[u_1 \quad 0][g_0, g_1]^{-1} = \begin{bmatrix} 0 & 0 \\ .5 & .5 \end{bmatrix}$$

This gives

$$F(d)-G(d)L(d) = \begin{bmatrix} -1.5 & .5 \\ 1-.5d & .5d \end{bmatrix}$$

By construction, $F(d)-G(d)L(d)$ is cyclic with generator $g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now let's compute a feedback matrix $B(d)$ so that $\det(DI - F(d) + G(d)B(d)) = D^2 + 3D + 2$. We have that

$$[g, (F-GL)g] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Using (4.17), we get

$$P = (.5) \begin{bmatrix} 1-d & 2 \\ d-5 & 2 \end{bmatrix}$$

and from (4.18),

$$B = L - uq(P^{-1})$$

where $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $q = (.5)\ [-d-5 \quad -3-d]$

This yields

$$B = (1/8) \begin{bmatrix} 5-d^2 & d^2+4d+7 \\ 4 & 4 \end{bmatrix}$$

Finally, we get

$$F(d) - G(d)B(d) = (1/8) \begin{bmatrix} d^2-17 & -d^2-4d-3 \\ d^2-4d+3 & -d^2-7 \end{bmatrix}$$

By construction, the closed-loop system is equivalent to a system whose free (unforced) behavior is given by

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t)$$

## 4.6 Observers.

Let (F,G,H) be an m-input p-output n-dimensional abstract system over A, and let L be a n×p matrix over A. The system $(F-LH,\hat{G},I)$, where $\hat{G} = [G \;\; L]$ and I is the n×n identity matrix, is an observer for the system (F,G,H). The system $(F-LH,\hat{G},I)$ can be interpreted in the usual manner. For example, if F,G,H are the coefficient matrices of a discrete-time system over a ring of scalars A, the observer is given by the dynamical equations

$$\hat{x}(t+1) = (F-LH)\hat{x}(t) + \hat{G}\begin{bmatrix} u \\ y \end{bmatrix}$$

$$\hat{y}(t) = \hat{x}(t)$$

where u (resp. y) is the input (output) of the given system.

Now let (F-LH,O,I) denote the free (unforced) system associated with $(F-LH,\hat{G},I)$, and consider the dual of (F-LH,O,I) given by (F'-(H')(L'),I,O). The key point to note here is that the dual (F'-(H')(L'),I,O) can be viewed as a closed-loop system constructed from the dual (F',H',O) of (F,O,H) with the feedback matrix equal to L'. Therefore, the design of observers can be approached by considering the design of state-feedback controllers for the dual system (F',H',O) using the above results. Note that in order to apply the above results in those cases for which $\theta$ is transcendental over A, it is necessary that the dual be reachable. As noted in the preceding chapter, reachability of the dual is a stronger condition than observability of the given system.

# 5. LINEAR TIME-VARYING SYSTEMS

## 5.1 Introduction.

In this chapter we present an algebraic theory for the class of linear time-varying discrete-time systems. The theory, which is taken from [35], is based on the concept of a semilinear transformation that is derived from the given system. Using the notion of a cyclic semilinear transformation, we develop a (control) canonical form which is then applied to the construction of state-feedback controllers.

## 5.2 System description and properties.

Let K be a fixed field and let A denote the set of all functions defined on Z with values in K. With addition and multiplication given by

$$\begin{aligned}(a + b)(t) &= a(t) + b(t) \\ (ab)(t) &= a(t)b(t)\end{aligned} \qquad a,b\in A$$

A is a commutative ring. It should be noted that A contains divisors of zero, so it is not an integral domain.

Let $\sigma$ denote the right-shift operator on A, defined by $(\sigma a)(t) = a(t-1)$. For any $a,b\in A$, $\sigma(a + b) = \sigma a + \sigma b$ and $\sigma(ab) = (\sigma a)(\sigma b)$. The operator $\sigma$ has an inverse $\sigma^{-1}$ = left-shift operator on A.

We then have the following notion of a system over A.

DEFINITION 5.1 Let m,n,p be fixed positive integers. A m-input p-output n-dimensional linear time-varying discrete-time system over the ring of time functions A is a triple (F,G,H) of $n\times n$, $n\times m$, $p\times n$ matrices over A, together with the dynamical equations

$$\begin{aligned}x(t + 1) &= F(t)x(t) + G(t)u(t) \\ y(t) &= H(t)x(t)\end{aligned} \qquad (5.1)$$

where $x(t) \in K^n$ is the state at time $t \in Z$, $u(t) \in K^m$ is the input at time t, and $y(t) \in K^p$ is the output at time t.

We shall work with a modified version of the standard equations (5.1): Apply the right-shift operator to both sides of the first equation given by (5.1) and let $D(t) = F(t-1)$, $E(t) = G(t-1)$. This gives the following equations

$$\begin{aligned} x(t) &= D(t)x(t-1) + E(t)u(t-1) \\ y(t) &= H(t)x(t) \end{aligned} \tag{5.2}$$

From here on, we shall work with the representation (5.2). The system given by (5.2) will be denoted by the triple (D,E,H).

Our first objective is to characterize systems over A in terms of a semilinear transformation. First, we need the following constructions.

Let $A^n$ denote the set of all n-element column vectors over A. With componentwise addition

$$(v_1 \;\; v_2 \; . \; . \; . \; v_n)' + (w_1 \;\; w_2 \; . \; . \; . \; w_n)' = (v_1 + w_1 \; . \; . \; . \; v_n + w_n)'$$

and scalar multiplication

$$a(v_1 \;\; v_2 \; . \; . \; . \; v_n)' = (av_1 \;\; av_2 \; . \; . \; . \; av_n)', \; a \in A$$

$A^n$ is a <u>module</u> over the ring A. A module is similar to a vector space, except that the scalars come from a ring rather than a field.

Given a n×n matrix M over A, define the operator $S: A^n \to A^n : v \mapsto S(v) = M(\sigma v)$, where $(\sigma v)(t) = v(t-1)$. For any $v,w\in A^n$, we have that $S(v + w) = S(v) + S(w)$, so S is additive. But for $a\in A, v\in A^n$,

$$S(av) = M\sigma(av) = M(\sigma a)(\sigma v) = (\sigma a)S(v)$$

which shows that S is <u>not linear</u> with respect to the A-module structure on $A^n$ (linearity requires that $S(av) = aS(v)$). The operator S is called a semilinear transformation relative to $\sigma$.

Now given an n-dimensional system (D,E,H) over A, the operator $S:A^n \to A^n : v \mapsto D(\sigma v)$ will be referred to as the semilinear transformation (s.l.t.) of the system (D,E,H). We shall first show that the state and output responses can be expressed in terms of the system's s.l.t.

Given $t_o \in Z$ and $x_o \in K^n$, let $\hat{x}_o$ denote the element of $A^n$ defined by $\hat{x}_o(t) = x_o$ when $t = t_o$ and $\hat{x}_o(t) = 0$ when $t \neq t_o$. Let $S^0 = I$ = identity operator on $A^n$, and for $i = 1,2,\ldots$, define $(S^iE)(t) = D(t)(S^{i-1}E)(t-1)$. Then the solution $x(t)$ of (5.2) resulting from initial state $x_o \in K^n$ at initial time $t_o$ and input $u(t)$, $t \geq t_o$, is given by

$$x(t) = (S^{t-t_o}\hat{x}_o)(t) + \sum_{i=t_o}^{t-1} (S^{t-i-1}E)(t)u(i), \quad t > t_o \tag{5.3}$$

If the initial state $x_o$ is zero, the output response is

$$y(t) = \sum_{i=t_o}^{t-1} (HS^{t-i-1}E)(t)u(i), \quad t > t_o \tag{5.4}$$

Note that the expressions (5.3-4) for the state and output responses closely resemble the expressions (2.1-2) for the state and output responses in the time-invariant case. The primary difference between the two frameworks is that in the time-invariant case, dynamical behavior is specified in terms of the linear transformation defined by the matrix F; whereas, in the time-varying case dynamical behavior is given in terms of the semilinear transformation S.

From (5.4), we see that the input/output behavior of the time-varying system (D,E,H) is completely characterized by the sequence $(J_1(t), J_2(t), \ldots)$ where $J_i(t) = HS^{i-1}E(t)$. This observation leads to a realization theory based on a Hankel-matrix approach. The details are under development.

The next concept is reachability.

DEFINITION 5.2 The system (D,E,H) is reachable at time $t \in Z$ if, for any $x(t) \in K^n$, there is an integer $N > 0$ and inputs $u(t-N), u(t-N+1), \ldots, u(t-1)$ that drive the system from the zero state at time t-N to the state x(t) at time t. If there is a fixed N for all $x(t) \in K^n$, (D,E,H) is reachable in N steps at time t. The system (D,E,H) is reachable in N steps at all times if it is completely reachable in N steps at each $t \in Z$.

A necessary and sufficient condition for reachability at time t can be expressed in terms of the system's s.l.t. S as follows.

PROPOSITION 5.3 The following conditions are equivalent.

(1) (D,E,H) is reachable at time t.

(2) There is an integer $N > 0$ such that (D,E,H) is reachable in N steps at time t.

(3) There is an integer $N > 0$ such that the rank of $[E, SE, \ldots, S^{N-1}E](t)$ is equal to n.

Proof. Follows from (5.3) and the fact that $K^n$ is a finite-dimensional vector space.

5.3 Cyclicity.

In this section we shall develop a structure theory based on a notion of cyclicity. This concept is defined in terms of the system's s.l.t. as follows.

An s.l.t. $S:A^n \to A^n$ is cyclic with generator $g \in A^n$ if every $v \in A^n$ can be written as a finite A-linear combination of $g, Sg, \ldots$; that is, $A^n$ can be generated from $g, Sg, \ldots$. Since $A^n$ consists of n-element column vectors over A, it follows that S is cyclic with generator g if and only if there is a positive integer q such that the elements $g, Sg, \ldots, S^{q-1}g$ generate $A^n$. In constrast to the theory of linear transformations, it can happen that the smallest possible value of q is strictly greater than n. There is no Cayley-Hamilton theorem for s.l.t.'s, which would guarantee that q = n.

EXAMPLE 5.4 Suppose that $D = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$. Let $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then

$$S(g) = \begin{bmatrix} t \\ 1 \end{bmatrix} \text{ and } S^2(g) = \begin{bmatrix} 2t-1 \\ t \end{bmatrix}$$

The rank of

$$[g, Sg, S^2g] = \begin{bmatrix} 0 & t & 2t-1 \\ 1 & 1 & t \end{bmatrix}$$

is equal to 2 for all $t \in Z$, which shows that the elements $g, Sg, S^2g$ generate $A^2$. Since the determinant of $[g, Sg]$ is equal to $-t$, the elements $g, Sg$ do not generate $A^2$.

An s.l.t. $S:A^n \to A^n$ is <u>n-cyclic</u> with generator $g \in A^n$ if every $v \in A^n$ can be written as an A-linear combination of $g, Sg, \ldots, S^{n-1}g$. This is the case if and only if the $n \times n$ matrix $[g, Sg, \ldots, S^{n-1}g]$ is invertible over A.

The next result interconnects cyclicity and reachability.

PROPOSITION 5.5 Let (D,e,H) be a single-input (m = 1) n-dimensional system. Then (D,e,H) is reachable in N steps at all times for some N > 0 (resp. reachable in n steps at all times) if and only if S is cyclic (resp. n-cyclic) with generator e.

Proof. Follows from Proposition 5.3 and the definition of cyclicity.

As proved in [35], n-cyclicity is equivalent to the existence of a canonical form that is identical to the form considered in Section 4.3. The constructions are as follows.

Suppose that $S:A^n \to A^n : v \mapsto D(\sigma v)$ is n-cyclic with generator g. Let U denote the matrix $[g, Sg, \ldots, S^{n-1}g]$ and define

$$a = [a_0 \;\; a_1 \;\; . \; . \; . \;\; a_{n-1}]' = -U^{-1}(S^n g)$$

Let $\bar{S}$ denote the s.l.t. on $A^n$ defined by $\bar{S}(v) = \bar{D}(\sigma v)$ where

$$\bar{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \; . \; . \; . \; . & 0 \\ 0 & 0 & 1 & 0 \; . \; . \; . \; . & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & . \; . \; . \; . & 1 \\ -a_0 & -\sigma^{-1}a_1 & -\sigma^{-2}a_2 & . \; . \; . \; . & -\sigma^{-n+1}a_{n-1} \end{bmatrix} \qquad (5.5)$$

Finally, let $\bar{U} = [\bar{g}, \bar{S}\bar{g}, \ldots, \bar{S}^{n-1}\bar{g}]$ where $\bar{g} = [0 \quad 0 \; . \; . \; . \; 0 \quad 1]' \epsilon A^n$.

PROPOSITION 5.6 $\bar{D} = P^{-1}D(\sigma P)$ and $\bar{g} = P^{-1}g$ where $P = U(\bar{U})^{-1}$ and $(\sigma P)(t) = P(t-1)$.

If we apply the coordinate transformation $\bar{x}(t) = P^{-1}(t)x(t)$ to the dynamical equations (5.2), we have that

$$\bar{x}(t) = \bar{D}(t)x(t-1) + P^{-1}(t)E(t)u(t-1)$$

$$y(t) = H(t)P(t)\bar{x}(t)$$

where $\bar{D}$ is given by (5.5). Thus n-cyclicity implies that the given system (D,E,H) is equivalent to a system $(\bar{D},\bar{E},\bar{H})$ with $\bar{D}$ in the form (5.5). This result is useful in the study of state feedback, which we now consider.

Given the system (D,E,H) with the dynamical equations (5.2), let $u(t-1) = -B(t)x(t-1) + r(t-1)$, where $B(t)$ is a $m \times n$ feedback matrix over A and $r(t)$ is an external signal. The resulting closed-loop system is given by the triple (D-EB,E,H), with the dynamical equations

$$x(t) = [D(t) - E(t)B(t)]x(t-1) + E(t)r(t-1)$$

$$y(t) = H(t)x(t)$$

The s.l.t. of the closed-loop system will be denoted by $S_B$. Then $S_B(v) = (D-EB)(\sigma v)$ for all $v \epsilon A^n$.

Now suppose that there is a feedback matrix T over A such that $S_T$ is n-cyclic with generator $g = Eu$ for some $u \epsilon A^m$. Then [35] for any elements $b_0, b_1, \ldots, b_{n-1}$ belonging to A, there is a feedback matrix B over A such that

$$P^{-1}(D-EB)(\sigma P) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -\sigma^{-1}b_1 & -\sigma^{-2}b_2 & \cdots & -\sigma^{-n+1}b_{n-1} \end{bmatrix} \tag{5.6}$$

where P is a n×n invertible matrix over A.

This result implies that if we select the $b_i$ to be time independent, we can construct a closed-loop system that is equivalent to a system $(\bar{D},\bar{E},\bar{H})$ with $\bar{D}$ time independent and with any desired eigenvalues for $\bar{D}$. In particular, if we set $b_i = 0$ for all i, the free response of the closed-loop system will become zero after n steps (i.e., we have a dead-beat control system).

Let us first consider conditions for the existence of a feedback matrix T such that $S_T$ is n-cyclic with generator g = Eu. Then we will give the constructions for setting up (5.6).

As shown in [35], the existence of an n-cyclic $S_T$ requires that the given system be reachable in n steps at all times. However, reachability is not a sufficient condition. A necessary and sufficient condition is given in the following theorem, which is the time-varying version of Theorem 4.12.

THEOREM 5.7 There is a T over A such that $S_T$ is n-cyclic with generator g = Eu for some $u \epsilon A^m$ if and only if there are elements $u_0, u_1, \ldots, u_{n-1} \epsilon A^m$ such that $g_0, g_1, \ldots, g_{n-1}$ generate $A^n$, where $g_0 = Eu_0$ and $g_i = D(\sigma g_{i-1}) + Eu_i$ for $i = 1,2,\ldots,n-1$. Given $u_0, u_1, \ldots, u_{n-1}$ and $g_0, g_1, \ldots, g_{n-1}$

satisfying these conditions, T can be taken to be the matrix $-[u_1,u_2,\ldots,u_{n-1},0]$ $[(\sigma g_0),(\sigma g_1),\ldots,(\sigma g_{n-1})]^{-1}$.

The computation of the $u_i$ and $g_i$ can be carried out using a procedure corresponding to the one given in Section 4.4 for the time-invariant case. An example will be given shortly.

Now suppose that we have a T and a $g = Eu$, such that $S_T$ is n-cyclic with generator g. Given $b_0,b_1,\ldots,b_{n-1} \epsilon A$, we shall construct a feedback matrix B satisfying (5.6). First, let $U = [g,S_T,\ldots,S_T^{n-1}g]$ and define

$$d = [d_0 \quad d_1 \quad . \; . \; . \; d_{n-1}]' = -U^{-1}(S_T^n g)$$

Let $\bar{S}_T$ denote the s.l.t. on $A^n$ defined by $\bar{S}_T(v) = \bar{D}_T(\sigma v)$ where $\bar{D}_T$ is given by (5.5) with the $a_i$ equal to the $d_i$. Then by Proposition 5.6, $\bar{D}_T = P^{-1}(D-ET)(\sigma P)$ where

$$P = U(\bar{U}^{-1}) = [g,S_T g,\ldots,S_T^{n-1}g][\bar{g},\bar{S}_T\bar{g},\ldots,(\bar{S}_T)^{n-1}\bar{g}]^{-1} \tag{5.7}$$

Let

$$B = T - uq(\sigma P)^{-1} \tag{5.8}$$

where

$$q = [d_0-b_0 \quad d_1-b_1 \quad . \; . \; . \quad d_{n-1}-b_{n-1}]$$

Then [35]$P^{-1}(D-EB)(\sigma P)$ is equal to the matrix given in (5.6).

5.4 Example.

Consider the time-varying system given by

$$x_1(t) = t\, x_1(t-1) + (t+1)x_2(t-1) + t\, u_1(t-1)$$

$$x_2(t) = x_2(t-1) + u_2(t-1)$$

For this example,

$$D = \begin{bmatrix} t & t+1 \\ 0 & 1 \end{bmatrix} \quad \text{and } E = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

We shall first construct a matrix T such that $S_T$ is 2-cyclic with generator $g = Eu$. Following the time-varying version of the procedure given in Section 4.4, choose $u_0, u_1 \epsilon A^2$ so that $[Eu_0, S(Eu_0) + Eu_1]$ is invertible over A. A solution is

$$u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Then

$$T = - \begin{bmatrix} u_1 & 0 \end{bmatrix} \begin{bmatrix} (\sigma g_0) & (\sigma g_1) \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

This gives

$$D - ET = \begin{bmatrix} 2t & 1 \\ 0 & 1 \end{bmatrix}$$

By construction, $S_T$ is 2-cyclic with generator $g = Eu_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now let's compute a feedback matrix B such that

$$P^{-1}(D - EB)(\sigma P) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We have that

$$U = [g, S_T g] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$d = -U^{-1}(S_T^2 g) = \begin{bmatrix} 2t \\ -2t-1 \end{bmatrix}$$

Then

$$\bar{D}_T = \begin{bmatrix} 0 & 1 \\ -2t & 2t+3 \end{bmatrix}$$

which gives

$$\bar{U} = \left[\bar{g}, \bar{S}_T \bar{g}\right] = \begin{bmatrix} 0 & 1 \\ 1 & 2t+3 \end{bmatrix}$$

From (5.7),

$$P = U(\bar{U}^{-1}) = \begin{bmatrix} 1 & 0 \\ -2t-2 & 1 \end{bmatrix}$$

and from (5.8)

$$B = T - uq(\sigma P)^{-1} \quad \text{where } u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } q = [2t \quad -2t-3].$$

This yields

$$B = \begin{bmatrix} -1 & 1 \\ 4t(t+1) & 2t+3 \end{bmatrix}$$

Then

$$D - EB = \begin{bmatrix} 2t & 1 \\ -4t(t+1) & -2(t+1) \end{bmatrix}$$

By construction, the free system $x(t) = (D-EB)x(t-1)$ has the property that $x(t) = 0$ for $t \geq t_o+2$ for any initial state $x(t_o) \in K^2$ at time $t = t_o$.

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| 1. Report No. NASA CR-3016 | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| 4. Title and Subtitle "Lectures on Algebraic System Theory: Linear Systems Over Rings" | | 5. Report Date July 1978 |
| | | 6. Performing Organization Code |
| 7. Author(s) Edward W. Kamen | | 8. Performing Organization Report No. |
| | | 10. Work Unit No. |
| 9. Performing Organization Name and Address School of Electrical Engineering Georgia Institute of Technology Atlanta, GA 30332 | | 11. Contract or Grant No. A-43119-B |
| | | 13. Type of Report and Period Covered Contractor Report |
| 12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Ames Research Center Moffett Field, California | | 14. Sponsoring Agency Code |
| 15. Supplementary Notes | | |

16. Abstract

This report provides an introduction to the theory of linear systems defined over rings. The theory is presented with a minimum of mathematical details, so that anyone having some knowledge of linear system theory should find the work easy to read. The presentation centers on four classes of systems that can be treated as linear systems over a ring. . These are (i) discrete-time systems over a ring of scalars such as the integers; (ii) continuous-time systems containing time delays; (iii) large-scale discrete-time systems; (iv) time-varying discrete-time systems.

| 17. Key Words (Suggested by Author(s)) | | 18. Distribution Statement | |
|---|---|---|---|
| Systems Theory<br>Ring Algebra<br>Linear Control | | Unclassified, Unlimited<br><br>Star Category - 66 | |
| 19. Security Classif. (of this report) | 20. Security Classif. (of this page) | 21. No. of Pages | 22. Price* |
| Unclassified | Unclassified | 68 | $3.75 |

*For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1978