```
(NASA-CR-14533i) EFXCF 2 FOTCECRAFT N78-30042
SIPOLATICN MCDEL. YC TME 1: ENGINEEKING
DOCJMENTA:ICN Final Eechrical Seport
(Lockheed-Califcrnia こo.. בurtank.) <7% F Unclas
HC A12/MF :01 CeこL 0is G3/G2 28587
```


# REXDF II ROTORCRAFT SIMULATION MODEL <br> VOLUME I - ENGINEERING DOCUMENTATION 

d. S. Romer,
P. H. Kretsinger

LOCK HEED-CALIFORNIA CO.
P.O. BOX 551

BURBANK, CALIF. 91520

CONTRACT NAS1-14570
JUNE 1978
misn
National Aeronautics and


Space Administration
Langley Research Center
Hampton, Virginia 23665

## FORENORD

This report describes a nonlinear rotorcraft medel and associated computer software which has been developed and documented for NASA, Langley Research Center, Hampton, Virginia under contract NASl-14570 (July 1976). This work has been performed by the Lockheed-California Company, Burbank, California.
P. H. Kretsinger (Lockheed) performed the software impiementation. W. D. Anderson and Fox Conner (both of Lockheed) assisted in preparation of the program.

# patumer pact mane not miad  

## TABLE of COMrEArs

Section Page
FOREHORD ..... iii
LIST OF ILLUSTRATIOIS ..... xi
LIST OF TABLES ..... xili
SURYARY ..... 1
LIST OF SMBOLS ..... 1IIITRODUCTIOM12
1.2 RexOR II Capabilities ..... 12Scope of the REXOR II Progran12
1.3 Improper Application REXOR II ..... 15
1.4 ..... 15
2 BASIC COMPUTATIORAL IDEA ..... 17
2.1Modal Solution - Overview2.2Energy Methods Development172.3Calculation of Rotor Mode Displacements, Velocitiesand Accelerations18
2.4 Output ..... 19
3 SMBOLS ..... 203.1
3.1 .1 ..... 20Subscripting Notation20
3.1 .2 Mode number ..... 20
3.1 .3 Mode type ..... 20
3.1 .4 Generalized mass, damper, spring, forces
3.1 .5 Forces and moments ..... 21
4 COORDINATE SYSTEMS ARD TRANSFORMATIONS ..... 224.1
4.2 ..... 2Introduction22
4.2 .1 Fuselage coordinates
4.2 .2 Hub coordinates. ..... 22
4.2 .3 Shaft coordinates. ..... 26
4.2 .4 Rotor coordinates. ..... 26

## TABLE of Conimeins (Continued)

Section Page
4.2 .5 Blade coordinates ..... 26
4.2 .6 Blede element coordinates ..... 26
4.2 .7 Freestream (earth) set ..... 30
4.2 .8 Svashplate coordinates ..... 30
4.3 Degrees of Freedom ..... 30
4.3 .1 Vehicle or rigid body ..... 34
4.3 .2 Rotor ..... 34
4.3 .3 Shaft or transmission deflections ..... 34
4.3 .4 Blades ..... 36
4.3 .5 Swashplate ..... 40
4.4 General Mction and Coordinate Transformations ..... 40
4.4 .1 General case of space motion. ..... 41
4.4.2 Coordinate transformations - Euler angles ..... 44
4.4 .3 Angular velocities and accelerations - general ..... 48
4.5 Equations of Motion. ..... 51
4.5 .1 Fuselage motion in inertial space ..... 51
4.5 .2 Hub motions in inertial spece ..... 58
4.5 .3 Motion of rotor coordinate axis ..... 62
4.5 .4 Blade coordinate relative to rotor coordinates ..... 64
4.5 .5 Blade element motion ..... 66
4.5 .6 Swashplate motion ..... 105
4.5 .7 Blade feathering motion ..... 108
5 EQUATIONS OF MOTION ..... 113
5.1 Introduction ..... 113
5.2 Energy Approach to Development of Equations of Motion ..... 1135.3
Iterative Concept and Equation Set Solution Method ..... 119
5.4 Overview of Rotor-Blade Model ..... 1385.4 .1
5.4 .2 Blade bending - modal variable. ..... 138Concept of modes.138

## TABLE OF COMTEXTS (Continued)

Section Page
5.4 .3 Blade mode generation ..... 139
5.4 .4 Modal coefficients. ..... 139
5.4 .5 Independent blades. ..... 140
5.4 .6 Blade element aerodyamic forces - orerviev ..... 140
5.4 .7 Blade torsional response. ..... 141
5.4 .8 Radial integration. ..... 141
5.5 Equation System Development. ..... 141
5.5 .1 Reference to base operation matrix. ..... 141
5.5 .2 Organization by degrees of freedom ..... 142
5.5 .3 Partial derivatives ..... 143
5.5 .4 Generalized masses ..... 147
5.5 .5 Generalized forces ..... 148
5.6 Blade Bending and Torsion Equations. ..... 149
5.6 .1 Blade radial sumation. ..... 149
5.6 .2 Partial derivatives ..... 149
5.6.3 Generalized masses. ..... 160
5.6 .4 Generalized forces. ..... 167
5.6 .5 Quasi-static blade torsion. ..... 173
5.6 .6 Quasi-static pitch horn bending ..... 176
5.7 Shaft Axes Equations ..... 176
5.7 .1 Transmission isolation mount ..... 176
5.7 .2 Partial derivatives ..... 176
5.7 .3 Generalized masses. ..... 177
5.7 .4 Generalized forces. ..... 178
5.8 Principal Reference Axis Equations ..... 179
5.8 .1 Nonzero contributions from most vehicle mass elements. ..... 179
5.8.2 Partial derivatives ..... 180
5.8 .3 Generalized masses. ..... 181

## TABLE OF COATEATS (Continued)

Section Page
5.8 .4 Generalized forces. ..... 183
5.9 Swashplate Equations ..... 187
5.9.1 Partial derivatives ..... 187
5.9 .2 Generalised masses. ..... 189
5.9.3 Generalized forces. ..... 190
5.9 .4 Control inputs ..... 196
5.10 Engine Equations ..... 197
5.10 .1 Rotor azimuth and rotation rate ..... 197
5.10 .2 Engine model ..... 197
5.10 .3 Partial derivatives ..... 199
5.10 .4 Generalized masses. ..... 201
5.10 .5 Generalized forces. ..... 203
< AERODYIAMICS ..... 204
6.1 Introduction ..... 204
6.1 .1 Aerodynamic forces producing surfaces considered ..... 204
6.1 .2 Use of forces generated ..... 204
6.2 Main Rotor ..... 204
6.2 .1 Overview. ..... 204
6.2 .2 Concept of rotor inflow model ..... 205
6.2 .3 Blade element velocity components ..... 215
6.2 .4 Coefficient table lookup - overview ..... 227
6.2 .5 Blade element and rotor aerodynamic loads summary ..... 227
6.3 Interference Terms ..... 228
6.3 .1 Nature of the phenomenon ..... 228
6.3 .2 Rotcr to wing/fuselage. ..... 229
6.3 .3 Rotor to horizontal tail. ..... 230
6.3 .4 Data sources. ..... 230
6.3 .5 Empennage velocity components ..... 231
6.4 Body Loads ..... 233
6.4 .1 Nonrotating airframe airloads ..... 233

## table of Comtemrs (Continued)

Section Page
6.4 .2 Component additional airloads. ..... 238
6.5 Tail Rotor. ..... 239
6.5 .1 Formulations ..... 239
6.5 .2 Airloads - control settings. ..... 247
6.6 Auxiliary Thrustors ..... 247
6.6.1 Formulations and airloads ..... 248
7 COBTZAOL SYSIEM ..... 249
7.1 Overview. ..... 249
7.2 Pilot Controls. ..... 249
7.3 Stability Augmentation Systems. ..... 251
8 beferrinces CITED ..... 260

## PRECEDING PAGE BLANK NOT FLLMEU

## LIST OF ILLUSTRATIOMS

Figure Page
1 Block diagram model description ..... 13
2 Coordinate systems fuselage set ..... 23
3 Coordinate systems fuselage axis to airmass ..... 24
4 Coordinate systems - hub (nonrotating shaft top to fuselage axis (flexible shaft) ..... 25
Coordinate systems - hub axis to airmass ..... 27
Coordinate systems - rotor, blade, and blade element sets ..... 28
Coordinate systems - blade element set ..... 29
Coordinate systems - freestream (earth) to principal reference axis ..... 31
Coordinate systems - trajectory path to freestream axis ..... 32
10
Swashplate coordinate system ..... 33
11
Degrees of freedom ..... 35
First inplane mode ..... 38
First flap mode ..... 38
Second flap mode ..... 38
Blade, pitch horn and feather hinge geometry ..... 39
General case of space motion in terms of moving coordinate axes $x, y, z$ and inertial axes $X, Y, Z$ ..... 41
Rotational displacement of a coordinate system ..... 45
Relationship of Euler angle and coordinate system angular rates ..... 49
Blade element c.g./origin location in blade coordinates ..... 68Effect of blade twist on location of blade elementc.g./axis system origin68
Blade precone angle, $\boldsymbol{\beta}_{0}$ ..... 70
Blade sweep, $T_{0}$, and blade droop, $Y$ ..... 70
Introduction of blade $1 / 4$ chord ofrset, Y jog and $Z_{\text {jog }}$ ..... 71
Point $p$ and feathering axis precone $\beta_{F A}$ ..... 73

## LIST OF ILLUSTRATIOAS (Continued)

Figure Page
25 Static feather bearing geometry ..... 76
26 Blade static pretwist, $\phi_{T W}$ and elastic twist, $\phi_{T}$ ..... 82
27 Heutral axis vs blade radius. ..... 100
28 Pitch horn blade feathering phase angle ..... 110
29
Equation solution loop. ..... 124
30 Swashplate friction ..... 192
31 Control axis. ..... 192
32 Bngine model and torque-speed characteristics ..... 198
33 Blade loading distributions in hover. ..... 206
34 or sake angle (forvard flight) ..... 20735
Incremental area for shaft moment integration ..... 209
36 Typical shape of longitudinal factor curve ..... 214
37 Dynamic stall-11ft coefficient vs angle-of-attack hysteresis loop ..... 224
38 Dynamic stall - moment coefficient vs angle-of-attack hysteresis loop ..... 226
Overall tail rotor geometry ..... 24139Tail rotor blade element detail241
41 Fixed aerodynamic surface ..... 252
42 Pilot controls. ..... 253
43 Longitudinal cyclic stability augmentation ..... 254
44 Lateral cyclic stability augmentation ..... 255
45Elevator stability augmentation256
46 Rudder stability augmentation ..... 257
47 Tail rotor stability augmentation ..... 258
48 Aileron stability auguentation. ..... 259

## LIST OF TABLES

Table
Page
1 Blade Generalized Masses ..... 161
2 Generalized Masses ..... 177
3 Generalized Forces ..... 178
4 Reference Axis Generalized Masses. ..... 181
5 Reference Axis Generalized Forces. ..... 184
6 Swashplate Generalized Masses. ..... 190
7 Engine Generalized Masses. ..... 201

## REXOR II ROTORCRAFT SIMULATION MODEL*

Volume I - Engineering Documentation
J. S. Reaser and P. H. Kretsinger

Lockheed-California Company
SUMMARY
This report describes a generalized format rocorcraft nonlinear simulation called REXOR II. The program models single main rotor vehicles with up to seven main rotor bludes. Wings, two horizontal tail planes, and auxiliary thrustors may be included to model a variety of compound helicopter configurations.

Program output is primarily in the form of machine plotted time histories specified from a signal list. This list is, in turn, user selected from a set of computation variables used by the program.

LIST OF SIMBOLS

The symbols used in the REXOR II equations are quite numerous. In order to keep the catalog of symbols to manageable proportions the following list is divided according to the discussion in Section 3. Namely, a list of basic symbols is given, followed by subscripts, superscripts, and postscripts. Nunconforming cases of usage together with complicated or obscure subscripting are fully annotated in the basic list.

SYMBOLS

| $u$ | arbitrary vector |
| :--- | :--- |
| $a_{s}$ | speed of sound, $\mathrm{m} / \mathrm{s}$ |
| $\ddot{a}_{0}$ | acceleration vector, $\mathrm{m} / \mathrm{s}\left(\mathrm{ft} / \mathrm{s}^{2}\right)$ |
| $\mathrm{a}_{1}$ | longitudinal component of blade first harmonic flapping, iad |
| $[\mathrm{A}]$ | generalized mass element matrix |
| $A_{1,2,3}$ | modal variables |
| $A_{l n}$ | generalized displacement of nth blade, first mode |
| \#The contract research effort which has lead to the results in this report |  |
| was financially supported by USARTL (AVRADCOM) Structures Laboratory. |  |


| $\mathrm{A}_{2 n}$ | generalized displacement of nth blade, second mode |
| :---: | :---: |
| $\mathrm{A}_{3 n}$ | generalized displacement of nth blade, third mode |
| ${ }^{1} 15$ | cosine component of blade first harmonic cyclic, rad |
| $b$ | number of main rotor blades; arbitrary vector |
| B | dissipation function |
| $\mathrm{B}_{1 S}$ | sine component of blade first harmonic cyclic, rad |
| c | blade segment chord, m (ft) |
| [c] | damping matrix . |
| $C_{\text {D }}$ | aerodynamic drag coefficient |
| $C_{L}$ | aerodynamic lift coefficient |
| $\mathrm{C}_{\mathrm{M}}$ | aerodynamic pitching moment coefficient |
| $C_{P}$ | power coefficient |
| $c_{T}$ | thrust coefficient |
| $C_{X, Y, Z}$ | linear damping, $\mathrm{N} / \mathrm{m} / \mathrm{s}(\mathrm{lb} / \mathrm{ft} / \mathrm{s}$ ) |
| $C_{\phi, \theta, \psi}$ | rotary damping, $\mathrm{N}-\mathrm{m} / \mathrm{rad} / \mathrm{s}$ ( $\mathrm{ft}-\mathrm{lb} / \mathrm{rad} / \mathrm{s}$ ) |
| $c_{1,2,3}$ | blade bending to feathering couplings |
| C(k) | lift deficiency function |
| d | infinitesimal increment |
| $d r$ | increment in rotor, radius, m (ft) |
| $d t$ | increment in time |
| $d / d t$ | derivative with respect to time |
| $(d / e)_{0}$ | swashplate to feather gear ratio, zero collective |
| $(d / e)_{1}$ | swashplate to feather gear ratio slope with collective |
| e | pitch horn effective crank arm, m (ft) |
| EI | blade bending stiffness distribution, $N-m^{2}\left(1 b-\mathrm{ft}^{2}\right)$ |
| $f_{i M R}$ | ground effect factor for main rotor |
| 2 |  |


| $F$ | factor; force, N (lb) |
| :---: | :---: |
| $F_{X, Y, Z}$ | force components along $X, Y, Z$ directions, $N$ (lb) |
| $F_{\phi, \theta, \psi}$ | generalized force about $\phi, \theta, \psi$ axis |
| $F_{B P H}$ | feathering mode generalized force |
| E | gravity, $\mathrm{m} / \mathrm{s}^{2}\left(\mathrm{ft} / \mathrm{s}^{2}\right.$ ) |
| $\mathrm{E}_{X, Y, Z}$ | gravity components along $X, Y, Z$ directions |
| $G$ | gear ratio |
| \{c\} | generalized force vector |
| $\ddot{G}$ | gyro angular acceleration partial product |
| GJ | blade torsional stiffness, $\mathrm{N}-\mathrm{m}^{2}\left(\mathrm{lb}-\mathrm{ft}^{2}\right.$ ) |
| $\mathrm{I}_{\mathrm{X}}$ | $=\sum_{m_{i}} X_{i}{ }^{2}, k g-m^{2}$ (slug- $\mathrm{ft}^{2}$ ) |
| $\mathrm{I}_{\mathbf{Y}}$ | $=\Sigma m_{i} Y_{i}{ }^{2}, k g-m^{2}$ (slug-ft ${ }^{2}$ ) |
| $\mathrm{I}_{\mathrm{Z}}$ | $=\sum m_{i} Z_{i}^{2}, k g-m^{2}\left(s l u g-f t^{2}\right)$ |
| $\mathrm{I}_{\mathrm{XX}}$ | $=\sum m_{i}\left(Y_{i}{ }^{2}+\sum_{i}{ }^{2}\right), k g-m^{2}$ (slug-ft ${ }^{2}$ ) |
| $I_{Y Y}$ | $=\sum_{i} m\left(X_{i}{ }^{2}+z_{i}{ }^{2}\right), k g-m^{2}\left(s l u g-f t^{2}\right)$ |
| $I_{\text {ZZ }}$ | $=\Sigma m_{i}\left(X_{i}^{2}+Y_{i}^{2}\right), k g-m^{2}\left(s l u g-f t^{2}\right)$ |
| $\mathrm{I}_{X Y}$ | $=\Sigma m_{i} X_{i} Y_{i}, k g-m^{2}\left(o l u g-f t^{2}\right)$ |
| $\mathrm{I}_{\mathrm{XZ}}$ | $=\Sigma m_{i} X_{i} z_{i}, k g-m^{2}\left(s l u g-f t^{2}\right)$ |
| $I_{Y Z}$ | $=\Sigma m_{i} Y_{i} Z_{i}, k g-m^{2}\left(s l u g-f t^{2}\right)$ |
| i | unit vector |
| j | unit vector |
| J | advance ratio |
| k | number of blade radial stations; recuced frequency, rad/s; unit vector |
| [K] | spring matrix |
| $K_{\text {mj }}$ | blade spring matrix element |


| E,Y,2 | spring conrtants along $X_{5} Y_{2} 2$ direction, $1 / 8$ ( $1 \mathrm{~b} / \mathrm{ft}$ ) |
| :---: | :---: |
| $\mathrm{K}_{4,0,0}$ | spring retes about $\dagger, 0, \psi$ axis, H -a/rad ( $\mathrm{ft}-1 \mathrm{l} / \mathrm{rad}$ ) |
| 18 | location inboard feather bearing, in (ft) |
| ${ }_{108}$ | location outboard feather beering, (ft) |
| 1 | redial iccetion of intersection of precone and feather ads. E (ft) |
| L | rolling moment, E-I (rt- lb) |
| - | mass of element, ks (sluga) |
| $F_{F}$ | sumed fuselage coordinate mass, kg (slugs) |
| 且 | sumed hub axis mass, hg (slugs) |
| $\underline{5}$ | mass of ith particle or blade segnent, kg (slugs) |
| ${ }_{\text {SP }}$ | suashplate sumed mass, kg (slugs) |
| M |  |
| [ M ] | Generalized mass matrix |
| Mt | generalized mass matrix element |
| ${ }^{\mathbf{M}}$ | $=\Sigma_{n_{i}} X_{i}, \mathrm{~kg}-\mathrm{m}$ (slug-rt) |
| ${ }_{\mathbf{M}}^{\mathbf{I}}$ |  |
| ${ }^{1}$ | $=\sum_{m_{i}} Z_{i}$, $\operatorname{Lg}-\mathrm{m}$ (slug- ft ) |
| M, Y,Z | moments about $X, Y, Z$ axis, m-m ( $\mathrm{ft}-1 \mathrm{l}$ ) |
| M |  |
| 1 | number of system particles |
| p | angular velocity about X axis, rad/s; particle |
| $\mathbf{p}_{\text {iM }}{ }^{\text {R }}$ | main rotor pitch moment inflow, $\mathbf{1 / s}$ ( $\mathrm{ft} / \mathrm{s}$ ) |
| $q$ | generalized coordinate; angular velocity about $Y$ axis, red/s |


| $q_{\text {iNR }}$ | main rotor roll moment inflow, $\mathrm{m} / \mathrm{s}$ ( $\mathrm{ft} / \mathrm{s}$ ) |
| :---: | :---: |
| Q | generalized forcing function |
| $Q_{\text {A }}$ | aerodynamic pressure times reference wing area, kg ( lb ) |
| QLonds | total nonsain rotcr aerodynanic loads matrix |
| $Q_{\text {TR }}$ | tail rotor torque, $\mathrm{ll}-\mathrm{m}$ ( $\mathrm{ft}-\mathrm{lb}$ ) |
| $\mathbf{r}$ | general vector; radius of curveture, ft ; angular velocity abcut axis, rad/sec; notation for ( $X, Y, Z$ ) |
| $\mathrm{r}_{S}$ | static blade shape |
| R | vector displacement of particle $p$ in $X, Y, Z$ axis system |
| ${ }^{\text {R }}$ C | vector displacement of $x, y, z$ origin in $X, Y, Z$ system |
| $\mathrm{R}_{\mathrm{z} \dagger \text {, } \mathrm{z}}$ | eyro damper coupling ratios |
| 5 | Laplace variable, path of motion of particle $p$ |
| $S_{\text {MA }}$ | blade spline length along neutral axis locii, m (ft) |
| t | time |
| T | kinetic energy, $\mathrm{N-m}$ (ft-lb) |
| [T] | transformation of coordinates matrix |
| u | velocity in X direction, $\mathrm{m} / \mathrm{s}$ ( $\mathrm{ft} / \mathrm{s}$ ) |
| U | potential energy function, $\mathrm{N}-\mathrm{m}$ ( $\mathrm{ft-lb}$ ); strain energy, 1 l-m ( $\mathrm{ft}-\mathrm{lb}$ ) |
| $\mathrm{U}_{2, \mathrm{P}, \mathrm{S}, \mathrm{T}}$ | air velocity on blade element, $\mathrm{m} / \mathrm{s}$ ( $\mathrm{ft} / \mathrm{sec}$ ) |
| $v$ | velocity in $Y$ direction, $\mathrm{m} / \mathrm{s}(\mathrm{ft} / \mathrm{sec}$ ) |
| $\mathrm{V}_{\mathrm{T}}$ | trajectory velocity |
| w | velocity in 2 direction, $\mathrm{m} / \mathrm{s}$ ( $\mathrm{ft} / \mathrm{sec}$ ) |
| $w_{\text {iMR }}$ | main rotor collective inflow, m/s (ft/sec) |
| ${ }_{\text {iTR }}$ | tail rotor sollective inflow, $\mathrm{m} / \mathrm{s}(\mathrm{ft} / \mathrm{sec})$ |
| $\times$ | motion in X direction, m (ft); blade span location |


| X | ```coordinate direction; axds; deflection, (ft); location, m(ft); cross product``` |
| :---: | :---: |
| ${ }^{\text {SN }}$ | blade radial station of sweep and jog, m (ft) |
| $x_{5}$ | trajectory path, $m$ ( ft ) |
| ${ }_{\text {IR }}$ | tail rotor longitudinal force, m (1b) |
| J | motion in $Y$ direction, $\times$ ( ft ) |
| $\mathbf{Y}$ | coordinate direction; axis; deflection, m (ft); location, ( ft ) |
| $\mathrm{Y}_{\mathrm{TIO}}^{1,2,3}$ | tension torsion peck outboard end modal coefficients |
| $\mathbf{Y}_{\text {OXA }}$ | difference between $Y$ direction locations of cg and neutral axis points of blade element, (ft) |
| 2 | motion in 2 direction |
| 2 | coordinate direction; axis; deflection, ( ft ); location, ( ft ) |
| $\mathbf{Z}_{\text {SP }}$ | relative swashplate vertical displacenent rith respect to the hub, ( ft ) |
| $\mathrm{KNO}_{1,2,3}$ | tension-torsion pack outboard end modal coefficients |
| $\mathrm{Z}_{\text {ORL }}$ | teetering rotor undersiling, ( ft ) |
| $\mathrm{Z}_{\text {OP }}$ | hub set distance above fuselage set, m ( ft ) |
| ${ }^{2} 0$ OPP | hub set distance above swashplate set, (ft) |
| a | angle of attack, rad |
| $a_{2}$ | angle of attack with hub set, rad |
| B | sideslip angie, rad |
| $\boldsymbol{B}_{\mathrm{FA}}$ | blade feathering angle, rad |
| ${ }^{\text {Prinn }}$ | feathering/pitch-horn bending or dynamic torsion generalized coordinate displacement |
| $\boldsymbol{B}_{0}$ | blace droop relative to precone angle, rad |


partial derivative, derivation
SUBSCRIPTS

| a | arbitrary coordinate set a |
| :---: | :---: |
| A | due to aerodynanics |
| $b$ | arbitrary cocrdinate set b |
| BEAD | associated with blade elastic bending |
| BLE | blade element coordinate system |
| BLa | blade reference axis system for the nth blade |
| C | associated vith pilot control input, chordrise |
| cos | associated with center of gravity location |
| CORR | ccrrective, correction |
| DH | referring to downrash |
| DYM | referring to dynamic component |
| E | earth axis |
| ExIG | associated vith powerplant - engine |
| EST | estimated |
| F | fuselage axis; associated with blade feathering |
| FA | referring to blade feather axis |
| FB | associated with feedback |
| Fn | associated with feathering of the nth blade |
| FR | due to friction |
| G | referring to gyro or gyro coordinate system |
| GEN | associated with gas generator section of powerplant |
| GFB | associated with gyro control feedback |
| GSP | gyro to swashplate connection |

GUB
H
HT
i

IB
j
jog
J
relating to gyro gimbal unbalance referring to hub or principal reference axis system associated with horizontal tail referring to inflow, particle referring to inboard feather bearing location spring matrix index associated with blace attachment joggle associsted with gyro end of feedback rod linkage associated with feedback rod coming from the nth blade generalized mass index associated with lead-lag damper signifying limiting value
blade mode index, spring matrix index associated with main rotor blade number index referring to blade segment neutral axis newiy dotermined value normal (to airflow) component pertaining to nonrotating value referring to outboard feather bearing location value from previous time step associated with propeller; perpendicular blade component referring to pitch horn generalized mass index referring to rotor axis system

| REF | associated with blade feather reference value |
| :---: | :---: |
| RM | referring to control gyro feedtack lever moment |
| S | referring to blade spanwise velocity; general mode; static; structural; shaft |
| SC | referring to blade segment shear center |
| SP | referring to swashplate |
| $\mathrm{SP}_{\mathrm{c}}$ | command to swashplate |
| S, SP | referring to swashplate limit stop |
| STEADY | steady component |
| SW | referring to blade sweep angle location |
| T | associated with trajectory path relating to E axis; tangential blade component; blade torsion; blade twist |
| TR | associated with the tail rotor |
| TRIM | initial or trim value |
| TW | associated with blade twiss (built in) |
| UB | relating to control gyro unbalance |
| UNSTEADY | associated with unsteady component |
| VT | associated with vertical tail |
| WING | associated with the wing |
| X | relatins to component in X direction |
| $\mathbf{Y}$ | relating to component in $Y$ direction |
| YA | relating to aerodynamic component in $Y$ direction |
| Z | relating to component in Z direction |
| ZA | relating to aerodjnamic component in $Y$ direction |


| 0 | (nought) associated with collective value, courdinate axis value, with respect to principal reference axis, blade root summation |
| :---: | :---: |
| 1,2,3 | with respect to blade modes 1,2 , or 3 |
| 15 | first harmonic component shaft axis feathering |
| $1 / 4 \mathrm{c}$ | with respect to blade $1 / 4$ chord |
| 3/4c | with respect to blade $3 / 4$ chord |
| ${ }^{8} \mathrm{PHn}$ | associated with the feathering mode of the nth blade |
| ¢ | relating to component in the $\phi$ direction |
| $\theta$ | relating to component in the $\theta$ direction |
| $\psi$ | relating to component in the $\psi$ direction |
| SUPERSCRIPTS |  |
| I | referring to inertial reference |
| T | matrix transpose |
| (-) | (bar) average quantity |
| (') | (prime) slope with respect to blade span |
| (-) | (dot) time derivative of basic quantity |
| (-.) | (double dot) second time derivative |
| $(-1)$ | matrix inverse |
| $(\rightarrow)$ | vector quantity |
| POSTSCRIPTS |  |
| (i) | blade radial station index |
| ( n ) | blade number index |

## 1. INTRODUCTION

### 1.1 Scope of the REXOR II Program

REXOR II is a rotoreraft analysis tool which has resulted from applying an interdisciplinary math modeling philosophy. The REXOR II math model is written for a single rotor helicopter vith capability for analysis of hingeless or hinged rotor systems with conventional controls. This helicopter may be conventional in design, winged, or compounded. The main rotor may have a maximum of seven blades. The model is broken dow: into the three major categories show in Figure l. These categcries are the sontrol system, the rotor, and the body.

Figure 1 indicates the manner in which these components are related to one another as utilized in the $\varepsilon$ 'rsis. The analysis is the simulation of a: entire aircraft, which incluc detailed dynamic description of the :otor and control system as weli . conventional six-degree-of-freedom body dynamic description which operates in two modes identified as TRIM and FLY. In the TRIM mode, the aircraft is constrained to a prescribed static flight condition while the controls are activated and the rotor is allowed to respond to obtain a force and moment equilibrium of the aircrafi at that static condition. In the FLY rode the entire aircraft is free to respond dynamically to control inputs or to any other arbitrary inputs such as gusts. Pilot inpuis can be any single or multiple control manipulation in the form of simple steps or pulses, doublets, stick stirs, or other transient input within the capabilities of the control system simulated. As a result, transient loads and resulting aircraft and rotor dynamic response can be obtained. For correlation purposes, actual flight test control motions can be used as input to provide comparative response data. Additionally, gust inputs and other types of external excitations can be applied directly to the rotor and/or airframe.

### 1.2 REXOR II Capabilities

REXOR II is a detailed rotorcraft math model simulation with particular emphasis on the main rotor mechanics. The program is particularly valuable in a detailed exploration of rotor characteristics of proposed designs, in identifying problem areas and verifying fixes in flight test development programs. A case history is given in Reference 1.


## Typical REXOR II applications are listed below

Dynamics:

- Rotor stability as a function of flight speed, maneuvers, rotor rpm , nonlinear blade aerodynamics
- Rotor/body sensitivity and dissipation capacity as a function of gusts and oilot contiol inputs
- Effects of design parameters (mechanical and elastic couplings, controls, etc.) on rotor stability and load sensitivity
- Correlation and check of specialized dynamic models.


## Handling Qualities:

- Vehicle response to pilot control inputs for vehicle flight conditions, speed, altitude, rotor rpm, design parameter variations
- Vehicle stability as function of speed, rotor rpm, flight conditions, design parameters
- Effect of design parameter variations on hancling qualities
- Development and checking of handing qualities models.

Failure Analysis:

- Effect of loss of one inplane damper on subsequent flight time history
- Blade projectile hit and ensuing events
- Blade strike and resulting rotor track.

Ferformance:

- Correlation and independent check of performance models, particularly in regions of righly nonlinear blade aerodynamic operation (retreating blade stall and compressibility effects)
- Develop data for performance models for use in nonlinear areas


## Loads:

- Steady-state rotor loads as a function of rotor rpm, fiight velocity, control trim settings
- Dynamic rotor ioads as a function of rotor rpm, flight velccity, vehicle maneuvers, pilot control inputs
- Rotor/fuselage clearances as a function of speed, vehicle maneuvers, rotor rpm, pilot control inputs, fiight configuration
- Rotor/fuselage/wing design characteristics requirements as functions of maneuver load factor, control commands (see Reference 2).


### 1.3 Improper Application REXOR II

While REXOR II is capable of performing a number of analysis tasks, the progrem range of use is certainly not all inclusive. Examples of types of use where REXUR iI either wouldn't work well or would be impractical are given below.

REXOR II is an extensive math model and, as such, may consume a considerable amount of computer time to execute a case. Therefore, the program is not intended as a paiametric design analysis tool, but rather as a device to verify the correctness of a parametric selection process.

REXOR II does not treat blade-to-blade vortex interaction. This condition limits the validity of the vibration solution in the transition flight regime.

REXOR II typically uses twenty or less blade radial stations. The computer blade deflections show good correlation to measured data with this modeling. However, since shear is a first derivative, and moment is a second derivative of deflection ata, care needs to be exercised in their use (Reference 3).

### 1.4 The REXOR II Report and Its Use

This report is presented in three volumes.

- Volurne I

A develofment of rotorcraft mechanics and aerodynamics including a derivation of the equations of motion from first principles.

- Volume II

The deveiopment and explanation of the computer code required to implement the equations of motion.

- Volune III

A user's manual containing a description of code input/output end instructions to operate the program.

Volume $I$ is intended to be a self-sur $\because$ cient guide to the math development of the equations of motion and is the reference background as such. Volume II gives the location of computation elements, and serves to locate elements for inspection or modification. Volume III presents rormal program operation plus troubleshooting gu: ie material required for day-to-day program use.

## 2. BASIC COMPUTATIONAL IDEA

### 2.1 Modal Solution - Overviev

The aircraft is described dynamically by an array of fully-coupled degrees of freedom. In addition to the six degrees of freedom of the faselage principal reference axes, six degrees of frepicm describe rotor hub to fuselage deflection due to shaft bending and transmission mount motion. Rotor/engine speed is a degree of freedom. The contrci swashplate has three degrees of freedom. Motion of each of the main rotor blades is described by three coupled flapwise and inplane modes and a pitch horn bending degree of freedom which couples blade feathering to the swasholate. The total number of degrees of freedom possible is $16+4 b$, where $b$ is the number of slades.

The blade modes are primitive modes in that they are dejermired from a lumped parameter analysis at a select: 3 rotor speed and collective blade angle, hereafter referred to as the reference feather angle. The generalized stiffness matrix is computed using these rotating modes and contains only the structural stiffness of the blades and hub. This formulation ensures proper internal and external force and moment balance. The model deflections outboard of the feather hinge are rotated through the actual feather angle less the reference feather angle. Thus, blade element deflections outboard of the feathering hinge due to modal displacements are defined to remain aligned with a coordinate axis system which is orthogonal to a plane containing the instaneous deformed feather axis and rotated through the instantaneous feather angle less the reference $f$ ather angle. As a result, the internal strain energy in the blade due to unit model displacements is in.. variant with variation in blade angle. This technique permits the highest resolution of motion and forces fcr the blade with an assumed mode solution for a given number of modes.

### 2.2 Energy Methods Development

The equations of motion for REXOR II are developed from Lagrange's equations, which is an energy approach. If one can express the kinetic, potential, and jissipative energies of a system in addition to the work done by external forces, then Lagrange's equations $\dot{\sim} ニ \because \approx$ de a powerful method for developing ine equations of motion.

The dynamic equations of motion are $k$-itten in matrix form as

$$
\begin{equation*}
-[A]\{\ddot{q}\}+\{G\}=0 \tag{1}
\end{equation*}
$$

vhere [A] is a square matrix of generalized mass elements, $\{\ddot{q}\}$ is a colum vector of accelerations of the generalized coordinates and $\{G\}$ is a column vector derived fros the Lagrangian energy functions, dissipation function and generalized forces, which take the form:

$$
\begin{equation*}
\{\mathrm{a}\}=-[\mathrm{E}]\{\mathrm{a}\}-[\mathrm{c}]\{\mathrm{a}\}+[\mathrm{Q}]\{(\mathrm{t})\} \tag{2}
\end{equation*}
$$

The equations of motion are solved using a ime history sclution with rotor azimuth angie increments required to provide a stable solution at the highest frequency mode present.

### 2.3 Calculation of Rotor Vode Displacements, Velocities, and Accelerations

In a rotor simulation of this type, it is difficult to compute the proper displacement velocities and accelerations and associated inertia and aerodymanic forces and moments which are required for high resolution of the blade feathering moments. This requires exacting aerodynamic data as well as a precise statement of the inertial loadings. To establish the feathering moments due to these loads, the relationship between the ieather axis and the point of application of the loads must be preciseiy determined. This is accomplished by a very accurate analytic construction of the undeformed blade and a superposition $2=$ the blade elastic bending on this shape. In order to achieve the highest resolution of the predicted blade shape and feather axis position, the blade modes are defined at approximately the trim collective blade angle. The blade static position is also constructed at this blade angle. Blade element displacements, velocities, and accelerations are then ccmputed from the combined static shape, the eiestic blade motion, and blade feathering with respect to the reference feather angle.

The aerodyamic description used in the analysis is composed of a rotor inflow model, nonlinear steady and unsteady blade element aeroaynamics, nonlinear fuselage aerodynamic characteristics, rotor/body aerodynamic interference, and auxiliary airloads from the tail rotor and tail surfaces. The main rotor downwash effect on the wing and horizontal tail angles of attack is an empirical functicn of zotor thrust and forward velocity. The nonlinear fuselage aerodynamics a ay be inputted as tables of actual wind tunnel test data.

The sircraft primary control systems are simulated from the pilot control levers operating through $e$ boost systemi in all cintrol axes. Gearing and gains in the cortrol path are inputs to the analysis: and m:y be easily changed for studiving the effects of design changes in the cont.rol system.

Control servos are simulated by first-order lags with rate limits and with soft and hard physical stops. Contr - stiffnesses in collective and cyclic pitch axes of the main rotor are included in the dynamic equations of motion.

## 24 Output

The analysis is a time history solution of the equations of motion. REXOR II does not directly process the results of the solution process for output, creates an output file of user selected paraneters which are correlated by the computation time step. From this data bank the recorded signals can be selected for tabular or plotted output. Assuming a good selection of parameters is chosen to be recorded, the user in an interactive mode may select as little or as much of the information for viewing as is needed. Thus a configuration can be examined thoroughly without having to rerun the case to select additiona? output.

## 3. SYNBOLS

The notation used in REXOR II generally follows what could be termed RASA notation. In general:

- Axis systems use a right-hand triad $X, Y, Z$
- Rotations about these axes are also a right-hand triad $\theta, \varphi, \psi$
- Rotation rates, again a right-hand triad,are $p, q, r$
- Velocity components of $X, Y, Z$ are $u, v, w$.


### 3.1 Subscripting Fotation

Subscripting is used as a rule in REXOR II to further identify variable. Superscripts except in a fer colum vectors are reserved to denote raising to a power. The subscripting can mean:

- Type of element; $F$ for fuselage, $S P$ fcr swashplate, TR for tail rotor, $R$ for rotor, etc.
- Coordinate system reference; BLa for blade axis, $H$ for hub axis, $R$ for rotor axis, ete.
- Modal identifiers.
3.1.1 Biade number. - The blade mcdal identifier typically is of the form $A_{\mathrm{mn}}$. Where n is the blade number.
3.1.2 Mcde number. - Also from $F_{m n}$, $m$ is the mode number, and is keyed to the symbol A. A represents blade bending modes (3). Therefore $m$ can be 1 to 3.
3.1.3 Mode type. - Other than blade bending the remairing blade mode is torsion, and is separately identified as $\beta_{p: n n}$. Nonblade modes are identified by the direction and subscripted axis of motion. Examples are $\mathrm{N}_{\mathrm{R}}$ for rotation of the rotor and $\theta_{S}$ for shaft pitching.
3.1.4 Generalized mass, damper, spring, forces. - The generalized masses are denoted as $M$ doubly subscripted by the two modes active for that mass. Examples are $M_{S} \phi_{S}$ and $M_{A_{m n}} \theta_{H}$. This scheme is also used for other elements of the equations of motion, dampers (C), springs ( $K$ ), forces ( $F$ ). Note the forces are a column vector and singly subscripted.
3.1.5 Forces and moments. - In the process of forming the equations of motion many subelements of forces and moments are formed, translated and combined. Several levels of subscripting may exist in performing this process. The guidelines to the layering are:
- First level denotes the direction or axis system that the quantity is formed in. Examples are $X$ and BLE.
- Second is the axis system invoived or axis system being translated to, depending on the specification of the first level. The second level may also be specified as 0 or nought, to indicate the value is $a$ : the coordinate system origin. This notation is used to show an inertial reference and blade root sumation quantities.
- The third level, usually outside a series of bracketed quantities, shows the blade number being computed, or the overall coordinate system in use for the computation at hand.


## 4. COORDINATE SYSTEMS AND TRANSFORMATIONS

### 4.1 Introduction

Prior to developing the equations of motion, a system of coordinate sets with a description of the elements of the system in these sets and the interrelationship of the sets is required.

### 4.2 Coordinate Sets

4.2.1 Puselage coordinates ( $X_{F}, Y_{F}, Z_{F}$ ). - The fuselage $X$ and $Z$ axes lie in the fuselage plane of symetry. The location of the origin is arbitrary. See Figure 2. The coordinates form a right-hand triad $X_{F}, Y_{F}, Z_{F}$. Notations for velocities with respect to earth of these cocrdinates are either $\dot{X}_{F}, \dot{Y}_{F}, \dot{X}_{F}$ or $u_{F}, v_{F}, u_{F}$. A conventional double dot notation is used for acceleration. Euler rotations of the set follow conventional practice of roll right $\phi_{F}$, pitch up $\theta_{F}$, and yau right $\psi_{F}$. Rates of rotation are either denoted by dot notation or $P_{F}, G_{F}, r_{F}$. Angular acceleration is double dot notation of the rotation or dot notation of the rates, $\ddot{\phi}_{F}, \ddot{\theta}_{F}, \ddot{\phi}_{F}$ or $\dot{p}_{F}, \dot{q}_{F}, \dot{r}_{F}$,

Numerous aerodynamic terms are referenced to the fuselage set. Figure 3 shous the relationship of airflou to this set. The components of airflow, also noted as $u_{F}, v_{F}, W_{F}$, are defined with respect to the fuselage set by an angle of attack $a$, and a sideslip angle $B$. The angle of attack is the arcsin of the ratio of the vertical corponent and the vector sum of the $X$ and $Z$ components. The sideslip is the $Y$ component of airflow in relation to the total vector airflow sum. The angle of attack is positive (pitch up) of the fuselage set with respect to the airflow. The sideslip is positive (yaw ieft) for the airfiou relative to the set. The airflow is the vector sum of the fuselage set inertial motion and flow fields from other parts of the vehicle, such as main rotor downash.
4.2.2 Hub coordinates ( $X_{H}, Y_{H}, Z_{H}$ ). - The hub set origin is at the top of the main rotor mast, but does nct rotate with the mast.

Airflow information is referenced to the hub set for use in the main rotor aerodyamic calculations. The reference scheme is shown on Figure 4. For components of airflow $u_{H}, v_{H},{ }_{H}$ with respect to the hub set, an angle of


Figure 2. - Coordinate systems-fuselage set.


Figure 3. - Coordinate systems fuselage axis to airmass.


$$
a_{2}=\sin ^{-1} \frac{w_{H}}{\sqrt{u_{H}{ }^{2}+v_{H}{ }^{2}+w_{H}{ }^{2}}} ; \psi_{w}=\sin ^{-1} \frac{v_{H}}{\sqrt{u_{H}{ }^{2}+v_{H}{ }^{2}}}
$$

Figure 4. - Coordinate systems - hub axis to airmass.
attack $\alpha_{2}$, and sideslip $\psi$ are defined. The generatic conventions are different from the fuselage airflow reference in order to clearly separate the inplane and outplane airflow components.
4.2 .3 Shaft coordinates $\left(X_{s}, Y_{s}, Z_{s}\right)$. The shaft axes are an intermediate set between the hub and fuselage sets, see Figure 5. The geometry is determired by the distances $\left(X_{0}, Y_{0}, Z_{0}\right)_{F-H}$ and $\left(X_{0}, Y_{0}, Z_{O}\right)_{S-H}$ that the hub origin is located from the fuselage and shaft axes and by the rotation ( $\phi_{0}, \theta_{0}$, $\left.\psi_{0}\right)_{S-H}$ of the shaft axes from the fuselage axes. The hub axes are parallel to the shaft axes.

The elastic deflections due to motions of the shaft and transmissicn suspension are given by the set of coordinates ( $X_{s}, Y_{s} Z_{s}, \phi_{s}, \theta_{s}, \psi_{s}$ ). The hub is assumed to move as a rigid body with respect to the shaft axis orizin.
4.2 .4 Rotor coordinates $\left(X_{R}, Y_{R}, Z_{R}\right)$. The undeflected rotor set has the same crigin as the huo set. See Figure 6. The $X_{R}$ and $Y_{R}$ axes rotate with the blade number 1 reference axis system. At $\psi_{R}=0$, the $X_{R}$ and $Z_{R}$ axes are aligned but point in a direction coposite to the $X_{H}$ and $Z_{H}$ axes. The rotation of the rotor set is measured counterclockwise (CCW) from the $-X_{H}$ axis by the angle $\psi_{R}$.
4.2.5 Blade coordinates ( $X_{B L n}, Y_{B T n}, Z_{B L n}$ ). - To bookkeep the deflecticns properly for all the main rotor blades, sets equivalent co the rotor set are created for each blade. These are the BLn sets, where $n$ is the biade number (counted clockwise from blade number one). All BLn sets are identical except for an azimuthal rotation $(n-1) \Delta \psi$, where $\Delta \psi$ is the interblade angular spacing. The rotation is about the $Z_{R}$ axis. Note that BLn sets are rotating coordinates and have a common $Z$ axis.
4. 2.6 Blade element coordinates ( $X_{B L E}, Y_{B L E}, Z_{E L E}$ ). - The blade element set origin is located at the center of gravity of an element of a particular blade. See Figure 7. Reference to a column vector subscripted by BLE is used to dencte the blade element located by the blade element set origin. The right-hand coordinate triad of this set has the $X$ axis parallel to the iocal quarter chord line, the $Y$ axis along the chord line toward the leading edge. The $Z$ axis is mutually perpendicular and pointed up. The BLE set is used to track the local feather angle, to develop aerodynamic and dynamic loading terms.

(b) SHAFT AND TRANSMISSION DEFLECTIONS

Figure 5. - Coordinate systems - hub, shaft, and fuselage sets.
a. ROTOR AND HUB AXIS SETS

b. ROTOR AND BLADE AXIS SETS (DOWN)


Figure 6. - Coordinate systems - rotor, blade,
and blade element sets.


Figure 7. - Coordinate systems - blade element set.

The BLE set origin for each blade element specifies the element c.g. with respect to the quarter chord, and in terms of the BLE directions, i.e., for the Kth element the position coordinates are $S Y(K)$ and $S X(K)$ where $S X(K)$ is the blade radial station. Transformations to the neutral, nostretch axis are made for $X$ deflections. Note: The quarter chord is merely a convenient reference datum, and does not convey any model limitations or assumptions.
4.2.7 Freestream (earth) set $\left(X_{E}, Y_{E}, \ddot{Z}_{E}\right)$. - The freestream set is essentially the earth or inertial set inasmuch as the axis alignments are the seme. However, the freestream set can assume any origin. Thus the use of the set is to reference the local gravity vector and/or an absolute angular displacement or linear velocity acceleration of another set. As shown on Figure 8, the $Z_{E}$ axis points down toward local gravity. Other sets reference to the $E$ set, as the $F$ set shown here, may assume any starting value of roll and pitch such as the trim initial conditions. The relative orientation changes with progressing time of flight.

With the freestream set origin located coincident with the fuselage set, the components of fuselage set velocity in $E$ set are $u_{E}, v_{E}, w_{E}$. These components combine into a trajectory velocity $u_{T}$ and path $X_{T}$. The trajectory path is yawed right $\psi_{T}$ and pitched up $\gamma_{T}$ from the $E$ set. See Figure 9.
4.2.8 Swashplate coordinates $\left(X_{S P}, \underline{v}_{S P}, Z_{S P}\right)$. - As shown on Figure 10 , the SP set origin is located in line with the $Z_{H}$ axis and above the hub set a distance $Z_{O S P}$. The SP set does not rotate with the rotor shaft. For no deflection of the $S P$ set, the $X$ and $Y$ axes have the same alignment as the $X$ and $Y$ of the hub set.

### 4.3 Degrees of Freedom

The degrees of freedom of the REXOR II equations are defined as the generalized coordinate variables of the set of equations of motion to be developed in Section 5. These degrees of freedom fully describe the motion of the physical elements of the modeled helicopter, but each direction of otion of the helicopter may not have a degree of freedom directly assoiated with it. The physical motions may be described by a series of modal variables (Section 5.4) or through a set of transformations and combinations of the degrees of freedom as developed in Sections 4.4 ard 4.5.


[^0]

BY DEFINITION: VT. WT $\equiv 0$

Figure 9. - Coordinat systers - trajectory path to freestream axis.


Figure 10. - Swashplate coordinate system.

The REXOR II rotorcraft simulation analysis can be applied to describe the vehicle-rotor-control system dynamic iesponse for up to $16+4 b$ (where b is the number of blades) fully-coupled degrees of freedom. These include the normal six rigid body or vehicle degrees of freedom; rotor speed; provisions for up to twenty-eight degrees of freedom defining rotor blade motion (four mode degrees of freedom for seven blade maximum). Three swashplate degrees-of-freedom and six for describing shaft/transmission deflection. The equations of motion are uritten in a general form so that additional degrees-offreedom can be added if desired. The current degrees-of-freedom are listed in Figure ll. The discussion following describes ther in eetail.
4.3.1 Vehicle or rigid body. - The six rigid body degrees-of-freedom; three translations, and three rotations, are defined as motions of the fuselage or principal reference axis system, Section 4.2.2, relative to freestream (inertiai) reference datum. Translational displacements ( $X, Y, Z$ ) of of the origin of the fuselage coordinate, and rotational displacenents ( $\theta, \theta,{ }_{i}{ }^{\prime}$
about the fuselage axes describe these degrees of freedom. See Figure 8. As mentioned in Section 4.2.7, the freestresm set may instantaneously assume any reference point; therefore, only the time derivatives of $(X, Y, Z)$ and
$(\phi, \theta, \psi)_{H}$ have significance. In order to locate the direction of the gravity vector relative to the hub, a running calculation of the Euler angles ${ }_{E},{ }^{\theta}{ }_{E}, \psi_{E}$ must be made. Since these are not degrees of freedom and therefore not calculated in the equations of motion, they must be calculated outside the dynaric equations as the time history proceeds. When the initial orientation of the huo is defined, $\phi_{E}, \theta_{E}$, and ${ }_{E}$ are known and their charging values nay be calculated by integrating the hub rotaticn rates in the earth or freestream axes.
4.3.2 Rotor. - The rotation for the rotor degree of freedom $i_{R}$ is derine as motion of the rotor coordinate system relative to the hub axis system. This is show in Figure 6. This figure also indicates the change fror $Z$ down to $Z$ up axis, which is equivalent to a 280 -degree positive rotation about the $Y$ axis. Note: Rotor rotation alsc includes blade feathering from swashplate rotatior in addition to blade root rotation.
4.3.3 Shaft or transmission deflections. - Shaft or transmission degrees-of-freedom are defined as motions of the hut coordinates relative to the shaft axis system. Hence, as show in Figure 4, hub motions are dependent variables which are functions of the shaft deflecticns.

ITEM

## fuselage <br> PRINCIPAL AXIS

ROTOR

ShaFt OR TRANSMISSION DEFLECTION

Blades
( $n=7$ MAXIMUM)

SWASHPLATE

SYMBEL
$X_{\text {OF }}, Y_{\text {OF }}, Z_{\text {OF }}$
$\phi_{F}, \theta_{F} \cdot{ }_{F}$
${ }^{\succ}$ R
$\mathbf{X}_{\mathbf{s}}, \mathbf{Y}_{\mathbf{s}}, \mathbf{Z}_{\mathbf{s}}$.
$\theta_{s}, \theta_{s}, \iota_{s}$
$A_{1 n} \cdot A_{2 n} \cdot A_{\mathbf{3 n}}$
$\hat{A}_{\text {PH }}$
$\phi_{\text {SP }}, \theta_{\text {SP }}$
$Z_{S P}$

TYPE OF MOTION
TRANSLATION AND ROTATION WITH RESPECT TO INERTIAL REFEHENCE

ROTATION OF ROTOR SET WITH RESPECT TO HUB AXES

DEFLECTION OF HUB SET WITH RESPECT TO SHAFT AXES
blade rending modes and FEATHER/PITCH HORN BENDING OR TORSION WITH RESPECT TO BLADE ROOT AXES

SWASHFLATE AXES MOTIONS WITH RESPECT TO HUB AXES

Figure :1. - Segrees of freedom.
4.3.4 歼ades. - Each tlade's mution reiative to the rotor coordinate system is defined in terms of four generalized coordinates. These consist of three blade bending modes and a combined feathering, pitch arm bending mode, or a torsion mode.
4.3.4.1 Blade bending. - Blade motion due to blade bending is defined by the generailized medal coordinates $A_{m n}$ which typically represent a coupled first irplane bending mcie, a coupled first flapwise bending mode, and a ccupled second flaprise bending mode. Ordinarily in a model analysis, the effects of centrifugal and structural stiffness are lumped together into a generaiized stiffness which is simply the modal natural frequency squared times the generalized mass. In contrast to this, the REXOR II analysis separately treats the strain energy or structural stiffness in each mode and the stiffeaing due to the centrifugal Corce field. This provides the capability of being able to account for the periocic variation of stiffness in the modes due to the reorientation of the centrifugal force field with respect to the blade principal axis due to variations in blade angle. This feature can be inportant in the study of suoharcionic stability where the periodic variation of coefficients may be important, but it also permits being able to make rather large changes in rotor speed and collective blade angle without heving to change blace modal data.

Yoie shapes and natural frequencies are initially determined for a twisted blace at or near the collective biade angle and rotor speed to be analyzed. Suck effects as precone, blade sweep, biade droop, and blade angle variation are included in the REXOR II anaiysis and couple the initially orthogonal rodes. The elastic bending contribution aue to the modal deflections is calculated relative to the blade's static shape.

As previously noted, the blade modes are initially defined at some reference feathering ansle, $\phi_{\text {REF }}$. As time progresses in the anaiysis, the blade feather angle varies about this reference positicn. The mode shapes are correspondingly transformed to accourt for the difference between the instantaneous feathering angle and the reference feathering angle, at the same time accounting for other efiects such as the static and instantaneous shape of the blades. This yieles the modal coefficients (partial derivatives: that relate biade element motion to the blade bending generaiized coordinates as a function of ine.

The verticai and inplane blade element variational motions, $\delta Y_{i}$ and $\delta Z_{i}$, can be written as follows:

$$
\begin{equation*}
\delta Y_{i n}=\frac{\partial Y_{i}}{\partial A_{l n}}\left(q_{r}, t\right) \delta A_{l n}+\frac{\partial Y_{i}}{\partial A_{2 n}}\left(q_{r}, t\right) \delta A_{2 n}+\frac{\partial Y_{i}}{\partial A_{3 n}}\left(q_{r}, t\right) \delta A_{3 n} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta Z_{i n}=\frac{\partial Z_{i}}{\partial f_{1 n}}\left(q_{r}, t\right) \delta A_{1 n}+\frac{\partial Z_{i}}{\partial A_{2 n}}\left(q_{r}, t\right) \delta A_{2 n}+\frac{\partial Z_{i}}{\partial A_{3 n}}\left(q_{r}, t\right) \delta A_{3 n} \tag{4}
\end{equation*}
$$

where the given or input partial aerivatives are the true modal coefficients of the orthogonal modes for the blade in an undeformed shape, with no static geometry accounted for, and at the rotor speed and collective angle for which the blade modes vere initially calculated.

The orthogonal bending modes used in the analysis are illustrated in Figures 12, 13, and 14. Observe that the root boundary conditions for the modes may be cantilevered or articulated.

Note that in addition to the normal bending responses, $Y_{i}$ and $Z_{i}$, the spanvise motion of each blade element is alsc determined, and biade feathering due tJ pitch-lag and pitch-fiap kinematic coupl $\quad \mathrm{g}$ effects are also accounted :or in each blade bending mode. This feathering is added to that due to sweshplate motion as is blade feathering due to flexibility.

This nodal data is developed to the form used in the blade equations in Section 4.5.5. The discussion of modes is carried on from a math viewpoint in Section 5.4.
4.3.4.2 Pitch horn bending - dynamic torsion. - The remaining mode per blade, pitch horn tending, is comprised of either a blade feathering drive flexibility with a torsionally rigid blade or an uncoupled torsion mode. Examining the first alternative, the swashplate position determines the primary blade featherirg motion. In addition, the linkage between the swashplate and the blade (see Figure 15) has flexibility in the pitch link, pitch horn, and cuff. The feathering or pitch horn bending degree-offreedor therefore can be rigid klade featiering motion outboard of the blade cuff coupied with a net inboard stiffness. Inbourd of the blade cuff, feathering flexibility results from the pitch link, pitch link bearings, pitch horn, and cuff. The relationship between blade feathering, ${ }^{\phi_{\mathrm{Fn}}}$, and motion of this degree-of-freedom, $B_{\mathrm{PHn}}$, is defined as the partial derivative, $\left(\frac{\partial \phi_{F}}{\partial \beta_{P H}}\right)_{n}$.


Figure 12. - First inplane mode.


Figure 13. - First flap mode.


Figure 14. - Second flap mode.


Figure 15. - Blade, pitch horn, and feather hinge geometry.

Alternatively, this degree of freedom, $B_{\mathrm{PHn}}$, can be a distributed torsional response of the blade based upon defining an uncoupled dynamic torsion mode. The selection of the degree-of-freedom representation is made on the basis of the type of analysis being performed. The mode defined is uncoupled in the sense that it is not a function of the flapping or lead-lag modes.

An optional quasi-steady torsional response of the blade may be used in conjunction with pitch horn bending. This is superimposed on the rigid blade feathering and permits a distributed torsional response alternative of the blade reacting the spanwise variation of applied torsional momen s from aerodynamics, coriolis, and centrifugai force terms. The blade torsional response at the ith blade station is computed from the following equation:

$$
\begin{equation*}
\phi_{T i}=\frac{1}{\tau_{T}}{ }^{s+1} \int_{\text {root }}^{x_{i}} \frac{d x}{\operatorname{GJ}(\alpha)} \int_{r_{i}}^{\text {tip }} M_{\phi}(x) d x \tag{5}
\end{equation*}
$$

where $S$ is the Laplace operator, and $\tau_{T}$ is the time constant associated with blade torsional response. This equation is implemented numerically in the PEXOR IS progran.

To aid in program trouble shooting the pitch horn bending representation (with or without quasi-static torsion) may also be operated as a quisistatic degree of freedom without second-order response.
1.3.5 Swashplate. - The swashplate has three degrees of freedom: $\Phi_{S P}$, $\theta_{S P}$, and $Z_{S P}$. Rotations $\phi_{S P}$ and $\theta_{S P}$ are Euler angles defining the orientation of swashplate coordinates relative to the hub. Likewise, the translation $Z_{S p}$ defines vertical displacement of the swashplate relative to the hub axis. These -re show schematically in Figure 10.

### 4.4 General Motion and Coordinate Transformations

In uevelorment of the equations of motion, it is convenient tu write the forces, moments, velocities, and accelerations in coordinate systems related to separate elements of the system. Consider tine concept of general space motion of a particle.
4.4.1 General case of space motion. - For the general case of space motion, a particle, $p$, moves with respect to a reference axis system which is, in turn, in motion with respect to a fixed coordinate system. This is illustrated in Figure 16 where the fixed or inertial coordinate system is designated by capital letters $X, Y, Z$, and the moving coordinate system is designated by lower case letters $x, y, z$. The moving coordinate system is rotating at an angular velocity, $\vec{\omega}$. The vector $\vec{\omega}$ may, in general, vary in magnitude and direction, both of which can be referenced with respect to the fixed $X, Y, Z$ axes.

Thus, the absolute motion of the particle $p$, referred to the inertial $X, Y, Z$ axes, is equal to the motion of the particle relative to the moving coordinate axes $x, y, z$ plus the motion of the moving axis system with respect to inertial space.


Figure 16. - General case of space motion in terms of moving coordinate axes $X, y, z$ and inertial axes $X, Y, Z$.

To visualize the motion of the particle $p$, let its motion with respect tc the roving axis system be indicated along a curve $s$ fixed in the moving axis system, $x, y, z$. An observer sitting on the moving axis system would therefore see only the motion of $p$ along the curve $s$.

From Figure 16, the position of $p$ relative to the $x, y, z$ axes is represented by the vector

$$
\begin{equation*}
\vec{r}_{R}=x \vec{i}+y \vec{j}+z \vec{k} \tag{6}
\end{equation*}
$$

where $i, j$, and $k$ are unit vectors along $x, y, z$, and therefore must be treated as variables due to their changing direction. Differentiating $\vec{r}_{R}$ results in

$$
\begin{equation*}
\overrightarrow{\dot{r}}=\dot{x} \dot{i}+\dot{y} \vec{j}+\dot{z} \vec{k}+x \frac{d \vec{i}}{d t}+y \frac{d \vec{j}}{d t}+z \frac{d \vec{k}}{d t} \tag{7}
\end{equation*}
$$

Since $\frac{\overrightarrow{d i}}{d \pm}=\vec{\omega} \times \vec{i}, \frac{\overrightarrow{d j}}{d t}=\vec{\omega} \times \vec{j}$ and $\frac{\overrightarrow{d k}}{d t}=\vec{\omega} \times \vec{k}$, this expression can be written as

$$
\begin{equation*}
\overrightarrow{\dot{r}}=\dot{\mathrm{x} i}+\dot{\mathrm{y} j}+\overrightarrow{i \mathrm{k}}+\vec{\omega} \mathrm{x}(\mathrm{xi}+\overrightarrow{\mathrm{j}}+\overrightarrow{\mathrm{j}}+\overrightarrow{\mathrm{k}}) \tag{8}
\end{equation*}
$$

or

$$
\begin{equation*}
\overrightarrow{\dot{r}}=\vec{r}+\vec{\omega} \times \vec{r} \tag{9}
\end{equation*}
$$

In this equation, the first term, $\overrightarrow{\dot{r}}$ represents the velocity $p$ relative to the rotating axis, $x, y, z$. The second term, $\vec{\omega} x \vec{r}$, is the velocity of the point in the moving coordinate system due to the rotation $\omega$. The absolute or inertial velocity $\overrightarrow{\dot{R}}$ of the point $p$ is obtained by adding the velocity of the origin $\overrightarrow{\dot{R}}_{0}$ of the moving axis system to $\overrightarrow{\dot{r}}$, or:

$$
\begin{equation*}
\overrightarrow{\dot{R}}=\overrightarrow{\dot{R}}_{0}+\overrightarrow{\dot{r}}+\vec{\omega} \tag{10}
\end{equation*}
$$

where

$$
\vec{\omega}=p \vec{i}+q \vec{j}+r \vec{k}
$$

and

$$
\dot{\omega}=\dot{p} \vec{i}+\dot{q} \vec{j}+\dot{r} \vec{k}
$$

42

The inertial accelerations of the point $p$ can now be determined by simply differentiating this expression with respeci to time. Performing this differentiation yields

$$
\begin{equation*}
\overrightarrow{\ddot{R}}=\overrightarrow{\ddot{R}}_{0}+\overrightarrow{\ddot{r}}+\vec{\omega} \times \vec{\omega} \times \vec{r}+\overrightarrow{\dot{\omega}} \times \vec{r}+2 \omega \times \overrightarrow{\dot{r}} \tag{11}
\end{equation*}
$$

where the terms $\vec{\omega} \times \vec{\omega} \times \vec{r}$ and $\overrightarrow{\dot{\omega}} \times r$ represent accelerations of the coincident point in the moving axis system, $\ddot{r}$ is the acceleration of $p$ relative to the moving axes, $x, y, z$, and $2 \vec{\omega} \times \vec{r}$ is the coriolis acceleration which is directed normal to the plane containing the vectors $\vec{\omega}$ and the relative velocity $\vec{r}$, as given by the right-hand rule.

The vectors expressed in the preceding equations are in the most general form for defining the motion of a particle moving in a moving coordinate system. All special cases can be deduced from these equations.

For convenience, the time derivative equations can be expanded in matrix form. The inertial or absolute velocity and accelerations of the particle $p$, written in expanded matrix form, are given by:

$$
\left\{\begin{array}{l}
\dot{x}  \tag{12}\\
\dot{y} \\
\dot{z}
\end{array}\right\}^{I}=\left\{\begin{array}{l}
\dot{y}_{0} \\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}^{I}+\left\{\begin{array}{c}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right\}+\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

and

$$
\begin{align*}
& \left\{\begin{array}{l}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{array}\right\}=\left\{\begin{array}{l}
\ddot{x}_{0} \\
\ddot{x}_{0} \\
\ddot{z}_{0}
\end{array}\right\}+\left\{\begin{array}{l}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{array}\right\}+\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\} \\
& +\left[\begin{array}{rrr}
0 & -\dot{r} & \dot{q} \\
\dot{r} & 0 & -\dot{p} \\
-\dot{q} & \dot{p} & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}+2\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]\left\{\begin{array}{l}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right\} \tag{13}
\end{align*}
$$

Performing the indicated matrix multiplication gives:

$$
\left\{\begin{array}{c}
\dot{x}  \tag{14}\\
\dot{y} \\
\dot{z}
\end{array}\right\}=\left\{\begin{array}{c}
\dot{x}_{0} \\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}+\left\{\begin{array}{l}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right\}+\left\{\begin{array}{l}
z q-j \underline{n} \\
x r-z p \\
y p-x q
\end{array}\right\}
$$

and

$$
\begin{align*}
\left\{\begin{array}{c}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{array}\right\}= & \left\{\begin{array}{l}
\ddot{x}_{0} \\
\ddot{y}_{0} \\
\ddot{z}_{0}
\end{array}\right\}+\left\{\begin{array}{c}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right\} \\
& +\left\{\begin{array}{l}
x\left(-r^{2}-q^{2}\right)+y(p q-\dot{r})+z(p r+\dot{q})+2 \dot{z} q-2 \dot{y} r \\
x(p q+\dot{r})+y\left(-r^{2}-p^{2}\right)+z(q r-\dot{p})+2 \dot{x} r-2 \dot{z} p \\
x(p r-\dot{q})+y(q r+\dot{p})+z\left(-p^{2}-q^{2}\right)+2 \dot{y} p-2 \dot{x} q
\end{array}\right\} \tag{15}
\end{align*}
$$

4.4.2 Coordinate transformations - Euler angles. - To describe motions in one coordinate system in terms of motions in another coordinate system, Euler angles $\phi, \theta$, and $\psi$ with the appropriate subscripts are introduced. These angles can be applied to define the rctation of one coordinate system, $x, y, z$, relative to another coordinate reference frame, $X, Y, Z$. Since the development contained in this report utilizes these angles in relating csordinate systems, a brief explanation is given here.

Kotational displacement of a coordinate system can be represented by the three rotational displacements $\phi, \theta$, and $\psi$, as shown in Figure 17. The order of rotation is not important as long as the sequence selected remains consistent and the reverse order is used when zotating back to the


AXES $(X, Y, Z)_{b}$ DEFINED RELATIVE TO REFERENCE
AXES $(X, Y, Z), B Y$ EULER ANGLES $\phi, \theta, \psi$

Figure 17. - Rotational displacement of a coordinate system.
original position. In this analysis, the rotations start witn alsplacement $\quad$ about the $x$ bis, then a rotation $\theta$ about the new $y$ axis, followed by a rotation $\psi$ about the new or final $z$ axis unless geometry or physical considerations of the modeled part dictates another order.

This means the $(X, Y, Z)$ cocrilinates can be rotated into the ( $X, Y, Z)_{b}$ axis systen as follows:

$$
\left\{\begin{array}{l}
X  \tag{16}\\
Y \\
z
\end{array}\right\}_{b}=\left[\begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
c & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{l}
x \\
Y \\
Z
\end{array}\right\}
$$

or:

$$
\left\{\begin{array}{l}
x  \tag{17}\\
y \\
z
\end{array}\right\}_{b}=\left[\begin{array}{r}
T_{a-b}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

and the inverse transformation can be uritten as

$$
\left\{\begin{array}{l}
x  \tag{18}\\
y \\
z
\end{array}\right\}_{a}=\left[T_{a-b}\right]^{-]}\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}_{b}=\left[\begin{array}{l}
T_{b-a}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}_{b}
$$

where
$\left[\mathrm{T}_{\mathrm{a}-\mathrm{b}}\right]=\left[\begin{array}{ccc}\cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1\end{array}\right]\left[\begin{array}{ccc}\cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta\end{array}\right]\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi\end{array}\right]$
and

$$
\left[T_{a-b}\right]^{-1}=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{20}\\
0 & \cos \phi & -\sin \phi \\
ن & \sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{array}\right]\left[\begin{array}{ccc}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right]
$$

By inzpection, then, it can be seen that

$$
\text { inversc of }[T]=\text { transpose of }[T]
$$

or

$$
\begin{equation*}
\left[T_{a-b}\right]^{-1}=\left[T_{a-b}\right]^{T}=\left[T_{b-a}\right] \tag{21}
\end{equation*}
$$

Carrying out the indicated matrix multiplication yields the transformation matrix [ $T$ ]:
$\left[T_{a-b}\right]=\left[\begin{array}{ccc}(\cos \psi \cos \theta) & (\sin \phi \sin \theta \cos \psi+\cos \phi \sin \psi) & (-\sin \theta \cos \phi \cos \psi+\sin \phi \sin \psi) \\ (-\sin \psi \cos \theta) & (-\sin \psi \sin \phi \sin \theta+\cos \psi \cos \phi) & (\sin \psi \sin \theta \cos \phi+\cos \psi \sin \phi) \\ (\sin \theta) & (-\cos \theta \sin \phi) & (\cos \phi \cos \theta)\end{array}\right]$

Using this transformation, the inertial veiocities and accelerations of a point or particle be written in one coordinate system in terms of those in the other coordirate system as follows:

$$
\left\{\begin{array}{c}
\dot{X}  \tag{23}\\
\dot{Y} \\
\dot{Z}\}_{b}^{I}=\left[T_{a-b}\right]\left\{\begin{array}{l}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{array}\right\}_{a}^{I}, ~
\end{array}\right.
$$

and:

$$
\left\{\begin{array}{c}
\ddot{X}  \tag{24}\\
\ddot{Y} \\
\ddot{Z}
\end{array}\right\}_{b}=\left[\begin{array}{l}
T_{a-b}
\end{array}\right]\left\{\begin{array}{c}
\ddot{X} \\
\ddot{Y} \\
\ddot{Z}
\end{array}\right\}_{a}^{I}
$$

and inversely,

$$
\left\{\begin{array}{l}
\dot{X}  \tag{25}\\
\dot{Y} \\
\dot{Z}
\end{array}\right\}=\left[\begin{array}{l}
I \\
b-a
\end{array}\right]\left\{\begin{array}{l}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{array}\right\}
$$

and

$$
\left\{\begin{array}{c}
\ddot{X}  \tag{26}\\
\ddot{Y} \\
\ddot{Z}
\end{array}\right\}=\left[\begin{array}{l}
I \\
b-a
\end{array}\right]\left\{\begin{array}{c}
\ddot{X} \\
\ddot{Y} \\
\ddot{Z}
\end{array}\right\}_{b}^{I}
$$

4.4.3 Angular velosities and accelerations - general. - For the general case, ccnsider the coordinates in the previous section, and let ( $p, q, r$ ) a ana ( $p, q, r)_{b}$ be the respective angular velocities of and about the $(x, y, z)_{a}$ and $(s, y, a)_{b}$ axis systems. Also, assume that the Euler angles are varying with time $(\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$, and let $(x, y, z)$ be the reference coordinate set $w i t h(x, y, z)_{b}$ coordinate set moving relative to it. This is illustrated in Figure 13.


Figure 18. - Relationship of Euler angle and coordinate systen argular rates.

From this figure, the following can be written.

$$
\begin{aligned}
& \left\{\begin{array}{l}
\mathbf{p} \\
q \\
\mathbf{q}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
\dot{\phi}
\end{array}\right\}+\left[\begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right]\left\{\left\{\begin{array}{c}
0 \\
\dot{\theta} \\
0
\end{array}\right\}+\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left\{\begin{array}{l}
\dot{i} \\
0 \\
0
\end{array}\right\}\right. \\
& \left.+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\} a\right\}
\end{aligned}
$$

Differentiating tcis expression with respect to time results in angular accelerations $(\dot{p}, \dot{q}, \dot{r})_{b}$ in terms of the reference coordinate system angular velocities and accelerations. This results in the following:

$$
\begin{align*}
& \left\{\begin{array}{c}
\dot{\mathbf{p}} \\
\dot{q} \\
\dot{\mathbf{r}}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
\ddot{\varphi}
\end{array}\right\}+\dot{\dot{\psi}}\left[\begin{array}{ccc}
-\sin \psi & \cos \psi & 0 \\
-\cos \psi & -\sin \psi & 0 \\
0 & 0 & 0
\end{array}\right]\left\{\left\{\begin{array}{l}
0 \\
\dot{\theta} \\
0
\end{array}\right\}+\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left\{\begin{array}{l}
\dot{\phi} \\
0 \\
0
\end{array}\right\}\right. \\
& \left.\left.\left.+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}, \mathbf{a}\right\}\right\}\right\}+\left[\begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{array}\right]\left\{\begin{array}{l}
0 \\
\ddot{\theta} \\
0
\end{array}\right\} \\
& \left.+\dot{\theta}\left[\begin{array}{ccc}
-\sin \theta & 0 & -\cos \theta \\
0 & 0 & 0 \\
\cos \theta & 0 & -\sin \theta
\end{array}\right]\left\{\begin{array}{l}
\dot{\phi} \\
0 \\
0
\end{array}\right\}+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{l}
\mathrm{p} \\
q \\
r
\end{array}\right\}\right\} \\
& +\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left\{\left\{\begin{array}{l}
\ddot{\phi} \\
0 \\
0
\end{array}\right\}+\dot{\phi}\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & -\sin \phi & \cos \phi \\
0 & -\cos \phi & -\sin \phi
\end{array}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}\right. \\
& \left.+\left\{\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{c}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}, a\right\} \tag{28}
\end{align*}
$$

These equations represent a general form for defining angular velocities and accelerations of one axis system rotating relative to another axis system, whick in turn is in mation.

A special case is the angular velocities of system $b$ with zero Euler angles.

$$
\left\{\begin{array}{c}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{b}=\left\{\begin{array}{c}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}+\left\{\begin{array}{c}
\ddot{\phi} \\
\ddot{\theta} \\
\ddot{\psi}
\end{array}\right\}+\left\{\begin{array}{c}
\ddot{\psi}+\dot{\psi} q-\dot{\theta} r \\
-\dot{\psi} \dot{\phi}+\dot{\phi} r-\dot{\psi} p \\
\ddot{\theta} \dot{\phi}+\dot{\theta} p-\dot{\phi} q
\end{array}\right\} a
$$

4.5 Relative Motions and Transformations Used in the Equations of Motion

In this section the inertial linear and angular velocities and accelerations of major components of the vehicle, are presented. Also included is the aevelopment of coordinate transformations that relate motion in one axis system to another. Motion of the principal reference axis system in reiation to the earth is described. Motion of each component or reference axis system is then defined in terms of the degrees of freedom.
4.5.1 Fuselage motion in inertial space. - At each instant in time the fuselage axis (Section 4.2.1) is related to an inertial coordinate axis system. Inertial accelerations of the fuselage axis system are defined by the vector

$$
\left\{\begin{array}{c}
\ddot{x}_{0}  \tag{29}\\
\ddot{x}_{0} \\
\ddot{z}_{0} \\
\ddot{\phi}_{0} \\
\ddot{\theta}_{0} \\
\ddot{\psi}_{n}
\end{array}\right\}
$$

where the quantities represent the total inertial acceleration of the generalized conrdinates of the vehicle as defined by motion of the principal coordinate axis system.

Orientation of this system relative to the earth is specified by Euler angles $\phi_{E}, \theta_{E}$, and $\psi_{E}$ as seen in Figure 8. The sequence and definition of these angles is $\psi_{E}$ (yaw), $\theta_{E}$ (pitch), ${ }_{E}$ (roll). Note that the sequence of rotations is opposite to that given by Figure 17. The angular rates, $p_{F}, q_{F}, r_{F}$, of the fuselage or principal reference axis system with respect to the ineriial coordinate system can be uritten as

$$
\begin{align*}
\left\{\begin{array}{l}
\mathrm{p} \\
\mathrm{q} \\
\mathrm{r}
\end{array}\right\}_{\mathrm{F}} & =\left\{\begin{array}{l}
\dot{\phi}_{E} \\
0 \\
0
\end{array}\right\}+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{E} & \sin \phi_{E} \\
0 & -\sin \phi_{E} & \cos \phi_{E}
\end{array}\right]\left\{\left\{\begin{array}{l}
0 \\
\dot{\theta}_{E} \\
0
\end{array}\right\}\right. \\
& \left.+\left[\begin{array}{ccc}
\cos \theta_{E} & 0 & -\sin \theta_{E} \\
0 & 1 & 0 \\
\sin \theta_{E} & 0 & \cos \theta_{E}
\end{array}\right]\left\{\begin{array}{l}
0 \\
0 \\
\dot{\psi_{E}}
\end{array}\right\}\right\} \tag{30}
\end{align*}
$$

This equation can be rewritten to solve for $\phi_{E}, \theta_{E}$, and $\gamma_{E}$ as

$$
\left\{\begin{array}{l}
\dot{\phi}_{E}  \tag{31}\\
\dot{\theta}_{E} \\
\dot{\psi}_{E}
\end{array}\right\}=\left[\begin{array}{ccc}
1 & \sin \phi_{E} \tan \theta_{E} & \cos \phi_{E} \tan \theta_{E} \\
0 & \cos \phi_{E} & -\sin \phi_{E} \\
0 & \sin \phi_{E} \sec \theta_{E} & \cos \phi_{E} \sec \theta_{E}
\end{array}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}
$$

The Euler ances defining orientation of the principal reference axis system with respect to the earth is next obtained by integrating the rates with respect to time, or

$$
\begin{align*}
& \phi_{E}=\int_{0}^{t} \dot{\phi}_{E} d t+\phi_{t}=0, E  \tag{32}\\
& \theta_{E}=\int_{0}^{t} \dot{\theta}_{E} d t+\theta_{t}=0, E  \tag{33}\\
& \psi_{E}=\int_{0}^{t} \dot{\psi}_{F_{1}} d t+\psi_{t}=0, E \tag{34}
\end{align*}
$$

Angular velocities of the fuselage, with respect to the inertial axes reference system, $F_{F}, q_{F}, r_{F}$, are defined in terms of the degrees of freedom as

$$
\left\{\begin{array}{l}
p  \tag{35}\\
q \\
r
\end{array}\right\}=\left\{\begin{array}{l}
\dot{\phi}_{0} \\
\dot{\theta}_{0} \\
\dot{q}_{0}
\end{array}\right\}_{F}=\int_{0}^{t}\left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{F} d t+\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{t=0, F}
$$

where

$$
\left\{\begin{array}{c}
\dot{p}  \tag{36}\\
\dot{q} \\
\dot{r}
\end{array}\right\}=\left\{\begin{array}{l}
\ddot{\phi}_{0} \\
\ddot{\theta}_{0} \\
\ddot{\psi}_{0}
\end{array}\right\}_{F}^{I}
$$

Linear velocities of the fuselage or principal axis system are now determined. The first three quantities of the fuselage axis acceleration vector represent the linear interial accelfration of the fuselage. For a system in motion, the inertiai acceleration, $\ddot{a}_{0}{ }^{I}$, at the origin of the system is defined, based on the vector algebra of Section 4.4.1, as

$$
\begin{equation*}
\overrightarrow{\ddot{a}}_{0} I=\vec{a}_{0}+\vec{\omega} \times \vec{v}_{0} \tag{37}
\end{equation*}
$$

where $\vec{a}_{0}$ is $\frac{d \vec{V}_{0}}{d t}$ is the rate of change of velocity, $V_{0}$, of the origin of the moving ccordinate $\varepsilon_{\cdot} \cdot$ stem and $\omega$ is the rotational velocity of the moving coordinate system, both relative to the earth. Now defining:

$$
\left\{v_{0}\right\}_{F}^{I}=\left\{\begin{array}{l}
u  \tag{38}\\
v \\
w
\end{array}\right\}_{F}
$$

gives

$$
\left\{\begin{array}{l}
\ddot{x}_{0}  \tag{39}\\
\ddot{y}_{0} \\
\ddot{z}_{0}
\end{array}\right\}_{F}^{I}=\left\{\begin{array}{l}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{array}\right\}_{F}+\left[\begin{array}{rrr}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]_{F}\left\{\begin{array}{l}
u \\
v \\
w
\end{array}\right\}
$$

From this equetion, then, the rate of change of velocity of the moving coordinate reference system becomes

$$
\left\{\begin{array}{l}
\dot{u}  \tag{40}\\
\dot{v} \\
\dot{v}
\end{array}\right\}_{F}=\left\{\begin{array}{l}
\ddot{x}_{0}^{I}-q w+r v \\
\ddot{y}_{0}^{I}-r u+p w \\
\ddot{z}_{0}^{I}-p v+q u
\end{array}\right\}_{F}
$$

This set of accelerations and the time integral represent airflow acceleration and velocity incident on the helicopter.
m separate set of accelerations is carried through the analysis which contain the acceleration due to gravity. Ordinarily, gravity is treated as a force of mg on the right-hand side of the equations. However, the gravi,ational term can be accounted for by defining

$$
\left\{\begin{array}{c}
\ddot{x}_{0}  \tag{41}\\
\ddot{y}_{0} \\
\ddot{z}_{0}
\end{array}\right\}_{F}=\left\{\begin{array}{l}
\ddot{x}_{0} \\
\ddot{x}_{0} \\
\ddot{z}_{0}
\end{array}\right\}_{F}-\left\{\begin{array}{l}
g_{X} \\
g_{Y} \\
g_{Z}
\end{array}\right\} F F
$$

where $g_{X F}, g_{Y F}, g_{Z F}$ ere the three components of the gravity vector to be defined. The acceleration on the left may be defined as being in earthinertial, EI, axes.

The logic behind this substitution is as follows. For a rigid body in motion, the equilibrium equations can be written as

$$
\begin{align*}
& m \ddot{X}=m(\dot{u}+q w-r v)=F_{x} \\
& m \ddot{Y}=m(\dot{v}+r u-p v)=F_{Y}  \tag{42}\\
& m \ddot{Z}=m(\dot{w}+p v-q u)=F_{Z}
\end{align*}
$$

where

$$
\left\{\begin{array}{l}
\bar{F}_{X}  \tag{43}\\
F_{Y} \\
F_{Z}
\end{array}\right\}=\left\{\begin{array}{l}
\bar{F}_{X} \\
\bar{F}_{Y} \\
\bar{F}_{Z}
\end{array}\right\}+\left\{\begin{array}{l}
g_{X} \\
g_{Y} \\
g_{Z}
\end{array}\right\} m
$$

$\bar{F}_{X}, \bar{F}_{Y}$, and $\bar{F}_{Z}$ represent the external forces acting on the body, exclusive of gravitational forces.

Subtracting the gravitational vector from each side of the previous equations yields:

$$
\begin{align*}
& m\left(\ddot{X}-g_{X}\right)=m\left(\dot{u}+q w-r v-g_{X}\right)=\bar{F}_{X}  \tag{44}\\
& m\left(\ddot{Y}-g_{Y}\right)=m\left(\dot{v}+r u-p w-g_{Y}\right)=\bar{F}_{Y}  \tag{45}\\
& m\left(\ddot{Z}-g_{Z}\right)=m\left(\dot{w}+p v-q u-g_{Z}\right)=\bar{F}_{Z} \tag{46}
\end{align*}
$$

Which by inspection gives

$$
\begin{align*}
& \ddot{X}-g_{X}=\dot{u}+q w-r v-g_{X}  \tag{47}\\
& \ddot{y}-g_{Y}=\dot{v}+r u-p w-g_{Y}  \tag{48}\\
& \ddot{Z}-g_{Z}=\dot{w}+p v-q u-q_{Z} \tag{49}
\end{align*}
$$

Rearranging these equations yields:

$$
\begin{align*}
& \dot{u}=\left(\ddot{X}-g_{X}\right)-q w+r v+g_{X}  \tag{50}\\
& \dot{v}=\left(\ddot{Y}-g_{Y}\right)-r u+p w+g_{Y}  \tag{51}\\
& \dot{w}=\left(\ddot{z}-g_{Z}\right)-p v+q u+g_{Z} \tag{52}
\end{align*}
$$

The first terms on the right side of the equation are identified with the proposed gravitational acceleration definition of Equation 41.

Making the sibstitution:

$$
\left\{\begin{array}{c}
\dot{u}  \tag{53}\\
\dot{v} \\
\dot{w}
\end{array}\right\}_{F}=\left\{\begin{array}{l}
\ddot{X_{0}^{E I}}-q w+r v+g_{X} \\
\ddot{Y}_{0}^{E I}-r u+p w+g_{Y} \\
\ddot{z}_{0}^{E I}-p v+q u+g_{Z}
\end{array}\right\} F
$$

In this equation, the accelerations $\ddot{X}_{O F}^{E I}, \ddot{Y}_{O F}^{E I}$, and $\ddot{\mathrm{Z}}_{\mathrm{OF}}^{\mathrm{EI}}$ are the degree-of-freedom accelerations of the principal reference axis system used in the REXOR II analysis. These accelerations represent the inertial accelerations plus the equivalent accelerations of the reaction force to gravity. Thus, gravity is an equivalent acceleration applied to the reference coordinate axis system. Via coordinate system referencing, every mass element on the vehicle is therefore acted upon by this acceleration. This avoids including gravitational force as an external force individually applied to each mass element.

The gravitational vector at the fuselage is simply the gravity vector in earth axis transformed to the fuselage axis system through the Euler angle rotations $\phi_{E}, \theta_{E}$, and $\psi_{E}$. or

$$
\left\{\begin{array}{l}
g_{X}  \tag{54}\\
g_{Y} \\
g_{Z}
\end{array}\right\}_{F}=\left[T_{E-F}\right]\left\{\begin{array}{l}
0 \\
0 \\
g
\end{array}\right\}
$$

where

$$
\left[T_{E-F}\right]=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{55}\\
0 & \cos \phi_{E} & \sin \phi_{E} \\
0 & -\sin \phi_{E} & \cos \phi_{E}
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta_{E} & 0 & -\sin \theta_{E} \\
0 & 1 & 0 \\
\sin \theta_{E} & 0 & \cos \theta_{E}
\end{array}\right]\left[\begin{array}{ccc}
\cos \psi_{E} & \sin \psi_{E} & 0 \\
-\sin \psi_{E} & \cos \psi_{E} & 0 \\
0 & 0 & 1
\end{array}\right]
$$

The velocities of the principal axis system are obtained by integrating the rates of change of velocity with time, or

$$
\left\{\begin{array}{l}
\dot{x}_{0}  \tag{56}\\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{F}^{I}=\left\{\begin{array}{l}
u \\
v \\
w
\end{array}\right\}_{F}^{I}=\int_{0}^{t}\left\{\begin{array}{l}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{array}\right\}_{F} d t+\left\{\begin{array}{l}
u \\
v \\
w
\end{array}\right\}
$$

These velocities in earth coordinates can be written as

$$
\left\{\begin{array}{l}
\dot{x}_{0}  \tag{57}\\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{E}^{I}=\left[T_{F-E}\right]\left\{\begin{array}{c}
\dot{x}_{0} \\
\dot{x}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{F}=\left[T_{E-F}\right]^{T}\left\{\begin{array}{l}
\dot{x}_{0} \\
\dot{x}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{F}
$$

which can be integrated to give the position of the system relative to the earth. Doing this yields

$$
\left\{\begin{array}{l}
x_{0}  \tag{58}\\
y_{0} \\
z_{0}
\end{array}\right\}_{E}=\int_{0}^{t}\left\{\begin{array}{l}
\dot{x}_{0} \\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{E}^{I} d t+\left\{\begin{array}{l}
x_{0} \\
y_{0} \\
z_{0}
\end{array}\right\}_{t=0, E}
$$

4.5.2 Hub motions in inertial space. - The geometry of the hub, shaft and fuselage is described in Sections 4.2.1 through 4.2.3. Forming the motions of the hub first requires knowledge of the shaft set motions and shaft generalized coordinates. The transform from fuselage to shaft set, $\left[\mathrm{T}_{\mathrm{F}-\mathrm{S}}\right]$, involves rotor tilt alignment data $\phi_{O_{S}}$ and $\theta_{O_{S}}$. Elastic motions of the transmission suspension, and consequently the hub, are described by the generalized coordinates $(X, Y, Z, \phi, \theta, \psi)_{S}$ which are measured with respect to the shaft (S) set. The transform from shaft to hub, $[\mathrm{H}, \mathrm{S}-\mathrm{H}]$ is a function of the generalized coordinate angles $(\phi, \theta, \psi)_{S}$.
The development starts with fuselage to shaft set relations:

$$
\begin{align*}
& \left\{\begin{array}{l}
\dot{x}_{0} \\
\dot{Y}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{S}^{I}=\left[T_{F-S}\right]\left\{\left[\begin{array}{l}
\dot{x}_{0} \\
\dot{Y}_{O} \\
\dot{Z}_{O}
\end{array}\right]_{F}^{I}+\left[\begin{array}{rrr}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]\left\{\begin{array}{l}
x_{O_{S}} \\
r_{O_{S}} \\
z_{O_{S}}
\end{array}\right\}\right\}  \tag{59}\\
& \left\{\begin{array}{l}
\ddot{X}_{0} \\
\tilde{q}_{0} \\
\ddot{Z}_{0}
\end{array}\right\}_{S}^{E I}=\left[T_{F-S}\right]\left\{\left[\begin{array}{l}
\ddot{x}_{0} \\
\ddot{\mathrm{Y}}_{0} \\
\ddot{Z}_{0}
\end{array}\right\}_{F}^{E I}+\left[\left[\begin{array}{rrr}
0 & -r & q \\
r & 0 & -p \\
-q & p & c
\end{array}\right]_{F}\left[\begin{array}{rrr}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]{ }_{F}\right.\right. \\
& \left.+\left[\begin{array}{rrr}
0 & \dot{r} & \dot{q} \\
\dot{\mathbf{r}} & 0 & -\dot{e} \\
-\dot{q} & \dot{p} & 0
\end{array}\right]_{F}\right]\left\{\begin{array}{l}
x_{O_{S}} \\
\mathrm{y}_{\mathrm{O}_{S}} \\
\mathrm{z}_{\mathrm{O}_{S}}
\end{array}{ }_{F}\right\} \tag{60}
\end{align*}
$$

Noting the following transformations:

$$
\left\{\begin{array}{c}
\mathrm{p}_{\mathrm{F}}  \tag{61}\\
\mathrm{q}_{\mathrm{F}} \\
\mathrm{r}_{\mathrm{F}}
\end{array}\right\}=\left[\mathrm{T}_{\mathrm{F}-\mathrm{S}}\right]\left\{\begin{array}{l}
\mathrm{p} \\
\mathrm{q} \\
\mathrm{r}
\end{array}\right\}_{\mathrm{F}}
$$

$$
\left\{\begin{array}{l}
\dot{p}_{F}  \tag{62}\\
\dot{q}_{F} \\
\dot{r}_{F}
\end{array}\right\}=\left[{ }_{S}{ }_{F-S}\right]\left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{F}
$$

Applied to hub set equations:

$$
\begin{aligned}
& =\left\{\begin{array}{c}
u \\
v \\
w
\end{array}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& +\left[\left[\begin{array}{ccc}
0 & -r_{F} & q_{F} \\
r_{F} & 0 & -p_{F} \\
-q_{F} & p_{F} & 0
\end{array}\right]_{S}\left[\begin{array}{ccc}
0 & -r_{F} & q_{F} \\
r_{F} & 0 & -p_{F} \\
-q_{F} & p_{F} & 0
\end{array}\right]_{S}+\left[\begin{array}{ccc}
0 & -\dot{r}_{F} & \dot{q}_{F} \\
\dot{r}_{F} & 0 & -\dot{p}_{F} \\
-\dot{q}_{F} & \dot{p}_{F} & 0
\end{array}\right]_{S}\right]\left\{\begin{array}{c}
x_{H} \\
Y_{H} \\
z_{H}
\end{array}\right\}
\end{aligned}
$$

(64)
where

Forming the relative location of the hub ir fuselage coordinates:

$$
\left\{\begin{array}{l}
x_{H}  \tag{65}\\
y_{Y} \\
Z_{H}
\end{array}\right\}_{F}\left\{\begin{array}{l}
x_{O_{S}} \\
y_{O_{S}} \\
Z_{O_{S}}
\end{array}\right\}+\left[T_{F-S}\right] T\left\{\begin{array}{c}
x_{H} \\
y_{Y} \\
Z_{H}
\end{array}\right\}
$$

Locking at enguiar information in the hub set:

$$
\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{\dot{L}}=\left\{\begin{array}{c}
c \\
c \\
\dot{c} s
\end{array}\right\}+\left[\begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right] s\left\{\begin{array}{l}
0 \\
\dot{\theta} \\
0
\end{array}\right\}_{S}
$$

$$
\begin{align*}
& +\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]_{S}\left\{\left\{\begin{array}{l}
\dot{\phi} \\
0 \\
0
\end{array}\right\}_{S}+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{c}
p_{F} \\
q_{F} \\
\left.r_{F}\right\}_{S}
\end{array}\right\}\right\}  \tag{69}\\
& \left\{\begin{array}{l}
\dot{\dot{p}} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{H}=\left\{\begin{array}{l}
0 \\
0 \\
\dot{\psi}
\end{array}\right]_{S}+\dot{\psi}\left[\begin{array}{ccc}
-\sin \psi & \cos \psi & 0 \\
-\cos \psi & -\sin \psi & 0 \\
0 & 0 & 0
\end{array}\right]_{S}\left\{\left[\begin{array}{l}
0 \\
\dot{\theta} \\
0
\end{array}\right]_{S}+\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]_{S}\left[\left\{\begin{array}{l}
\dot{\phi} \\
0 \\
0_{S}
\end{array}\right.\right.\right. \\
& \left.\left.+\left[\begin{array}{ccc}
i & 0 & 0 \\
0 & \cos \phi \sin \phi \\
0 & -\sin \phi & \cos \psi
\end{array}\right]\left\{\begin{array}{l}
P_{F} \\
q_{F} \\
r_{F}
\end{array}\right\}\right\}\right\}+\left[\begin{array}{ccc}
\cos \psi \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right]_{S}\left\{\left\{\begin{array}{l}
0 \\
\bar{\theta} \\
0
\end{array}\right\}\right. \\
& +\dot{\theta}_{S}\left[\begin{array}{ccc}
-\sin \theta & 0 & -\cos \phi \\
0 & 0 & 0 \\
\cos \theta & 0 & -\sin \theta
\end{array}\right]_{S}\left\{\left\{\begin{array}{l}
\dot{\phi} \\
0 \\
0
\end{array}\right\}_{S}+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{l}
p_{F} \\
q_{F} \\
r_{P}
\end{array}\right\}_{S}\right\} \\
& +\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]_{S}\left\{\left\{\begin{array}{l}
\phi \\
0 \\
0
\end{array}\right\}_{S}+\dot{\phi}_{S}\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & -\sin \phi & \cos \phi \\
0 & -\cos \phi & -\sin \phi
\end{array}\right]\left\{\begin{array}{l}
p_{F} \\
q_{r} \\
r_{F}
\end{array}\right\}\right. \\
& \left.+\left[\begin{array}{ccc}
1 & c & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{c}
\dot{P}_{F} \\
\dot{q}_{F} \\
\dot{r}_{F}
\end{array}\right\}\right\}
\end{align*}
$$

4.5.3 Motion of rotor coordinate axis. - The rotor coordinate axis system is shown in Figure 6. Note that the rotor coordinate axis syster is rotated 180 degrees about the $\mathbf{v}$ axis relative to the hub axis system at the tine when the rotor is at azimuth position zero. That is, $X$ and $Z$ change directinns. The rotor coordinate system then rotates through the angle $\psi_{R}$ from this position.

The sequence of rotation in going from hub to rotor coordinates consists of first a 180 -degree $\theta$ rotation, fcllowed by the $\psi_{R}$ rotation. Foliouing the convention estabiished in Section 4.4 .2 for Euler angles:

$$
\begin{align*}
{\left[T_{H-R}\right] } & =\left[\begin{array}{ccc}
\cos \psi_{R} & \sin \psi_{R} & 0 \\
-\sin \psi_{R} & \cos \psi_{R} & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
\cos \pi & 0 & \sin \pi \\
0 & 1 & 0 \\
-\sin \pi & 0 & \cos \pi
\end{array}\right] \\
& =\left[\begin{array}{ccc}
\cos \psi_{R} & \sin \psi_{R} & 0 \\
-\sin \psi_{R} & \cos \psi_{R} & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right] \tag{71}
\end{align*}
$$

where the last matrix represents the 180 -degree $\theta$ rotation. The next matrix is the rotor rotation, $\psi_{R}=\int_{0}^{t} s_{R} d t$.

Since the origins of the rotor coordinate system and the principal referenct axis system are ccincicient, the linear velocities and accelerations of the origin of the rotor conrdinate system can be directly written as:

$$
\left\{\begin{array}{l}
x_{0}  \tag{72}\\
y_{0} \\
\dot{z}_{0}
\end{array}\right\}_{R}^{I}=\left[T_{H-R}\right]\left\{\begin{array}{l}
\dot{x}_{0} \\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{H}^{I}
$$

and

$$
\left\{\begin{array}{l}
\dot{x}_{0} \\
\ddot{y}_{0} \\
\ddot{z}_{0}
\end{array}\right\}_{R}^{E I}=\left[T_{H-R}\right]\left\{\begin{array}{l}
\ddot{x}_{0} \\
\ddot{x}_{0} \\
\ddot{z}_{0}
\end{array}\right\}_{H}^{E I}
$$

Moting gravity has been treated as an equivalent acceleration in the hub generalized coordinate accelerations. This same equivalent acceleration is included in $\left(\ddot{X}_{0}, \ddot{Y}_{0}, \ddot{Z}_{0}\right)_{R I}^{E I}$ the rotor coordinate accelerations.
The angular velocities, $p_{R}, q_{R}, r_{R}$, and accelerations, $\dot{p}_{R}, \dot{q}_{R}, \dot{r}_{R}$ of the rotor coordinate system are determined; again noting the rotation order. The rotor coordinate system angular velocities are:

$$
\left\{\begin{array}{l}
\mathrm{p}  \tag{74}\\
\mathrm{q} \\
r
\end{array}\right\}_{R}=\left\{\left\{\begin{array}{l}
0 \\
0 \\
\dot{\psi}_{R}
\end{array}\right\}_{H}+\left[\begin{array}{ccc}
\cos \psi_{R} & \sin \psi_{R} & 0 \\
-\sin \psi_{R} & \cos \psi_{R} & 0 \\
0 & 0 & 1
\end{array}\right]\left\{\begin{array}{c}
-\mathrm{p} \\
q \\
-r
\end{array}\right\}_{H}\right\}
$$

I.ikewise, accelerations of the rotor coordinate system are:

$$
\begin{align*}
& \left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{R}=\left\{\left\{\begin{array}{l}
0 \\
0 \\
\ddot{\psi}_{R}
\end{array}\right\}_{H}+\dot{\psi}_{R}\left[\begin{array}{ccc}
-\sin \psi_{R} & \cos \psi_{R} & 0 \\
-\cos \psi_{R} & -\sin \psi_{R} & 0 \\
0 & 0 & 0
\end{array}\right]\left\{\begin{array}{c}
-\mathrm{p} \\
q \\
-r
\end{array}\right\}\right. \\
& +\left[\begin{array}{ccc}
\cos \psi_{R} & \sin \psi_{R} & 0 \\
-\sin \psi_{R} & \cos \psi_{R} & 0 \\
0 & 0 & 1
\end{array}\right]\left\{\begin{array}{c}
-\dot{p} \\
\dot{q} \\
-\dot{r}
\end{array}\right\}, H \tag{75}
\end{align*}
$$

The above equations then define the coordinate transformation from hub to rotor coordinates; and rotor axis system linear and angular velocities and accelerations in terms of velocities and accelerations of hub and the rotor iegrees of freedom ${\hbar_{R}}$.
4.5.4 Blade coordinate relative to rotor coordinates. - Since each blade has its own blade reference system, as shown in Figure 6, the $X_{B L n}$ and $Y_{B L n}$ axes are rotated with respect to the $X_{R}$ and $Y_{R}$ axes azimuthally by en angle $*_{B L n}$ derined by the equation

$$
\begin{equation*}
\phi_{\mathrm{BL}, \mathrm{n}}=\frac{-2 \pi(\mathrm{n}-1)}{\mathrm{b}} \tag{76}
\end{equation*}
$$

where $b$ is the number of blades and $n$ is the blade number. This equation staies that the $X_{B L I}$ and $X_{R}$, and the $Y_{B L I}$ and the $Y_{R}$ axes are coincident.
The transformations betweer the rotor coordinate axis system and the blade coordinate axis systems are deifined by the equation

$$
\left[\mathrm{T}_{\mathrm{R}-\mathrm{BLn}}\right]=\left[\begin{array}{ccc}
\cos \psi_{\mathrm{BLn}} & \sin \psi_{B L n} & 0  \tag{77}\\
-\sin \psi_{\mathrm{BLn}} & \cos \psi_{\mathrm{BLn}} & 0 \\
0 & 0 & 1
\end{array}\right]
$$

Note that these equations define blaqe one as being straight aft at time zero.

In the blade referense axes, the velocities and accelerations of the origin of the blade reference axis system become:

$$
\left\{\begin{array}{l}
\dot{X}_{O B L n}  \tag{78}\\
\dot{Y}_{O B L n} \\
\dot{z}_{O B L n}
\end{array}\right\}_{\text {BLn }}=\left[\mathrm{T}_{\mathrm{R}-\mathrm{BLn}}\right]\left\{\begin{array}{c}
\dot{\mathrm{x}}_{0} \\
\dot{\mathrm{Y}}_{0} \\
\dot{\mathrm{Z}}_{0}
\end{array}\right\}_{R}^{I}
$$

and

$$
\left\{\begin{array}{l}
\ddot{x}_{0 B L n}  \tag{79}\\
\ddot{\mathrm{y}}_{\text {OBLn }} \\
\ddot{z}_{\text {OBLn }}
\end{array}\right\}_{\text {BLn }}=\left[\mathrm{T}_{\mathrm{R}-\mathrm{BLn}}\right]\left\{\begin{array}{c}
\ddot{\mathrm{x}}_{0} \\
\ddot{\mathrm{y}}_{0} \\
\ddot{\bar{z}}_{0}
\end{array}\right\}_{R}
$$

Likevise, the angular velccities and accelerations of the blade reference axis systems become:

$$
\left\{\begin{array}{l}
\mathrm{p}  \tag{80}\\
q \\
r
\end{array}\right\}_{B L n}=\left[\mathrm{T}_{\mathrm{R}-\mathrm{BLn}}\right]\left\{\begin{array}{l}
\mathrm{p} \\
q \\
r
\end{array}\right\}_{\mathrm{R}}
$$

and

$$
\left\{\begin{array}{c}
\dot{p}  \tag{81}\\
\dot{q} \\
\dot{r}
\end{array}\right\}_{B L n}=\left[T_{R-B L n}\right]\left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{R}
$$

4.5.5 Blade element motion. - The following blade motion description, due to the involved nature $\mathrm{c}^{2}$ the geometry, is rather leagthy. First, in this development, the motion of the blade with respect to the relative blade coordinates is given. This motion is the sum of static and modal deflections. Then the relation to freestream coordinates is computed. Partial derivatives are extracted from the transformations for use in the equations of motion of the blade in Section 5.6.

The blade element motions for the nth blade are defined relative to the blade (BLa) conrdinate reference axes (Figure 6). The blade element relative motions are functions of the static shepe, of blade feathering and torsional deflection, and of blade bending of the coupled inplane and flapping modes.

The static shape includes such items as blade twist, ${ }_{T W}$, hub precone angle, $B_{0}$, blade droop angle relative to the precone angle, $\gamma$, blade sweep angle, $\tau_{0}$, feathering axis precone, $B_{F A}$, the blade feathering angle, and the blade element center of gravity location.

The blade motions about this static shape include the effects of tie three blade berding mudes, $A_{1 n}, A_{2 n}$ and $A_{3 n}$, blade feathering, $\phi_{F}$, and blade torsional deflection, $\dagger_{t}$.

The blade element motions are now defined. The blade static position in the blade reference axis systen is first developed. The blade bending and feathering deflections are then introduced. Both deflections and slopes are developed and then these equations are differentiated with respect to time to obtain the blade element linear and angular velocities and accelerations.

The blade element linear motions are developed in blade (BLn) coordinates and the blade element angular velocities and accelerations are jeveloped in blade element (BLE) coordinates. The coordinate transformation matrix $\left[T_{B L n-B L E}\right]$ is also defined to permit the transformation of the inertial velocities and accelerations from one axis system to the other. The development of the blade relative motion equations now starts with the description of the shape of the blade.
4.5.5.1 Blade static shape. - Blade elemental motion is defined as motion of tie blade glement reference axis system which has its origin at the blade element center of gravity. The blade eerodynamic reference axis is selected as the $1 / 4$ chord. Likewise, the geometry and dynamics are referenced to the $1 / 4$ chord, though any reference line could have been used. Starting with the straight untwisted blade vith the blade $1 / 4$ chord lying along the $X_{\text {BLn }}$ axis as in Figure 19, the blade element cg and blade element coordinate axis system origin are defined by the coincident point defined by the vector

$$
\left\{\begin{array}{l}
X_{C G}(i)  \tag{82}\\
Y_{C G}(i) \\
Z_{C G}(i)
\end{array}\right\}_{B L n}
$$

in blade coordinates. Thd dimension $X_{C G}(i){ }_{B L n}$ is the undeformed spanwise location of the $\mathrm{cg} / \mathrm{blade}$ element origin. The dimension $Y_{C G}(i)_{B L n}$ is the chordwise location of the c.g./blade element axis system origin forward of the blade $1 / 4$ chord and $Z_{C G}(i)_{B L n}$ is any vertical offset of the c.g./blade element origin with respect to the reference chord plane oi the blade.

Now, introducing blade tuist by rotating about the $X_{\text {BLn }}$ axis through the local blade twist angles, Figure 20, results in:

$$
\left\{\begin{array}{l}
X(i)_{B L E}  \tag{83}\\
Y(i)_{B L E} \\
Z(i)_{B L E}
\end{array}\right\}_{I}=\left\{\begin{array}{l}
X(i)_{B L E} \\
Y(i)_{B L E} \\
Z(i)_{B L E}
\end{array}\right\}_{B L n}=\left\{\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{T W} & -\sin \phi_{T W} \\
0 & \sin \phi_{T W} & \cos \phi_{T W}
\end{array}\right\}\left\{\begin{array}{l}
X_{C G} \\
Y_{C G} \\
Z_{C G}
\end{array}\right\}
$$

The Roman numeral subscript $I$ denotes the first of a sequence of static line transformations.


Figure 19. - Blade element c.g./origin location in blade coordinates.


Figure 20. - Effect of blade twist on location of blade element c.g./axis system origin.

At this point the subscripting, BLE will be dropped to simplify the deveiopment. Rewriting the above equation, we have:

$$
\left\{\begin{array}{l}
X(i)  \tag{84}\\
Y(i) \\
Z(i)
\end{array}\right\}_{I}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{TW}} & -\sin \phi_{\mathrm{TW}} \\
0 & \sin \phi_{\mathrm{TW}} & \cos \phi_{\mathrm{TW}}
\end{array}\right\}\left\{\begin{array}{l}
X_{\mathrm{CG}} \\
\mathrm{Y}_{\mathrm{CG}} \\
Z_{\mathrm{CG}}
\end{array}\right\}
$$

Introducing clade coning, $B_{0}$, results in the location of the blade as shown in Figure 21. This results in:

$$
\left\{\begin{array}{l}
X(i)  \tag{85}\\
Y(i) \\
Z(i)
\end{array}\right\}_{I I}=\left[\begin{array}{ccc}
\cos \beta_{0} & 0 & -\sin \beta_{0} \\
0 & 1 & 0 \\
\sin \beta_{0} & 0 & \cos \beta_{0}
\end{array}\right]\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}
$$

The next item of static geometry that is considered is blade droop, $r$, and then blade sweep, ${ }^{T} 0^{\text {. }}$ These rotations are shown in Figure 22. Note that since the blade sweep and droop angles are introduced at a distance $X_{S W}$ out on the blade, it is first necessary to transfer axes to this location before making the rotations. Therefore, the blade displacements outboard of Station $X_{\text {SW }}$ become:

$$
\begin{align*}
\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}= & {\left[\begin{array}{ccc}
\cos \tau_{0} & -\sin \tau_{0} & 0 \\
\sin \tau_{0} & \cos \tau_{0} & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
\cos \gamma & 0 & \operatorname{sin\gamma } \\
0 & 1 & 0 \\
-\sin \gamma & 0 & \cos \gamma
\end{array}\right]\left\{\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}\right.} \\
& \left.-\left\{\begin{array}{c}
X_{S W} \cos \beta_{0} \\
0 \\
X_{S W} \sin \beta_{0}
\end{array}\right\}\right\} \tag{86}
\end{align*}
$$



Figure 2l. - Blade precone angle, $6_{0}{ }^{\circ}$


Figure 22. - Blade sweep, $\tau_{0}$, and blade droop, $\gamma$.

At this same station, provisions are introduced to allow for offsets of the blade in both the vertical and horizontal directions by $Z_{j o g}$ and $Y_{j o g}$, respectively, These offsets are shown in Figure 23. These offsets represent displacement of the blade $1 / 4$ chord with respect to the blade precone line at blade station $X_{S W}$.

Introduction these offsets, then, and transferring back to the center of rotation through $X_{S W}$ results in the description of the blade displacements outboard of station 'SW, including the effects of precone, sweep, droop, and offset of the blade from the precone line.

$$
\left\{\begin{array}{l}
X(i)  \tag{87}\\
Y(i) \\
Z(i)
\end{array}\right\}_{I V}=\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}_{I I I}+\left\{\begin{array}{c}
0 \\
Y_{j \circ g} \\
Z_{j \circ g}
\end{array}\right\}+\left\{\begin{array}{c}
X_{S W} \cos B_{0} \\
0 \\
X_{S W} \sin B_{0}
\end{array}\right\}
$$



Figure 23. - Introduction of blade $1 / r$ chord offset, $Y_{j o g}$ and $Z_{j o g}$
with respect to precone line.

At this point, a reminder that the rrior development represents the blade displacement inboard of Station $X_{S W}$ and the above equation outboard of Station $X_{S W}$. Therefore, iniooard of Station $X_{S W}$ :

$$
\left\{\begin{array}{l}
X(i)  \tag{88}\\
Y(i) \\
Z(i)
\end{array}\right\}=\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}_{I I}
$$

Outboard of Station $X_{S W}$ :

$$
\left\{\begin{array}{l}
X(i)  \tag{89}\\
Y(i) \\
Z(i)
\end{array}\right\}=\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z!i)
\end{array}\right\}
$$

With this in mind, the remaining developing of inciuding the efiects of feathering axis static precone and blade reference feather angle in describing the static blade position continues. No distinction will be made in the following developments between inboard of Station $X_{S W}$ and outboard of Station $X_{S W}$.

Figure 24 shows how blade feathering is introduced. The axis system is translated to a point $p$ which is located at the intersection of the precone line and the feathering axis. The location of this point is a distance $\ell_{p}$ along the cone line, as shown in this figure. The blade is first rotated to the feather axis; then rotated about the reference feathering angle, $\phi_{\text {REF }}$, the feathering angle for which the blade modes are defined. Doing this results in:

$$
\begin{align*}
\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}= & {\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{R E F} & -\sin \phi_{R E F} \\
0 & \sin \phi_{R E F} & \cos \phi_{R E F}
\end{array}\right]\left[\begin{array}{ccc}
\cos \beta_{F A} & 0 & \sin \beta_{F A} \\
0 & 1 & 0 \\
-\sin \beta_{F A} & 0 & \cos \beta_{F A}
\end{array}\right]\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\} } \\
& \left.-\left\{\begin{array}{c}
\ell_{p} \cos \beta_{0} \\
0 \\
\ell_{p} \sin \beta_{0}
\end{array}\right\}\right\} \tag{90}
\end{align*}
$$



Figure 24. - Point $p$ and fecthering axis precon: $\beta_{F A}$.

This equation defines the lonatior of the static shape of the blade in an axis system with the $y$-axis horizontal and the $x$-axis aligned with the blace static feathering axis. Transforming now back through the feathering axis precone angle and translating back to the rotor shaft centerline results in the static shape of the blade defined in blade coordinates, or

$$
\left.\left\{\begin{array}{c}
X_{S}(i)  \tag{91}\\
Y_{S}(i) \\
Z_{S}(i)
\end{array}\right\}_{E L n}=\left[\begin{array}{ccc}
-\quad \cos \beta_{F A} & 0 & -\sin \beta_{F A} \\
0 & 1 & 0 \\
\sin \beta_{F A} & 0 & \cos \beta_{F A}
\end{array}\right]\left\{\begin{array}{l}
X(i) \\
Y(i) \\
Z(i)
\end{array}\right\}, \begin{array}{c}
0 \\
\ell_{p} \cos \beta_{0} \\
\ell_{p} \sin \beta_{0}
\end{array}\right\}
$$

where subscript $S$ refers to blade static or undeformed shape. Combining equations developed so far results, then, in the follcwing two equations which represent the static shape of the blade for boti inboard and outboard of blade station $X_{S W}$.

Inboard of Station $X_{S k}$ :

$$
\begin{aligned}
& \left\{r_{S L L r .}=\left\{\begin{array}{l}
x_{S} \\
Y_{S} \\
Z_{S}
\end{array}\right\}=\left[\begin{array}{ccc}
\cos \beta_{F A} & 0 & -\sin \beta_{F A} \\
0 & 1 & 6 \\
\sin B_{F A} & 0 & \cos \beta_{F A}
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{R E F} & -\sin \phi_{R E F} \\
0 & \sin \phi_{R E F} & \cos \phi_{\mathrm{DEF}}
\end{array}\right]\right. \\
& {\left[\begin{array}{ccc}
\cos \beta_{F A} & 0 & \sin \beta_{F A} \\
0 & 1 & 0 \\
-\sin \beta_{F A} & 0 & \cos \beta_{F A}
\end{array}\right]\left\{\left[\begin{array}{ccc}
\cos \beta_{0} & 0 & -\sin \beta_{0} \\
0 & 1 & 0 \\
\sin \beta_{0} & 0 & \cos \beta_{0}
\end{array}\right]\right.}
\end{aligned}
$$

$$
\begin{aligned}
& +\left\{\begin{array}{c}
e_{p} \cos 3_{0} \\
0 \\
e_{p} \sin 8_{0}
\end{array}\right\}
\end{aligned}
$$

Outboard of Station $\mathrm{K}_{\mathrm{SW}}$ :

$$
\begin{aligned}
& \left\{r_{B L n}=\left\{\begin{array}{c}
X_{S} \\
Y_{S} \\
Z_{S}
\end{array}\right\}=\left[\begin{array}{ccc}
\cos B_{F A} & 0 & -\sin B_{F A} \\
0 & 1 & 0 \\
\sin B_{F A} & 0 & \cos B_{F A}
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & \cos \phi_{R E F} \\
0 & -\sin \phi_{R E F} \\
0 & \cos \phi_{R E F}
\end{array}\right]\right. \\
& {\left[\begin{array}{ccc}
-0 s B_{F A} & 0 & \sin \hat{B}_{F A} \\
0 & 1 & 0 \\
-\sin B_{F A} & 0 & \cos B_{F A}
\end{array}\right]\left\{\left[\begin{array}{ccc}
\cos \tau_{0} & -\sin \tau_{0} & 0 \\
\sin \tau_{0} & \cos \tau_{0} & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
\cos \gamma & 0 & \sin Y \\
0 & 1 & 0 \\
-\sin \gamma & 0 & \cos \gamma
\end{array}\right]\right.} \\
& \left\{\left[\begin{array}{ccc}
\cos \beta_{0} & 0 & -\sin \beta_{0} \\
0 & 1 & 0 \\
\sin \beta_{0} & 0 & \cos \beta_{0}
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{T}} & -\sin \phi_{T W} \\
0 & \sin \phi_{T W} & \cos \phi_{T W}
\end{array}\right]\left\{\begin{array}{c}
x_{C G} \\
{ }_{C G} \\
Z_{C G}
\end{array}\right\}\right.
\end{aligned}
$$

$$
\begin{align*}
& +\left\{\begin{array}{c}
\hat{q}_{p} \cos \beta_{0} \\
0 \\
e_{p} \sin \beta_{0}
\end{array}\right\} \tag{93}
\end{align*}
$$

These two equations ther define completely the static shape of the 'lade. The de: ijopment will not proseed to include the blade bending or elastic deformaticr. fowever, before proceeding with this, the static location of the blade feathering bearing is define since these will be usec in the development that follows.

Referring to Figure 25 , it san be seen that the static position of the inboard feather vearing location car be written as:

$$
\left\{\begin{array}{c}
x_{S_{I B}}  \tag{94}\\
y_{S_{I B}} \\
Z_{S_{I B}}
\end{array}\right\}=\left\{\begin{array}{c}
\ell_{I B} \cos \beta_{0} \\
0 \\
{ }_{I B} \sin \beta_{0}-\left({ }^{\ell}{ }_{p}-\ell_{I B}\right)\left(\tan \left(B_{F A}-\beta_{0}\right) \cos _{0}\right.
\end{array}\right\}
$$

The static location of the outboard feather bearing is:

$$
\left\{\begin{array}{c}
x_{S_{O B}}  \tag{95}\\
y_{S_{O B}} \\
z_{S_{O B}}
\end{array}\right\}=\left\{\begin{array}{c}
\hat{e}_{O B} \cos B_{0} \\
0 \\
\ell_{O B} \sin B_{0}+\left(\ell_{V B}-\ell_{p}\right)\left(\tan \left(s_{F A}-B_{0}\right) \cos R_{0}\right.
\end{array}\right\}
$$

With these definitions, the analysis will not proceed to include the effects of blade bending, blace feathering, and torsional deflection.


Figure 25. - Static Eather bearing gecmetry.
4.5.5.2 Blade shape - elastic deformation. - In the foregoing development, the analysis has proceeded in a completely rigorous fashion. At this point, though, a departure from a completely rigorous simulation of the elemental blade motions will be made. It will be assumed, as far as blade elastic deformation is concerned, that the cosine of angles, like precone less droop, blade sueep, elastic flapping, and elastic inplane slopes, but not blade feathering is approximately equal to 1 , and therefore, the blade elastic deflections, $y$ and $z$, in blade coordinates, will be assumed to be equal to those in the static blade element coorcinates. This assumption is a reasonably valid assumption anc is completely consistent with standard practice in the matiematical representation of blace element motions.

Additionally, as far as the effect on structural axis reorientation due to biade rotation, the effect due to blade legstic twist is considered to be small comparec to that due to blade cyclic and collective feathering. Also it will be assumed that the contributions to blade $Y$ and $Z$ motion are small due to blade torsional motion, other than that due to local center of gravity offset.

With these $a s=$ mptions in mind, blade elastic tnang will now be introduced. The contribution to elastic blade bending is siorly

$$
\left\{\begin{array}{c}
0  \tag{96}\\
Y_{\text {BEND }} \\
Z_{B E N D}
\end{array}\right\}_{\text {BLn }}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
Y_{1} & y_{2} & y_{3} \\
Z_{1} & Z_{2} & z_{3}
\end{array}\right]\left\{\begin{array}{l}
A_{1 n} \\
A_{2 n} \\
A_{3 n}
\end{array}\right\}
$$

Note that $X$ or spanwise motions are not included in this equation. Blade spanwise motion till be determinea separately by utilizing blade slope data to determine the change in the profected blade length upon the blade $X$ axis. With this in mind, the total y anc 2 blade motions including blade bending, but not yet including blade feathering or blede elastic twist, is strictly the sur of the previous static line expressions and the modal deflection. Blade torsional deflection is treated as an independent degree of freedc 7 , and therefore is not inciuced as art of these blade modes. Combining the previous static deflection with the modal deflections gives:

$$
\left\{\begin{array}{c}
(B+S)  \tag{97}\\
y_{(B+S)} \\
y_{(B+S)}
\end{array}\right\}_{B L n}=\left\{\begin{array}{c}
y_{S} \\
y_{S} \\
z_{S}
\end{array}\right\}_{\text {BLn }}+\left\{\begin{array}{c}
0 \\
y_{B E N D} \\
z_{B E N D}
\end{array}\right\}_{\text {BLn }}
$$

4.5.5.3 Blade feathering. - Blade feathering is relative to the reference feathering angle ${ }_{\text {REF }}$. The feather angle, then, as far as blade motion is concerned, is due to the difference in the total Seather angle $p_{F}$ and the reference feetiner angle ${ }_{\text {REF }}$ -

The blade feathering motion is introduced similarly to the way the blade reference feathering angle was introduced, except that the feather axis slopes are due to the static position as vell as due to elastic ceformation in both the flapuise and inplane deflection.

If we let $z^{\prime}{ }_{F A}$ and $Y^{\prime}{ }_{F A}$ represent the instantaneous vertical and inplane slopes of the feathering axis, then transferring to the inboard feathering bearing, making the rotations through $Z_{F A}$ and $Y^{\prime} F_{F A}$ to the feathering axis, rotating through the delta feather angle $-\left(\phi_{F^{-t}}{ }_{R E F}\right)$ of $-\Delta \phi_{F}$, rotating back through $-Y^{\prime} F_{A}$ and $-Z^{\prime}{ }_{F A}$, and then transferring back to the BLn axis system results in the definition of the disflacements in blade axis coordinates.

However, before proceeding with this, the feathering axis slopes Y'FA and $z^{\prime}{ }_{F A}$ are defined. The slopes are simply defined as the difference in the total static ard elastic defiection of the outboard and inboard feather bearings divided by the spanwise distance between the bearings. Then from Figure 25 and the bearing static location equation:

$$
\begin{equation*}
Y_{F A}=\sin ^{-1}\left(\frac{Y_{O S}-Y_{I S}}{\ell_{B}}\right) \tag{98}
\end{equation*}
$$

and

$$
\begin{equation*}
Z_{F A}=\sin ^{-1}\left(\frac{Z_{O B}-z_{I B}}{i_{S}}\right) \tag{99}
\end{equation*}
$$

where in terms of the static an modal deflections

$$
\left\{\begin{array}{c}
x_{I B}  \tag{100}\\
y_{I B} \\
z_{I B}
\end{array}\right\}=\left\{\begin{array}{l}
x_{S_{I B}} \\
y_{S_{I B}} \\
z_{I B} \\
{ }_{I B}
\end{array}\right\}+\left[\begin{array}{ccc}
0 & 0 & 0 \\
y_{I B_{1}} & y_{I B_{2}} & y_{I B_{3}} \\
z_{I B_{1}} & z_{I B_{2}} & z_{I B_{3}}
\end{array}\right\}\left\{\begin{array}{l}
A_{I n} \\
A_{2 n} \\
A_{3 n}
\end{array}\right\}
$$

and

In the development that followr, the time derivatives of $Y^{\prime}{ }_{F A}$ and $Z^{\prime}{ }_{F A}$ are required, so therefore, they are now defined. Taking the first and second time derivatives of the slope equations yields

$$
\begin{align*}
& \dot{Y}_{F A}=\left(\dot{Y}_{O B}-\dot{Y}_{I B}\right) / \cos \left(Y_{F A}^{\prime}\right) l_{B}  \tag{102}\\
& \left.\dot{Z}_{F A} \approx \dot{Z}_{\because}-\dot{Z}_{I B}\right) / \cos \left(Z_{F A}^{\prime}\right) l_{B} \tag{103}
\end{align*}
$$

and

$$
\begin{align*}
& \ddot{Y}_{F A} \approx\left(\ddot{Y}_{O B}-\ddot{Y}_{I B}\right) / \cos \left(Y_{F A}^{\prime}\right) \ell_{B}+\sin \left(Y_{F A}^{\prime}\right) \dot{Y}_{F A}^{\prime} 2 / \cos \left(Y_{F A}^{\prime}\right)  \tag{104}\\
& \ddot{Z}_{F A}^{\prime} \approx\left(\ddot{Z}_{O B}-\ddot{z}_{I B}\right) / \cos \left(z_{F A}^{\prime}\right) \ell_{E}+\sin \left(Z_{F A}^{\prime}\right) \dot{Z}_{F \cdot}^{\prime}{ }^{2} / \cos \left(Z_{F A}^{\prime}\right) \tag{105}
\end{align*}
$$

where

$$
\begin{align*}
& \left\{\begin{array}{l}
\dot{y}_{i \underline{E}} \\
\dot{Z}_{I B}
\end{array}\right\}=\left[\begin{array}{lll}
y_{I B_{1}} & Y_{I B_{2}} & Y_{I B_{3}} \\
z_{I B_{1}} & z_{I B_{2}} & z_{I B_{3}}
\end{array}\right]\left\{\begin{array}{l}
\dot{A}_{1 n} \\
\dot{A}_{2 n} \\
\dot{A}_{3 n}
\end{array}\right\}  \tag{106}\\
& \left\{\begin{array}{l}
\dot{Y}_{O B} \\
\dot{z}_{O B}
\end{array}\right\}=\left[\begin{array}{lll}
y_{O B_{1}} & y_{O B_{2}} & y_{O B_{3}} \\
z_{O B_{1}} & z_{\mathrm{OB}_{2}} & z_{O B_{3}}
\end{array}\right]\left\{\begin{array}{l}
\dot{A}_{1 n} \\
\dot{A}_{2 n} \\
\dot{A}_{3 n}
\end{array}\right\} \tag{107}
\end{align*}
$$

and where

$$
\begin{align*}
& \left\{\begin{array}{l}
\ddot{y}_{J B} \\
\ddot{z}_{O B}
\end{array}\right\}=\left[\begin{array}{ccc}
y_{O B_{1}} & y_{O B_{2}} & y_{O B_{3}} \\
z_{O S_{1}} & z_{O B_{2}} & z_{O B_{3}}
\end{array}\right]\left\{\begin{array}{l}
\ddot{A}_{1 n} \\
\ddot{A}_{2 n} \\
\ddot{A}_{3 n}
\end{array}\right\} \tag{109}
\end{align*}
$$

Transferring the blade displacements as indicated above to the inboard feather bearing, trensforming to the feathering axis, and verforming the feathering rotaticn as discussed earlier, yields the following equation which defines the blade displacements in blade axis coordinates:

$$
\begin{aligned}
& \cdot\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \Delta \phi_{F} & -\sin \Delta \phi_{F} \\
0 & \sin \Delta \phi_{F} & \cos \Delta \phi_{F}
\end{array}\right]\left[\begin{array}{ccc}
\cos Y^{\prime} & \sin ^{\prime} Y_{F A} & 0 \\
-\sin Y^{\prime}{ }_{F A} & \cos Y^{\prime} & 0 \\
0 & 0 & 1
\end{array}\right] \\
& {\left[\begin{array}{ccc}
\cos ^{\prime}{ }_{F A} & 0 & \sin Z^{\prime} \\
0 & 1 & 0 \\
-\sin Z^{\prime} & 0 & 0
\end{array}\right]\left\{\left\{\begin{array}{l}
X_{(B A} \\
\left.Y_{(B)}\right) \\
Z_{(B+S)}^{-}
\end{array}\right\}\right.} \\
& \left.-\left\{\begin{array}{c}
x_{I B} \\
y_{I B} \\
z_{I B}
\end{array}\right\}\right\}+\left\{\begin{array}{c}
x_{I B} \\
y_{I B} \\
z_{I B}
\end{array}\right\}
\end{aligned}
$$

This equation then gives the blade displacement in blade coordinates, including the effects of the static shape, blade bending, and biade static twist. The effect of blade elastic twist is now considered.
4.5.5.4 Blade elastic twist. - Blade motion due to blade elastic twist is accountec for $b_{j}$ going back to the static twist equation. Blade elastic iwist, $\psi_{T}$ is assumed to te directly superpositionable with blade static or blade pretwist, ${ }_{T W}$, except that the static pretwist takes place about the $1 / 4$ chord, and the blace elastic twist takes place about the blade element shear center. This is shown in Figure 26. From this figure it can be seen that previous siatic twist equation car be rewritten as:

a) BLADE PRETWIST, $\phi(i) \mathbf{T w}$. ABOUT BLADE REFERENCE AXIS

b) BLADE ELASTIC TWIST. $\varphi(i)$ T ABOUT BLADE SHEAR CENTER

Figure 26. - Blade static pretwi:\%, $\phi_{T W}$ and elastic twist, $\phi_{T}$.

$$
\begin{align*}
\left\{\begin{array}{l}
\mathrm{X}_{\mathrm{BLE}} \\
\mathrm{Y}_{\mathrm{BLE}} \\
\mathrm{Z}_{\mathrm{BLE}}
\end{array}\right\}= & {\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{T}} & -\sin \phi_{\mathrm{T}} \\
0 & \sin \phi_{\mathrm{T}} & \cos \phi_{\mathrm{T}}
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{TW}} & -\sin \phi_{\mathrm{TW}} \\
0 & \sin \phi_{\mathrm{TW}} & \cos \phi_{\mathrm{TW}}
\end{array}\right]\left\{\left\{\begin{array}{l}
0 \\
\mathrm{Y}_{\mathrm{CG}} \\
\mathrm{z}_{\mathrm{CG}}
\end{array}\right\}\right.} \\
& -\left\{\begin{array}{l}
\mathrm{X}_{\mathrm{SC}} \\
0
\end{array}\right\}+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{TW}} & -\sin \phi_{\mathrm{TW}} \\
0 & \sin \phi_{\mathrm{TW}} & \cos \phi_{\mathrm{TW}}
\end{array}\right\}\left\{\begin{array}{l}
0 \\
\mathrm{Y}_{\mathrm{SC}} \\
0
\end{array}\right\} \tag{ill}
\end{align*}
$$

If we let $\Phi_{T}=\left(\phi_{T}+\phi_{T W}\right)$ then this equation becunes

$$
\begin{align*}
& =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \Phi_{\mathrm{T}} & -\sin \Phi_{\mathrm{T}} \\
0 & \sin \Phi_{\mathrm{T}} & \cos \Phi_{\mathrm{T}}
\end{array}\right]\left\{\begin{array}{c}
X_{\mathrm{CG}} \\
Y_{C G} \\
Z_{C G}
\end{array}\right\}+Y_{\mathrm{SC}}\left\{\begin{array}{c}
0 \\
\operatorname{sos} \phi_{T W}-\cos \Phi_{\mathrm{T}} \\
\sin \phi_{\mathrm{TW}}-\sin \Phi_{\mathrm{T}}
\end{array}\right\} \tag{112}
\end{align*}
$$

4.5.5.5 Final blade element $y, Z$ displacement equation. - Suostituting the above equation in the previous development sequence yields the blade displacement equation which includes the effect of the static shape of blade bending, of blade feathering, and of blade elastic twist.

However, befre proceeding with these substitutions, the following column vector is defilad to simplify the notation.

$$
\left\{\begin{array}{l}
X_{a}  \tag{113}\\
X_{a} \\
Y_{a} \\
Z_{a}
\end{array}\right\}
$$

The total blade element displacement equation becomes:

$$
\begin{align*}
& \cdot\left\{[ T _ { \beta _ { 0 } } ] ^ { T } \left\{\left[T_{\Phi_{T}}\right]^{T}\left\{r_{C G}\right\}+\left\{\left[\mathrm{T}_{\phi_{T W}}\right]^{T}-\left[\mathrm{T}_{\Phi_{T}}\right]^{T}\right\}\left\{r_{S C}\right\}\right.\right. \\
& \left.-\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{S W}\right\}\right\}+\left\{\left\{r_{j 0 g}\right\}+\left[\mathrm{T}_{B_{0}}\right]^{\mathrm{T}}\left\{r_{S W}\right\}\right. \\
& \left.\left.-\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{F}}\right\}\right\}\right\}+\left\{\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}} \quad\left\{\mathrm{r}_{\mathrm{p}}\right\}\right. \\
& \left.\left.\left.-\left\{r_{I B}\right\}\right\}\right\}\right\}+\left\{r_{I B}\right\} \tag{114}
\end{align*}
$$

where:

$$
\left[\frac{\partial r}{\partial A_{n}}\right]\left\{A_{j n}\right\}=\left[\begin{array}{ccc}
0 & 0 & 0  \tag{i15}\\
Y_{1 n} & Y_{2 n} & Y_{3 n} \\
Z_{1 n} & Z_{2 n} & Z_{3 n}
\end{array}\right]\left\{\begin{array}{l}
A_{1 n} \\
A_{2 n} \\
A_{3 n}
\end{array}\right\}
$$

Note that for convenience of using the condensed matrix notation discussed above, the most general vectors for such terms as $\ell_{p}, X_{S W}, Z_{j o g}$, and $Y_{S C}$ have been used. As can be seen in this equation, these have all been treated as full vectors. Making the appropriate substitutions of course will result in the expressions previously obtained.

It is noted that the equation is written for the relative displacement of points on the blade outboard of Station $X_{3 W}$. Inooard of that station, the displacements are determined from the previous anboard equation or simply by zeroing out such terms as $\left\{r_{j 0 g}\right\}$ and $\left\{r_{S W}\right\}$ and substituting unit diagonal transformations for $\left[T_{T_{0}}\right]$ and $\left[{ }^{T} Y\right.$ ] in the full equation. Following either approach yielcs the blade displacement equation for points inboard of Station $X_{S W}$; or

$$
\begin{align*}
& +\left\{[ [ \mathrm { T } _ { \beta _ { \mathrm { FA } } } ] ^ { \mathrm { T } } [ \mathrm { T } _ { \Phi _ { \mathrm { REF } } } ] ^ { \mathrm { T } } [ \mathrm { T } _ { \beta _ { \mathrm { FA } } } ] ] \left\{[ \mathrm { T } _ { \beta _ { 0 } } ] ^ { \mathrm { T } } \left\{\left[\mathrm{T}_{\Phi_{\mathrm{T}}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{CG}}\right\}\right.\right.\right. \\
& \left.\left.+\left\{\left[\mathrm{T}_{\Phi_{\mathrm{TW}}}\right]^{\mathrm{T}}-\left[\mathrm{T}_{\Phi_{\mathrm{I}}}\right]^{\mathrm{T}}\right\}\left\{\mathrm{r}_{S C}\right\}\right\}-\left\{\mathrm{r}_{I B}\right\}\right\}+\left\{\mathrm{r}_{I B}\right\} \tag{116}
\end{align*}
$$

The ith station blade displacements, $Y$ and $Z$, in blade coordinates for points on the blade both outtoard ana inboard of station $X_{S W}$ are then defined.
4.5.5.6 Blade element $Y$ and $Z$ relative velocities and accelerations. - The blade element coordinate axis system linear $Y$ and $Z$ velocities relative to the blade reference axis system can be found by differentiating the position equation with respect to time. Note that no distinction will be made at this point between outboard or inboard of station $X_{S W}$, but using the equation for displacements outboard of this station and as discussed earlier, zeroing out certain terms, results in the equations for velocities or accelerations of points inboard of that station.

$$
\begin{aligned}
& \left.+\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Delta \phi_{\mathrm{F}}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]\left[\dot{\mathrm{T}}_{\mathrm{Z}}{ }^{\prime}{ }_{\mathrm{FA}}\right]\right]
\end{aligned}
$$

$$
\begin{align*}
& \left.+\left[\mathrm{T}_{\mathrm{Z}^{\prime}}{ }_{\mathrm{FA}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Delta \phi_{\mathrm{F}}}\right]^{\mathrm{T}}\left[\dot{\mathrm{~T}}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]\left[\mathrm{T}_{\mathrm{Z}^{\prime}}{ }_{\mathrm{FA}}\right]\right] \\
& \left.+\dot{\phi}_{\mathrm{F}}\left[\left[\mathrm{~T}_{Z^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]\left[\dot{\mathrm{T}}_{\Delta \phi_{\mathrm{F}}}\right]\left[\mathrm{T}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]\left[\mathrm{T}_{Z^{\prime}{ }_{\mathrm{FA}}}\right]\right]\right] \\
& \text { - }\left\{\left\{\left[\frac{\partial r}{\partial A_{n}}\right]\left\{A_{j n}\right\}\right\}+\left\{\left[\left[T_{B_{F A}}\right]^{T}\left[T_{\phi_{R E F}}\right]^{T}\left[T_{B_{F A}}\right]\right]\right.\right. \\
& \text { - }\left\{[ [ { } ^ { T _ { \tau _ { 0 } } } ] ^ { T } [ T _ { r } ] ^ { T } ] \left\{[ { } _ { T _ { B _ { 0 } } } ] ^ { T } \left\{\left[T_{\Phi_{T}}\right]^{T}\left\{r_{C G}\right\}\right.\right.\right. \\
& \left.\left.+\left\{\left[\mathrm{T}_{\Phi_{T W}}\right]^{\mathrm{T}}-\left[\mathrm{T}_{\Phi_{T}}\right]^{\mathrm{T}}\right\}\left\{\mathrm{r}_{\mathrm{SC}}\right\}\right\}-\left[\mathrm{T}_{\mathrm{B}_{0}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{SW}}\right\}\right\}+\left\{\left\{\mathrm{r}_{\mathrm{jOg}}\right\}\right. \\
& \left.\left.+\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{SW}}\right\}-\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{p}}\right\}\right\}\right\}+\left\{\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{p}}\right\}\right. \\
& \left.\left.\left.-\left\{r_{I B}\right\}\right\}\right\}\right\}+\left[\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Delta \phi_{\mathrm{F}}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}^{\prime}}{ }_{\mathrm{FA}}\right]\left[\mathrm{r}_{\mathrm{Z}^{\prime}}{ }_{\mathrm{FA}}\right]\right] \\
& \text { - }\left\{\left\{\left[\frac{\partial r}{\partial A_{n}}\right]\left\{\dot{A}_{j_{n}}\right\}!+\left[\left[\mathrm{T}_{\beta_{F A}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\phi_{R E F}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\beta_{F A}}\right]\right]\left[\left[\mathrm{T}_{\tau_{0}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{r}}\right]^{\mathrm{T}}\right]\right.\right. \\
& \text { - } \left.\left.\left[\mathrm{T}_{\beta_{0}}\right]^{\mathrm{T}} \quad \dot{\phi}_{\mathrm{T}}\left[\dot{\mathrm{~T}}_{\Phi_{\mathrm{T}}}\right]^{\mathrm{T}}\left\{\left\{\mathrm{r}_{\mathrm{CG}}\right\}-\left\{\mathrm{r}_{\mathrm{SC}}\right]\right\}\right\}-\left\{\dot{r}_{\mathrm{IB}}\right\}\right\}+\left\{\dot{\mathrm{r}}_{\mathrm{IB}}\right\} \tag{117}
\end{align*}
$$

Note in the above equation that the $\left[\begin{array}{c}1 \\ \zeta\end{array}\right]$ mat-ices are not time derivatives of the $\left[\mathrm{T}_{\zeta}\right]$ matrices but are derivatives of the transformation matrices with respect to the transformation angle 5 . This is arrived at by making the substitution that:

$$
\begin{equation*}
\frac{d}{d t}\left[\mathrm{~T}_{\zeta}\right]=\frac{d \zeta}{d t}\left[\frac{d T_{\zeta}}{\mathrm{d}_{\zeta}}\right]=\dot{\zeta}\left[\dot{\mathrm{T}}_{\zeta}\right] \tag{11.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}}\left[T_{r}\right]=\ddot{\zeta}\left[\dot{T}_{\zeta}\right]+\dot{\zeta}^{2}\left[\ddot{T}_{\zeta}\right] \tag{119}
\end{equation*}
$$

Taking the time derivative again of Equation 113 yields the blade element $Y$ and $Z$ linear accelerations relative to the blade reference axis systcu.

$$
\begin{aligned}
& +2\left[\dot{\mathrm{~T}}_{Z^{\prime}}{ }^{\prime}\right]^{T}\left[\mathrm{~T}_{Y^{\prime}}{ }^{\mathrm{FA}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Delta \varphi_{F}}\right]^{\mathrm{T}}\left[\mathrm{r}^{\prime} \mathrm{Y}^{\prime}{ }_{F A}\right]\left[\dot{\mathrm{T}}_{Z^{\prime}}\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left[\mathrm{i}_{\mathrm{r}}^{\mathrm{ran}} \mathrm{ma}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \cdot\left[\mathrm{T}_{B_{0}}\right]^{\mathrm{T}}\left\{\dot{\phi}_{\mathrm{T}}\left[\dot{\mathrm{~T}}_{\Phi_{\mathrm{T}}}\right]^{\mathrm{T}}\left\{\left\{\mathrm{r}_{\mathrm{CG}}\right\}-\left\{\mathrm{r}_{\mathrm{SC}}\right\}\right\}-\left\{\dot{r}_{\mathrm{IB}}\right\}\right\}
\end{aligned}
$$

$$
\begin{align*}
& +\left[\left[{ }^{T}{ }_{B_{F A}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\phi_{R E F}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\beta_{F A}}\right]\right]\left[\left[{ }^{\mathrm{T}} \tau_{0}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y}\right]{ }^{\mathrm{T}}\right]\left[\mathrm{T}_{B_{0}}{ }^{\mathrm{T}}\right. \\
& \left.\cdot\left\{\ddot{\phi}_{\mathrm{T}}\left[\ddot{\ddot{W}}_{\Phi_{\mathrm{T}}}\right] \mathrm{T}\left\{\left\{r_{C G}\right\}-\left\{r_{\mathrm{SC}}\right\}\right\}\right\}-\left\{\ddot{r}_{\mathrm{IB}}\right\}\right\}+\left\{\ddot{\mathrm{r}}_{\mathrm{IB}}\right\} \tag{120}
\end{align*}
$$

These equations define the blade element relative displacement velocities and accelerations, respectively, required by the blade inertial velocity equations developec shortly. Note that in the preceding development these equations are written for the nth blade, and with the exception of the $\left[\frac{\partial r}{\partial A_{n}}\right],\left[T_{B_{F A}}\right]$

matrices, tiae terns are all blade dependent. Remember, also, that inboard of $\left\{r_{S H}\right\}$ the $\left[\begin{array}{rl}T_{T} \\ \left.{ }_{0}\right]\end{array}\right]$ and $\left[T_{Y}\right]$ matrices are unit diagonal.
4.5.5.7 BLade element slopes. - The blade element $Y$ ' and $Z^{\prime}$ slopes are determined by differentiating the deflection equation with respect to the nth blade radial distance, $X_{B H n}$. These formulations are used for quasi-static torsion formulation and output. Performing the required differentiation for points along the blade reference line:

$$
\begin{align*}
& \left.\frac{\partial}{\partial X_{B L n}}\left\{\begin{array}{c}
- \\
Y_{B L E} \\
Z_{B L E}
\end{array}\right\}_{B L n}=\left\{\begin{array}{c}
- \\
Y^{\prime} \\
Z_{B L E}^{\prime}
\end{array}\right\}_{B L E}\right\}_{B L n}=\left\{r^{\prime}{ }_{B L E}\right\}_{B L n} \tag{121}
\end{align*}
$$

$$
\begin{aligned}
& \cdot\left\{\Phi_{\mathrm{T}^{\prime}}\left[\dot{T}_{\varphi_{\mathrm{T}}}\right\}^{\mathrm{T}}\left\{\mathbf{r}_{\mathrm{CG}}-\mathbf{r}_{\mathrm{SC}}\right\}+\left\{\mathrm{T}_{\Phi_{\mathrm{T}}}\right\}^{\mathrm{T}}\left\{\mathbf{r}_{\mathrm{CG}}{ }^{\prime}-\mathrm{r}_{\mathrm{SC}}^{\prime}\right\}\right\}
\end{aligned}
$$

where

$$
\left.\left\{\phi_{T^{\prime}}\left\{\vec{T}_{\phi_{T}}\right]^{T}\left\{r_{C G}-r_{S C}\right\}+\left[T_{\phi_{T}}\right]^{T}\left\{r_{C G}^{\prime}-r_{S C}^{\prime}\right\}\right\}=\left\{\begin{array}{l}
1  \tag{123}\\
0 \\
0
\end{array}\right\} \quad \text { as programmed }\right)
$$

4.5.5.8 Transformation from blade-to-blade element coordinates. - In this section, the transformation matrix for the blade root to the ith blade element motion will be developed. Each blade will have its own transformation matrix for each ith blade station. The transformation matrix will initially be developed as the transform from blade element to blade coordinates, [ T BLE-BLn].

The transformation matrix, $\left[T_{B L E-B L n}\right]$, can be de-eloped by referring to the development of the deflection equations. The first rotation from blade element to blade coordinates is through the combined iwist angle, ${ }^{-\Phi_{T}}$; the second rotation is through the negative of the precone, $B_{0}$; the third through the negative of thw sweep and droop angles, $\tau_{0}, Y$; the fourth through the feathering axis angle, $B_{F A}$; the fifth through the negative of the reference feathering angles $\phi_{\text {REF }}$; and the sixth back through the negative of the feathering axis Frecone angle, $\beta_{F A}$.

These rotations then define the transformation from blade element to blade coordinates, including the effects of the static shape of the blade, pretwist, precone, sweep, droop, etc. Also included is the effect of blade elastic trist. Again note that for stations inboard of Station $X_{S W}$, the sweep and droop angles, ${ }^{1} 0$ and $r$, respectively must be set to zero in the formulation of the transformation matrix as in the definition of the blade displacements and blade slopes. This portion of the transformation matrix which includes the static blade shape and combined twist is defined as follows:

$$
\begin{equation*}
\left[\mathrm{T}_{\text {BLE-BLn }}\right] \mathrm{S}=\left[\left[\mathrm{T}_{\beta_{F A}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\phi_{R E F}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\beta_{F A}}\right]\right]\left[\left[\mathrm{T}_{\mathrm{T}_{0}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\gamma}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\beta_{0}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Phi}\right]^{\mathrm{T}}\right] \tag{124}
\end{equation*}
$$

The next two rotations from blade element to blade coordinates are due to the elastic blade bending slopes. Since $Y^{\prime}{ }_{\text {BEND }}{ }^{\text {and }} Z_{\text {BEND }}$ are motions of the blade elements with respect to the blade, then to transform from blade element to blade coordinates requires negative rotations of $Y^{\prime}$ BEND and $Z^{\prime}$ BEND to be included. Finally, the blade feathering rotation from the reference feather angle must be included. The final transformation then, from blade element to blade coordinates, is defined by the following equation:

#  <br> $$
\cdot\left[\mathrm{T}_{\mathrm{B}_{\mathrm{FA}}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\phi_{\mathrm{REF}}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{B}_{\mathrm{FA}}}\right]\left[\mathrm{T}_{\tau_{0}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\gamma}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{B}_{0}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\phi_{\mathrm{T}}}\right]^{\mathrm{T}}
$$ 

(125)
where

$$
\left[\mathrm{r}_{Z^{\prime}}{ }_{\text {BEND }}\right]=\left[\begin{array}{ccc}
\cos \left(Z_{\text {BEND }}^{\prime}\right) & 0 & \sin \left(z_{\text {BEND }}^{\prime}\right)  \tag{126}\\
0 & 1 & 0 \\
-\sin \left(Z_{\text {BEND }}^{\prime}\right) & 0 & \cos \left(z_{\text {BEND }}^{\prime}\right)
\end{array}\right]
$$

and

$$
\left[T_{Y_{\text {BEND }}}\right]=\left[\begin{array}{ccc}
\cos \left(Y_{\text {BEND }}^{\prime}\right) & \sin \left(Y_{\text {BEND }}^{\prime}\right) & 0  \tag{127}\\
-\sin \left(Y_{\text {BEND }}^{\prime}\right) & \cos \left(Y_{\text {BEND }}^{\prime}\right) & 0 \\
0 & 0 & 1
\end{array}\right]
$$

and again where

$$
\left[\begin{array}{l}
\mathrm{T}^{\tau_{0}} \tag{128}
\end{array}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\gamma}\right]^{\mathrm{T}}=[\mathrm{I}]
$$

inboard of Station $X_{S W}$.

Also:

$$
\left\{\begin{array}{l}
Y_{\text {BEND }}^{\prime}  \tag{129}\\
Z^{\prime} \\
\text { BEND }
\end{array}\right\}=\left\{\begin{array}{ccc}
Y_{1} & Y_{2}^{\prime} & Y_{3}^{\prime} \\
Z_{1}^{\prime} & Z_{2}^{\prime} & Z_{3}^{\prime}
\end{array}\right\}\left\{\begin{array}{l}
A_{1 n} \\
A_{2 n} \\
A_{3 n}
\end{array}\right\}
$$

The inverse or transpose of this equation yields the transformation from blade to blade element coordinates, or:

$$
\begin{aligned}
& {\left[\mathrm{T}_{\mathrm{BLn} \text {-BLE }}\right]=\left[\mathrm{T}_{\mathrm{BLE}-\mathrm{BLn}}\right] \mathrm{T}}
\end{aligned}
$$

again where

$$
\left[\mathrm{T}_{\gamma}\right]\left[\begin{array}{l}
\left.\mathrm{T} \mathrm{~T}_{0}\right]=[\mathrm{I}] \tag{131}
\end{array}\right.
$$

inboard of station $X_{S W}$ *
4.5.5.9 Blade element angular velocities and accelerations. - From the foregoing discussion, the blade element angular velocity vector can be determined. Starting with the angular velocities $(p, q, r)_{B L n}$ of the blade reference axis system and systematically and progressively transforming these velocities through each axis rotation and adding the respective angular velocity associated with each of the indicated angular rotations, results in the following equation for the blade element angular velocities.

$$
\begin{aligned}
& +\left[T_{Y^{\prime}}{ }_{\text {BEND }}\right]\left\{\left\{\begin{array}{c}
0 \\
-\dot{Z}^{\prime} \text { BEND }^{0}
\end{array}\right\}+\left[T_{Z^{\prime}}{ }_{\text {BEND }}\right]\left\{\left\{\begin{array}{c}
0 \\
\dot{Z}^{\prime}{ }_{\text {FA }} \\
0
\end{array}\right\}\right.\right. \\
& \left.+\left[\mathrm{T}_{Z^{\prime}}\right]^{\mathrm{FA}}\right]^{\mathrm{T}}\left\{\left\{\begin{array}{c}
0 \\
0 \\
\dot{Y}^{\prime},{ }_{F A}
\end{array}\right\}+\left[\mathrm{T}_{Y^{\prime}}{ }_{\mathrm{FA}}\right]^{\mathrm{T}}\left\{\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{F}} \\
0 \\
0
\end{array}\right\}+\left[\begin{array}{l}
\left.\mathrm{T}_{\Delta \phi_{\mathrm{F}}}\right]
\end{array}\left\{\begin{array}{l}
0 \\
0 \\
\mathrm{Y}_{\mathrm{FA}}^{\prime}
\end{array}\right\}\right.\right.\right.
\end{aligned}
$$

Hote that in this equation, starting on the right-hand side with the quantities in the innermost brackets, the blade reference system angular velocities are first transformed through the increment of feathering axis flapping slope due to bending, $\mathrm{Z}^{\prime} \mathrm{FA}$, and then the feathering axis flapping angular velocity, $-Z^{\prime}{ }_{F A}$, is added. Minus is used since $Z$ ' is a negative $\theta$ rotation. Next, the resultant $W$ vector is transformed through $Y^{\prime} F A$ and $Y^{\prime}{ }_{F A}$ is added. This is then transformed through the delta feathering angle, $\Delta \phi_{F}$, and the feathering angular velocity, $\dot{\phi}_{F}$, is added. This is then transformed back through the increments of feathering axis slopes due to blade bending, giving the vector:

$$
\begin{align*}
& \left.+\left[\mathrm{T}_{Y^{\prime}}{ }_{F A}\right]\left\{\left\{\begin{array}{c}
0 \\
-\dot{Z}_{F A}^{\prime} \\
0
\end{array}\right\}+\left[\mathrm{T}_{Z^{\prime}}{ }^{\prime}\right]\left[\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{\mathrm{BLn}}\right\}\right\} \cdot \cdot\right\} \tag{133}
\end{align*}
$$

which represents the blade element angular velocities due to combined blade feathering and blade reference axis system angular velocities.

Next, the effects of blade bending e.t each blade station are introduced. The above vector is first transformed through the local blade element flapwise bending slope, $Z^{\prime}{ }_{\text {BEND }}$, and then the angular velocity, $-Z^{\prime}{ }_{\text {BEND }}$, is added. This result is transformed through the blade element inplane bending slope, $Y$ "BEND" and the inplane angular velocity due to blade bending, $Y$ ' ${ }_{\text {BEND }}$, is added, resulting in the total vector less the initial transformation string. This vector represents the blade eiement angular velocities due to the combined effects of the blade reference axis system angular velocities of the blade feathering angle and of the blade anguiar velocities due to blade elastic bending. The remaining transformations then include the static effects of the blade feathering axis precone, ${ }^{\prime} F_{F A}$, the blade reference feathering angle, $\phi_{\text {REF }}$, blade sweep, $\tau_{0}$, blade droop, $\gamma$, and blade or hub precone, $B_{0}$, and $\tau_{i}=$ combined effect of blade static and elastic twist, represented by $\Phi_{T}$. Finally, the blade elastic twist angular velocity, $\Phi_{T}$, is added, giving the total blade element anguiar velocities, $\left\{\begin{array}{l}p \\ q \\ r\end{array}\right\}_{\text {BLE }}$.

Also note, as indicated before, the matrix $\left[T_{\gamma}\right]\left[T_{\tau_{0}}\right]$ has the value calculated if $X$ is greater than $X_{S W}$ and has the value of unity if $X$ is inboard of station $X_{S W}$.

At this point is has been assumed that the contributions of $\dot{Y}{ }^{\prime}$ FA and $\dot{Z}^{\prime}$ FA
are small compared to the other contributions to $\{q\}$. This assumption
is supported by referring to the final form of the above development. First of all, both of these vectors are small compared to $\{q\}$, which is funda-
mentally the rotational speed of the rotor. Also, both of the feathering axis flapping ana inplane angular velocities are first added and then transformed through the delta feathering angle and then subtracted, meaning that fundamentally the principal magnitude or component contributions due to $\dot{Y}$ ' FA and $\dot{Z} '_{F A}$ are self-cancelling.

With the above assumption:

$$
\begin{align*}
& \left.\left.+\left[\mathrm{T}_{Z^{\prime}}{ }_{\text {BEND }}\right] \quad\left[\mathrm{T}_{Z^{\prime}}\right]_{\mathrm{FA}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}}\right]_{\mathrm{FA}}\right]^{\mathrm{T}}\left\{\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{F}} \\
0 \\
0
\end{array}\right\}\right. \\
& \left.\left.\left.+\left[T_{\Delta \phi_{F}}\right]\left[T_{Y_{F A}^{\prime}}\right]\left[T_{Z_{F A}^{\prime}}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{B L n}\right\}\right\}\right\} \tag{134}
\end{align*}
$$

where:

$$
[\mathrm{R}]^{T}=\left[\mathrm{T}_{\beta_{0}}\right]\left[\mathrm{T}_{Y}\right]\left[\begin{array}{l}
\left.\mathrm{T}_{\tau_{0}}\right]\left[\begin{array}{l}
\mathrm{T}_{\beta_{F A}}
\end{array}\right]^{T}\left[\begin{array}{l}
\mathrm{T}_{\phi_{R E F}}
\end{array}\right]\left[\mathrm{T}_{\beta_{F A}}\right] . \tag{135}
\end{array}\right.
$$

and:

$$
\left\{\begin{array}{l}
\dot{Y}_{\text {BEND }}^{\prime}  \tag{136}\\
\dot{Z}_{\text {BEND }}^{\prime}
\end{array}\right\}=\left[\begin{array}{ccc}
Y_{1}^{\prime} & Y_{2}^{\prime} & Y_{3}^{\prime} \\
Z_{1}^{\prime} & Z_{2}^{\prime} & Z_{3}^{\prime} \\
& &
\end{array}\right\}\left\{\begin{array}{l}
\dot{A}_{1 n} \\
\dot{A}_{2 n} \\
\dot{A}_{3 n}
\end{array}\right\}
$$

The blade element angular accelerations can now be determined by differentiating this equation with respect to time. Again, as in the case of the angular velocities, the contributions due to time derivatives of the feathering axis flapping and inplane slope changes due to bending are neglected. With this assumption, the time derivative is:

$$
\begin{aligned}
& \left\{\begin{array}{c}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}=\left\{\begin{array}{c}
\ddot{\phi}_{T} \\
0 \\
0
\end{array}\right\}+\dot{\phi}_{T}\left[\dot{T}_{\Phi_{T}}\right]\left[{ }^{R}\right]^{T}\left\{\left\{\begin{array}{c}
0 \\
0 \\
\dot{Y}^{\prime}{ }_{\text {BEND }}
\end{array}\right\}+\left[\mathrm{T}_{Y^{\prime}}{ }_{\text {BEND }}\right]\left\{\left\{\begin{array}{c}
0 \\
-\dot{Z}^{\prime}{ }_{\text {BEND }} \\
0
\end{array}\right\}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& \text { - }\left\{\left\{\begin{array}{c}
0 \\
-\dot{Z}_{\text {BEND }}^{\prime} \\
0
\end{array}\right\}+\left[\mathrm{T}_{Z^{\prime}}{ }_{\text {BEND }}\right]\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}}{ }^{\prime}\right]^{\mathrm{T}}\left\{\left\{\begin{array}{l}
\dot{\phi}_{F} \\
0 \\
0
\end{array}\right\}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.+\left[T_{\Delta \phi_{F}}\right]\left[T_{Y^{\prime}}\right]\left[T_{Z^{\prime}}{ }^{\prime}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{B L n}\right\}
\end{aligned}
$$

where:

$$
\left\{\begin{array}{l}
\ddot{Y}_{\text {BEND }}  \tag{1.38}\\
\ddot{Z}_{\text {BEND }}^{\prime}
\end{array}\right\}=\left[\begin{array}{ccc}
Y_{1}^{\prime} & Y_{2}^{\prime} & Y_{3}^{\prime} \\
Z_{1}^{\prime} & Z_{2}^{\prime} & Z_{3}^{\prime}
\end{array}\right\}\left\{\begin{array}{l}
\ddot{A}_{1 n} \\
\ddot{A}_{2 n} \\
\ddot{A}_{3 n}
\end{array}\right\}
$$

4.5.5.10 Blade element $X$ motions. - In the previous development, the equations did not account for the blade element displacement, velocity, and acceleration in the spanwise or X-direction. The method used to define these is one of taking the neutral axis as the axis of no stretch and determing the projection of this axis onto the X-axis as the blade bends. This projection, then, is the spanwise or $X$ location of the neutral axis in blade coordinates. The rate of change of this projection is the spanwise relative velocity and the second rate of change is the spanwise relative acceleration of the blade element neutral axis location or point. The motions are then transformed to the center of gravity to obtain the spanwise motion of the origin of the blade element reference axis.

In Figure 27, the deflected neutral axis is shows as a function of blade radius. The (i-1) and ith station are shown. It can be seen from this figure that as $X_{N A}(i-1)$ approaches $X_{N A}(i)$, then the delta length of the blade $\left(S_{N A}(i)-S_{N A}(i-1)\right)$, can be written as:

$$
\begin{align*}
\left(S_{N A}(i)-S_{N A}(i-1)\right)^{2}= & \left(X_{N A}(i)-X_{N A}(i-1)\right)_{B L n}^{2}+\left(Y_{N A}(i)-Y_{N A}(i-1)\right)_{B L n}^{2} \\
& +\left(Z_{N A}(i)-Z_{N A}(i-1)\right)_{B L n}^{2} \tag{139}
\end{align*}
$$

Rearranging this equation and summing from the blade root to the kth blade station yields:

$$
\begin{align*}
X_{N A}(k)= & \sum_{i=2}^{k}\left(X_{N A}(i)-X_{N A}(i-1)\right)_{B L n}=\sum_{i=2}^{k}\left[\left(S_{N A}(i)-S_{N A}(i-1)\right)^{2}\right. \\
& \left.-\left(Y_{N A}(i)-Y_{N A}(i-1)\right)_{B L n}^{2}-\left(Z_{N A}(i)-Z_{N A}(i-1)\right)_{B L n}^{2}\right]^{1 / 2} \tag{140}
\end{align*}
$$

and

$$
\begin{equation*}
S_{N A}(1) \approx X_{B L E}(1) \tag{141}
\end{equation*}
$$



Figure 27. - Neutral axis vs blade radius.

Likewise,

$$
\begin{equation*}
\dot{\mathrm{X}}_{\mathrm{NA}}^{\mathrm{BLn}}(1)=\ddot{\mathrm{X}}_{\mathrm{NA}}^{\mathrm{BLn}}(1)=0 \tag{142}
\end{equation*}
$$

$S_{N A}(i)$ is simply the blade length to the ith station measured along the neutral axis and $Y_{N A}(i)$ and $Z_{N L}{ }_{N L n}^{(i)}$ are the $Y$ and $Z$ locations of the neutral axis in the blade coordinate axis system for the nth blade. These displacementi, along with their derivatives, will be defined later. First, however, by taking the first and second time derivative of $X$ equation, the spanwise velocities and accelerations of the blade element neutral axis point are determined and are given by the following two equations.

$$
\begin{align*}
\dot{X}_{N A}(k)= & \sum_{i \ln }^{k}\left[\frac{-\left(Y_{N A}(i)-Y_{N A}(i-1)\right)_{B L n}\left(\dot{Y}_{N A}(i)-\dot{Y}_{N A}(i-1)\right)_{B L n}}{\left(X_{N A}(i)-J_{N A}(i-1)\right)_{B L n}}\right. \\
& \left.\frac{-\left(Z_{N A}(i)-Z_{N A}(i-1)\right)_{B L n}\left(\dot{Z}_{N A}(i)-\dot{Z}_{N A}(i-1)\right)_{B L n}}{\left(X_{N A}(i)-X_{N A}(i-1)\right)_{B L n}}\right] \tag{143}
\end{align*}
$$

$$
\begin{align*}
& \ddot{x}_{M A}(k)=\sum_{i=2}^{k}\left[\frac{-\left(\dot{y}_{M A}(i)-\dot{y}_{M A}(i-1)\right)_{B L n}^{2}-\left(\dot{z}_{M A}(i)-\dot{z}_{M A}(i-1)\right)_{B L n}^{2}}{\left(x_{M A} i()-x_{M A}(i-1)\right)_{B L n}}\right. \\
& \frac{-\left(Y_{K A}(i)-\ddot{i}_{M A}(i-1)\right)_{B L \Sigma}\left(\dot{Y}_{X A}(i)-\ddot{Y}_{M A}(i-1)\right)_{B L n}}{\left(X_{M A}(i)-X_{B A}(i-1)\right)_{B L n}} \\
& \frac{-\left(7_{H A}(i)-z_{K A}(i-i)\right)_{B L n}\left(\ddot{z}_{M A}(i)-\ddot{z}_{M A}(i-1)\right)_{B L n}}{\left(x_{M A}(i)-x_{N A}(i-1)\right)_{B L n}} \\
& -\frac{\left[\left(Y_{M A}(i)-Y_{N A}(i-1)\right)_{B L n}\left(\dot{Y}_{M A}(i)-\dot{Y}_{H A}(i-1)\right)_{B L n}\right.}{\left(X_{M A}(i)-x_{N A}(i-1)\right)_{B L n}^{3}} \\
& \left.\frac{\left.\left.+\left(z_{N A}(i)-z_{N A}(i-1)\right)_{B L n}\left(\dot{z}_{R A}(i)-\dot{z}_{R A}(i-1)\right)_{B L n}\right]^{2}\right]}{\left(x_{N A}(i)-x_{N A}(i-1)\right)_{B L n}^{3}}\right] \tag{144}
\end{align*}
$$

If $Y_{0 N A}(i)$ is the distance along the ith blade element chord line from the blade element reference axis origin or center of gravity to the blade element neutral axis, then the blade element neutral axis motions can be written in terms of the blade element motions as:

$$
\left\{\begin{array}{l}
\Delta X_{N A}(i)  \tag{145}\\
Y_{N A}(i) \\
Z_{N A}(i)
\end{array}\right\}_{B L n}=\left\{\begin{array}{c}
0 \\
Y_{B L E}{ }^{(i)} \\
Z_{B L E}{ }^{(i)}
\end{array}\right\}_{E L n}+\left[T_{B L E-B L n}\right]\left\{\begin{array}{c}
0 \\
y_{O N A}(i) \\
0
\end{array}\right\}_{B L E}
$$

Referring to Secticn 4.5, the time derivative of the above equation is:

$$
\left\{\begin{array}{l}
\Delta \dot{X}_{M A}(i)  \tag{146}\\
\dot{Y}_{M A}(i) \\
\dot{z}_{M A}(i)
\end{array}\right\}_{B L n}+\left\{\begin{array}{c}
0 \\
\dot{Y}_{B L E}{ }^{(i)} \\
\dot{z}_{B L E}(i)
\end{array}\right\}_{B L n}+\left[T_{B L E-B L n}\right]\left\{\begin{array}{c}
-r_{B L E} Y_{O N A} \\
0 \\
p_{B L E} Y_{O R A}
\end{array}\right\}_{B L E}{ }_{(i)}
$$

and likerise, the second tine derivative is:

These three equations, then, define the $Y$ and $Z$ displacements, velocities, and accelerations of the neutral axis point used in the $X$ equations and time derivatives. Also, the increments of spanwise motions due to the offset between the center of gravity and neutral axis are defined by these same three equations. This increment represents the motion of the neutral axis relative to the blade reference axis origin, therefore, the span motion at the center of gravity is determined by subtracting $\Delta X_{M A}$ (i) from the spanwise metion of the neutral axis, or:

$$
\begin{align*}
& X_{B L E}^{(i)}=X_{N L n}(i)-\Delta X_{N L n}  \tag{148}\\
& \dot{X}_{B L E}^{(i)}  \tag{149}\\
& B L n  \tag{150}\\
& (i)=\dot{X}_{N A}(i)-\dot{\Delta i n}_{B L n}^{\prime} A_{B L n}^{(i)} \\
& \ddot{X}_{B L E}^{(i)}=\ddot{X}_{N A}(i)-\ddot{\Delta X_{N L}}(i)
\end{align*}
$$

These equations, then, along with the previous expressions for $X$ and $i$, define the blade element relative displacement, velcaity, and acceleration rectors required for the total inertial vectors which follow.
4.5.5.11 Blade motion in absolute coordimates. - To this point the blade element motion has been defined in terms of the blade axis or relative coordinates. The elemenis defined are:


$$
\text { blade element relative velocities }\left\{\begin{array}{l}
\dot{X}_{B L E}{ }^{(i)}  \tag{152}\\
\dot{Y}_{B L E} \\
\dot{Z}_{B L E} \\
(i)
\end{array}\right\}_{B L n}
$$

$$
\text { and blade element relative accelerations }\left\{\begin{array}{l}
\ddot{x}_{B L \Sigma^{(i)}}  \tag{153}\\
\ddot{\mathrm{y}}_{\mathrm{BLE}}{ }^{(i)} \\
\ddot{z}_{\mathrm{BLE}}{ }^{(i)}
\end{array}\right\}_{\mathrm{BL}}
$$

Using the method of Section 4.4.1, expressions in freestream (absolute) coordinates can be written for use in the equations of motion. The blade element velocity becomes:

The blade element accelerations are:

$$
\begin{align*}
& +\left[\begin{array}{rrr}
0 & -\dot{\mathbf{r}} & \dot{\mathbf{q}} \\
\dot{\mathbf{r}} & \dot{-} & -\dot{\mathbf{p}} \\
-\dot{\mathbf{q}} & \dot{\mathbf{p}} & 0
\end{array}\right]_{\mathrm{BLn}}\left\{\begin{array}{c}
\mathrm{X}_{B L E}(\mathrm{i}) \\
\mathrm{Y}_{\mathrm{BLE}}(\mathrm{i}) \\
\mathrm{Z}_{\mathrm{BLE}}(\mathrm{i})
\end{array}\right\}_{\mathrm{BLn}} \\
& +\left[\begin{array}{rrr}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]_{B L n}\left[\begin{array}{rrr}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]_{B L n}\left\{\begin{array}{l}
x_{B L E}(i) \\
y_{B L E}(i) \\
z_{B L E}(i)
\end{array}\right\}_{B L n} \\
& +2\left[\begin{array}{rrr}
0 & -\mathbf{r} & \mathbf{q} \\
\mathbf{r} & 0 & -\mathrm{p} \\
-\mathrm{q} & \mathrm{p} & 0
\end{array}\right]_{\mathrm{BLn}}\left\{\begin{array}{c}
\dot{\mathrm{X}}_{\mathrm{BLE}} \\
\dot{\mathrm{Y}}_{\mathrm{BLE}} \\
\dot{\mathrm{z}}_{\mathrm{BLE}}
\end{array}\right\}_{\mathrm{BLn}} \tag{155}
\end{align*}
$$

where

$$
\left\{\begin{array}{c}
\ddot{x}_{0}  \tag{156}\\
\ddot{y}_{0} \\
\ddot{z}_{0}
\end{array}\right\}_{B L n}, \quad\left\{\begin{array}{c}
\dot{x}_{0} \\
\dot{\mathrm{Y}}_{0} \\
\dot{z}_{0}
\end{array}\right\}_{B L n}
$$

and matching rotation terms are defined in Section 4.5 .4 in terms of rotor axis terms which are in turn related to the principal (hub) reference axis.
4.5.6 Swashplate motion. - As shown in Figure 10, the swashplate reference axis system is defined with the 2 -axis down. The motion of the swashplate reference system is definec by three generalized coordinate displacements, $Z_{S P}{ }^{\prime} \|_{S P}$, and $\theta_{S P}$, which move relative to the hub axis syster.

The rotations $\phi_{S P}$ and $\theta_{S P}$ are taken in the same order as shown in Figure 16 and therefore, from Section 4.4.3 the angular velocities are:

$$
\begin{align*}
\left\{\begin{array}{l}
\mathbf{p} \\
\mathbf{q} \\
\mathbf{r}
\end{array}\right\} \mathrm{SF}_{\mathrm{R}} & \left(\begin{array}{c}
0 \\
0 \\
\dot{\psi}_{S P}
\end{array}\right\}+\left[\begin{array}{ccc}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right]\left\{\left\{\begin{array}{c}
0 \\
\dot{\theta}_{\mathrm{SP}} \\
0
\end{array}\right\}\right. \\
& +\left[\begin{array}{ccc}
\cos _{\mathrm{SP}} & 0 & -\sin \theta_{\mathrm{SP}} \\
0 & 1 & 0 \\
\sin \theta_{\mathrm{SP}} & 0 & \cos \theta_{\mathrm{SP}}
\end{array}\right]\left\{\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{SP}} \\
0 \\
0
\end{array}\right\}\right. \\
& \left.+\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{SP}} & \sin \phi_{\mathrm{SP}} \\
0 & -\sin \phi_{\mathrm{SP}} & \cos \phi_{\mathrm{SP}}
\end{array}\right]\left\{\begin{array}{l}
\mathrm{p} \\
\mathrm{q} \\
\mathrm{r}
\end{array}\right\}\right\} \tag{157}
\end{align*}
$$

where $\dot{\psi}_{S P}$ is the rotational speed of the swashplate, and

$$
\begin{equation*}
\dot{\psi}_{\mathrm{SP}}=-\dot{\psi}_{\mathrm{R}} \tag{158}
\end{equation*}
$$

where $\dot{\psi}_{R}$ is the rotational speed of the rotor. Note no coupling is provided for shaft motion, the assumption being that swashplate motions relative to the hub due to shaft motions have been designed out of the system.

As indicated before, the chosen swashplate axes do not rotate at the rotational speed $\dot{\psi}_{S P}$. Hovever, the total angular velocities reflect the rotational
rate $\dot{W}_{S P}$. Therefore, the total angular rates of the swashplate in swashplate axes are obtained with $\psi=0$. This gives

$$
\begin{align*}
& \left\{\begin{array}{l}
\mathrm{p} \\
\mathrm{q} \\
\mathrm{r}
\end{array}\right\} \mathrm{SP}=\left\{\begin{array}{l}
0 \\
0 \\
\dot{\phi}_{\mathrm{SP}}
\end{array}\right\}+\left\{\begin{array}{l}
0 \\
\dot{\theta}_{\mathrm{SP}} \\
0
\end{array}\right\}+\left[\begin{array}{ccc}
\cos _{\mathrm{SP}} & 0 & -\sin \theta_{\mathrm{SP}} \\
0 & 1 & 0 \\
\sin \theta_{\mathrm{SP}} & 0 & \cos \theta_{\mathrm{SP}}
\end{array}\right]\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{SP}} \\
0 \\
0
\end{array}\right\} \\
& \left.+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{S P} & \sin \phi_{S P} \\
0 & -\sin \phi_{S P} & \cos \phi_{S P}
\end{array}\right]\left\{\begin{array}{c}
n \\
q \\
r
\end{array}\right\}\right\} \tag{159}
\end{align*}
$$

Honrotating swashplate angular velocities, subscripted SP, are obtained by deleting $\dot{\psi}_{S P}$ above.

The swasaplate angular accelerations can be similarly determined by evaluating the general expression at $\psi=\psi_{S P}=0$. This yields:

$$
\begin{align*}
& \left\{\begin{array}{l}
\dot{\mathrm{p}} \\
\dot{q} \\
\dot{q}
\end{array}\right\}=\left\{\begin{array}{c}
0 \\
\ddot{\theta}_{S P} \\
0
\end{array}\right\}+\left\{\begin{array}{c}
\dot{\psi}_{S P} q_{S P} \\
-\dot{\psi}_{S P}{ }_{S P} \\
\ddot{\psi}_{S P}
\end{array}\right\}+\dot{\theta}_{S P}\left[\begin{array}{ccc}
-\sin \theta_{S P} & 0 & -\cos \theta_{S P} \\
0 & 0 & 0 \\
\cos \theta_{S P} & 0 & -\sin \theta_{S P}
\end{array}\right\}\left\{\left\{\begin{array}{l}
\dot{\phi}_{S P} \\
0 \\
0
\end{array}\right\}\right. \\
& \left.+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{\mathrm{SP}} & \sin \phi_{\mathrm{SP}} \\
0 & -\sin \phi_{\mathrm{SP}} & \cos \phi_{\mathrm{SP}}
\end{array}\right]\left\{\begin{array}{l}
\mathrm{p} \\
q \\
r
\end{array}\right\} \mathrm{H}^{2}\right\}+\left[\begin{array}{ccc}
\cos \theta_{\mathrm{SP}} & 0 & -\sin \theta_{\mathrm{SP}} \\
0 & 1 & 0 \\
\sin \theta_{\mathrm{SP}} & 0 & \cos \theta_{\mathrm{SP}}
\end{array}\right]\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{SP}} \\
0 \\
0
\end{array}\right\} \\
& +\dot{\phi}_{S P}\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & -\sin \phi_{S P} & \cos \phi_{S P} \\
0 & -\cos \phi_{S P} & -\sin \phi_{S P}
\end{array}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{S P} & \sin \phi_{S P} \\
0 & -\sin \phi_{S P} & \cos \phi_{S P}
\end{array}\right]\left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\} \tag{160}
\end{align*}
$$

The vertical velocities and accelerations of the swashplate are simply defined as:

$$
\begin{equation*}
\dot{\mathrm{z}}_{\mathrm{OSP}}^{I}=\dot{\mathrm{Z}}_{\mathrm{SP}}+\dot{\mathrm{z}}_{\mathrm{OH}}^{I} \tag{161}
\end{equation*}
$$

and

$$
\begin{equation*}
\ddot{z}_{O S P}^{I}=\ddot{z}_{S P}+\ddot{z}_{O H}^{I} \tag{162}
\end{equation*}
$$

It is noted in these equations that the Z-axis motion is assumed to remain parallel to the hub Z axis.

The swashplate angular displacements are obtained by integrating the angular velocities, or:

$$
\begin{equation*}
\phi_{S P}=\int_{0}^{t} \dot{\phi}_{S P} d t+\phi_{t=0, S P} \tag{163}
\end{equation*}
$$

and

$$
\begin{equation*}
\theta_{S P}=\int_{0}^{t} \dot{\theta}_{S P} d t+\theta_{t=0, S P} \tag{164}
\end{equation*}
$$

Likewise, the vertical displacement of the swashplate relative to the hub is:

$$
\begin{equation*}
\mathrm{z}_{\mathrm{SP}}=\int_{0}^{t} \dot{\mathrm{z}}_{S P} d t+\mathrm{z}_{\mathrm{t}=0, \mathrm{SP}} \tag{165}
\end{equation*}
$$

4.5.7 Blade feathering motion. - The feathering occurring at the feather bearings, Figure 15, is taken to be the sum of the motions of the following dynamic and kinematic elements:

- Swashpiate - collective command
- Srashplate - cyclic comand
- Blade bending to feathering couplings
- Elastic pitch horn and associated components

The total feathering response is:

$$
\begin{align*}
\phi_{\mathrm{Fn}}= & \theta_{0}-A_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{\mathrm{R}} \prime-\mathrm{B}_{1 \mathrm{~S}} \sin \left(\psi_{\mathrm{BLn}}+\psi_{\mathrm{R}}\right)+\frac{\partial \phi_{\mathrm{Pn}}}{\partial A_{1}} A_{1 n}\right. \\
& +\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{2}} A_{2 \mathrm{n}}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{3}} A_{3 n}+\frac{\partial \phi_{\mathrm{F}}}{\partial \beta_{\mathrm{Pi}}} \beta_{\mathrm{PHn}} \tag{166}
\end{align*}
$$

Velocities and accelerations are formed by differentiation. The desired relations are:

$$
\begin{align*}
\dot{\phi}_{\mathrm{Fn}}= & \dot{\theta}_{0}-\dot{A}_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)-\dot{B}_{1 S} \sin \left(\psi_{\mathrm{BLn}}+\psi_{R}\right) \\
& +\left[A_{1 S} \sin \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)-\mathrm{B}_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)\right] \dot{\psi}_{R} \\
& +\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1}} \dot{A}_{1 n}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{2}} \dot{A}_{2 n}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{3}} \dot{A}_{3 n}+\frac{\partial \phi_{F}}{\partial \beta_{\mathrm{PH}}} \dot{\beta}_{\mathrm{PHn}} \tag{167}
\end{align*}
$$

and for accelerations:

$$
\begin{align*}
\ddot{\phi}_{\mathrm{Fn}}= & \ddot{\theta}_{0}-\ddot{A}_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)-\ddot{B}_{1 S} \sin \left(\psi_{\mathrm{BLn}}+\psi_{R}\right) \\
& +2\left[\dot{A}_{1 S} \sin \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)-\dot{B}_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)\right] \dot{\psi}_{R} \\
& +\left[A_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)+B_{1 S} \sin \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)\right] \dot{\psi}_{R} \\
& +\left[A_{1 S} \cos \left(\psi_{\mathrm{BLn}}+\psi_{R}\right)-B_{1 S} \sin \left(\psi_{\mathrm{BLn}}+\psi_{K}\right)\right] \ddot{\psi}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1}} \ddot{A}_{1 n} \\
& +2 \ddot{A}_{2 n}+\frac{\partial \Phi_{\mathrm{Fn}}}{\partial A_{3}} \ddot{A}_{3 n}+\frac{\partial \phi_{F}}{\partial \beta_{P H}} \ddot{\beta}_{\mathrm{PHn}} \tag{168}
\end{align*}
$$

The commanded cyclic blade angles are:

$$
\left\{\begin{array}{c}
\mathrm{A}_{1 \mathrm{~S}}  \tag{169}\\
\mathrm{~B}_{1 \mathrm{~S}}
\end{array}\right\}=\left(\frac{\mathrm{d}}{\mathrm{e}}\right)\left[\begin{array}{cc}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}} \\
\cos \psi_{\mathrm{PH}} & -\sin \psi_{\mathrm{PH}}
\end{array}\right]\left\{\begin{array}{l}
\phi_{\mathrm{SP}} \\
\theta_{\mathrm{SP}}
\end{array}\right\}
$$

where the angle $\psi_{\mathrm{PH}}$ is the pitch horn-swashplate connection lead to feather axis. See Figure 28. This angle is computed as a static value. It should be noted that some hub configurations carry the pitch horn toward the blade trailing edge. These configurations are entered in REXOR II by forming the supplement of $\psi_{\mathrm{PH}}$.

$$
\begin{equation*}
\psi_{\mathrm{PH}}=180-\psi_{\mathrm{PH}}(\text { degrees }) \tag{170}
\end{equation*}
$$

This angle gives the correct modeling of the sense of rotation reversal with the trailing pitch horn geometry.

The velocities and accelerations of the command cyclic are obtained by differentiation.


Figure 28. - Pitch horn blade feathering phase angle.

$$
\begin{align*}
\left\{\begin{array}{c}
\dot{\mathrm{A}}_{1 \mathrm{~S}} \\
\dot{\mathrm{~B}}_{1 \mathrm{~S}}
\end{array}\right\}= & \left(\frac{\mathrm{d}}{\mathrm{e}}\right)\left[\begin{array}{lr}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}} \\
\cos \psi_{\mathrm{FH}} & -\sin \psi_{\mathrm{PH}}
\end{array}\right]\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{SP}} \\
\dot{\theta}_{\mathrm{SP}}
\end{array}\right\} \\
& +\left(\frac{\mathrm{d}}{\mathrm{e}}\right)_{1} \dot{\theta}_{0}\left[\begin{array}{ll}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}} \\
\cos \psi_{\mathrm{PH}} & -\sin \psi_{\mathrm{FH}}
\end{array}\right]\left\{\begin{array}{l}
\phi_{\mathrm{SP}} \\
\theta_{\mathrm{SP}}
\end{array}\right\} \tag{171}
\end{align*}
$$

and

$$
\begin{align*}
\left\{\begin{array}{l}
\ddot{\mathrm{A}}_{1 S} \\
\ddot{\mathrm{~B}}_{1 \mathrm{~S}}
\end{array}\right\}= & \left(\frac{\mathrm{d}}{\mathrm{e}}\right)\left[\begin{array}{lr}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}} \\
\cos \psi_{\mathrm{PH}} & -\sin \psi_{\mathrm{FH}}
\end{array}\right]\left[\begin{array}{l}
\ddot{\phi}_{\mathrm{SP}} \\
\ddot{\theta}_{\mathrm{SP}}
\end{array}\right\} \\
& +2\left(\frac{d}{\mathrm{e}}\right)_{1} \dot{\theta}_{0}\left[\begin{array}{rr}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}} \\
\cos \psi_{\mathrm{PH}} & -\sin \psi_{\mathrm{PH}}
\end{array}\right]\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{SP}} \\
\dot{\theta}_{\mathrm{SP}}
\end{array}\right\} \\
& +\left(\frac{\mathrm{d}}{\mathrm{e}}\right)_{1} \ddot{\theta}_{0}\left[\begin{array}{ll}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}} \\
\cos \psi_{\mathrm{PH}} & -\sin \psi_{\mathrm{PH}}
\end{array}\right]\left\{\begin{array}{l}
\phi_{\mathrm{SP}} \\
\theta_{\mathrm{SP}}
\end{array}\right\} \tag{172}
\end{align*}
$$

The overall coupling (swashplate to feathering) gear ratio, d/e, is expressed as a static term plus a first-order collective correction.

$$
\begin{equation*}
\left(\frac{d}{e}\right)=\left(\frac{d}{e}\right)_{0}+\left(\frac{d}{e}\right)_{1} \theta_{0} \tag{173}
\end{equation*}
$$

The collective is:

$$
\begin{equation*}
\theta_{0}=-Z_{S P} / e \tag{174}
\end{equation*}
$$

The swashplate vertical motion, $Z_{S P}$, is developed in Section 4.5.6. The value $e$ is the static effective crank (pitch horn) arm about the blade feather axis. This crank length is entered as a negative number for a trailing pitch horn geometry to give the proper sense of collective for swashplate vertical translation.

Taking time derivatives:

$$
\begin{equation*}
\dot{\theta}_{0}=-\dot{z}_{S P} / e \tag{175}
\end{equation*}
$$

and

$$
\begin{equation*}
\ddot{i}_{0}=-\ddot{z}_{S P} / e \tag{176}
\end{equation*}
$$

The blade bending to feathering coupling fE=tors are $\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1}}, \frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{2}}$, and
$\frac{\partial \phi_{\mathrm{Fn}}}{\partial \mathrm{A}_{3}}$ for the first, second, and third blade modes. The blade bending modes are described without a torsion component; this allows freedom in varying the blade sweep, droop, jog , or other geometric parameters without new input data for the blade mode shape. The torsion either is calculated separately along the blade proper or as a blade root component by pitch horn bending. The coupling factors are intended to add a feathering component to the blade mode which would exist even with no torsion or feathering moments. As such, they are in effect the $\delta_{3}, a_{2}$, etc., coupling usually described in the
literature. These couplings are usually determined as a function of the distance from the flap or inplane mechanical or vertical hinge to a pitch horn projection.

## 5. EQUATIONS OF MOTION

### 5.1 Introduction

With the coordinate systems and transformation between systems well in hand, the development can proceed to the equations of motion. The development yields a set of second-order differential equations with time varying coefficients. These equations are formulated using the energy approach in a form credited to Lagrange. The solution to the system of equations is in the time domain by numerical integration. The result is a time history of the displacements, velocities, accelerations, and loads of the components of the helicorter modeled in detail, and the program treats each blade separately.

In the fillcwing development a Lagrangian approach to system modeling is applied to a set of point masses and then extended to discrete masses and inertias. The result is a set of generalized mass and force expressions. In REXOR (Reference 4) these expr sions are programmed directly, element by element. In REXOR II extensive us is made of matrix notation both in cescription and programing. The transition to matrix notation is given at the end of section 5.3.

### 5.2 Energy Approach to Development of Equations of Motion

There are two basic approaches to developing the equations of motion for a physical system. These are:

- Vector summation of forces
- Energy approach

Given an equal set of conditions, limitations, and assumptions, both procedures should result in equivalent sets of equations. The difference is in the ease of arriving at a complete set of equations. Note that force is a vector, whereas energy is a scalar quantity. Therefore, in dealing in terms of energy, less information regarding direction needs to be handled. Also the systematic nature of the energy approach reduces the risk of error. As stated by Lagrange (Mecanique Analytique, 1788), "The methods which I present here do not require either constructions or reasonings of geometrical or mechanical nature, but only algebraic operations proceeding after a regular and uniform plan".

The starting point of this development is Lagrange's equation. It may be derived by postulating Newton's second law, or from Mai ,on's principle. Lagrange's equation may be written in the following form:

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial T}{\partial \dot{q}_{r}}\right)-\frac{\partial r}{\partial q_{r}}+\frac{\partial B}{\partial \dot{q}_{r}}+\frac{\partial U}{\partial q_{r}}=Q_{r} \tag{177}
\end{equation*}
$$

where
$T$ is kinetic energy
$q$ is a generalized coordinate
$B$ is dissipation function
$U$ is potential energy function
$Q_{2}$ is the generalizea force, derised from the virtual work, $\delta \mathrm{W}$, and
is defined by the equation

$$
\begin{equation*}
Q_{r}=\frac{\partial C_{U}}{\partial q_{r}} \tag{178}
\end{equation*}
$$

Equation 177 will now be developed into the form as applied in REXOR II. This form bears a close resemblance to a force balance equation, but is derived from energy considerations. For clarity, the aevelopment is first shown for a set of discrete mass particles, then, in the section that follows, is extended to the distributed elemental masses of the PEXOR II modeling and to the iterative solution scheme used.

In a conventional manner the equation is formulated in terms of generalized coordinates. These coordinates are a function of time, and completely define the system. They are generally not directly identifiable as a physical quantity.

The physical parameters or elemental coordinates are defined to be functions of the generalized coordinates and in turn a function of time. Consider a system to be composed of particles whose physicsl coordinates are a function of $n$ generalized coordinates. For the ith particle:

$$
\begin{align*}
& x_{i}=x_{i}\left(q_{1}, q_{2}, \cdots, q_{i} ; t\right)  \tag{179}\\
& y_{i}=y_{i}\left(z_{1}, q_{2}, \cdots, q_{n} ; t\right)  \tag{180}\\
& z_{i}=z_{i}\left(q_{1}, q_{2}, \cdots, q_{n} ; t\right) \tag{181}
\end{align*}
$$

Note: a Cartesian coordinate set is selected, and used in REXOR II. However, the argument is true for an arbitrary coordinate set.

The functional relationship of the physical or constrained coordinates and generalized coordinates yields:

$$
\begin{align*}
& \delta x_{i}=\frac{\partial x_{i}}{\partial q_{1}} \delta q_{1}+\frac{\partial x_{i}}{\partial q_{2}} \delta q_{2}+\cdots+\frac{\partial x_{i}}{\partial q_{n}} \delta q_{n}  \tag{182}\\
& \delta y_{i}=\frac{\partial y_{i}}{\partial q_{1}} \delta q_{1}+\frac{\partial y_{i}}{\partial q_{2}} \delta q_{2}+\cdots+\frac{\partial y_{i}}{\partial q_{n}} \delta q_{n}  \tag{183}\\
& \delta z_{i}=\frac{\partial z_{i}}{\partial q_{1}} \delta q_{1}+\frac{\partial z_{i}}{\partial q_{2}} \delta q_{2}+\cdots+\frac{\partial z_{i}}{\partial q_{n}} \delta q_{n} \tag{184}
\end{align*}
$$

The tise dependence is inf' $c i t$ in the increments of the generalized coordinates. The equation is strictly true for infinitesimal increments. In REXOR II the generalized coordinates are distinct from physical coordinates in the main rotor blade descriptions. Here the generalized coordinates are blade modal variables. The modal variables represent tangible ieflections of the blade from a reference position, and as such are small but not infinitesimal variables.

As the variables are a function of time:

$$
\begin{align*}
& \dot{x}_{i}=\frac{\partial x_{i}}{\partial q_{1}} q_{1}+\frac{\partial x_{i}}{\partial q_{2}} \dot{q}_{2}+\cdots+\frac{\partial x_{i}}{\partial q_{n}} q_{n}  \tag{185}\\
& \dot{y}_{i}=\frac{\partial y_{i}}{\partial q_{1}} \dot{q}_{1}+\frac{\partial y_{i}}{\partial q_{2}} \dot{q}_{2}+\cdots+\frac{\partial y_{i}}{\partial q_{n}} \dot{q}_{n}  \tag{186}\\
& \dot{z}_{i}=\frac{\partial z_{i}}{\partial q_{1}} \dot{q}_{1}+\frac{\partial z_{i}}{\partial q_{2}} \dot{q}_{2}+\cdots+\frac{\partial z_{i}}{\partial q_{n}} \dot{q}_{n} \tag{187}
\end{align*}
$$

In terms of the ith particle the kias+ic energy for the system may be identified as:

$$
\begin{equation*}
T=\sum_{i=1}^{N} \frac{1}{2} m_{i}\left(\dot{x}_{i}^{2}+\dot{y}_{i}^{2}+\dot{i}_{i}^{2}\right) \tag{1.88}
\end{equation*}
$$

Toward the particular formulation of Lagrange's equation used in REXOR II, the first two terms of the previously stated form, Equation 177, are developed:

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial T}{\partial \dot{\dot{\theta}}_{r}}\right)^{-}-\frac{\partial T}{\partial q_{r}} \tag{189}
\end{equation*}
$$

Performing these operations for the ith particle case and the rth gencralized coordinate and summing over the system yields:

$$
\begin{align*}
\frac{d}{d t}\left(\frac{\partial T}{\partial \dot{q}_{r}}\right)-\frac{\partial T}{\partial q_{r}}= & \sum_{i=1}^{N}\left(\frac{1}{2} m_{i} \frac{d}{d t} \frac{\partial}{\partial \dot{q}_{r}}\left(\left(\dot{x}_{i}^{2}+\dot{y}_{i}^{2}+\dot{i}_{i}^{2}\right)\right)\right. \\
& \left.-\frac{1}{2} m_{i} \frac{\partial}{\partial q_{r}}\left(\dot{\dot{x}}_{i}^{2}+\dot{y}_{i}^{2}+\dot{i}_{i}^{2}\right)\right) \tag{190}
\end{align*}
$$

A useful math operation of cancellation of the dots is developed prior to proceeding. Recall:

$$
\begin{equation*}
\delta x_{i}=\frac{\partial x_{i}}{\partial q_{1}} \delta q_{1}+\frac{\partial x_{i}}{\partial q_{2}} \delta q_{2}+\cdots+\frac{\partial x_{i}}{\partial q_{n}} \delta q_{n} \tag{191}
\end{equation*}
$$

Ther also

$$
\begin{equation*}
\dot{x}_{i}=\frac{\partial x_{i}}{\partial q_{1}} \dot{q}_{1}+\frac{\partial x_{i}}{\partial q_{2}} \dot{q}_{2}+\cdots+\frac{\partial x_{i}}{\partial q_{n}} \dot{q}_{n} \tag{192}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial \dot{x}_{i}}{\partial \dot{q}_{r}}=\frac{\partial x_{i}}{\partial q_{r}} \tag{193}
\end{equation*}
$$

This is also true for $y$ and $z$ and for the double dot terms in $x, y$ and $z$.

An operation to reverse the order of spacial and temporal differentiation is required. To show this the time derivative of a partial is taken as

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial x_{i}}{\partial q_{r}}\right)=\frac{\partial}{\partial q_{1}}\left(\frac{\partial x_{i}}{\partial q_{r}}\right) \dot{q}_{1}+\frac{\partial}{\partial q_{2}}\left(\frac{\partial x_{i}}{\partial q_{r}}\right) \dot{q}_{2}+\cdots+\frac{\partial}{\partial q_{n}}\left(\frac{\partial x_{i}}{\partial q_{r}}\right) \dot{q}_{n} \tag{194}
\end{equation*}
$$

Hext the spacial derivative of $\dot{x}_{i}$ is given as

$$
\begin{equation*}
\frac{\partial \dot{x}_{i}}{\partial q_{r}}=\frac{\partial}{\partial q_{r}}\left(\frac{\partial x_{i}}{\partial q_{1}} \dot{q}_{1}+\frac{\partial x_{i}}{\partial q_{2}} \dot{q}_{2}+\cdots+\frac{\partial x_{i}}{\partial q_{n}} \dot{q}_{n}\right) \tag{195}
\end{equation*}
$$

Kow since

$$
\begin{equation*}
x_{i}=x_{i}\left(q_{1}, q_{2}, \cdots, q_{n}\right) \tag{196}
\end{equation*}
$$

the order of spacial differentiation is reversible

$$
\begin{equation*}
\frac{\partial^{2} x_{i}}{\partial g_{r} \partial q_{s}}=\frac{\partial^{2} x_{i}}{\partial q_{s} \partial q_{r}} \tag{197}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial x_{i}}{\partial q_{r}}\right)=\frac{\partial \dot{x}_{i}}{\partial q_{r}} \tag{198}
\end{equation*}
$$

Similar relations exist for $y_{i}$ and $z_{i}$.
Proceeding on with the kinetic energy terms:

$$
\begin{align*}
\frac{d}{d t}\left(\frac{\partial T}{\partial \dot{q}_{r}}\right)-\frac{\partial T}{\partial q_{r}}= & \sum_{i=1}^{H} m_{i}\left[\ddot{x}_{i} \frac{\partial \dot{x}_{i}}{\partial \dot{q}_{r}}+\ddot{y}_{i} \frac{\partial \dot{y}_{i}}{\partial \dot{q}_{r}}+\ddot{z}_{i} \frac{\partial \dot{z}_{i}}{\partial \dot{\dot{q}}_{r}}+\dot{x}_{i} \frac{d}{d t}\left(\frac{\partial \dot{x}_{i}}{\partial \dot{q}_{r}}\right)+y_{i} \frac{d}{d t}\left(\frac{\partial \dot{y}_{i}}{\partial \dot{q}_{r}}\right)\right. \\
& \left.+\dot{i}_{i} \frac{d}{d t}\left(\frac{\partial \dot{z}_{i}}{\partial \dot{q}_{r}}\right)-\dot{x}_{i} \frac{\partial \dot{x}_{i}}{\partial q_{r}}-\dot{y}_{i} \frac{\partial \dot{y}_{i}}{\partial q_{r}}-\dot{i}_{i} \frac{\partial \dot{z}_{i}}{\partial q_{r}}\right] \tag{199}
\end{align*}
$$

Using the relationship for cancelling dots in pertials, reversing the order of differentiation and cancelling terms gives

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial T}{\partial \dot{q}_{r}}\right)-\frac{\partial T}{\partial q_{r}}=\sum_{i=1}^{H} m_{i}\left[\ddot{x}_{i} \frac{\partial x_{i}}{\partial q_{r}}+\ddot{y}_{i} \frac{\partial y_{i}}{\partial q_{r}}+\ddot{z}_{i} \frac{\partial z_{i}}{\partial q_{r}}\right] \tag{200}
\end{equation*}
$$

Then from Equation 177, Lagrange's Equation in constrained coordinates vith point masses becomes

$$
\begin{equation*}
\sum_{i=1}^{n} m_{i}\left[\ddot{x}_{i} \frac{\partial r_{i}}{\partial q_{r}}+\ddot{y}_{i} \frac{\partial y_{i}}{\partial q_{r}}+\ddot{z}_{i} \frac{\partial z_{i}}{\partial q_{r}}\right]+\frac{\partial B}{\partial \dot{q}_{r}}+\frac{\partial U}{\partial q_{r}}=q_{r} \tag{201}
\end{equation*}
$$

Aiso, in the same vein of defining the generalized coordinates, the relationship between the elemental and generalized forces can be developed. This relatic $=$ ihip is developed from the definition of virtual work on a particle as the scalar product or the applied force and an infinitesimal displacement. There ore for the total system of X elements,

$$
\begin{equation*}
\delta W=\sum_{i=1}^{1}\left[F_{x_{i}} \delta x_{i}+F_{y_{i}} \delta y_{i}+F_{z_{i}} \delta z_{i}\right] \tag{202}
\end{equation*}
$$

Using the definition of $Q_{r}$ from Equation 178 gives:

$$
\begin{equation*}
Q_{r}=\sum_{i=1}^{F}\left({ }_{x_{i}} \frac{\partial x_{i}}{\partial q_{r}}+F_{y_{i}} \frac{\partial y_{i}}{\partial q_{r}}+F_{z_{i}} \frac{\partial z_{i}}{\partial q_{i}}\right) \tag{203}
\end{equation*}
$$

Substituting Equation 203 into Equation 201 yields the final form of the Lagrange energy equation in constrained coordinates for point masses, which is in the form from which the REXCR II Equations of motion are developed. Making this substitution and rearranging the equation yields

$$
\begin{align*}
\sum_{i=1}^{n}\left[\left(m_{i} \ddot{x}_{i}-F_{x_{i}}\right) \frac{\partial \ddot{i}_{i}}{\partial q_{r}}\right. & \left.+\left(m_{i} \ddot{y}_{i}-F_{y_{i}}\right) \frac{\partial y_{i}}{\partial q_{r}}+\left(m_{i} \ddot{z}_{i}-F_{z_{i}}\right) \frac{\partial z_{i}}{\partial q_{r}}\right] \\
& +\frac{\partial B}{\partial \dot{q}_{r}}+\frac{\partial U}{\partial q_{r}}=0 \tag{204}
\end{align*}
$$

The above equation is the basis for the entire derivation of the equations of motion of REXOR II. Note that this equation is vxitten for discrete element masses and discrete forces. Also, at any instant in time all of the ingredients required to define the elemental accelerations, $\ddot{\boldsymbol{x}}_{i}, \ddot{\mathbf{y}}_{i}, \ddot{\mathbf{z}}_{i}$, are not known. Specifically, the generalized coordinate displacements and velocities, $q_{r}$ and $\dot{q}_{r}$, are knewn at any instant in time but the generalized coordinate accelerations, $\ddot{q}_{r}$, remain to be determined at the time the elemental accelerations are computed.

The following section presents the manner in which the foregoing equation set is adapted to the REXOR II numerical solution to solve the equilibrium equations or equations of motions for the generalized coordinate accelerations. This development is first presented in the simpler form, for clarity, for discrete mass elements and forces and then in expanded form to include elemental distributed masses and applied moments.

### 5.3 Iterative Concept and Equation Set Solution Method

Given a set of equations as developed in the previous section, the next step is to establish a method of solution. The solution process is defined as solving the equation set for the accelerations, integrating the accelerations for updated velocity, and position; then substituting the integrands back into the equations to determine new values of accelerations.

The first step of the process is to define exdicitlv the arcelpratinns from the equation set. In the process of implementing the REXOR II equations, it is desirable to handle the accelerations as an estimated plus a corrective term. In generalized coordinates then we can write

$$
\left\{\begin{array}{c}
\ddot{q}_{1}  \tag{205}\\
\cdot \\
\cdot \\
\cdot \\
\ddot{q}_{n}
\end{array}\right\}_{\mathrm{NEW}}=\left\{\begin{array}{c}
\ddot{q}_{1} \\
\cdot \\
\cdot \\
\cdot \\
\ddot{q}_{n}
\end{array}\right\}_{\mathrm{CORR} .}+\left\{\begin{array}{c}
\ddot{q}_{1} \\
\cdot \\
\cdot \\
\cdot \\
\ddot{q}_{\mathrm{n}}
\end{array}\right\}_{\mathrm{OLD}}
$$

This operatior proceeds on a sequential time basis. For each increment advance in time, the previous 'NEW' becomes the 'OLD'. In REXOR II, the time increment corresponds with a step azimuthal advance of the main rotor
blades. However, this need not be the case. The 'NEW' accelerations must be used in the numerical integration process to define the generalized coordinate velocities and displacements. But if some form of a predictor on accelerations is used then the 'OLD' would be this predicted value and in this case it would be an estimated, 'EST', value.

Using the notation 'OLD' and 'EST' interchangeably the linear elemental accelerations can be written at time $t$ as

$$
\left\{\begin{array}{c}
\ddot{x}_{i}  \tag{206}\\
\ddot{y}_{i} \\
\ddot{z}_{i}
\end{array}\right\}_{i}=\left\{\begin{array}{c}
\ddot{x}_{i} \\
\ddot{y}_{i} \\
\ddot{z}_{i}
\end{array}\right\}_{\text {CORR }}+\left\{\begin{array}{c}
\ddot{x}_{i} \\
\ddot{y}_{i} \\
\ddot{z}_{i}
\end{array}\right\}_{\text {EST }}^{I}
$$

where the estimated accelerations are determined using the generalized displacements and velocities, $q_{r}$ and $\dot{q}_{r}$, at iime $t$, and the generalized coordinate accelerations $\ddot{q}_{r}$, either estimated or from one previous time step in the numerical integration process.

Then, at any given instant in time where the 'EST' elemental accelerations are thusly determined, the corrective elemental accelerations, $(\ddot{x}, \ddot{y}, \ddot{z})_{i} \operatorname{CORR}$ can be written as a function of the generalized coordinate corrective accelerations.

Or

$$
\left\{\begin{array}{l}
\ddot{x}_{i}  \tag{207}\\
\ddot{y}_{i} \\
\ddot{z}_{i}
\end{array}\right\}=\left\{\begin{array}{l}
\frac{\partial x_{i}}{\partial q_{1}} \ddot{q}_{l_{\text {CORR }}}+\cdots+\frac{\partial x_{i}}{\partial q_{n}} \ddot{q}_{n_{C O R R}} \\
\frac{\partial y_{i}}{\partial q_{l}} \dot{q}_{1_{C O R R}}+\cdots+\frac{\partial y_{i}}{\partial q_{n}} \ddot{q}_{n_{C O R R}} \\
\left.\frac{\partial z_{i}}{\frac{\partial q_{1}}{} \ddot{q}_{1_{C O R R}}+\cdots+\frac{\partial z_{i}}{\partial q_{n}} \ddot{q}_{n_{C O R R}}}\right\}
\end{array}\right\}
$$

or

$$
\left\{\begin{array}{l}
\ddot{x}_{i}  \tag{208}\\
\ddot{y}_{i} \\
\ddot{z}_{i}
\end{array}\right\}_{C O R R}=\sum_{k=1}^{n}\left\{\begin{array}{ll}
\frac{\partial x_{i}}{\partial q_{k}} & \ddot{q}_{k_{C O R R}} \\
\frac{\partial y_{i}}{\partial q_{k}} & \ddot{q}_{k_{C O R R}} \\
\frac{\partial z_{i}}{\partial q_{k}} & \ddot{q}_{k} \\
{ }_{C O R R}
\end{array}\right\}
$$

Now making the substitution of Equations 203 and 207 into Equation 201 from the previous section and rearranging terms yields the Lagrange equation for the $q_{r}$ ccordinate in terms of the estimated elemental accelerations and the corrective generalized coordinate accelerations.

$$
\begin{align*}
& \sum_{i=1}^{N}\left[\left(m_{i} \ddot{x}_{i_{E S T}}-F_{x_{i}}\right) \frac{\partial x_{i}}{\partial q_{r}}+\left(m_{i} \ddot{y}_{i_{E S T}}-F_{y_{i}}\right) \frac{\partial y_{i}}{\partial q_{r}}+\left(m_{i} \ddot{i}_{i_{E S T}}-F_{z_{i}}\right) \frac{\partial z_{i}}{\partial q_{r}}\right] \\
& +\sum_{i=1}^{N} m_{i}\left[\frac{\partial x_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial x_{i}}{\partial q_{k}} \ddot{q}_{k_{C O R R}}+\frac{\partial y_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial y_{i}}{\partial c_{k}} \ddot{q}_{k_{k}}+\frac{\partial z_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial z_{i}}{\partial q_{k}} \ddot{q}_{k_{k}}\right] \\
& +\frac{\partial B}{\partial \dot{q}_{r}}+\frac{\partial U}{\partial q_{r}}=0 \tag{209}
\end{align*}
$$

The equations of motion for the system can now be combined and presented in matrix form.

where the matrices, $M_{r k}, C_{r k}$ and $K_{r k}$ will be defined in the following discussion. However before proceeding with this, Equation 210 is now rearranged into the form actually used in the numerical process in REXOR II. The equation is solved in terms of the corrective accelerations.

The correction terms come from an inversion (or simultaneous equation, Cholesky method) operation on the model equation set.


As indicated before estimated accelerations in physical coordinates come from the 'EST' or 'OLD' generalized coordinate acceleratiors and the current generalized coordinate velocities and displacements. The integration part of the solution operation supplies the ( $\dot{q}$ ) and ( $q$ ) data.

$$
\begin{equation*}
\dot{q}=\int \ddot{\mathrm{q}}_{\mathrm{NEW}} d t \quad ; \quad \mathrm{q}=\int \dot{\mathrm{q}} \mathrm{dt} \tag{212}
\end{equation*}
$$

The whole package operates in a cyclical fashion, as shown in Figure 29. Arranging the solution sequence as such gives it some important attributes and advantages.

First, to determine the corrective acceleration, the inverted mass matrix premultiplies the difference of applied and estimated reactive forces represented by the quantity in the large brackets on the right-hand side


Figure 29. - Equation solution locp.
of Equation 211. With the usual, small, integration steps these differences will be relatively small. Therefore, inaccuracies in the mass matrix or its inversion process only slightly affect the total acceleration determination. This means approximations and simplifications to the mass matrix are acceptable. In some instances, a diagonal mass matrix will give convergence to the required solution.

Second, as will be shown in the Section 5.4 , (blade equations section), carrying the runnirg acceleration in elemental coordinates allows for the simple separation of the centrifugal and structural stiffness of the rotor blades which has important advantages which have been discussed. Also, the aerodynamic loading terms, already by nature in physical coordinates, are easily accounted for.

In the actual application of Equation 211 to REXOR II, distributed elemental rigid body masses are associated with each coordinate point and applied moments in addition to forces at each coordinate point are accounted for.

Referring back to Equation 203 the generalized force, $Q_{r}$, from virtual work can be simply written in the following form to account for applied moments at each of the itiz grid points as

$$
\begin{equation*}
Q_{r}=\sum_{i=1}^{N}\left[F_{x_{i}} \frac{\partial x_{i}}{\partial q_{r}}+F_{y_{i}} \frac{\partial y_{i}}{\partial q_{r}}+F_{z_{i}} \frac{\partial z_{i}}{\partial q_{r}}+M_{x_{i}} \frac{\partial \phi_{i}}{\partial q_{r}}+M_{y_{i}} \frac{\partial \theta_{i}}{\partial q_{r}}+M_{z_{i}} \frac{\partial \psi_{i}}{\partial q_{r}}\right] \tag{213}
\end{equation*}
$$

The terms of Equation 200 in Equation 204 can be developed for the distributed masses by going back to the elemental acceleration equation, Equation 13 of Section 4.4 .1 wnich is repeated here, in a rearranged form, for clarity of this development.

$$
\begin{align*}
\left\{\begin{array}{l}
\ddot{x} \\
\dot{y} \\
\ddot{z}
\end{array}\right\}^{I} & =\left\{\begin{array}{c}
\ddot{x}_{0} \\
\ddot{y}_{0} \\
z_{0}
\end{array}\right\}^{I}+\left\{\begin{array}{l}
\ddot{x} \\
\ddot{y} \\
\dot{z}
\end{array}\right\}+\left[\begin{array}{ccc}
\left(-r^{2}-q^{2}\right) & p q & p r \\
p q & \left(-r^{2}-p^{2}\right) & q r \\
p r & q r & \left(-p^{2}-q^{2}\right)
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\} \\
& +2\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}+\left[\begin{array}{ccc}
0 & z & -y \\
-z & 0 & x \\
y & -x & 0
\end{array}\right]\left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\} \tag{214}
\end{align*}
$$

For distributed masses of rigid body with coordinate point and system embedded in the body:

$$
\begin{equation*}
\dot{x}=\dot{y}=\dot{z}=0 \tag{215}
\end{equation*}
$$

and Equation 214 oecomes:

$$
\left\{\begin{array}{l}
\ddot{x}  \tag{216}\\
\ddot{y} \\
\ddot{z}
\end{array}\right\}^{I}=\left\{\begin{array}{l}
\ddot{x}_{0} \\
\ddot{y}_{0} \\
\ddot{z}_{0}
\end{array}\right\}^{I}+\left\{\begin{array}{l}
-x\left(r^{2}+q^{2}\right)+y p q+z p r \\
x p q-y\left(r^{2}+p^{2}\right)+z q r \\
x p r+y q r-z\left(p^{2}+q^{2}\right)
\end{array}\right\}+\left\{\begin{array}{c}
z \dot{q}-y r \\
-z \dot{p}+x \dot{r} \\
y \dot{p}-x \dot{q}
\end{array}\right\}
$$

Now, remembering that for a point mass,

$$
\begin{align*}
& \frac{\partial x}{\partial q_{r}}=\frac{\partial \ddot{x}}{\partial \ddot{q}_{r}}  \tag{217}\\
& \frac{\partial y}{\partial q_{r}}=\frac{\partial \ddot{y}}{\partial \dot{q}_{r}}  \tag{218}\\
& \frac{\partial z}{\partial q_{r}}=\frac{\partial \ddot{z}}{\partial \dot{q}_{r}}  \tag{219}\\
& \frac{\partial \phi}{\partial q_{r}}=\frac{\partial \dot{p}}{\partial \dot{q}_{r}}  \tag{220}\\
& \frac{\partial \theta}{\partial q_{r}}=\frac{\partial \dot{q}^{\prime}}{\partial \dot{q}_{r}} \tag{221}
\end{align*}
$$

and

$$
\begin{equation*}
\frac{\partial \psi}{\partial q_{r}}=\frac{\partial \dot{r}}{\partial \dot{q}_{r}} \tag{222}
\end{equation*}
$$

The total partial derivatives relating the motion of the coordinate point and set imbedded within each elemental body and the motion of the generalized coordinate becomes

$$
\begin{align*}
& \frac{\partial x_{r}}{\partial q_{r}}=\frac{\partial x_{0}}{\partial q_{r}}-y \frac{\partial \psi}{\partial q_{r}}+z \frac{\partial \theta}{\partial q_{r}}  \tag{223}\\
& \frac{\partial y}{\partial q_{r}}=\frac{\partial y_{0}}{\partial q_{i}}+x \frac{\partial \psi}{\partial q_{r}}-z \frac{\partial \phi}{\partial q_{r}}  \tag{224}\\
& \frac{\partial z}{\partial q_{r}}=\frac{\partial z_{0}}{\partial q_{r}}-x \frac{\partial \theta}{\partial q_{r}}+y \frac{\partial \phi}{\partial q_{r}} \tag{225}
\end{align*}
$$

where on the right side of these equations, $x, y$, and $z$ represent the location of the distributed masses within the rigid body elemental mass, and $x_{0}, y_{0}$ and $z_{0}$ represent the motiin of the mass element reference point.

For each jth coordinate of the system, the elements of Equation 200 can be written by substitution of Equations 216, 223, 224 and 225. This gives

$$
\begin{aligned}
& {\left[\sum_{i=1}^{N} E_{i}\left\{\begin{array}{c}
\ddot{x}_{i} \\
\ddot{y}_{i} \\
\ddot{z}_{i}
\end{array}\right\}_{E S T} \cdot \frac{\partial}{\partial q_{r}}\left\{\begin{array}{l}
x_{i} \\
y_{i} \\
z_{i}
\end{array}\right\}\right]_{j}} \\
& =\left[\begin{array}{l}
N=\left[\begin{array}{l}
\dot{x}_{0} \frac{\partial x_{0}}{\partial q_{r}}-\ddot{x}_{0} y_{i} \frac{\partial \psi}{\partial q_{r}}+\ddot{x}_{0} z_{i} \frac{\partial \theta}{\partial q_{r}}-x_{i}\left(r^{2}+q^{2}\right) \frac{\partial x_{0}}{\partial q_{r}}+x_{i} y_{i}\left(r^{2}+q^{2}\right) \frac{\partial y}{\partial q_{r}} \\
\ddot{y}_{0} \frac{\partial y_{0}}{\partial q_{r}}+\ddot{y}_{0} x_{i} \frac{\partial \psi}{\partial q_{r}}-\ddot{y}_{0} z_{i} \frac{\partial \phi}{\partial q_{r}}+x_{i} p q \frac{\partial y_{0}}{\partial q_{r}}+x_{i}{ }^{2} p q \frac{\partial \psi}{\partial q_{r}}-x_{i} y_{i} p q \frac{\partial \phi}{\partial q_{r}} \\
\ddot{z}_{0} \frac{\partial z_{0}}{\partial q_{r}}-\ddot{z}_{0} x_{i} \frac{\partial \theta}{\partial q_{r}}+\ddot{z}_{0} y_{i} \frac{\partial \phi}{\partial q_{r}}+x_{i} p r \frac{\partial z_{0}}{\partial q_{r}}-x_{i}{ }^{2} p r \frac{\partial 0}{\partial q_{r}}+x_{i} y_{i} p r \frac{\partial \phi}{\partial q_{r}}
\end{array}\right]
\end{array}\right.
\end{aligned}
$$

$$
\begin{align*}
& \dot{i}+y_{i} q_{r} \frac{\partial z_{0}}{\partial q_{r}}-x_{i} y_{i} q_{r} \frac{\partial \theta}{\partial q_{r}}+y_{i}{ }^{2} q_{r} \frac{\partial \phi}{\partial q_{r}}-z_{i}\left(p^{2}+q^{2}\right) \frac{\partial z_{0}}{\partial q_{r}} \\
& \sum_{i}^{-}-z_{i} y_{i} p r \frac{\partial \psi}{\partial q_{r}}+z_{i}{ }^{2} p r \frac{\partial \theta}{\partial q_{r}}+z_{i} \dot{q} \frac{\partial x_{0}}{\partial c_{r}}-y_{i} z_{i} \dot{q} \frac{\partial \psi}{\partial q_{r}}+z_{i}{ }^{2} \dot{q} \frac{\partial \theta}{\partial q_{r}} \\
& +\dot{i}^{+} x_{i} z_{i} q_{r} \frac{\partial \psi}{\partial q_{r}}-z_{i}{ }^{2} q_{r} \frac{\partial \phi}{\partial q_{r}}-z_{i} \dot{p} \frac{\partial y_{0}}{\partial q_{r}}-x_{i} z_{i} \dot{p} \frac{\partial \psi}{\partial q_{r}}-z_{i}{ }^{2} \dot{p} \frac{\partial \phi}{\partial q_{r}} \\
& x_{i} z_{i}\left(p^{2}+q^{2}\right) \frac{\partial \theta}{\partial q_{r}}-y_{i} z_{i}\left(p^{2}+q^{2}\right) \frac{\partial \phi}{\partial q_{r}}+y_{i} \dot{p} \frac{\partial z_{0}}{\partial q_{r}}-x_{i} y_{i} \dot{p} \frac{\partial \theta}{\partial q_{r}} \\
& \dot{e}_{i}^{r}-y_{i} \dot{r} \frac{\partial x_{0}}{\partial q_{r}}+y_{i}{ }^{2} \dot{r} \frac{\partial \psi}{\partial q_{r}}-z_{i} y_{i} \dot{r} \frac{\partial \theta}{\partial q_{r}} \\
& +x_{i} \dot{r} \frac{\partial y_{0}}{\partial q_{r}}+x_{i}{ }^{2} \dot{r} \frac{\partial \psi}{\partial q_{r}}+x_{i} z_{i} \dot{r} \frac{\partial \phi}{\partial q_{r}}  \tag{226}\\
& \left.\dot{\vdots}+y_{i}^{2} \dot{p} \frac{\partial \phi}{\partial q_{r}}-x_{i} \dot{q} \frac{\partial z_{0}}{\partial q_{r}}+x_{i}{ }^{2} \dot{q} \frac{\partial \theta}{\partial q_{r}}-x_{i} y_{i} \dot{q} \frac{\partial \phi}{\partial q_{m}}\right] \text {, }
\end{align*}
$$

Expanding and identifying mass moment and moment of inertia terms:
$\left[\sum_{i=1}^{N} m_{i}\left\{\begin{array}{l}\ddot{x}_{i} \\ \ddot{y}_{i} \\ \ddot{z}_{i}\end{array}\right\}_{E S T} \cdot \frac{\partial}{\partial_{r}}\left\{\begin{array}{c}x_{i} \\ y_{i} \\ z_{i}\end{array}\right\}\right]$,

$$
\left[\begin{array}{c}
{\left[+I_{Y} \dot{r} \frac{\partial \psi}{\partial q_{r}}-I_{Y Z} \dot{r} \frac{\partial \theta}{\partial q_{r}}\right.}  \tag{227}\\
+I_{X} \dot{r} \frac{\partial \psi}{\partial q_{r}}-I_{X Z} \dot{r} \frac{\partial \phi}{\partial q_{r}} \\
\vdots \\
-I_{X Y} \dot{q} \frac{\partial \phi}{\partial q_{r}}
\end{array}\right]
$$

and finally collecting and grouping teras yields the final and conplete definition of the terms of Eipuation 211 for the estimated elemental accelerations.

$$
\left[\sum_{i=1}^{n} m_{i}\left\{\begin{array}{l}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{array}\right\}_{E S T} \cdot \frac{\partial}{\partial q_{T}}\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right)\right]_{J}=
$$

$$
\left[M\left\{\begin{array}{l}
\ddot{x} \\
\dot{y} \\
\ddot{z}
\end{array}\right\}+M \bar{R}\left\{\begin{array}{l}
-\left(r^{2}+q^{2}\right) \\
p q+\dot{r} \\
p r-\dot{q}
\end{array}\right\}+M \bar{V}\left\{\begin{array}{c}
p q-\dot{r} \\
-\left(r^{2}+p^{2}\right) \\
q r+\dot{p}
\end{array}\right\}+M \bar{z}\left(\begin{array}{c}
p r+\dot{q} \\
q r-\dot{p} \\
-\left(p^{2}+q^{2}\right)
\end{array}\right\}\right] \frac{\partial}{\partial q_{r}}\left\{\begin{array}{l}
x_{0} \\
y_{0} \\
z_{0}
\end{array}\right\}
$$

$$
+M\left[\frac{\partial \phi}{\partial q^{\prime}}\left\{\begin{array}{c}
0 \\
-\overline{z \ddot{y}} \\
\bar{y} \ddot{z}
\end{array}\right\}+\frac{\partial \theta}{\partial q_{r}}\left\{\begin{array}{c}
\bar{z} \ddot{x} \\
0 \\
-\bar{x} \ddot{z}
\end{array}\right\}+\frac{\partial \phi}{\partial q_{r}}\left\{\begin{array}{c}
-\bar{y} \ddot{x} \\
\bar{x} \dot{y} \\
0
\end{array}\right]\right]
$$

$$
+\left[\begin{array}{ccc}
I_{X}(0) & +I_{Y}\left(-\frac{\partial \phi}{\partial q_{r}}(p q-\dot{r})\right) & +I_{Z}\left(\frac{\partial \theta}{\partial q_{r}}(p r+\dot{q})\right) \\
I_{X}\left(\frac{\partial \phi}{\partial q_{r}}(p q+\dot{r})\right) & +Z_{Y}(0) & -I_{Z}\left(\frac{\partial \phi}{\partial q_{r}}(q r-\dot{p})\right) \\
-I_{X}\left(\frac{\partial \theta}{\partial q_{r}}(p r-\dot{q})\right) & +I_{Y}\left(\frac{\partial \psi_{q}}{\partial q_{Y}}(q r+\dot{p})\right) & +I_{Z}(0)
\end{array}\right]
$$

$$
\left[\begin{array}{l}
I_{X Y}\left(r^{2}+q^{2}\right) \frac{\partial \phi}{\partial q_{r}}-I_{X Z}\left(r^{2}+q^{2}\right) \frac{\partial \theta}{\partial q_{r}}+I_{Y Z}\left((p q-\dot{r}) \frac{\partial \theta}{\partial q_{r}}-(p r+\dot{q}) \frac{\partial \psi}{\partial q_{r}}\right) \\
-I_{X Y}\left(r^{2}+p^{2}\right) \frac{\partial \psi}{\partial q_{r}}+I_{X Z}\left(-(p q+\dot{r}) \frac{\partial \phi}{\partial q_{r}}+(q r-\dot{p}) \frac{\partial \psi}{\partial q_{r}}\right)+I_{Y Z}\left(r^{2}+p^{2}\right) \frac{\partial \phi}{\partial q_{r}} \\
I_{X Y}\left((p r-\dot{q}) \frac{\partial \phi}{\partial q_{r}}-(q r+\dot{p}) \frac{\partial \theta}{\partial q_{r}}\right)+I_{X Z}\left(p^{2}+q^{2}\right) \frac{\partial \theta}{\partial q_{r}}-I_{Y Z}\left(p^{2}+q^{2}\right) \frac{\partial \phi}{\partial q_{r}}
\end{array}\right]_{J}
$$

where in this case the sumation $\sum_{i=1}^{n}$ represents sumation over the jth rigia bosy eiement. With this in mind, suostituting Exuations 228 and $2-3$ back into Equation 201 yields the complete form of the Lagrange energy equation in constrained coordinates vith distributed elemental masses and forces from which the REXOR II equations of motion are all developed. Also including these terms as well as the moment terms of Equation 213 in Equation 211 yields the final form of the equation as used in REXOR II. This form will be presented following the development of the generalized mass damping and stiffness matrices.

From Equation 209 it is easily seen, by examining the coefficients of the corrective accelerations that generaiized mass matriz elements,
$M_{r k}$, can be written as

$$
\begin{equation*}
M_{r k}=\sum_{i=1}^{N} m_{i}\left(\frac{\partial x_{i}}{\partial q_{r}} \frac{\partial x_{i}}{\partial q_{k}}+\frac{\partial y_{i}}{\partial q_{r}} \frac{\partial y_{i}}{\partial q_{k}}+\frac{\partial z_{i}}{\partial q_{r}} \frac{\partial z_{i}}{\partial q_{k}}\right) \tag{229}
\end{equation*}
$$

This equation is for point masses. Actually, as discussed earlier, the REXOR II equations model a set of distributed masses characterized by an overall mass, center of gravity, and moment of inertia values. As shown in the previous section, extension to the distributed mass form is made by describing the particle absolute cocrdinates in terms of the position of a relative cocrdinate set in inertial space and the particle position in terms of this relative set as developed in Section 4.4. For a rigid body the associated relative set and the particle associated with the body maintain a fixed relationship. The suming over the particles of the system then becomes a sum over produc: $\rho$ masses and lengths yielding mass moment and moment of inertia terrs.

The mass elements car be d.ieloped by substituting the partial derivatives deveioped in the preceding discussion. These partials describe both the motion of tie mass element reference and also the distributed masses within the rigid body elemental masses.

Substituting these partials, Equations 223, 224 and 225 in the generalized mass expression, Equation 229 yields:

$$
\begin{align*}
M_{r k}= & \sum_{i=1}^{N} m_{i}\left(\frac{\partial x_{0}}{\partial q_{r}} \frac{\partial x_{0}}{\partial q_{k}}-y_{i} \frac{\partial x_{0}}{\partial q_{r}} \frac{\partial x_{0}}{\partial q_{k}}+z_{i} \frac{\partial x_{0}}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}-y_{i} \frac{\partial \psi}{\partial q_{r}} \frac{\partial x_{0}}{\partial q_{k}}\right. \\
& +y_{i}^{2} \frac{\partial \psi}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}-y_{i} z_{i} \frac{\partial \psi}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}+z_{i} \frac{\partial \theta}{\partial q_{r}} \frac{\partial x_{0}}{\partial q_{k}}+y_{i} z_{i} \frac{\partial \theta}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}} \\
& +z_{i}^{2} \frac{\partial \theta}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}+\frac{\partial y_{0}}{\partial q_{r}} \frac{\partial y_{0}}{\partial q_{k}}+x_{i} \frac{\partial y_{0}}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}-z_{i} \frac{\partial y_{0}}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}} \\
& +x_{i} \frac{\partial \psi}{\partial q_{r}} \frac{\partial y_{0}}{\partial q_{k}}+x_{i}^{2} \frac{\partial \psi}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}-x_{i} z_{i} \frac{\partial \psi}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}-z_{i} \frac{\partial \phi}{\partial q_{r}} \frac{\partial y_{0}}{\partial q_{k}} \\
& -z_{i} x_{i} \frac{\partial \phi}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}+z_{i}^{2} \frac{\partial \phi}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}+\frac{\partial z_{0}}{\partial q_{r}} \frac{\partial z_{0}}{\partial q_{k}}-x_{i} \frac{\partial z_{0}}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}} \\
& +y_{i} \frac{\partial z_{0}}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}-x_{i} \frac{\partial \theta}{\partial q_{r}} \frac{\partial z_{0}}{\partial q_{k}}+x_{i}^{2} \frac{\partial \theta}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}-x_{i} y_{i} \frac{\partial \theta}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}} \\
& \left.+y_{i} \frac{\partial \phi}{\partial q_{r}} \frac{\partial z_{0}}{\partial q_{k}}-x_{i} y_{i} \frac{\partial \phi}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}+y_{i}^{2} \frac{\partial \phi}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}\right) \tag{230}
\end{align*}
$$

and using moment of inertia and mass moment definitions:

$$
\begin{align*}
M_{r k}= & M\left(\frac{\partial x}{\partial q_{r}} \frac{\partial x}{\partial q_{k}}+\frac{\partial y}{\partial q_{r}} \frac{\partial y}{\partial q_{k}}+\frac{\partial z}{\partial q_{r}} \frac{\partial z}{\partial q_{k}}\right)+I_{Z Z}\left(\frac{\partial \psi}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}\right) \\
& +I_{X X}\left(\frac{\partial \phi}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}\right)+I_{Y Y}\left(\frac{\partial \theta}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}\right)+I_{X Z}\left(-\frac{\partial \psi}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}-\frac{\partial \phi}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}\right) \\
& +I_{X Y}\left(-\frac{\partial \theta}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}-\frac{\partial \phi}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}\right)+I_{Y Z}\left(-\frac{\partial \psi}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}-\frac{\partial \theta}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}\right) \\
& +M \bar{x}\left(\frac{\partial y}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}+\frac{\partial \psi}{\partial q_{r}} \frac{\partial y}{\partial q_{k}}-\frac{\partial \theta}{\partial q_{r}} \frac{\partial z}{\partial q_{k}}-\frac{\partial z}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}\right) \\
& +M \bar{M}\left(-\frac{\partial x}{\partial q_{r}} \frac{\partial \psi}{\partial q_{k}}-\frac{\partial \psi}{\partial q_{r}} \frac{\partial x}{\partial q_{k}}+\frac{\partial \phi}{\partial q_{r}} \frac{\partial z}{\partial q_{k}}+\frac{\partial z}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}\right) \\
& +M \bar{z}\left(\frac{\partial x}{\partial q_{r}} \frac{\partial \theta}{\partial q_{k}}+\frac{\partial \theta}{\partial q_{r}} \frac{\partial x}{\partial q_{k}}-\frac{\partial \phi}{\partial q_{r}} \frac{\partial y}{\partial q_{k}}-\frac{\partial y}{\partial q_{r}} \frac{\partial \phi}{\partial q_{k}}\right) \tag{231}
\end{align*}
$$

and is identified as a generalized mass. For orthogonal systems $M_{r k}$ is zero except for $r=k$. The development of REXOR II is mostly nonorthogonal coordinates, therefore, the generalized mass matrix has many off-diagonal terms.

Similarly, terms can be developed for the strain (potential) energy and damping functions.

$$
\begin{align*}
U= & \frac{1}{2} \sum_{i=1}^{N}\left[\left(k_{x_{i}} x_{i}^{2}+k_{y_{i}} y_{i}^{2}+k_{z_{i}} z_{i}^{2}\right)\right. \\
& \left.+\left(k_{\phi_{i}} \phi_{i}{ }^{2}+k_{\theta_{i}}{ }_{i}{ }^{2}+k_{\psi_{i}} \psi_{i}^{2}\right)\right] \tag{232}
\end{align*}
$$

$$
\begin{align*}
\frac{\partial U}{\partial q_{r}}= & \sum_{i=1}^{N}\left(k_{x_{i}} x_{i} \frac{\partial x_{i}}{\partial q_{r}}+k_{y_{i}} y_{i} \frac{\partial y_{i}}{\partial q_{r}}+k_{z_{i}} z_{i} \frac{\partial z_{i}}{\partial q_{r}}\right. \\
& \left.+k_{\phi_{i}} \phi_{i} \frac{\partial \phi_{i}}{\partial q_{r}}+k_{\theta_{i}} \theta_{i} \frac{\partial \theta_{i}}{\partial q_{r}}+k_{\psi_{i}} \psi_{i} \frac{\partial \psi_{i}}{\partial q_{r}}\right) \\
= & \sum_{i=1}^{N}\left[k_{x_{i}} \frac{\partial x_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial x_{i}}{\partial q_{k}} q_{k}+k_{y_{i}} \frac{\partial y_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial y_{i}}{\partial q_{k}} q_{k}\right. \\
& +k_{z_{i}} \frac{\partial z_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial z_{i}}{\partial q_{k}} q_{k}+k_{\phi_{i}} \frac{\partial \phi_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial \phi_{i}}{\partial q_{k}} q_{k} \\
& \left.+k_{\sigma_{i}} \frac{\partial \theta_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial \theta_{i}}{\partial q_{k}} q_{k}+k_{\psi_{i}} \frac{\partial \Psi_{i}}{\partial q_{r}} \sum_{k=1}^{n} \frac{\partial \psi_{i}}{\partial q_{k}} q_{k}\right] \tag{233}
\end{align*}
$$

Derine

$$
\begin{align*}
& \sum_{i=1}^{N}\left[k_{x_{i}} \frac{\partial x_{i}}{\partial q_{r}} \frac{\partial x_{i}}{\partial q_{k}}+k_{y_{i}} \frac{\partial y_{i}}{\partial q_{r}} \frac{\partial y_{i}}{\partial q_{k}}+k_{z_{i}} \frac{\partial z_{i}}{\partial q_{r}} \frac{\partial z_{i}}{\partial q_{k}}\right. \\
&  \tag{234}\\
& \left.\quad+k_{\phi_{i}} \frac{\partial \phi_{i}}{\partial q_{r}} \frac{\partial \phi_{i}}{\partial q_{k}}+k_{\theta_{i}} \frac{\partial \theta_{i}}{\partial q_{r}} \frac{\partial \theta_{i}}{\partial q_{k}}+k_{\psi_{i}} \frac{\partial \psi_{i}}{\partial q_{r}} \frac{\partial \psi_{i}}{\partial q_{k}}\right]=K_{r k}
\end{align*}
$$

Similarly for damping:

$$
\begin{align*}
& \sum_{i=1}^{N}\left[c_{x_{i}} \frac{\partial x_{i}}{\partial q_{r}} \frac{\partial x_{i}}{\partial q_{k}}+c_{y_{i}} \frac{\partial y_{i}}{\partial q_{r}} \frac{\partial y_{i}}{\partial q_{k}}+c_{z_{i}} \frac{\partial z_{i}}{\partial q_{r}} \frac{\partial z_{i}}{\partial q_{k}}\right. \\
& \left.\quad+c_{\phi_{i}} \frac{\partial \phi_{i}}{\partial q_{r}} \frac{\partial \phi_{i}}{\partial q_{k}}+c_{\theta_{i}} \frac{\partial \theta_{i}}{\partial q_{r}} \frac{\partial \theta_{i}}{\partial q_{k}}+c_{\psi_{i}} \frac{\partial \psi_{i}}{\partial q_{r}} \frac{\partial \psi_{i}}{\partial q_{k}}\right]=c_{r k} \tag{235}
\end{align*}
$$

The stiffness and damping matrix terms in REXOR II are defined vith reference to relative coordinates; which parallels the physical configuration. The coordinates used with these terms then should be on a relative basis. This statement at first appears to be contradictory to the premise of the equation development. However, if these matrix terms were defined on an absolute basis the terms other than those associated with a relative motion would be identically zero. The integration of the accelerations produces changes in velocity and position. These changes vith the proper starting reference are the relative coordinates and velocities.

Equation 211 is now repeated here in a slightly expanded form to include the effect of appiled elemental monents, Equation 213, and distributed elemental masses, Equation 228.

(236)

Even though $Q_{r}$ was defined as the generalized forces of the system, for the purpose of further development and of the application of the above equation in the KEXCR II aralysis, each line in the large brackets on the right side of Equation 236 will hereafter be referred to as a generalized force or a generalized delta force and will be referred to by the symbol, FR, in the following development.

In REXOR II use is made of matrix notation to produce compact partial derivative, generalized mass and generalized force expressions. The partial derivative set

$$
\frac{\partial x}{\partial q_{r}}, \frac{\partial y}{\partial q_{r}}, \frac{\partial z}{\partial q_{r}}, \frac{\partial \phi}{\partial q_{r}}, \frac{\partial \theta}{\partial q_{r}}, \frac{\partial \psi}{\partial q_{r}}
$$

is replaced by

$$
\left\{\frac{\partial \tau}{\partial q_{r}}\right\}
$$

Usually the generalized coordinates $q$ are grouped as three linear plus three rotational motions. The full partial derivative for system coordinate $A$ and generalized coordinate set $B$ is:

$$
\left[\frac{\partial r_{O A}}{\partial \gamma_{B}}\right]
$$

The generalized mass expression (Equation 231) can be rewritten as:

$$
\mathbf{M}_{\mathbf{r k}}=\left\{\frac{\partial T}{\partial q_{r}}\right\}^{T}\left[\begin{array}{cccccc}
M & 0 & 0 & 0 & M \bar{Z} & -M \bar{Y}  \tag{237}\\
0 & M & 0 & -M \bar{Z} & 0 & M \bar{X} \\
0 & 0 & M & \overline{M Y} & -M \bar{X} & 0 \\
0 & -M \bar{Z} & M \bar{Y} & I_{X Y} & -I_{X Y} & -I_{X Z} \\
M \bar{Z} & 0 & -M \bar{X} & -I_{X Y} & I_{Y Y} & -I_{Y Z} \\
-M \bar{Y} & M \bar{X} & 0 & -I_{X Z} & -I_{Y Z} & I_{Z Z}
\end{array}\right]\left\{\frac{\partial T}{\partial q_{k}}\right\}
$$

The generalized delta force, $\Delta Q_{B}$, can be expressed in matrix notation for contribution of coordinate set $\mathcal{A}^{\prime}$ as:

$$
\begin{align*}
& -\left\{\begin{array}{l}
{\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]_{A}\left[\begin{array}{ccc}
0 & M \bar{Z} & -M \bar{Y} \\
-M \bar{Z} & 0 & M \bar{X} \\
M \bar{Y} & -M \bar{X} & 0
\end{array}\right]_{A}\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right]_{A}} \\
{\left[\begin{array}{ccc}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{array}\right]_{A}\left[\begin{array}{cc}
I_{X X} & -\overline{1} X Y \\
-I_{X Z} \\
-I_{X Y} & I_{Y Y} \\
-I_{X Z} \\
-I_{X Z} & -I_{Y Z} \\
I_{Z Z}
\end{array}\right]_{A}\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}}
\end{array}\right\} \\
& -\left[C_{A}\right]\left\{\begin{array}{c}
x \\
\dot{Y} \\
\dot{z} \\
p \\
q \\
Y
\end{array}\right\}-\left[K_{A}\right]\left\{\begin{array}{l}
x \\
y \\
z \\
q \\
\theta \\
\Psi
\end{array}\right\} \tag{238}
\end{align*}
$$

### 5.4 Overview of Rotor-Blade Model

Many elements of the rotorcraft can be directly modeled following the methods developed in Sections 5 through 5.2 and systematized in Section 5.5. However, there are enough special considerations and concepts involved in modeling the individual blades and combined rotor to justify a separate section to address these topics.
5.4.1 Concept of modes. - The basic textbook principles governing solutions for eigenvalues (natural frequencies) and eigenvectors (mode shapes) for systems of several degrees of freedom can be applied to those of many degrees of freedom. For each independent degree of freedom there is an additional natural frequency and mode shape.

Free vibrations of continuous systems such as beams, or for example the helicopter fuselage, or rotor blades, are generally analyzed mathematically by reducing the system to a system of discrete masses and elastic constraints.
5.4.2 Blade bending - modal variable. - The blade is a twisted rotating beam and its analysis requires considering the coupled flaprise-chordwisetorsional response of the blade. For the REXOR II analysis, coupled flapwisechordwise mode shapes are used, upon which is superimposed one of a number of torsional response representations of varying complexity (Sections 4.3.4, $5.4 .7,5.6 .5$, and 5.6 .6 ).

If one applies generalized coordinates, each blade mode in the analysis may be treated as a single degree of freedom. The generalized coordinates are salled normal coordinates for the special case when the modes are orthogonal, in which case the generalized mass matrix reduces to a diagonal matrix, as $\dot{c} c e s$ the generalized stiffness matrix.

The REXOR II analysis uses blade modes calculated for the blade at a fixed rpm, fixed coll active, $\because d$ in an unswept, unconed orientation. Since the program allows for variation of all of these parameters, which is accounted for in the overall REXOR II analysis, the predetermined modes become nonorthogonal as used in the program. Thus, blade motion is effectively described by a set of modal variables, each representing a characteristic frequency, and a set of modal coefficients that describe the relative amplitude of oscillation for each blade segment and each frequency.

Since the modes are nonorthogonal, we will find in REXOR II, as would be expected in such a case, off-diagonal coupling terms in both the generalized mass and stiffness matrices. It can readily be shown in cases
where generalized or normal coordinates are applied, that relatively few modes need be taken to define accurately the time-history of blade deflection. This assumes that the primary frequencies of excitation fall within the range of modes considered.
5.4.3 Blade mode generation. - The blade modes can be determined by any appropriate classical method of analysis for coupled flapwise-inplane bending beams. The only requirements is a cantilever (hinge or hingeless) boundary condition for the modes and that the terms included in the homogenous part of equations 28 and 29 of Reference 5 be accounted for. These equations are repeated here for convenience. Flapwise:

$$
\begin{align*}
& {\left[\left(E I_{1} \cos ^{2} \beta+E I_{2} \sin ^{2} \beta\right)_{v^{\prime \prime}}+\left(E I_{2}-E I_{1}\right) \sin \beta \cos \beta v^{\prime \prime}\right] "} \\
& -!\left(\mathrm{IV}^{\prime}\right)^{\prime}-\Omega^{2} m v+m \ddot{\mathrm{v}}=0 \tag{239}
\end{align*}
$$

and inplanewise:

$$
\begin{align*}
& {\left[\left(E I_{2}-E I_{1}\right) \sin \beta \cos \beta w^{\prime \prime}+\left(E I_{1} \sin ^{2} \beta+E I_{2} \cos ^{2} \beta\right) v^{\prime \prime}\right] "} \\
& -\left(T v^{\prime}\right)^{\prime}-\Omega_{m v}^{2}+m \ddot{v}=0 \tag{240}
\end{align*}
$$

5.4.4 Modal coefficients. - Several additional points need to be made regarding modes in order that the equation development be properly understood. First, the same modal coefficients apply to the first and second time derivatives of the function, since

$$
\begin{equation*}
\frac{\partial Z_{S}}{\partial A_{i}}=f(x) \tag{241}
\end{equation*}
$$

Then

$$
\begin{align*}
& \dot{Z}_{S}=\frac{\partial Z_{S}}{\partial A_{1}} \dot{A}_{1}+\frac{\partial Z_{S}}{\partial A_{2}} \dot{A}_{2}+\cdots+\frac{\partial Z_{S}}{\partial A_{n}} \dot{A}_{n}  \tag{241}\\
& \ddot{Z}_{S}=\frac{\partial Z_{S}}{\partial A_{1}} \ddot{A}_{1}+\frac{\partial Z_{S}}{\partial A_{2}} \ddot{A}_{2}+\cdots+\frac{\partial Z_{S}}{\partial A_{n}} \ddot{A}_{n} \tag{243}
\end{align*}
$$

Second, the motion is not necessarily confined to one direction. A given modal frequency may excite or couple with motions in other directions. For example:

$$
\begin{align*}
& Y_{S}=\frac{\partial Y_{S}}{\partial A_{1}} A_{1}+\frac{\partial Y_{S}}{\partial A_{2}} A_{2}+\cdots+\frac{\partial Y_{S}}{\partial A_{n}} A_{n}  \tag{244}\\
& \dot{Y}_{S}=\frac{\partial Y_{S}}{\partial A_{1}} \dot{A}_{1}+\frac{\partial Y_{S}}{\partial A_{2}} \dot{A}_{2}+\cdots+\frac{\partial Y_{S}}{\partial A_{n}} \dot{A}_{n}  \tag{245}\\
& \ddot{Y}_{S}=\frac{\partial Y_{S}}{\partial A_{1}} \ddot{A}_{1}+\frac{\partial Y_{S}}{\partial A_{2}} \ddot{A}_{2}+\cdots+\frac{\partial Y_{S}}{\partial A_{n}} \ddot{A}_{n} \tag{246}
\end{align*}
$$

5.4.5 Independent blades. - In REXOR II the blade motions are computed and tracked individually. One set of equations operates on a blade in BLE coordinates as explained: Section 5.4.11. The result for a time step is stored in BLn coordinates for that blade. The operating set in BLE coordinates then performs the computation: for the next blade in turn.
5.4.6 Blade element aerodynamic forces - overview. - The functions $F_{X_{i}}$, $F_{Y_{i}}, F_{Z_{i}}$, and moment terms from Section 5.2 are primarily aerodynamic loads for the blade equations. These loads are derived from blade inertial velocity (equivalent to air velocity) and table lookup aerodynamic coefficients as given in Section 6.

### 5.4.7 Blade torsional response.

5.4.7.1 Pitch horn bending, - Several alternate approaches to modeling blade feathering dynamics exist in REXOR II. One approach is to assume the blade is torsionally rigid, and that the ilexibility is in the pitch horn.
5.4.7.2 Quasi-static blade torsion. - The blade pitch horn bending description is improved by the addition of a blade twist dependent on the moment loading. This quasi-static torsion is computed by integrating the blade pitching moment times the torsional flexibility from tip into the root. (Developed in Sections 4.3 .4 and 5.6.5.)
5.4.7.3 Dynamic blade torsion. - A third approach to blade torsional response in REXOR is an uncoupied torsional mode which operates as additional blade twist. This material is developed in Section 5.6.
5.4.8 Radial integration. - For each element of a rotor blade the equations of motion are formed per Section 3.2.9. As briefly touched on in Section 5.4 .6 these data are formed in bLE axis. These elements are summed to total equations for each blade in BLn coordinate at the blade root. This is explained in Sections 5.6.3 and 5.6.4. These blade root summations are also used in the fuselage axis (Section 5.8).

### 5.5 Equation System Development

5.5.1 Reference to base operation matrix. - The equation of motion, as developed in Sections 5.2 and 5.3 and as presented in most general form by Equation 236, may be given in abbreviated form as

$$
\begin{equation*}
\left\{\ddot{q}_{r}\right\}_{C O R R}=\left[M_{r k}\right]^{-1}\left\{F_{r}\right\}_{\mathrm{EST}} \tag{247}
\end{equation*}
$$

The $M_{r k}$ 's are the generalized mass matrix elements, the $F_{r}$ 's are the generalized forces, and the $q_{r}$ 's the generalized coordinates or degrees of freedom. As explained previously, the $F_{r}$ 's are the complete set of
external forces and internal reactions computed with estimated values of the accelerations, $\ddot{\ddot{q}}_{r_{E S T}}$ 's, at the next time point. The $\ddot{\ddot{q}}_{r_{C O R R}}$ 's are then corrections to the estimated values.

The generalized mass, $M_{r k}$, is developed in Section 5.3. The gen-ralized force may be expanded as (using the point mass form):

$$
\begin{align*}
F_{r}= & -\sum_{i=1}^{N}\left[m_{i}\left(\ddot{x}_{i} \frac{\partial X_{i}}{\partial q_{r}}+\ddot{Y}_{i} \frac{\partial Y_{i}}{\partial q_{r}}+\ddot{z}_{i} \frac{\partial Z_{i}}{\partial q_{r}}\right)\right]-\frac{\partial B}{\partial \dot{q}_{r}}-\frac{\partial U}{\partial q_{r}} \\
& +F_{F R_{r}}+F_{A_{r}}+F_{c_{r}} \tag{248}
\end{align*}
$$

The inertia, damping, and elastic terms are developed further in Section 5.3 (see Equation 236). The friction force $F_{F R_{r}}$, the aerodynamic external forces $F_{A_{r}}$, and the pilot control forces $F_{c_{r}}$ are described as needed. Note that the potential energy and dissipation terms have been directly included in the force expression. Where the stiffness and damping matrices are simple diagonals, this is done. In the case of the blade equations the distinct stiffness and damping matrix form (Section 5.3) is computed before combining all the applied forces, internal reactions, stiffness and damping terms into an overall force.
5.5.2 Organization by degrees of freedom. - In developing the equations of motion there are three types of ingredients needed:

- Generalized masses
- Generalized forces
- Partial derivatives (used in both of the above items)

The equation development can then proceed with these ingredients along one of two lines of organization.

- For every major rotorcraft piece (fuselage, rotor, etc.), compute all the ingredients and sort according to degree oi freedom for equation use.
- For every degree-of-freedom group, sort through the rotorcraft pieces for applicable ingredients. Sorting is minimal because of close association of degrees of freedom and component parts.

The latter development is used here. The degrees of freedom modeled are given in Figure 11.

The fcllowing subsections will describe the appropriste partial differentiations, the generalized masses, and the generalized forces in detail. Each generalized mass couples the inertia of one generalized coordinate rith another or iiself. The algebraic equations for each generalized mass viil be given only once. If the reader cannot find a particular mass elesent under one subsection, he should look into the other subsection reiating to the coupled generalized coordinate.
5.5.3 Partial ierivatives. - Fine generalized masses and forces use partial derivatives which cescr se the variational motion of each physical aass element in rectangilar coordinates relative to the motion of each generalized coordinate. The partisi derivatives required are determined from the generalized mass and force expressions for distributed masses of Section 5.2. The partial lerivatives are easily sonstructed from the coordinate trunsformatic.s which bave been developed.

In develcping the zotions of a physical mass element relative to a generalize, coordinate, a number of transforms may be used. These can be categorized as eitner linear or Euler axes transforas which either displace without rotation or rotate dithout displacenent. The overall partial will be the product of partials associated vith each of these transforms. The typieal form of these partials vill now be illustrated.

To obtain tice partials, the equations reiating the velocities are obtained first. Reriewing Sections 4.4.1 ana 4.4.2, the velocity relations of interest are restated. For linear transforms:

$$
\left\{\begin{array}{c}
\dot{x}  \tag{237}\\
\dot{y} \\
\dot{z}
\end{array}\right\}_{b}=\left\{\begin{array}{l}
\dot{x}_{c} \\
\dot{y}_{0} \\
\dot{z}_{0}
\end{array}\right\}+\left\{\begin{array}{c}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right\}+\left[\begin{array}{ccc}
0 & z_{a} & -y_{a} \\
-z_{a} & 0 & x_{a} \\
y_{a} & -x_{a} & 0
\end{array}\right]\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}
$$

$$
\left\{\begin{array}{l}
p  \tag{238}\\
q \\
r
\end{array}\right\}=\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}
$$

and for Euner transforms

$$
\left.\begin{array}{c}
\left\{\begin{array}{l}
\dot{x} \\
\dot{Y} \\
\dot{z}
\end{array}\right\}_{b}=\left[T_{a-b}\right]\left\{\begin{array}{l}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right\}_{a} \\
\left\{\begin{array}{l}
p \\
\underline{a} \\
r
\end{array}\right\}_{b}=\left\{\begin{array}{l}
0 \\
0 \\
\dot{\psi}_{b}
\end{array}\right\}_{a}+\left[T_{\psi}\right]\left\{\begin{array}{l}
0 \\
\dot{\theta}_{b} \\
0
\end{array}\right\}_{a}+\left[T_{\theta}\right]\left\{\left\{\begin{array}{l}
\dot{\phi}_{b} \\
0 \\
0
\end{array}\right\}_{a}+\left[T_{\phi}\right]\left[\begin{array}{l}
\mathrm{p} \\
\mathrm{q} \\
\mathrm{r}
\end{array}\right\}\right\}
\end{array}\right\}
$$

The partials of interest are conveniently organized into 3 by 3 matrices. They are for the linear transform:

$$
\begin{align*}
& {\left[\left.\left.\frac{\partial}{\partial x_{0 a}}\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}_{b}\right|^{\left\lvert\, \frac{\partial}{\partial y_{0 a}}\right.}\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}_{b}\right|_{i} \frac{\partial}{\partial Z_{0 a}}\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right]_{b}\right]\left[=[I]=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]\right.} \tag{241}
\end{align*}
$$

For the Euler angles defined in Section 4.4.2:
(247)
(248)
(249)

$$
\left[\begin{array}{c}
\frac{\partial}{\partial \phi_{b}}\left\{\begin{array}{l}
\phi \\
\theta \\
\phi
\end{array}\right\}_{b}\left\{\begin{array}{c} 
\\
\end{array} \frac{\partial}{\partial \theta_{b}}\left\{\begin{array}{l}
\phi \\
\theta \\
\psi
\end{array}\right\}_{b}\right.
\end{array} \begin{array}{l} 
\\
\frac{\partial}{\partial \psi_{b}}\left\{\begin{array}{l}
\phi \\
\theta \\
\psi
\end{array}\right\}_{b}
\end{array}\right]=\left[\begin{array}{ccc}
\cos \psi \cos \theta & \sin \psi & 0 \\
-\sin \psi \cos \theta & \cos \psi & 0 \\
0 & 0 & 1
\end{array}\right]
$$

For Euler angles defined in reversed order or reversed sign the last matrix vill differ. Note by inspection that rotary to linear derivatives such as $\frac{\partial \theta}{\partial x}$ are $8-1$ zero. The derivatives can be strung together to get motion in a third axis $c$ relative to motion in axis a. Abbreviating the mat=ices:

$$
\begin{gather*}
{\left[\frac{\partial r_{c}}{\partial \zeta_{a}}\right]=\left[\frac{\partial r_{c}}{\partial r_{b}}\right]\left[\frac{\partial r_{b}}{\partial \zeta_{a}}\right]+\left[\frac{\partial r_{c}}{\partial \zeta_{b}}\right]\left[\begin{array}{l}
\partial \zeta_{b} \\
\partial \zeta_{a}
\end{array}\right]}  \tag{251}\\
{\left[\frac{\partial r_{c}}{\partial r_{a}}\right]=\left[\frac{\partial r_{c}}{\partial r_{b}}\right]\left[\frac{\partial r_{b}}{\partial r_{a}}\right]}  \tag{252}\\
{\left[\frac{\partial \zeta_{c}}{\partial r_{a}}\right]=0}  \tag{253}\\
{\left[\begin{array}{l}
\partial \zeta_{c} \\
\partial \zeta_{a}
\end{array}\right]=\left[\frac{\partial \zeta_{c}}{\partial \zeta_{b}}\right]\left[\begin{array}{l}
\partial \zeta_{b} \\
\partial \zeta_{a}
\end{array}\right]} \tag{254}
\end{gather*}
$$

assuming in general

$$
\begin{equation*}
r_{b}=r_{b}\left(r_{a}, \zeta_{a}\right) \tag{255}
\end{equation*}
$$

and

$$
\begin{equation*}
\zeta_{c}=\zeta_{c}\left(r_{b}, \zeta_{b}\right) \tag{256}
\end{equation*}
$$

The abbreviations used are

$$
\begin{equation*}
\mathbf{r}=\{X, Y, Z\} \tag{257}
\end{equation*}
$$

and

$$
\begin{equation*}
\zeta=\{\phi, \theta, \psi\} \tag{258}
\end{equation*}
$$

Transformations may involve linear and rotational operations. Partial derivatives showing the combined operations may be generated using a linear transform from set $a$ to $b$, followed by a rotation from $b$ to $c$. This sequence gives:

$$
\begin{align*}
& {\left[\frac{\partial r_{c}}{\partial \zeta_{a}}\right]=\left[\begin{array}{l}
T_{a-c}
\end{array}\right]\left[\begin{array}{ccc}
0 & Z_{a} & -Y_{a} \\
-z_{a} & 0 & x_{a} \\
Y_{a} & -x_{a} & 0
\end{array}\right]}  \tag{259}\\
& {\left[\frac{\partial r_{c}}{\partial r_{a}}\right]=\left[\begin{array}{l}
T_{a-c}
\end{array}\right]}  \tag{260}\\
& {\left[\frac{\partial \zeta_{c}}{\partial r_{a}}\right]=0}  \tag{2ól}\\
& {\left[\frac{\partial \zeta_{c}}{\partial \zeta_{a}}\right]=\left[\begin{array}{l}
T_{a-c}
\end{array}\right]} \tag{262}
\end{align*}
$$

5.5.4 Generalized masses. - As discussed before, the helicopter is assumed to consist of a finite number of mass elements. They are the

- Iuselage
- tail rotor
- engine rotor
- swashplate
- fixed hub (all parts that do not feather)
- k mass elements on each oî b blades.

The reader should realize the mass matrix is symetric from the definition of Equation (209) and interchange of the order of differentiation.

$$
\begin{equation*}
M_{k r}=M_{r k} \tag{263}
\end{equation*}
$$

Only the elements in the diagonal and the upper right triangle will be given in the following sections.

Bach of these mass elements must be summed for each of the generalized mass matrix elements. Each mass is handled vith the distributed mass Mrk relation
of Section 5.2. Fortunately, only the fuselage requires the full equation. The center-of-gravity terms irop out if the mass motion is determined at the center of gravity. This situation is true for the blade line which passes through the center of gravity of the blade section mass elements. Only the fuselage, the swashplate and the shaft/transmission have reference axis origin off the center of gravity. Another simplification is that cross products of inertia exist only for the ruselage and the shaft/transmission. Each blade mass element is considered to be in the shape of a rod lying along the chord at the blade station in question.

Certain small terms and factors are dropped from the generalized masses. As discussed in Section 5.3, the equations of motion are solved for small incrementa? corrections to the accelerations. With this formulation the messes car tolerate approximations as contrasted to the generalized forces.
5.5.5 Generalized forces. - The equation formulation, Section 5.3, requires that precision be used in compiling the generalized forces per Equation 247, expanded per Section 5.3, Equation 236, to include rigid body distributed mass elenents. This formulation includes for each degree of freedom:

- Sumation over all mass eiements of the mass times inertial acceleration times paitial derivaive. (Section 5.3 expression for distributed masses.)
- External (aerodynamic) loadings times a partial derivative.
- Potential energy and damping terms or assembled stiffness and damping terms with partial derivatives (Section 5.3).

For some degrees of freedom the applicable mass elements and the total integration are directly written as final results which can be verified by inspection. Degrees of Ireedom that properly include summation over the main rotor blades involve some extensive numerical integrations and complicated coordinate transformations.

### 5.6 Blade Bending and Torsion Equations

5.6.1 Blade radial sumation. - The contribution from all the individual blade sections are summed to give the blade generalized masses and forces. These are given for blade root, bending, feathering, and torsion motions. The blade root values are then transformed to the final degree of freedom variables by partial derivatives. The sumation is carried out over all elements of the rotorcraft, including the independent blades. Due to the relative isolation of one blade's modes from another, only the 4 by 4 submass matrices along the diagonal of the $4 b$ by $4 b$ rotor matrix are illed, where $b$ is the number of blades.
5.6.2 Partial derivatives. - The generalized masses and forces utilize partials relating the $X, Y, Z, \phi, \theta$, and innear and rotery motion of eash blade element to the blade bending, blade torsion, bcdy, rotor, and swashplate degrees of freedom. Caly the blade bending, torsion, and feathering partials are derived in this section; the blade partials for other degrees of freedom are to be found in their respective sections.

As developed in Section 4.3.4, the blade torsion may be modeled either as a pitch horn bending or an uncoupled dynamic torsion mode. For the former case the partial $\frac{\partial \phi_{\mathrm{Fn}}}{\partial \beta_{\mathrm{PHn}}}$ is a blade spanwise constant multiplier to summations which souple in the feather angle. In the latter case, $\frac{\partial \phi_{\mathrm{BLE}}}{\partial \beta_{\mathrm{PH}}}$ is a function of span and blade number.

The first partials to be sonsidered are those relating motions at any point $i$ on the blade to the rigid body motion of the blade root. These partials are:

$$
\left[\begin{array}{lll}
\partial\left(r_{B L E}\right)_{B L n} \\
\partial r_{O B L n}
\end{array}\right]=\left[\begin{array}{lll}
\frac{\partial\left(X_{B L E}\right)_{B L n}}{\partial X_{O B L n}} & \frac{\partial\left(X_{B L E}\right)_{B L n}}{\partial Y_{O B L n}} & \frac{\partial\left(X_{B L E}\right)_{B L n}}{\partial Z_{O B L n}} \\
\frac{\partial\left(Y_{B L E}\right)_{B L n}}{\partial X_{O B L n}} & \frac{\partial\left(Y_{B L E}\right)_{B L n}}{3 Y_{O B L n}} & \frac{\partial\left(Y_{B L E}\right)_{B L n}}{\partial Z_{O B L n}} \\
\frac{\partial\left(Z_{E L E}\right)_{B L n}}{\partial X_{O B L n}} & \frac{\partial\left(Z_{B L E}\right)_{B L n}}{\partial Y_{O B L n}} & \frac{\partial\left(Z_{B L E}\right)_{B L n}}{\partial Z_{O B L n}}
\end{array}\right]=[I]
$$

Hote that $X_{B L E}, Y_{B L E}$ and $\bar{Z}_{\text {BLE }}$ are expressed in blade root coordinates; while $\phi_{B L E},{ }^{\theta}{ }_{B L E}$ and $\phi_{B L E}$ are in terms of blade element axes aligned with the blade element principal axes.

Hext consider the blade $Y$ and $Z$ bending response with respect to the blade bending modes. A number of equations cas be used to develop the required expressions. The velocity equations from Section 4.5 .5 are selected for ease of analysis. Usirg cancellation of the dots (Seation 5.2):

$$
\begin{equation*}
\left(\frac{\partial \dot{Y}(i)_{B L E}}{\partial A_{m n}}\right)_{B L n}=\left(\frac{\partial Y(i)_{B L E}}{\partial A_{m n}}\right)_{B L n} \tag{267}
\end{equation*}
$$

$$
\begin{equation*}
\left(\frac{\partial \dot{Z}(i)_{\mathrm{BLE}}}{\partial A_{\mathrm{mn}}}\right)_{\mathrm{BLn}}=\left(\frac{\partial Z(i)_{\mathrm{BLE}}}{\partial A_{\mathrm{mn}}}\right)_{\mathrm{BLn}} \tag{268}
\end{equation*}
$$

gives:

$$
\begin{aligned}
& \left.\left[\frac{\partial Z_{B L E}}{\partial A_{m n}}\right\}_{B L n}+\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]^{T}\left[\mathrm{~T}_{Y^{\prime}}{ }_{F A}\right]^{T}\left[\mathrm{~T}_{\Delta \varphi_{\mathrm{f}}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}}{ }_{F A}\right]\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]\right] \\
& +\frac{\partial Y^{\prime}}{\partial A}{ }_{F A}\left[\left[\mathrm{~T}_{Z}{ }^{\prime}{ }_{F A}\right]^{\mathrm{T}}\left[\dot{\mathrm{~T}}_{Y^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Delta \phi_{F}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}}{ }_{F A}\right]\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]\right. \\
& \left.+\left[\mathrm{T}_{Z}:_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}}{ }_{F A}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\Delta \phi_{\mathrm{f}}}\right]^{\mathrm{T}}\left[\dot{\mathrm{~T}}_{Y^{\prime}}{ }_{F A}\right]\left[\mathrm{T}_{Z^{\prime}}{ }_{F A}\right]\right]
\end{aligned}
$$

$$
\begin{align*}
& \left\{\left[\frac{\partial r_{B L E}}{\partial A_{m n}}\right]\left\{A_{m n}\right\}+\left\{r_{s_{B L E}}\right\}\right\} \\
& +\left[\mathrm{T}_{Z^{\prime}{ }_{F A}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}}{ }_{F A}\right]^{T}\left[\mathrm{~T}_{\Delta \phi_{\mathrm{F}}}\right]_{\mathrm{T}}\left[\mathrm{~T}_{Y^{\prime}{ }_{F A}}\right]\left[\mathrm{T}_{Z^{\prime}{ }_{F A}}\right]\left\{\left\{\frac{\partial r_{\mathrm{BLE}}}{\partial A_{\mathrm{mn}}}\right\}\right. \\
& \left.-\left\{\frac{\partial r_{I B}}{\partial A_{m n}}\right\}\right\}+\left\{\frac{\partial r_{I B}}{\partial A_{m n}}\right\} \tag{269}
\end{align*}
$$

where

$$
\begin{align*}
& \left\{\mathrm{r}_{\mathrm{S}_{\mathrm{BLE}}}\right\}=\left[\mathrm{T}_{\mathrm{B}_{\mathrm{FA}}}\right]_{\mathrm{T}}\left[\mathrm{~T}_{\phi_{\mathrm{REF}}}\right]^{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{B}_{\mathrm{FA}}}\right]\left(\left[\mathrm{T}_{\mathrm{T}_{0}}\right]_{\mathrm{T}}\left[\mathrm{~T}_{\mathrm{Y}}\right]^{\mathrm{T}}\right. \\
& \text { - }\left\{\left[\mathrm{T}_{\mathrm{B}_{0}}\right]^{\mathrm{T}}\left\{\left[\mathrm{~T}_{\Phi_{\mathrm{T}}}\right]_{\mathrm{T}}\left\{\mathrm{r}_{\mathrm{CG}}{ }_{\mathrm{BLE}}\right\}-\left[\left[\mathrm{T}_{\phi_{\mathrm{TW}}}\right] \mathrm{T}-\left[\mathrm{T}_{\Phi_{\mathrm{T}}}\right]_{\mathrm{T}}\right]\left\{\mathrm{r}_{\mathrm{sc}}\right\}\right\}\right. \\
& \left.-\left[T_{B_{0}}\right] T\left\{r_{S W}\right\}\right\}+\left\{\left[T_{B_{0}}\right] T\left\{r_{S W}\right\}+\left\{r_{{ }_{\text {jog }}}\right\}\right. \\
& \left.\left.-\left[T_{B_{0}}\right] T\left\{r_{p}\right\}\right\}\right\}+\left\{\left[T_{B_{0}}\right] T\left\{r_{p}\right\}-\left\{r_{I B}\right\}\right\} \tag{270}
\end{align*}
$$

$\frac{\partial \phi_{F}}{\partial A_{1}}, \frac{\partial \phi_{F}}{\partial A_{2}}$, and $\frac{\partial \phi_{F}}{\partial A_{3}}$ are input data.
The angular derivatives with respect to the blade bending modes are also constructed in the velocity form.

$$
\left\{\begin{array}{c}
\frac{\partial p}{\partial A_{m n}}  \tag{271}\\
\frac{\partial Q}{\partial A_{m n}} \\
\frac{\partial r}{\partial \AA_{m n}}
\end{array}\right\}_{B L E}=\left\{\begin{array}{l}
\frac{\partial \phi_{B L E}}{\partial A_{m n}} \\
\frac{\partial \theta_{B L E}}{\partial A_{m n}} \\
\frac{\partial \psi_{B L E}}{\partial A_{m n}}
\end{array}\right\}_{B L E}
$$

Note that the angular derivatives, being applied to local segments, are presented in BLE axis. Referring to Equation 132:

$$
\begin{aligned}
& \left\{\begin{array}{l}
\frac{\partial \phi_{\mathrm{BLE}}}{\partial A_{\mathrm{mn}}} \\
\frac{\partial \theta_{\mathrm{BLE}}}{\partial A_{\mathrm{mn}}} \\
\frac{\partial \psi_{\mathrm{BLE}}}{}
\end{array}\right\}=\left[\mathrm{T}_{\Phi_{\mathrm{T}}}\right]\left[\mathrm{T}_{\mathrm{B}_{0}}\right]\left[\mathrm{T}_{\gamma}\right]\left[{ }^{\mathrm{T}} \mathrm{~T}_{0}\right]\left[\mathrm{T}_{B_{F A}}\right]\left[{ }^{\mathrm{T}} \phi_{\mathrm{REF}}\right]\left[{ }^{\left.\mathrm{T}_{B_{F A}}\right]}\right.
\end{aligned}
$$

From:

$$
\left\{\begin{array}{l}
Y_{\text {BEND }}^{\prime}  \tag{273}\\
Z^{\prime} \\
\text { BEND }
\end{array}\right\}=\left[\begin{array}{ccc}
Y^{\prime} & Y^{\prime} & Y^{\prime} \\
I_{3} & & \\
Z_{1}^{\prime} & Z^{\prime} & Z^{\prime} \\
Y_{3}
\end{array}\right\}\left\{\begin{array}{l}
A_{1 n} \\
A_{2 n} \\
A_{3 n}
\end{array}\right\}
$$

Gives:

$$
\left\{\begin{array}{l}
\frac{\partial Y^{\prime}}{{ }^{\partial E N D}}  \tag{274}\\
\partial A_{m n} \\
\frac{\partial Z^{\prime}{ }_{B E N D}}{\partial A_{m n}}
\end{array}\right\}=\left\{\begin{array}{l}
Y_{m}^{\prime} \\
Z_{m}^{\prime}
\end{array}\right\}
$$

Also note that in the same context and argument of Section 4.5.5 the feathering axis slopes, $Y^{\prime}{ }_{F A}$ and $Z_{F A}$ have been neglected in the above angular partials.

Derivatives with respect to the blade feathering are also constructed using cancellation of the dots.

$$
\left\{\begin{array}{l}
\frac{\partial Y_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}  \tag{275}\\
\frac{\partial Z_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}
\end{array}\right\}=\left\{\begin{array}{l}
\frac{\partial \dot{Y}_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}} \\
\frac{\partial \dot{Z}_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}
\end{array}\right\}
$$

gives:

Similarly, for angular motion:

$$
\begin{align*}
& \cong\left\{\begin{array}{l}
1 \\
0 \\
0
\end{array}\right\} \text { (as programmed) } \tag{277}
\end{align*}
$$

The partials developed with respect to $\phi_{\text {Fn }}$ are used directly in swashplate and rotor summations as well as some of the following mass and force terms. Some terms require a further compounding derivative, $\frac{\partial \phi_{\mathrm{Fn}}}{\partial \beta_{\mathrm{PHn}}}$, for the case of the pitch horn bending torsion option. Taking the ded of freedom to be pitch horn anpular deflection about the feather axis, the constant is approximately $1 / e$. For the dynamic torsion option a different set of mass formulations is used in terms of BLE axis, obviating the need for the compounded derivative. As indicated in Section 4.5.5, expressions inbriard of $X_{S W}$ equal expressions outboard of $X_{S W}$ with

$$
\begin{equation*}
\left[\mathrm{T}_{Y}\right]\left[\mathrm{T}_{\mathrm{T}_{0}}\right]=[\mathrm{I}] \tag{278}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{r_{S W}\right\}=\left\{r_{j \circ g}\right\}=\{0\} \tag{279}
\end{equation*}
$$

Blade $X$ motions must now be accounted for. The assumption of the neutral axis as the axis of no stretch is discussed in Section 4.5 .5 and the derivation of the $X$ motions shown. The equation for the partials in BLn axis for a point on the neutral axis is taken from the formulation for the $X$ velocities:

$$
\begin{aligned}
&\left(\frac{\partial X(i)_{N A}}{\partial A_{m n}}\right)_{B L n}=\sum_{i=2}^{k}\left(\frac{\left.-\left(Y(i)_{N A}-Y(i-1)_{N A}\right) \frac{\partial Y(i)_{N A}}{\partial A_{m n}}-\frac{\partial Y(i-1)_{N A}}{\partial A_{m n}}\right)}{X(i)_{N A}-X(i-1)_{N A}}\right. \\
&\left.\frac{-\left(Z(i)_{N A}-Z(i-1)_{N A}\right)\left(\frac{\partial Z(i)_{N A}}{\partial A_{m n}}-\frac{\partial Z(i-1)_{N A}}{\partial A_{m n}}\right)}{X(i)_{N A}-X(i-1)_{N A}}\right)
\end{aligned}
$$

and

$$
\left(\frac{\partial X(i)}{\partial A_{m n}}\right)_{\mathrm{Bin}}=0
$$

The program data, however, is at the blade center-of-gravity axis. The transfer is:
where the right-hand side elements are from Section 4.5.5 and the previous developent of this section. The distance $Y(i)_{\text {OMA }}=Y(i)_{M A}-Y(i)_{C G}{ }_{\text {BLE }}$ is the distance the neut:al axis is from the center-of-grarity axis, posisive corvard. The partials $\frac{\partial Y(i)_{M A}}{\partial A_{m n}}$ and $\frac{\partial Z(i)_{M A}}{\partial A_{m n}}$ are used in the preceding equation for $\frac{\partial X(i)_{H A}}{\partial A_{\operatorname{man}}} \cdot \frac{A \partial X(i)_{X A}}{\partial A_{m}}$ is then the difference in $x$ motions between the reference and the neutral axis, and is subtracted from the neutral axis motions:

$$
\begin{equation*}
\left(\frac{\partial X(i)_{B L E}}{\partial A_{\operatorname{man}}}\right)_{B L n}=\left(\frac{\partial X(i)_{M A}}{\partial A_{\operatorname{Bn}}}\right)_{B L n}-\left(\Delta \frac{\partial X(i)_{\mathrm{MA}}}{3 A_{\operatorname{ma}}}\right)_{\mathrm{BLn}} \tag{262}
\end{equation*}
$$

to obtain a center-of-gravity ralue.
The spanvise varietion of $x$ with reathering, $\frac{\partial X(i)_{B L E}}{\partial \rho_{\mathrm{Fn}}}$, can be derived in a mancer similar to $\frac{\partial X(i)}{\partial A_{m L E}}$. The formulations a:e :
wher:
and

$$
\begin{equation*}
\left(\frac{\partial X(1)_{\mathrm{FA}}}{\partial \phi_{\mathrm{Fa}}}\right)_{\mathrm{BLn}}=0 \tag{285}
\end{equation*}
$$

The program assumes $\left(\frac{\partial X_{\mathrm{BLE}}}{\partial \varphi_{\mathrm{Fn}}}\right)_{\mathrm{BLn}}$ to be zero fici generalized mass caicula-
tions. This is done because of the latitude possible in the generalized masses and the seconi order nature of the term. In contrast, the derivative $\left(\frac{\partial X_{B L E}}{\partial r_{\operatorname{man}}}\right)_{B L Q}$ is retained. The partial $\left(\frac{\hat{\partial}_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fa}}}\right)$ is set to unity ior the generalized mass terms. The full equations are used for all these terms in deteraining the generalized forces.

A simple partial derivative is also needed when $\boldsymbol{B}_{\text {PHa }}$ is defined as cynamic
torsion. Since torsion cecurs along the bent and twisted blade lire, in blade element axis BLE, oniy the verticai or normal to chord motion of the shear center is of interest, hence,

$$
\begin{aligned}
& \left.\frac{-\left(z(i)_{M A}-z(i-i)_{N A}\right)\left(\frac{\partial Z(i)_{N A}}{\partial \phi_{\mathrm{Fn}}}-\frac{\partial z(i-1)}{\partial \phi_{\mathrm{Fn}}}\right)}{X(i)_{N A}-X(i-i)_{\mathrm{NA}}}\right)_{E L .}
\end{aligned}
$$

$$
\begin{align*}
\left(\frac{\partial Z(i)_{B L E}}{\partial \beta_{\mathrm{PHn}}}\right)_{\mathrm{BLL}}= & \left(\frac{\partial Z(i)_{\mathrm{BLE}, \mathrm{BLn}}}{\partial Z(i)_{\mathrm{BLE}}} \frac{\partial Z(i)_{\mathrm{BLE}}}{\partial \phi_{\mathrm{BLE}}} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \beta_{\mathrm{PHn}}}\right)=  \tag{286}\\
& {\left[\mathrm{T}_{\mathrm{BLE}}-\mathrm{BLn}\right]\left(\mathrm{Y}(\mathrm{i})_{\mathrm{SC}}-\mathrm{Y}(i)_{\mathrm{CG}}\right)_{\mathrm{BLE}} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \beta_{\mathrm{PABn}}} }
\end{align*}
$$

$\frac{\partial \phi_{\mathrm{BLE}}}{\partial 6_{\mathrm{PHn}}}$ is program input for the torsion mode shape.

Use is made of a tlade mode to blade feathering partial derivative array to produce a compact development. This array is simply by definition:

$$
\left\{\begin{array}{c}
\partial \tau_{\mathrm{BL}} \\
\hline \partial \phi_{\mathrm{Fn}}
\end{array}\right\}=\left[\begin{array}{c:c} 
& {[\mathrm{c}]} \\
& 0 \\
\hdashline 0 & 0
\end{array} 0_{1}\right]
$$

(287)
where $\left|\tau_{B L}\right|=\left|A_{1 n}, A_{2 n}, A_{3 n}, \phi_{F n}\right|$.
Additional particl sets which are used to expedite the mass and force expression deveiopment are:

$$
\begin{align*}
& \left\{\begin{array}{c:c}
\frac{\partial^{\top} O_{R}}{\partial \partial^{T} H}
\end{array} \left\lvert\,=\left[\begin{array}{c:c}
{\left[T_{H-R}\right]} & {[0]} \\
\hdashline[0] & {\left[T_{H-R}\right]}
\end{array}\right]\right.\right. \tag{289}
\end{align*}
$$

5.6.3 Generalized masses. - The blade generalized masses in conjunction with partial derivatives couple blade feathering, blade torsion, blade bending, fuselage motions. All the blade generalized mnsses that assumed to
exist are given in the following table. As mentioned before, $\frac{\partial X_{B L E}}{\partial \phi_{\mathrm{Fn}}}$ is assumed zero and $\frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}$ is assumed one in the program, although it is given in the table. The blade has a rotary inertia about the center of gravity axis $I_{X X_{B L E}}$. The blade also has inertia $I_{Z Z_{B L E}}$ about a vertical axis.

Table $i$ lists all the terns coufling rotary motion at the blade root, M. and similar terms. Hovever, not all listed are used, as certain ${ }^{4}$ BLn ${ }^{6}$ BLn approximations are made in developing the principal axis generalized masses which reduce the number of blade coupling generalized masses needed. Since the mass matrix operates on the acceleration error term rather than the total acceleration, these approximations do not detract from the validity of the results produced.

Blade Root Coupling

$$
\begin{aligned}
& M_{Y_{O B L a}{ }^{*}{ }_{B L n}}=-M_{Z_{O B L n}{ }^{\theta}{ }_{B L n}}=m_{B L a} x_{C G_{B L n}}=\sum_{i=1}^{k}\left(x_{B L E}(i)\right)_{B L n}(m(i))_{(2,0)}
\end{aligned}
$$

$$
\begin{aligned}
& M_{X_{O B L n}} X_{O B L n}=M_{Y_{O B L n}} Y_{O B L n}=M_{Z_{O B L E}} Z_{O B L n}=n_{B L n}=\sum_{i=1}^{k} m(i)
\end{aligned}
$$

Feather Coupling

$$
\begin{align*}
& M_{X_{O B L n}{ }^{*}}=\sum_{i=1}^{k} m(i)\left(\frac{\partial x_{B L E}}{\partial \phi_{\mathrm{Fn}}}\right)_{\mathrm{BLn}}  \tag{204}\\
& M_{\mathrm{Y}_{\mathrm{OBLn}} \phi_{\mathrm{Fn}}}=\sum_{i=1}^{k} m(i)\left(\frac{\partial Y_{\mathrm{Bi}, \mathrm{E}}}{\partial \phi_{\mathrm{Fn}}}\right)_{\mathrm{BLn}}  \tag{295}\\
& \left.M_{Z_{O B L n}{ }^{(F n}}=\sum_{i=1}^{k} m_{i} i\right)\left(\frac{\partial z_{B L E}}{\partial \phi_{\mathrm{Fn}}}\right)_{\mathrm{BLn}}  \tag{296}\\
& M_{\phi_{B L n} \phi_{F n}}=\sum_{i=1}^{k}\left[m(i)\left(\frac{\partial Y_{B L E}}{\partial \phi_{B L n}} \frac{\partial Y_{B L E}}{\partial \phi_{\mathrm{Fn}}}+\frac{\partial Z_{B L E}}{\partial \phi_{B L n}} \frac{\partial Z_{B L E}}{\partial \phi_{\mathrm{Fn}}}\right)\right. \\
& \left.+I_{X X}(i)\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{BLn}}} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)\right]_{\mathrm{BLn}} \tag{297}
\end{align*}
$$

Feather Coupling (Continued)

$$
\begin{aligned}
& M_{\theta_{\mathrm{BLn}} \phi_{\mathrm{Fn}}}= \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial Y_{\mathrm{BLE}}}{\partial \theta_{\mathrm{BLn}}} \frac{\partial Y_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}+\frac{\partial Z_{\mathrm{BLE}}}{\partial \theta_{\mathrm{BLn}}} \frac{\partial Z_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)\right. \\
&+I_{\mathrm{XX}}^{\mathrm{BLE}} \\
&\left.(i)\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial \theta_{\mathrm{BLn}}} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)\right]_{\mathrm{BLn}} \\
& M_{\Psi_{\mathrm{BLn}} \phi_{\mathrm{Fn}}}= \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial Y_{\mathrm{BLE}}}{\partial \psi_{\mathrm{BLn}}} \frac{\partial Y_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}+\frac{\partial Z_{\mathrm{BLE}}}{\partial \psi_{\mathrm{BLn}}} \frac{\partial Z_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)\right.
\end{aligned}
$$

$$
\left.+I_{X X}(i)\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial \psi_{\mathrm{BLn}}} \frac{\partial \phi_{\mathrm{BLE}}}{2 \phi_{\mathrm{Fn}}}\right)\right]_{\mathrm{BLn}}
$$

$$
M_{\phi_{F n}{ }^{\dagger} F n}=\sum_{i=1}^{k}\left[m(i)\left(\left(\frac{\partial X_{B L E}}{\partial \phi_{\mathrm{Fn}}}\right)^{2}+\left(\frac{\partial Y_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)^{2}+\left(\frac{\partial Z_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)^{2}\right)\right.
$$

$$
\left.+I_{X X}(i)\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)\right]_{\mathrm{BLn}}
$$

$$
M_{A_{m n} \phi_{F n}}=\sum_{i=1}^{k}\left[m(i)\left(\frac{\partial X_{B L E}}{\partial A_{m n}} \frac{\partial X_{B L E}}{\partial \phi_{\mathrm{Fn}}}+\frac{\partial Y_{B L E}}{\partial A_{m n}} \frac{\partial Y_{B L E}}{\partial \phi_{F n}}+\frac{\partial Z_{B L E}}{\partial A_{m n}} \frac{\partial Z_{B L E}}{\partial \phi_{F n}}\right)\right.
$$

$$
\left.+I_{X X_{B L E}}^{(i)}\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial A_{\mathrm{mn}}} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\mathrm{Fn}}}\right)\right]
$$

Blade Bending Coupling

$$
\begin{align*}
M_{X_{O B L n} A_{m n}}= & \sum_{i=1}^{k} m(i)\left(\frac{\partial x_{B L E}}{\partial A_{m n}}\right)_{B L n}  \tag{302}\\
M_{Y_{O B L n} A_{m n}}= & \sum_{i=1}^{k} m(i)\left(\frac{\partial Y_{B L E}}{\partial A_{m n}}\right)_{B L n}  \tag{303}\\
M_{z_{O B L n} A_{m n}}= & \sum_{i=1}^{k} m(i)\left(\frac{\partial Z_{B L E}}{\partial A_{m n}}\right)_{B L n}  \tag{304}\\
M_{\psi_{B L n} A_{m n}}= & \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial X_{B L E}}{\partial \psi_{B L n}} \frac{\partial X_{B L E}}{\partial A_{m n}}+\frac{\partial Y_{B L E}}{\partial \psi_{B L n}} \frac{\partial Y_{B L E}}{\partial A_{m n}}+\frac{\partial Z_{B L E}}{\partial \psi_{B L n}} \frac{\partial Z_{B L E}}{\partial A_{m n}}\right)\right. \\
& \left.+I_{X X_{B L E}}\left(\frac{\partial \phi_{B L E}}{\partial \psi_{B L n}} \frac{\partial \phi_{B L E}}{\partial A_{m n}}\right)\right]_{B L n}
\end{align*}
$$

$$
\begin{align*}
M_{A_{m n} A}= & \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial X_{B L E}}{\partial A_{m n}} \frac{\partial X_{B L E}}{\partial A_{m n}}+\frac{\partial Y_{B L E}}{\partial A_{m n}} \frac{\partial Y_{B L E}}{\partial A_{m n}}+\frac{\partial Z_{B L E}}{\partial A_{m n}} \frac{\partial Z_{B L E}}{\partial A_{m n}}\right)\right. \\
& \left.+I_{x x_{B L E}}\left(\frac{\partial \phi_{B L E}}{\partial A_{m n}} \frac{\partial \Phi_{B L E}}{\partial A_{m n}}\right)\right]_{B L n} \tag{306}
\end{align*}
$$

${ }^{B_{\mathrm{FHn}} \text { Defined As Dynamic Pitch Horn Bending }}$

$$
\begin{equation*}
M_{\mathrm{A}_{\mathrm{mn}} \beta_{\mathrm{PHn}}}=M_{\mathrm{A}_{\mathrm{mn}} \phi_{\mathrm{Fn}}} \frac{\partial \phi_{\mathrm{Fn}}}{\partial \beta_{\mathrm{PHn}}} \tag{307}
\end{equation*}
$$

## TABLE 1. - Concluded

${ }^{B}$ PHn Defined As Dynamic Pitch Horn Bending (Continued)

$$
\begin{align*}
& M_{B_{\mathrm{PHn}} \beta_{\mathrm{PHn}}}=M_{\phi_{\mathrm{Fn}}{ }_{\mathrm{Fn}}\left(\frac{\partial \phi_{\mathrm{Fn}}}{\partial \beta_{\mathrm{PHn}}}\right)^{2}}^{M_{B_{\mathrm{PHn}} \phi_{\mathrm{Fn}}}=M_{\phi_{\mathrm{Fn}} \phi_{\mathrm{Fn}}}\left(\frac{\partial \phi_{\mathrm{Fn}}}{\partial \beta_{\mathrm{PHn}}}\right) \text { (used in swashplate) }} \tag{308}
\end{align*}
$$

$\underbrace{\text { PHn } \text { Defined As Dynamic Torsion }}$

$$
\begin{align*}
& M_{A_{m n} B_{P H n}} \cong \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial Z_{B L E}}{\partial A_{m n}}\right)\left(\frac{\partial Z_{B L E}}{\partial \phi_{B L E}}\right)+I_{X X_{B L E}(i)}\left(\frac{\partial \phi_{B L E}}{\partial A_{m n}}\right)_{B_{B L n}}\right]_{B L n}\left(\frac{\partial \phi_{B L E}}{\partial \beta_{P H n}}\right)_{B L n} \\
& \text { (310) } \\
& M_{\beta_{\mathrm{PHn}}{ }^{\mathrm{PKn}}} \cong \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial Z_{B L E}}{\partial \phi_{\mathrm{BLE}}}\right)^{2}+I_{X_{\mathrm{BLE}}(i)}\right]_{\mathrm{BLn}}\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial \beta_{\mathrm{PHn}}}\right)_{\mathrm{BLn}}^{2} \\
& M_{\beta_{P H n} \phi_{\mathrm{Fn}}} \equiv \sum_{i=1}^{k}\left[m(i)\left(\frac{\partial Z_{B L E}}{\partial \phi_{\mathrm{Fn}}}\right)\left(\frac{\partial Z_{\mathrm{BLE}}}{\partial \phi_{\mathrm{BLE}}}\right)+I_{X X}(i) \frac{\partial \phi_{\mathrm{BLE}}}{\partial \phi_{\cdot n}}\right]_{\mathrm{BLn}}\left(\frac{\partial \phi_{\mathrm{BLE}}}{\partial \beta_{\mathrm{PHn}}}\right)_{\mathrm{BLn}} \\
& \text { (used in swashplate) } \tag{312}
\end{align*}
$$

To save compitation time some of the masses generated by the blade intergration and summation process in Table 1 are saved as a pseudo mass associated rith a fictitioas rotor coordinate (R). Thus for any given time step the sum of the rotor blades can be treated as an equivalent mass matrix:

$$
\left[M_{O R}\right]=\sum_{n=1}^{M_{b}}\left\{\frac{\partial \tau_{B L n}}{\partial \tau_{R}}\right\} \sum_{i=1}^{T k}\left\{\frac{{ }^{\partial \tau_{B L E}}}{\partial \tau_{B L n}}\right\}\left[\begin{array}{ccccc}
m(i) & 0 & 0 & & {[0} \\
0 & m(i) & 0 & & \\
0 & 0 & m(i) & & \\
& & & I_{X X} & 0 \\
& 0 \\
& & 0 & 0 & 0 \\
& & 0 & 0 & I_{Z Z}
\end{array}\right]_{B L E}
$$

$$
\begin{equation*}
\bullet\left\{\frac{\partial \tau_{B L E}}{\partial \tau_{B L n}}\right\} \quad\left\{\frac{\partial \tau_{o_{B L n}}}{\partial \tau_{R}}\right\} \tag{313}
\end{equation*}
$$

The couplings of the pseudo rotor coordinate to blade and hub crordinates are also formed as an intermediate step to save repetitious blade integration.


Where the array $\left[M_{O_{B L B L}}\right]_{n}$ is the feather and bending coupling terms for the $\mathrm{n}^{\text {th }}$ blade. Ie.,

In addition to the rigid body motion blade coupling matrix, $\left[\mu_{O_{B L}-B L}\right]$, the blade mode coupling matrix, $\left[\mathrm{M}_{\mathrm{BL}-\mathrm{BL}}\right]$, will be used in the subsequent development

$$
\left[M_{\text {BBL }}\right]_{n}=\left[\begin{array}{lllllll}
M_{A_{1}} & A_{1} & M_{A_{1}} & A_{2} & M_{A_{1}} & A_{3} & M_{A_{1}}  \tag{316}\\
M_{f} \\
M_{A_{2}} & A_{1} & M_{A_{2}} & A_{2} & M_{A_{2}} & A_{3} & M_{A_{2}} \\
\phi_{f} \\
M_{A_{3}} & A_{1} & M_{A_{3}} & A_{2} & M_{A_{3}} & A_{3} & M_{A_{3}} \phi_{f} \\
M_{\phi_{f}} & M_{1} & M_{\phi_{f}} & A_{2} & M_{\phi_{f}} & A_{3} & M_{\phi_{f}} \\
\phi_{f}
\end{array}\right]_{n}
$$

5.5.4 Generalized forces. - The development herein proceeds by first deriving the equations for the loads on an individual blade element. The blade element loads are composed of aerodynamic and inertial components conveniently found in either the blade root axes BLn or the blade element axes BLE. The loads will be summed in Bln axes with the appropriate transformation. The desired equations are:

$$
\begin{aligned}
& \left\{\begin{array}{l}
M_{X}(i)_{B L E} \\
M_{Y}(i)_{B L E} \\
M_{Z}(i)_{B L E}
\end{array}\right\}_{B L n}=\left[{ }_{r_{B L n-B L E}}\right]^{T}\left\{-I_{X_{B L}}\left\{\begin{array}{c}
\dot{p}_{B L E}+q_{B L E} r_{B L E} \\
0 \\
\dot{r}_{B L E}-p_{B L E} q_{B L E}
\end{array}\right\}\right. \\
& +\left\{\begin{array}{c}
M_{X A}(i)^{B L E} \\
0 \\
0
\end{array}\right\}_{B L n}+\left\{\begin{array}{c}
-Y_{C G}(i)_{B L E}{ }^{F_{Z A}(i)_{B L E}} \\
0 \\
0
\end{array}\right\}
\end{aligned}
$$

The aerodynamic loads are in BLE axes alignment about the blade reference datum line which is the quarter chord. A transfer through the distance $Y_{C G}{ }_{B L E}$ is made to the aerodynamic moment. To put the data on a common
basis with dynamic terms. The blade aerodynamics is detailed in Section 6. Since only the blade section pitching moment is considered, $M_{Y A_{B L E}}=M_{Z A_{B L E}}=0$. Note the blade element is assumed configured as a chordwise rod for inertia; bence

$$
\begin{equation*}
\left(I_{X X_{B L E}}=I_{Z Z_{B L E}} \text { and } I_{Y Y_{B L E}}=0\right) \tag{319}
\end{equation*}
$$

A number of blade sumations are desired. All will be made in BLn axes along the center-of-gravity axis. The losds at the principal reference axes and for rotor tilt make use of the blade root shears and moment. These are simply the sum of the $k$ total blade elements,

$$
\left\{\begin{array}{l}
F_{X_{O B L n}}  \tag{320}\\
F_{Y_{O B L n}} \\
F_{Z_{O B L n}}
\end{array}\right\}=\sum_{i=1}^{k}\left\{\begin{array}{l}
F_{X^{(i)}} F_{B L E}(i)_{B L E} \\
F_{Z}(i)_{B L E}
\end{array}\right\}_{B L n}
$$

and likewise for root moments.

$$
\left\{\begin{array}{l}
M_{X_{B L n}}  \tag{321}\\
M_{Y_{B L n}} \\
M_{Z_{B L n}}
\end{array}\right\}=\sum_{i=1}^{k}\left\{\begin{array}{l}
M_{X}(i)_{B L E} \\
M_{Y}(i)_{B L E} \\
M_{Z}(i)_{B L E}
\end{array}\right\}_{B L n}
$$

The summations illustrated above are for the total inertial and aerodynamic components. In a similar manner, the blade root aerodynamic loads are derived. The blade root loads are summed over all the blade to give main asrodynamic loads for downwash computations (Section 6.2.2) in the manner the total main rotor loads are found in Section 5.9.

Total main rotor root loads are formed from the blade root shears and moments. Using the pseudo rotor coordinate:

$$
\left\{\begin{array}{l}
F_{X}  \tag{322}\\
F_{Y} \\
F_{Z} \\
F_{\phi} \\
F_{O} \\
F_{\Psi}
\end{array}\right\}=\sum_{n=1}\left\{\frac{N_{b}}{\partial \tau_{R}}\right\}\left\{\begin{array}{l}
F_{X_{O B L n}} \\
F_{Y_{O B L n}} \\
F_{Z_{O B L n}} \\
M_{X_{B L n}} \\
M_{Y_{B L n}} \\
M_{Z_{B L n}}
\end{array}\right\}
$$

Feathering moments are used by the swashplate equations of motion. These moments are:
(323)
where

$$
\left[\mathrm{T}_{\mathrm{BLn}-\mathrm{Fn}}\right]=\left[\begin{array}{ccc}
\cos ^{\prime} \mathrm{FA}_{\mathrm{FA}} & \sin ^{\prime}{ }_{\mathrm{FA}} & 0  \tag{324}\\
-\sin Y^{\prime} & \operatorname{cosA}^{\prime} & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
\cos ^{\prime} Z_{\mathrm{FA}} & 0 & \sin Z^{\prime} \\
0 & 1 & 0 \\
-\sin Z^{\prime} & 0 & \cos Z^{\prime} \\
\mathrm{FA}_{\mathrm{FA}}
\end{array}\right]
$$

Oniy the $X$ component is used by the program. The equations above transfer the summed blade loads to the inboard bearir.s, $\therefore$ ien transform them to feathering axes. Using the blade root loads is $\because$ icet when one recalls that the blade is defined as those portions that : . f feathered; the fixed hub is excluded.

The blade bendin: generalized forces are now presented. ", y are:

$$
\begin{align*}
F_{A_{m n}}= & \sum_{i=1}^{k}\left[\left(\frac{\partial X(i)_{B L E}}{\partial A_{m n}} F_{X}(i)_{B L E}+\frac{\partial Y(i)_{B L E}}{\partial A_{m n}} F_{Y}(i)_{B L E}+\frac{\partial Z(i)_{B L E}}{\partial A_{m n}} F_{Z}(i)_{B L E}\right)_{B L n}\right. \\
& \left.+\left(\frac{\partial \phi(i)_{B L E}}{\partial A_{m n}} M_{X}(i)_{B L E}+\frac{\partial \psi(i)_{B L E}}{\partial A_{m n}} M_{Z}(i)_{B L E}\right)_{B L n}\right]-\frac{\partial U}{\partial A_{m n}}-\frac{\partial B}{\partial \dot{A}_{m n}} \tag{325}
\end{align*}
$$

The potential energy is given as:

$$
\begin{equation*}
\frac{\partial U}{\partial A_{m n}}=\sum_{i=1}^{3} K_{m i} A_{j n} \tag{326}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{m j}=\int_{R O O T}^{T I P}\left(\frac{\frac{\partial M_{Y}}{\partial A_{m n}} \frac{\partial M_{Y}}{\partial A_{j n}}}{E I_{Y Y}}+\frac{\frac{\partial M_{Z}}{\partial A_{m n}} \frac{\partial M_{Z}}{\partial A_{j n}}}{E I_{Z Z}}\right) d S \tag{327}
\end{equation*}
$$

are inputs calculated external to the program from a berting beam model. $E I_{Y Y}$ and $E I_{Z Z}$ are the flapping and chord stiffness $a b s t$ axes aligned with the blade element principal axes. The chord and flapping moments, ${ }^{\prime} Y_{i}$ and $M_{Z_{i}}$ reflect the contribution of the bending moment from the $i$ (or j) mode. The integration goes from root tc tip. The K's are evaluated for whatever :ormalized modes are used as program input.

The last equation can be derived from the Bernoulli-Euler law for bending beams:

$$
\begin{equation*}
M=\frac{E I}{r} \tag{328}
\end{equation*}
$$

where $r$ is a radius of curvature. The strain dergy is

$$
\begin{equation*}
v=\int_{R O O T}^{T I P} \frac{1}{2}\left(M_{Y} \frac{d \sigma}{d S}+M_{2} \frac{d \psi}{d S}\right) d S \tag{329}
\end{equation*}
$$

Substititing in from the Bermoulli-Euler law and noting that $d S=r_{y} d \theta=$ $5=0 \psi$,

$$
\begin{equation*}
y=\int_{\mathrm{ROO}}^{\mathrm{ZIP}} \frac{1}{2}\left(\frac{M_{V}^{2}}{E_{Y Y}}+\frac{M_{Z}^{2}}{E I_{Z Z}}\right) d S \tag{330}
\end{equation*}
$$

Partial differentiation gives

Sonsidering the mements as a inear sum of components frow each bending mase, one has:

$$
\begin{equation*}
M_{Y}=\frac{\partial M_{Y}}{\partial A_{i n}} A_{2 n}+\frac{\partial M_{Y}}{\partial A_{2 n}} A_{2 n}+\frac{\partial M_{Y}}{\partial A_{3 n}} A_{3 n}=\sum_{j=1}^{3} \frac{\partial Y_{Y}}{\partial A_{j n}} A_{j n} \tag{332}
\end{equation*}
$$

and likewise for $f_{2}$. Then, by substitution, the desired eruation is obtained.
5.6.4.1 Blafe mution cismping. - The olade motion damping is modeled by structural dampire, aerodynari? Gempires, and a monhanical damprr for lead-lag motions. The aerodyamic dampins is accounted fo: in the aerodynamic blade loaùs develcred ir Section 6.2.

The stmictural factor is assumed proportional to the spring rate components of "cknticn 327 . The coefricients $\operatorname{Kan}_{\mathrm{m}}$ can be directiy identified with the
 gazpins cuef $\because$ icients, $C_{r k}$, are then tí: product of the proportionality factcr, Is, and $i_{i i^{\circ}}$ The structural dameine component is then:

$$
\begin{equation*}
\therefore \frac{\partial E}{\partial \dot{A}_{\mathrm{min}}}=\sigma_{S} \sum_{j=i}^{3} K_{m_{i}} \dot{A}_{j n} \tag{333}
\end{equation*}
$$

A lead-lag mechanical damper is usually a rotery or linear motion device enploying elestomeric, dry friciion, or viscous energy absorbing mechanism. The device is oupled about the blade lead-lag hinge or point of lead-lag motion slope by a linkage array. REXOR II models a rotary viscous damper mounted about a given blade location for which the inplane slope is specified as a function of the blade modal variables. For an articulated blade, this vill simply be the motion of the lead-lag hinge. The damper has a pressure relief valve so that an initial daping rate, $C_{\text {LAGI }}$, is replaced by CLAG2 abore a set motion rate, $\dot{Y}{ }_{1}$.
Given the slope rate data, $\bar{Y}_{n_{c}}$, for blade c at the damper location, $c$, the initial cakying rate sempent is defined by $\left|\dot{X}_{n_{c}}\right|<\dot{Y}_{\dot{1}}$. The damping is

$$
\begin{equation*}
\Delta \frac{\partial B}{\partial A_{m n}}=-\left(C_{I A G I}\right)\left(\dot{Y}_{\sum_{c}^{\prime}}^{\prime}\right)\left(\frac{\partial Y_{c}^{\prime}}{\partial A_{m}}\right) \tag{334}
\end{equation*}
$$

where

$$
\begin{equation*}
\dot{Y}_{c}^{\prime}=\sum_{j=1}^{3}\left(\frac{\partial Y_{c}^{\prime}}{\partial A_{j}} \dot{A}_{i n}\right) \tag{335}
\end{equation*}
$$

Beyond the pressure relisf opening point the damping contribution is

$$
\begin{aligned}
\Delta \frac{\partial B}{\partial \dot{A}_{m n}}= & -\left[\left(\dot{Y}_{i}^{\prime}\right)\left(\operatorname{SIGI}\left(\dot{Y}_{n_{c}^{\prime}}^{\prime}\right)\right) C_{\text {LAGI }}\right. \\
& +\left(C_{L A G 2}\right)\left(\dot{Y}_{n_{c}^{\prime}}^{\prime}-\left(\dot{Y}_{\underline{l}} j^{\prime}\right)\left(\operatorname{SIGI}\left(\dot{Y}_{n_{c}^{\prime}}\right)\right)\right]\left(\frac{\partial Y_{c}^{\prime}}{\partial A_{m n}}\right)
\end{aligned}
$$

The generalized force is developed for pitch horn bending and dynamic torsion. For pitch horn bending,

$$
\begin{equation*}
F_{B_{\mathrm{PHn}}}=M_{\mathrm{F}:} \frac{\partial \Phi_{\mathrm{Pn}}}{\partial 8_{\mathrm{PHn}}}-K_{\beta_{\mathrm{PHn}}} B_{\mathrm{PHn}} \tag{337}
\end{equation*}
$$

where $M_{F n}$ is the total feather moment as derived in Section 5.10. For the uncoupled dynamic torsion option,

$$
\begin{equation*}
F_{B_{P H n}}=\sum_{i=1}^{k}\left(\frac{\partial 2(i)_{B L E}}{{ }^{3} \phi_{B L E}} F_{Z}(i)_{B L E}+M_{X}(i)_{B L E}\right){ }_{B L E} \frac{\partial \phi_{\mathrm{BLE}}}{\partial \beta_{\mathrm{PHn}}}-K_{B_{\mathrm{PHr}}} B_{\mathrm{PYHn}} \tag{338}
\end{equation*}
$$

in BLE axes. Since the blade elements ioads are derived in $\operatorname{SLn} a x=3$, the transform

$$
\left\{\begin{array}{c}
-  \tag{339}\\
- \\
F_{Z}(i)_{B L E}
\end{array}\right\}=\left[T_{B L n-B L F}\right]\left\{\begin{array}{c}
F_{X}(i)_{B L E} \\
F_{Y}(i)_{B L E} \\
F_{Z}(i)_{B L E}
\end{array}\right\}_{B L n}
$$

is needed. The spring constant can be interpreted as

$$
\begin{equation*}
K_{S_{\mathrm{PH}}}=M_{\mathrm{E}_{\mathrm{PH}}} \omega_{\mathrm{B}_{\mathrm{PH}}}^{2} \tag{340}
\end{equation*}
$$

where the generalized mass is computed conimuously and $\omega_{B_{P H}}$ is the
nautrai frequency of the unccupled tursion mode, a program input constant.
5.6.5 Quasi-static blade torsion. - To improve the pitch horn bending blade feathering representation a quasi-static blade torsion distribution is introduced. Quasi-static torsion is computed from the structural stiffness, ${ }^{C} J_{S C}$, at each station and the torque ${ }^{3 X} X_{S C}$ at the shear center. The torque is s.med from the tip to the blade station in question as show in Section 4.3.4. The increment of twist produced at a blace station $j$ can be displayed as:

$$
\begin{equation*}
\tau_{T} \dot{\phi}_{T j}+\hat{t}_{T j}=\frac{M_{S C j}}{G_{S C J}} \tag{341}
\end{equation*}
$$

assuming a first-order lag represents the torsional dynamics. The time constant $r_{T}$ is chosen te be representative of the blade first torsional mode frequency.

To obtain this result, the available computation elements require some further operations. First, REXOR II conducts blade integrations from root to tip, in BLa axes. To obtain tip to root values:


Note the sumation is conducted from root to the station : in question. Thus the $J$ represents a sumation whereas the $i$ represents a blade station.

Second, these data are used to form the required torque at the shear center.

$$
\begin{align*}
& +\left\{\begin{array}{ccc}
0 & Z_{S C} & -Y_{S C} \\
-Z_{S C} & 0 & X_{S C} \\
Y_{S C} & -X_{S C} & 0
\end{array}\right]_{B L n}\left\{\begin{array}{c}
F_{X_{B L E J}} \\
F_{Y_{B L E J}} \\
F_{B L n}
\end{array}\right\} \tag{343}
\end{align*}
$$

Small angles are assumed. The moments $\left({ }^{M} X_{B L E}\right)_{B L n}$ etc., act along the BLn axes and hence the matrix of lengths $\left(X_{S C}\right)_{B L n}$ etc., are employed to obtain moments at the shear center which are then transformed into shear center axes, subscripted SC, parallel tc blade element center-of-gravity axes, subscripted BLE.

The blade deflections and siope in BLn are also needed for the above expressions.

$$
\left\{\begin{array}{c}
x_{S C}  \tag{344}\\
Y_{S C} \\
z_{S C}
\end{array}\right\}_{B L n}=\left\{\begin{array}{c}
X_{B L=} \\
Y_{B L E} \\
z_{B L E}
\end{array}\right\}_{B L n}+\left[T_{B L n-B L E}\right]^{T}\left\{\begin{array}{c}
0 \\
Y_{S C}-Y_{C G} \\
0
\end{array}\right\}_{B L E}
$$

and

$$
\left.\left\{\begin{array}{c}
0  \tag{345}\\
Y^{\prime} \\
Z_{S C}^{\prime}
\end{array}\right\}_{\mathrm{SC}}\right\}_{\mathrm{BL} \mathrm{\sim}} \equiv\left\{\begin{array}{c}
0 \\
Y^{\prime} \\
Z_{B L E}^{\prime} \\
B L E
\end{array}\right\}_{\mathrm{BLn}}
$$

5.6.6 Quasi-static pitch horn bending. - To facilitate troubleshooting numerical instability problems as optional quasi-static pitch horn bending degree of freedom is available. The computation elements are the same as developed in Section 5.6 .4 except that the solution does not use generalized masses, is therefore an uncoupled mode, and is calculated externally to the main computation flow. The formulation used is:

$$
\begin{equation*}
{ }^{\tau} \mathrm{PH} \dot{\phi}_{\mathrm{FnPH}}+\phi_{\mathrm{FnPH}}=\frac{\mathrm{M}_{\mathrm{Fn}}}{\mathrm{~K}_{\mathrm{B}_{\mathrm{PH}}}} \tag{346}
\end{equation*}
$$

The dynamics are assumed represented by a first-order lag vith $B_{P_{H}}$ as the time constant. The variable $\phi_{\mathrm{FnP}}$ is used to distinguish this formulation from the isual $\beta_{\mathrm{PHn}}$ symbology.

### 3.7 Shaft Axes Equations

5.7.1 Transmission isolation mount. - The shait equations couple the spring mionted transmissirn, swashplate and rotor to the ground side of the mounting springs (fuselage). The Euselage is the reference coordinate set hence derivatives with respect to the rotor, hub, swashplate and transmission masses exist for the shaft axes equatic:.s.
5.7.2 Partial derivatives. - E using rotor pse . Jordinate masses only a few opera*ions are required to assemble the cour . ass terms using one derivative vector, $\left\{\left.{ }^{2}{ }^{O_{H}}{ }^{/ \partial T_{S}}\right|^{\prime}\right.$.
E., nacing:

This expression is a compact notation form of the development of section 5.5.3. Note that the angle to angle portion of the array is not a full transformation, but rather reflects the relation of a dependent coordinate to an independent Euler angle.

The hub to rirashplate partial $\left\{{ }^{2 \tau_{O_{S P}}} / \partial \tau_{H}\right\}$ is developed in the next section.
5.7.3 Gemeralized masses. - The shaft axes matrix elements couple to the blase i $_{\mathrm{mn}}$ " $\mathrm{B}_{\mathrm{PHn}}$ ) and swashplate generalized coordinates as well as to itself. Use is made of matrix notation and the rotor pseudo coordinate to produce a compact notation.


5.7.4 Generalized forces. - The shaft axes exercise the transmission mount springs $\left[\mathrm{K}_{\mathrm{S}}\right]$ and dampers $\left[\mathrm{C}_{\mathrm{S}}\right]$. Rotor ioads, reflected through the hub coordinates also appear in the shaft generalized forces.


5.8 Frincipal Reference Axis Equations
5.8.1 Nonzero contributions from most vehicie mass elements. - The principal reference axis equations of motion consider contributions from all of the physical elements of the rotorcraft. The elements involved are:

- Main rotor - defined as all portions that can be feathered
- Rotor hub - includes all portions of the main rotor assembly that cannot be feathered, and is treated as a rigid bcjy
- Swashplate
- iail rotor
- Fuselage
- Engine

The six rigid degrees of freedom: $X, Y, Z, \phi, \theta, \psi$ are taken with respect to the stationary fuselage axes which are also the principal axes. The other elements considered are then referenced to the fuselage axes. The hub is subject to shaft bending motions relative to the principal axes. The tail rotor is installed on the fuselage and rotates at the main rotor speed times an appropriate gear ratio. Positive rotations are defined as:
$\left.\begin{array}{ll}\text { - Hub } \\ \text { - Swashplate }\end{array}\right\}$ same as main rotor

- Tail rotor - Clockwise looking right
- Engine - Counterclockwise looking forward

The engire is treated as a rigid rotating body but the tail rotor is allowed to flap (teetering hinge, etc.). This flapping is considered secondary and enters only into the aerodynamic computations. The main rotor is allowed feathering, bending and twisting.
5.8.2 Partial derivatives. - Elements used the reference axis masses and forces can by in large be conveniently related to either the fuselage or hub coordinates. Since the reference set is taken to be the fuselage coordinate set, no partials are required in this instance.

Partials relating hub coordinate motior to reference generalized follow the scheme given in section 5.5.3.

$$
\begin{aligned}
& \left\{\frac{\partial \tau^{\tau} O_{H}}{\partial \tau_{R E F}}\right\}=\left\{\frac{\partial \tau_{H}}{\partial \tau_{F}}\right\}\left\{\frac{\partial \tau_{O_{F}}}{\partial \tau_{R E F}}\right\}
\end{aligned}
$$

where

$$
\begin{equation*}
\left\{\frac{{ }^{\partial \tau^{O_{F}}}}{\partial \tau_{R E F}}\right\} \equiv[I] \tag{356}
\end{equation*}
$$

The swashplate system is physically connected with the hub structure, and partial derivatives describing its motion are taken through this intermediate, hub, coordinate. Note due to a parallelogram linkage the swashplate vertical motion is assumed to be unaffected by tilt angle.

$$
\left\{\begin{array}{c}
{ }^{\partial \tau_{0}}{ }_{\mathrm{SP}}  \tag{357}\\
{ }^{\partial \tau_{\mathrm{H}}}
\end{array}\right\}=\left[\begin{array}{ccc:ccc}
1 & 0 & 0 & 0 & \mathrm{Z}_{\mathrm{SP}} & 0 \\
0 & 1 & 0 & -\mathrm{Z}_{\mathrm{SP}} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hdashline[0] & & {\left[\mathrm{T}_{\mathrm{H}-\mathrm{SP}}\right]}
\end{array}\right]
$$

5.8.3 Generalized masses. - Use is made of matrix notation and the pseudo rotor coordinate to express the reference generalized masses given in Table 4.

- 'BLE 4. - REFEREITCE AXIS JENERALIZED MASSES

$$
\begin{align*}
& {\left[M_{R E F-B L}\right]=\left\{\frac{\partial \tau_{H}}{\partial \tau_{R E F}}\right\}^{T}\left\{\frac{\partial \tau_{0}}{\partial \tau_{H}}\right\}^{T}\left[{ }^{M_{O_{R-B L}}}\right]}  \tag{358}\\
& \text { where }\left[{ }^{M_{0}}{ }_{R-B L}\right] \text { is given in Table } 1 . \\
& {\left[M_{R E F-S P}\right]=\left\{\frac{\partial \tau_{O_{H}}}{\partial \tau_{R E F}}\right\}^{T}\left\{\frac{{ }^{\partial \tau_{O}}}{{ }_{S P}}{ }^{\partial \tau_{H}}\right\}^{T}\left[{ }^{M_{O_{S P}}}\right]\left[\frac{{ }^{\partial \tau_{O_{S P}}}}{\partial \tau_{S P}}\right\}} \tag{359}
\end{align*}
$$

$$
\begin{align*}
& +\left\{\frac{\partial \tau_{O_{H}}}{\partial \tau_{R E F}}\right\}\left[\left[{ }^{T}\left[{ }^{T} O_{H}+M_{O_{T}}\right]\right]\left\{\frac{\partial \tau_{H}}{\partial \tau_{S}}\right\}\right. \\
& +\left\{\frac{\partial \tau_{H}}{\partial \tau_{R E F}}\right\}^{T}\left\{\frac{\partial \tau_{O_{S P}}}{\partial \tau_{H}}\right\}^{T}\left[M_{O_{S P}}\right]\left\{\frac{\partial \tau_{O_{S P}}}{\partial \tau_{H}}\right\}\left\{\frac{\partial \tau_{H}}{\partial \tau_{S}}\right\} \tag{360}
\end{align*}
$$

TABLE 4. - Continued

$$
\begin{align*}
& {\left[M_{R E F}\right]=\left\{\frac{\partial \tau_{O_{H}}}{\partial \tau_{R E F}}\right\}^{T}\left\{\frac{\partial \tau_{O_{R}}}{\partial \tau_{H}}\right\}^{T}\left[M_{O_{R}}\right]\left\{\frac{\partial \tau_{O_{R}}}{\partial \tau_{H}}\right\}\left\{\frac{\partial \tau_{O_{H}}}{\partial \tau_{R E F}}\right\}} \\
& +\left\{\frac{{ }^{\partial \tau_{O}}{ }_{H}}{\partial \tau^{\tau}{ }_{R E F}}\right\}^{T}\left[\left[M_{U_{T}}\right]+\left[{ }^{M_{O_{H}}}\right]\right]\left\{\frac{\partial \tau O_{H}}{\partial \tau_{R E F}}\right\} \\
& +\left\{\frac{\partial \tau_{O_{H}}}{\partial \dot{\tau}_{R E F}}\right\}^{T}\left\{\frac{\partial \tau_{O_{S P}}}{\partial \tau_{H}}\right\}^{T}\left[{ }^{T}{ }^{0_{S P}}\right]\left\{\frac{\partial \tau_{S P}}{\partial \tau_{H}}\right\}\left\{\frac{\partial \tau_{0}}{\partial \tau_{R E F}}\right\} \\
& +\left[\left[{ }^{M_{O_{F}}}\right]+\left[M_{E N G}\right]+\left[M_{T R}\right]\right] \tag{361}
\end{align*}
$$

where


The mass of the fuselage is considered to contail. whe engine and tail rotor masses, although the moment of inertias is treated separately.
5.8.4 Generalized forces. - The lcads issociated with the six reference axis degrees of freedom are listed in Table 5. The tail rotor and engine are assumed to have shafts parallel to the fuselage axes. The transfer of the zerodynamic loads from tail rotor axes with origin at hub center and parallel to the fuselage reference axes is shown in the table. The fuselage aerodynamic loads include tail rotor and propulsion terms. Further developmert of the main rotor blade component loads is in Section 5.6 .1 and the aerodynamics for all rotcrs and fix ${ }^{-}$surfaces is left to Section 6.

$$
\left\{\begin{array}{l}
F_{X} \\
F_{Y} \\
F_{Z} \\
F \phi \\
F_{\theta} \\
F \psi
\end{array}\right\}_{R E F}=\left\{\frac{\partial O_{H}}{\partial \tau_{R E F}}\right\}^{T}\left\{\frac{\partial \tau_{O_{H}}}{\partial \tau_{i}}\right\}^{T}\left\{\begin{array}{c}
F_{X} \\
F_{Y} \\
F_{Z} \\
F \phi \\
F \theta \\
F \psi
\end{array}\right\}_{M R}
$$

$$
+\left\{\begin{array} { l } 
{ \partial \tau _ { F } \sigma ^ { T } \tau _ { R _ { L } } \} ^ { T } }
\end{array} \left\{\begin{array}{l}
\left\{\begin{array}{l}
F_{X} \\
F_{Y} \\
F_{Z} \\
F \phi \\
F \theta \\
F \psi
\end{array}\right\}_{F_{A}}-\left[\begin{array}{l}
M_{O_{F}}
\end{array}\right]\left\{\begin{array}{l}
\ddot{x}_{0} \\
\ddot{Y}_{0} \\
\ddot{z}_{0} \\
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{F},
\end{array}\right.\right.
$$

$$
\left[\mathrm{M}_{\mathrm{O}}\right]
$$




The angular velocities and accelerations associated with the engine and tail rotor require special consideration. Here the terms consist of a reference motion plus the turning due to the geared main rotor rate.

Using the rate and acceleration Euler transforms for zero Euler angles (Section 4.4.3):

$$
\left\{\begin{array}{l}
\mathrm{p}  \tag{366}\\
q \\
r
\end{array}\right\}_{\mathrm{TR}-\mathrm{REF}}=\left\{\begin{array}{l}
\mathrm{p} \\
q \\
r
\end{array}\right\}_{\mathrm{REF}}+G_{T R}\left\{\begin{array}{l}
0 \\
\dot{\psi}_{R} \\
0
\end{array}\right\}
$$

$$
\left\{\begin{array}{l}
\dot{\mathrm{p}}  \tag{367}\\
\dot{q} \\
\dot{r}
\end{array}\right\}_{\mathrm{TR}-\mathrm{REF}}=\left\{\begin{array}{l}
\dot{\mathrm{p}} \\
\dot{q} \\
\dot{i}
\end{array}\right\}_{\mathrm{REF}}+\mathrm{c}_{\mathrm{TR}}\left\{\begin{array}{l}
-\dot{\psi}_{\mathrm{R}} r_{\mathrm{REF}} \\
\dot{\psi}_{\mathrm{R}} \\
\dot{\psi}_{R}{p_{R E F}}
\end{array}\right\}
$$

and

$$
\begin{align*}
& \left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{\text {ENG-REF }}=\left\{\begin{array}{l}
p \\
q \\
r
\end{array}\right\}_{\text {REF }}-G_{E N G}\left\{\begin{array}{l}
\psi_{R} \\
0 \\
0
\end{array}\right\}  \tag{368}\\
& \left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{\text {ENG-REF }}=\left\{\begin{array}{l}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\}_{\text {REF }}-G_{E N G}\left\{\begin{array}{l}
\dot{\psi}_{R} \\
\dot{\psi}_{R} \\
r_{\text {REF }} \\
\dot{\psi}_{R} \\
q_{\text {REF }}
\end{array}\right\} \tag{369}
\end{align*}
$$

### 5.9 Swashplate Equations

5.9.1 Partial derivatives. - The swashplate partial derivatives are readily obtained from Section 4.3.s. Using matrix notation

$$
\left\{\frac{{ }^{\partial \tau_{0}}{ }_{S P}}{{ }^{\partial \tau_{S P}}}\right\}=\left[\begin{array}{cccccc}
0 & 0 & 0 & & &  \tag{370}\\
0 & 0 & 0 & & & \\
0 & 0 & 1 & & & \\
\hdashline[ & \cdots & \cos & \theta_{S P} & 0 & 0 \\
{[0]} & i & 0 & 1 & 0 \\
& & & & \\
& & \sin \theta_{S P} & 0 & 0
\end{array}\right]
$$

Since the swashplate axis is directly referenced to the principal (hub) set, the above derivatives are complete. The lack of translation to angular derivatives is explained by the parallelogram linkages used with swashplates to isolate the collective and cyclic inputs. The terms left out of the matrix indicate that the swashplate does not have a yaw degree of freedom.

The reader shouid be aware that the angular notation $\phi, \theta$, and $\psi$ have two meanings, depending on whether they are in the numerator or the denominator of the partial. The numerator is the displacement of the mass element with
respect to the hub axis, whereas the denominator is the degree of freedom incremental variable.

Swashplate motions pick up large inertia loads from the rotcr due to blade feathering. Partials relating feathering to swashplate motions are assembled by first relating the feathering motion in the rotating system rith feathering in the stationary system:

$$
\begin{gather*}
\frac{\partial \phi_{\mathrm{Fn}}}{\partial \theta_{0}}=1  \tag{371}\\
\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1 S}}=-\cos \left(\psi_{\mathrm{BLn}}+\psi_{\mathrm{R}}\right)  \tag{372}\\
\frac{\partial \phi_{\mathrm{Fn}}}{\partial \mathrm{~B}_{1 S}}=-\sin \left(\psi_{\mathrm{BLn}}+\psi_{\mathrm{R}}\right) \tag{373}
\end{gather*}
$$

From Section 4.5.8, equations relating swashplate motions to the stationary feather angles give

$$
\left[\begin{array}{ll}
\frac{\partial A_{1 S}}{\partial \phi_{S P}} & \frac{\partial A_{1 S}}{\partial \theta_{S P}}  \tag{374}\\
\frac{\partial B_{1 S}}{\partial \phi_{S P}} & \frac{\partial B_{1 S}}{\partial \theta_{S P}}
\end{array}\right]=\frac{d}{e}\left[T_{\Psi_{P H}}\right]
$$

and

$$
\left\{\begin{array}{l}
\frac{\partial A_{1 S}}{\partial \theta_{0}}  \tag{375}\\
\frac{\partial B_{1 S}}{\partial \theta_{0}}
\end{array}\right\}=\left(\frac{d}{e}\right)_{1}\left[{ }^{T_{\Psi_{P H}}}\right]\left\{\begin{array}{l}
\phi_{S P} \\
\theta_{S P}
\end{array}\right\}
$$

where

$$
\left[T_{\psi_{\mathrm{PH}}}\right]=\left[\begin{array}{ll}
\sin \psi_{\mathrm{PH}} & \cos \psi_{\mathrm{PH}}  \tag{376}\\
\cos \psi_{\mathrm{PH}} & -\sin \psi_{\mathrm{PH}}
\end{array}\right]
$$

Also

$$
\begin{equation*}
\frac{\partial \theta_{0}}{\partial Z_{S P}}=-\frac{1}{e} \tag{377}
\end{equation*}
$$

The $\left[T_{T}{ }^{T H}\right]$ matrix does not follow the conventional Euler angle notation since a desire existed to define $\psi_{P H}$ as the angle the pitch horn to pitch link attachment point leads the blade. The overall derivatives can be put together as:

$$
\begin{gather*}
\frac{\partial \phi_{\mathrm{Fn}}}{\partial \phi_{\mathrm{SP}}}=\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1 S}} \frac{\partial A_{1 S}}{\partial \phi_{\mathrm{SP}}}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial \mathrm{~B}_{1 S}} \frac{\partial \mathrm{~B}_{1 S}}{\partial \phi_{\mathrm{SP}}}  \tag{378}\\
\frac{\partial \phi_{\mathrm{Fn}}}{\partial \theta_{\mathrm{SP}}}=\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1 S}} \frac{\partial A_{1 S}}{\partial \theta_{\mathrm{SF}}}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial \mathrm{~B}_{1 S}} \frac{\partial \mathrm{~B}_{1 S}}{\partial \theta_{\mathrm{SP}}}  \tag{379}\\
\frac{\partial \phi_{\mathrm{Fn}}}{\partial Z_{\mathrm{SP}}}=\left(\frac{\partial \phi_{\mathrm{Fn}}}{\partial \theta_{0}}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial A_{1 S}} \frac{\partial A_{1 S}}{\partial \theta_{0}}+\frac{\partial \phi_{\mathrm{Fn}}}{\partial \mathrm{~B}_{1 S}} \frac{\partial \mathrm{~B}_{1 S}}{\partial \theta_{0}}\right) \frac{\partial \theta_{0}}{\partial Z_{\mathrm{SP}}} \tag{380}
\end{gather*}
$$

These partials are the elements of the $\left\{\left.\frac{{ }^{\partial \phi_{\mathrm{Fn}}}}{{ }^{\partial \tau_{S P}}} \right\rvert\,\right.$ vector.
5.9.2 Generalized masses. - Table 6 presents the generalized masses which couple the swashplate motions with one and another and with the blade fuselage degrees of freedom. The table uses summations of the blade that air described in detail in Section 5.6.1.
table 6. - SWAShPLate generalized masses

$$
\begin{align*}
& +\left|\frac{\partial \tau_{o S P}}{\partial \tau_{S P}}\right|^{\mathrm{T}} \quad\left(M_{o S P} \left\lvert\, \quad\left\{\left.\frac{\partial \tau_{o S P}}{\partial \tau_{S P}} \right\rvert\,\right.\right.\right. \tag{382}
\end{align*}
$$

5.9.3 Generalized forces. - The generalized forces are: (assuming a constant speed drive)

$$
\begin{align*}
& F_{\phi_{S P}}=-\frac{\partial \phi_{O S P}}{\partial \phi_{S P}}\left(\dot{p}_{S P} I_{X X}+q_{S P} r_{S P}\left(I_{Z Z}-I_{X X}\right)_{S P}\right)-\frac{\partial_{\psi_{O S P}}}{\partial_{S P}} \dot{r}_{S P} I_{Z z_{S P}} \\
& +\sum_{n=1}^{b} M_{F n} \frac{\partial \phi_{\mathrm{Fn}}}{\partial \phi_{\mathrm{SP}}}-\frac{\partial U}{\partial \phi_{\mathrm{SP}}}-\frac{\partial B}{\partial \dot{\phi}_{\mathrm{SP}}}-M_{\mathrm{FR}, \phi_{\mathrm{SP}}}  \tag{383}\\
& F_{\theta_{S P}}=-\frac{\partial \theta_{O S P}}{\partial \theta_{S P}}\left(\dot{q}_{S P} I_{Y Y_{S P}}-r_{S P} p_{S P}\left(I_{Z Z}-I_{X X}\right)_{S P}\right) \\
& +\sum_{n=1}^{b} M_{F n} \frac{\partial \phi_{F n}}{\partial \theta_{S P}}-\frac{\partial U}{\partial \theta_{S P}}-\frac{\partial B}{\partial \theta_{S P}}-M_{F R}, \theta_{S P} \tag{384}
\end{align*}
$$

Note $p, q$ terms are the same for $R$ and $N R$ systems.

$$
\begin{equation*}
F_{Z_{S P}}=-\left(\ddot{z}_{S P}+\ddot{z}_{F}\right) m_{S P}+\sum_{n=1}^{b} M_{F n} \frac{\partial \phi_{F n}}{\partial Z_{S P}}-\frac{\partial U}{\partial Z_{S P}}-\frac{\partial B}{\partial \hat{Z}_{S P}} \tag{385}
\end{equation*}
$$

where

$$
\left\{\begin{array}{c}
\dot{p}  \tag{386}\\
\dot{q} \\
\dot{r}
\end{array}\right\}_{S P, N R}=\left\{\begin{array}{c}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{array}\right\} \Delta P=\left\{\begin{array}{c}
\dot{\psi} q \\
-\dot{\psi} p \\
\dot{\psi}
\end{array}\right\} S P
$$

The moments used in these formulas are developed below.

The feathering moment, $M_{F n}$, is taken to be composed of blade and friction 10ads.

$$
\begin{equation*}
M_{\mathrm{Fn}}=\mathrm{M}_{\mathrm{Fn}}+\mathrm{M}_{\mathrm{FR}}^{\mathrm{Fn}} \tag{387}
\end{equation*}
$$

The detailing of ${ }^{M} X_{F n}$, feathering moments due to blade loads, is accomplished in Section 5.6.4. The friction load, $M_{F R ~ F n}$ follows the function shown in Figure 30. By reducing $\psi_{\mathrm{Fn}, \mathrm{BK}}$ to near zero, stiction
is obtained. Otherwise, if $\phi_{\mathrm{Fn}, \mathrm{BK}}$ is large, the ratio $\frac{\mathrm{M}_{\mathrm{FR}} \mathrm{Fn}, \mathrm{BK}^{\phi_{\mathrm{Fn}, \mathrm{BK}}}}{}$
determines the amount of viscous friction.
The remaining portions of the generalized force are the potential energy and dissipation functions. First consider the angular potential energy terms which model the swashblate tilt spring rate. This spring rate has a center dead-band, an operating range spring rate, and a high spring rate to simulate a travel limit stop.

Consider the normal operating range spring rate first. The swashplate springs are defined in control axes (Figure 31 ) as $K_{\phi_{S P}}$ and $K_{\theta_{S P}}$ (can be unequal in size). To find the elastic spring loads, the swashplate motions are first found in control axes as:


Figure 30. - Swashplate friction.


Figure 31. - Control axis.

$$
\left\{\begin{array}{l}
\phi_{\mathrm{SP}}  \tag{388}\\
\theta_{\mathrm{SP}}
\end{array}\right\}_{\mathrm{C}}=\left[\begin{array}{cc}
\cos \psi_{\mathrm{C}} & \sin \psi_{\mathrm{C}} \\
-\sin \psi_{\mathrm{C}} & \cos \psi_{\mathrm{C}}
\end{array}\right]\left\{\begin{array}{l}
\phi_{\mathrm{SP}} \\
\theta_{\mathrm{SP}}
\end{array}\right\}
$$

The geometric interpretation of $\psi_{C}$ is shown in Figure 31.
Taking the swashplate deflections in the control axes, subtracting control inputs $\phi_{C}$ and $\theta_{C}$, and using the inverse transform, the swashplate spring terms in swashplate axes become:

$$
\left\{\begin{array}{c}
\frac{\partial U}{\partial \phi_{S P}}  \tag{389}\\
\frac{\partial U}{\partial \theta_{S P}}
\end{array}\right\}_{1}=\left[\begin{array}{rr}
\cos \psi_{C} & -\sin \psi_{C} \\
\sin \psi_{C} & \cos \psi_{C}
\end{array}\right]_{\mathrm{C}}\left\{\begin{array}{ll}
K_{\phi_{S P}} & \left(\phi_{\mathrm{SP}}-\phi_{\mathrm{C}}\right)_{\mathrm{C}} \\
K_{\theta_{S P}} & \left(\theta_{\mathrm{SP}}-{ }^{\theta_{C}}\right)_{C}
\end{array}\right\}
$$

where the subscript ( 1 ; is used to distinguish these values (used in subsequent logic calculations) from the final expressions developed below.

Substituting for the swashplate motions in terms of the swashplate axes and rearranging:

$$
\left\{\begin{array}{c}
\frac{\partial \mathrm{U}}{\partial \phi_{S P}}  \tag{390}\\
\frac{\partial \mathrm{U}}{\partial \theta_{\mathrm{SP}}}
\end{array}\right\}_{1}=\left|\mathrm{K}_{\mathrm{SP}}\right|\left\{\begin{array}{l}
{ }_{\mathrm{SP}} \\
\theta_{\mathrm{S}}
\end{array}\right\}-\left|\mathrm{T}_{\psi_{C}}\right|^{\mathrm{T}}\left\{\begin{array}{l}
\mathrm{K}_{\phi_{\mathrm{SP}}{ }_{\mathrm{C}}} \\
{ }_{\mathrm{C}} \\
\mathrm{~K}_{\theta_{S P} \theta_{\mathrm{C}}}
\end{array}\right\}
$$

where

$$
\left[T_{\psi_{C}}\right]^{T}=\left[\begin{array}{cc}
\cos \psi_{C} & -\sin \psi_{C}  \tag{391}\\
\sin \psi_{C} & \cos \psi_{C}
\end{array}\right]
$$

and

Note: $\left[K_{S P}\right]$ is a symmetric matrix of constants.
The center dead band is modeled by the following logic.

$$
\begin{equation*}
\frac{\partial U}{\partial \phi_{\mathrm{SP}}}=0 \text { if }\left|\left(\frac{\partial \mathrm{U}}{\partial \phi_{\mathrm{SP}}}\right)_{1}\right| \leq \mathrm{K}_{\mathrm{SP}}(1,1) \delta \phi_{\mathrm{SP}} \tag{393}
\end{equation*}
$$

otherwise

$$
\begin{align*}
& \frac{\partial U}{\partial \phi_{S P}}=\left(\frac{\partial U}{\partial \phi_{S P}}\right)_{1}-K_{S P}(1,1) \delta \phi_{S P} \operatorname{SIGN}\left(\frac{\partial U}{\partial \phi_{S P}}\right)_{1}  \tag{394}\\
& \frac{\partial U}{\partial \theta_{S P}}=0 \text { if }\left|\left(\frac{\partial U}{\partial \theta_{S P}}\right)_{1}\right| \leq K_{S P}(2,2) \delta \theta_{S P} \tag{395}
\end{align*}
$$

otherwise

$$
\begin{equation*}
\frac{\partial U}{\partial \theta_{S P}}=\left(\frac{\partial U}{\partial \theta_{S P}}\right)-K_{S P}(2,2) \delta \theta_{S P} \operatorname{SIGN}\left(\frac{\partial U}{\partial \theta_{S P}}\right)_{1} \tag{396}
\end{equation*}
$$

$\delta \phi_{S P}$ and $\delta \theta_{S P}$ are input constants giving swashplate angular freeplay. Swashplate stops are also allowed with spring rate $K_{S, S P}$. A load

$$
\mathrm{K}_{\mathrm{S}, \mathrm{SP}}\left(\left(\phi_{\mathrm{SP}}^{2}+\theta_{\mathrm{SP}}^{2}\right)^{1 / 2}-\delta_{\mathrm{S}, \mathrm{SP}}\right)\left\{\begin{array}{l}
\phi_{\mathrm{SP}}  \tag{397}\\
\theta_{\mathrm{SP}}
\end{array}\right\} /\left(\phi_{\mathrm{SP}}^{2}+\theta_{\mathrm{SP}}^{2}\right)^{1 / 2}
$$

$$
\left\{\begin{array}{c}
\frac{\partial U}{\partial \phi_{S P}}  \tag{398}\\
\frac{\partial U}{\partial \theta_{S P}}
\end{array}\right\}
$$

to account for a limit travel stop. The limit deflection for the swashplate is

$$
\begin{equation*}
\left(\phi_{S P}^{2}+\theta_{S P}^{2}\right)^{1 / 2} \leq \delta_{S, S P} \tag{399}
\end{equation*}
$$

where $\delta_{S, S P}$ is the circular stop swashplate deflection limit.
The angular damping term is analogous to the spring load:

$$
\left\{\begin{array}{c}
\frac{\partial \mathrm{B}}{\partial \dot{\phi}_{\mathrm{SP}}}  \tag{400}\\
\frac{\partial \mathrm{~B}}{\partial \dot{\xi}_{\mathrm{SP}}}
\end{array}\right\}=\left[\mathrm{C}_{\mathrm{SP}}\right]\left\{\begin{array}{l}
\dot{\phi}_{\mathrm{SP}} \\
\dot{\theta}_{\mathrm{SP}}
\end{array}\right\}
$$

where $\left[{ }^{C_{S P}}\right]$ has the same formulation as $\left[K_{S P}\right]$.
Control friction is treated as having rotating and nonrotating components. The rotating component has already been discussed as part of the feathering moment. The nonrotating component is applied to the swashplate. It has the formulation shown in Figure 31 with a change in labels such that $\dot{\phi}_{\text {Fn }}$ is either $\dot{\phi}_{S P}$ or $\dot{\theta}_{S P}$ and $M_{F R}$ is either $M_{F R, \phi}$ or $M_{F R}, \theta_{S P}$

The vertical potential energy term is described as:

$$
\begin{equation*}
\frac{\partial U}{\partial Z_{S P}}=K_{1 Z_{S P}} Z_{S P}+F_{C} \quad \text { if }\left|z_{S P}\right|<Z_{1_{S P}} \tag{401}
\end{equation*}
$$

Otherwise

$$
\begin{equation*}
\frac{\partial U}{\partial Z_{S P}}=K_{1 Z_{S P}} Z_{1_{S P}}+K_{2 Z_{S P}}\left(z_{S P}-Z_{1_{S P}}\right)+F_{C} \tag{402}
\end{equation*}
$$

$F_{C}$ is a constant to center the gyro springs.
The spring rate is taker to be $K_{1 Z_{S P}}$ out to deflection $Z_{1_{S P}}$ and $K_{2 Z_{S P}}$ beyond.

A simple coupling from the rotary dampers gives the vertical dissipation function.

$$
\begin{equation*}
\frac{\partial B}{\partial Z_{S P}}=C_{Z_{S P}} \dot{z}_{S P}-R_{Z \phi} C_{\phi_{S P}} \dot{\phi}_{S P} \phi_{S P}-R_{Z \theta} C_{\theta_{S P}} \cdot \dot{\theta}_{S P} \theta_{S P} \tag{403}
\end{equation*}
$$

where $R_{Z \phi}, R_{Z \theta}$ are coupling ratios. Note the effect of vertical motion on the swashplate tilt loads through the rotary dampers is assumed zero.

To correlate with flight test records and/or to force the swashplate vertical response to cross the spring rate changeover a force offset constant is used. Introducing this constant into the swashplate vertical degree of freedom equation line, causes the variables, primarily the swashplate vertical motion, to shift and rebalance the equation.
5.9.4 Control inputs. - The swashplate input is controlled by the pilot's cyclic stick. The input torques are:

$$
\left\{\begin{array}{ll}
K_{\phi_{C}} & \phi_{C}  \tag{404}\\
K_{\theta_{C}} & { }^{\theta_{C}}
\end{array}\right\}=\left\{\begin{array}{ll}
-K_{X} & \\
{ }_{C} & { }_{C} \\
K_{Y_{C}} & Y_{C}
\end{array}\right\}
$$

The inputs are aligned with the control axis (Figure 31).
Note the equivalence of forms in terms of angular commands $\phi_{C}, \theta_{C}$, or
longitudinal stick (aft) $X_{C}$ and lateral stick (right) $Y_{C}$.

The controls are frequently linked to the swashplate through actuators which, as a first-order approximation, can be simulated by a first-order lag. See Section 7.2.

### 5.10 Engine Equations

5.10.1 Rotor azimuth and rotation rate. - The program allows a variation of rotor speed in maneuvers due to variations in the torque required by the various rotors and in the torque supplied by the engine. The dynamic system rotates as a rigid, geared unit. That is, the shafts are not allowed elastic windup. The main rotor speed, $\dot{\psi}_{R}$ and hence the engine speed, is referenced to the fuselage, and not to inertial space. The displacement $\psi_{R}$ is the azimuth of the number one blade.
5.10.2 Engine model. - Figure 32 illustrates the engine model used in the program. The figure also plots typical engine torque characteristics. The model represents the first-order lag power response characteristics of the free turbine powerplants commonly used in rotorcraft applications.

Being a perturbation model, the engine is referenced to its trim position. The change in engine torque in a maneuver is

$$
\begin{equation*}
M_{X A_{E N G}}-M_{X A_{E N G, T R I M}}=\frac{\partial M_{E N G}}{\partial \dot{\psi}_{G E N}} \dot{\psi}_{G E N}-\frac{\partial M_{E N G}}{\partial \dot{\psi}_{E N C}}\left(\dot{\psi}_{E N G}-\dot{\psi}_{E N G, T R I M}\right) \tag{405}
\end{equation*}
$$

where $0 \leq M_{X A_{E N G}} \leq M_{X A_{E N G, M A X}}$. The zero limit occurs if the overrunning clutch disconnects the engine in the transition to autorotation. The maximum value corresponds to the engine shaft torque limit.

The gas generator, speed, $\dot{\psi}_{\text {GEN }}$, is a degree of freedom. It is considered a secondary degree of freedom in that the coupling through the generalized masses with the primary degrees of freedom can be neglected. An equation for the generation speed can be supplied from its torque characteristics:

$$
\begin{equation*}
I_{G E N} \ddot{\psi}_{G E N}+C_{G E N} \dot{\psi}_{G E N}=-K_{E N G I} \ddot{\psi}_{E N G}-K_{E N G 2}\left(\dot{\psi}_{E N G}-\dot{\psi}_{E N G, T R I M}\right) \tag{4+6}
\end{equation*}
$$



Figure 32. - Engine model and torque-speed characteristics.

The terms on the left represent acceleration inertia torque and steadystate torque. On the right, the fuel control causes torque to be added if the engine speed drops below the trim value. The $\ddot{\psi}_{\text {ENG }}$ term exists since the control is modeled with simple lag. Restating t.is equation, using rotor speed and a generator time constant, gives:

$$
\begin{equation*}
\ddot{\psi}_{G E N}=\frac{-\dot{\psi}_{G E N}-K_{R 1} \ddot{\psi}_{R}-K_{R 2}\left(\dot{\psi}_{R}-\dot{\psi}_{R, T R I M}{ }^{\prime}\right.}{{ }_{\mathrm{GEN}}} \tag{407}
\end{equation*}
$$

where $\psi_{G E N}=\frac{I_{G E N}}{C_{G E N}}$ is the order of a second.
The engine droop characteristic can be used to size the engine constants. With $\ddot{\psi}_{\text {GEN }}=\ddot{\psi}_{R}=0$, substituting the generator equation ..... ae engine equation and rearranging,

$$
\begin{equation*}
\frac{\Delta\left({ }^{M_{X A}}{ }_{E N G}\right)_{R}}{\Delta \dot{\psi}_{R}}=\frac{\partial\left(M_{E N G}\right)_{R}}{\partial \psi_{G E N}} K_{R 2}-\frac{\partial\left(M_{\mathrm{ENG}}\right)_{R}}{\partial \dot{\psi}_{\mathrm{R}}} \tag{408}
\end{equation*}
$$

Only $\Delta$ incremental changes are of interest. The bracket subscripted $R$ indicates the torque is determined at the rotor speed and includes the engine gear ratio. The term on the right is the static droop line shown in Figure 32. This plot also reometrically interprets the partial derivatives on the left.

The generator speed $\dot{\psi}_{G E N}$ is not given a reference. Its value is zero when trim is completed.
5.10.3 Partial derivatives. - Shaft rotation not oniy invoives biade root rotation $\psi_{R}$, but also feathering motions. The feathering partial is obtained by differentiatiog the feather angle equation in Section 4.5 .8 :

$$
\begin{equation*}
\left.\frac{\partial \phi_{\mathrm{Fn}}}{\partial \psi_{\mathrm{R}}}=A_{1 S} \sin \left(\psi_{E L n}+\psi_{R}\right)-B_{1 S} \cos \psi_{B L n}+\psi_{R}\right) \tag{4,09}
\end{equation*}
$$

A set of partials are defined to relate the various rotating components to the rotor shaft and to the reference set.

$$
\begin{align*}
& \left\{\begin{array}{c}
\partial \tau_{O R} \\
\partial \tau_{R}
\end{array}\right\}=\left(\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right\}  \tag{410}\\
& \left\{\begin{array}{c}
\partial \tau_{\mathrm{OH}} \\
\partial \phi_{\mathrm{R}}
\end{array}\right\}=\left\{\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
-1
\end{array}\right\}  \tag{411}\\
& \left\{\begin{array}{c}
\partial \tau_{\mathrm{EMG}} \\
\partial \phi_{\mathrm{R}}
\end{array}\right\}=\left\{\begin{array}{c}
0 \\
0 \\
0 \\
-\mathrm{C}_{\mathrm{EIG}} \\
0 \\
0
\end{array}\right\}  \tag{412}\\
& \left\{\frac{\partial \tau_{\mathrm{TR}}}{\partial \phi_{\mathrm{R}}}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
\mathrm{G}_{\mathrm{T}_{\mathrm{R}}} \\
0
\end{array}\right\}  \tag{413}\\
& \left\{\frac{\partial \tau_{\mathrm{EMG}}}{\partial{ }^{\mathrm{REF}}}\right\}=[I]  \tag{414}\\
& \left\{\frac{\partial \tau_{T R}}{\partial \tau_{R E F}}\right\}=[\mathrm{I}] \tag{415}
\end{align*}
$$

5.10.4 Generaiized masses. - The engine degree of freedom couples with every other degree of freedor. Equations for the engine generalized masses are given in Table T. Matrix notation is again used for compactness. Hote the transmission is modeled as a non rotating mass, and therefore does not appear in these masses. The engine iegree of freedom contains not oniy rigid body motion of the roter blades, but also biade feathering. The feathering contribution is a minor contributor for some of the mass matrices and has been neglected.

TAELE 7. - ENGINE GEvERALIZED MASSES

$$
\left.\left.+\sum_{n=1}^{\mathrm{Mb}}\left\{\frac{\partial \phi_{\mathrm{FN}}}{\partial \psi_{\mathrm{R}}}\right)^{\mathrm{T}} \right\rvert\, \frac{\partial \tau_{\mathrm{BL}}}{\partial \tau_{\mathrm{FN}}}\right) \left.^{\mathrm{T}}\left[M_{\mathrm{BL}-\mathrm{BL}}\right]_{\mathrm{n}}\left(\frac{\partial \tau_{\mathrm{BL}}}{\partial \varphi_{\mathrm{FN}}}\right) \right\rvert\,\left(\frac{\partial \phi_{\mathrm{Fi}}}{\partial \tau_{S P}}\right)
$$

$\left[M_{R-S}\right]=\left\{\frac{\partial \tau_{O R}}{\partial \psi_{R}}\right\}^{T}\left[M_{O R}\right]\left\{\frac{{ }^{T} \tau_{O R}}{\partial \tau_{H}} \left\lvert\,\left\{\left.\frac{\partial \tau_{O H}}{\partial \tau_{S}} \right\rvert\,\right.\right.\right.$
$+\left\{\frac{\partial \tau_{\mathrm{OH}}}{\partial \Psi_{\mathrm{R}}}\right\}^{\mathrm{T}}\left[\mathrm{M}_{\mathrm{OH}}\right]\left\{\frac{\partial \tau_{\mathrm{OH}}}{\partial \tau_{\mathrm{S}}}\right\}$
$\left.+\left\{\frac{\partial \tau_{O H}}{\partial \psi_{R}}\right\}^{T} \quad\left(\frac{\partial \tau_{O S P}}{\partial \tau_{H}}\right\} \quad\left[M_{O S P}\right]\left\{\frac{\partial \tau_{O S P}}{{ }^{T} \tau_{H}}\right) \right\rvert\,\left\{\begin{array}{l}\frac{\partial \tau_{O H}}{\partial \tau_{S}}\end{array}\right.$

$$
\begin{align*}
& {\left[M_{R-B L}\right]_{n}=\left\{\frac{\partial \tau_{O R}}{\partial \psi_{R}}\right\}^{T}\left[M_{O R-B L}\right]_{n}} \tag{416}
\end{align*}
$$


5.10.5 Generalized forces. - Only one generalized force is needed.

$$
\begin{align*}
F_{R} & =\left(F_{\psi}\right)_{M R}-G_{E N G} I_{X X}{ }_{E N G}\left(\dot{P}_{F}-G_{E N G} \ddot{\psi}_{R}\right) \\
& +G_{T R}\left(F_{\phi_{T R}}-I_{Y Y_{T R}}\left(\dot{q}_{F}+G_{T R} \ddot{\psi}_{R}\right)\right) \\
& -\left(I_{Z_{Z}}+I_{Z Z_{S P}}\right) \ddot{\psi}_{R}+\sum_{n=1}^{N b} \frac{\partial \phi_{F n}}{\partial \psi_{R}} M_{F n}+G_{E N G} M_{X_{i E N G}} \tag{421}
\end{align*}
$$

The main rotor contribution (inertial and aerodmamic loads) is given in Section 5.6.4. The taii rotor aercdynamics are described in Section 6.5.

## 6. AERODYNAMICS

### 6.1 Introduction

Other than gravity, the external loadings acting on the REXOR II equations of motion can be traced to aerodynamic sources. The following subsections trace the source, nature and use oi these aerodynamic loads.
6.1.1 Aerodynamic forces producing surfaces considered. - The aerodynamic loads considered in REXOR II are divided into the categories of (1) associated rith the main rotor, or (2) the rest of the rotorcraft (nonrotating surfaces and tail rotor). In viev of the stated objectives of REXOR II, the program development emphasis is on the main rotor which is considered in Section 6.2.

The nonrotating components consist of the fuselage, ving, vertical tail, lower horizontal tail, upper horizontal tail, tail rotor, auxiliary thrustors, movable surfaces on the wings and empennage, and dive brakes. Wake effects from the main rotor and ving are addressed in Section 6.3. The nonrotating load elements are mostly jeveloped and assembled in Section 6.4. The tail rotor equations, in integrated form, are developed in Section 6.5, and the auriliary thrustor formulation is in Section 6.6.
6.1.2 Use of forces generated. - As mentioned, the aerodynamic loads are in essence the external forcing functions of the equations of motion. Generally the developed loads are in the axis of the apparent air velocity of the loaded element. Thus transformations are required to put the loads into the reference axes of the equation of motion considered.

### 6.2 Main Fntor

6.2.1 Overview. - To generate a main rotor model with sufficient detail to lo dynamic investigations, a reasonably good quality aerodynamics presentation is required. To this end a table lookup of blade section properties, multifunction inflow model, quasi-steady aerodynamics, and dynamic stall are used in REXOR II.
6.2.1.1 Blade flow field. - As developed in the following subsections, the instantaneous blade airflow is the inertial velocity of the blade element. This velocity includes the motion of the principal reference set and the motion of the blade element with respect to the principal reference set. The calculation assumes the airmass is at rest, which is reasonable for dynamics investigations.
6.2.1.2 Air pressure and angle of attack. - The dynamic pressure used for these calculations is based on sea level standard density. The loads are ratioed to the actual air density.

The angle of attack is the sum of geometric pitch argle and the instantaneous air velocity. The rate of angle of attack is also calculated and used for the transient blade aerc loads, Sections 6.2.3.3 and 6.2.3.4.
6.2.1.3 Forces and moments produced. - The steady blade loads are produced from the air velocity components of Section 6.2.3.1 and the coefficient tata $\left(C_{L}, C_{D}, C_{M}\right)$ of Section 6.2.4. The transiect lift and moment effects are developea in Sections 6.2.3.3 (quasi-steady aerodynamics) and 6.2.3.4 (dyramic stall).
6.2.2 Concept of rotor inflow model. - The main rotor inflow model used in REXOR II is based on the air flow incident upon the rotor disc plus the air velocity imparted due to momentum exchange due to integrated blade span loading. This is to be contrasted vith a formulation which tracks the rotor blade positions and the attendant trailing vorticies.

The incident air flow is the inertial velocity of the rotor coardinates, and is directly available from the preceding mechanical development. However a number of assumptions need tc be stated and utilized to arrive at the induced velocity component of the inflow model.

### 6.2.2.1 Induced velocity assumptions. -

1. Only the vertical cownwash and its variations radially and azimuthally over the rotor disk are considered. Induced swirl and lateral doumash components are neglected.
2. Dowmash effects due to unsteady aerodynamics are not treated here as an overall effect, but as a blade segment condition in Section 6.2.3.3.
3. Rotor-induced flow distribution in hover and forward flight is patterned after Reference 6. This reference assumes a uniform loading in hover. Figure 33, from Reference 7 , shows this distribution compared with typical loading and a triangular loading model. Figure 34 from Reference 6 shows the theoretical induced velocity distribution in forward flight as a consequence of a uniform hover distribution. This data is fitted to slopes or a longitudinal skew as a function of speed in REXOR II. Lateral distribution remains uniform in accord with Rererence 8 , which corrects the lateral distribution work of Reference 6.
4. A variation in lateral and longitudinal induced velocity is included to account for roll and pitch aerodynamic shaft moments.
5. Lifting line theory correction is accounted for by an effective rotor radius, BR.


Figure 33. - Blade loading distributions in hover.


Figure 34. - Induced velocity distribution as a function of wake angle (forward flight).
6. Root cut out effects are ignored.
7. Transient effects are simulated by a single time lag.
6.2.2.2 Steady state values. - The starting point for determining the downwash is momentum theory as applied to an elementary dA:

$$
\begin{align*}
d T & =\left(\frac{d m}{d t}\right) 2 w_{i}=\left({ }^{\rho} V_{i M R} d A\right) 2 u_{i} \\
& =\rho \sqrt{u_{H}^{2}+v_{H}^{2}+\left(u_{H}-w_{i}\right)^{2}} d A 2 w_{i} \tag{422}
\end{align*}
$$

The thrust increment is dT , $\mathrm{dm} / \mathrm{dt}$ is the flow of air through the rotor disk with resultant velocity $V_{i M R}$, $\rho$ is the air density and $w_{i}$ the downwash velocity. The velocities are taken in hub coordinates and no effort is made to account for rotor tilt.

The thrust expression above is used to define the following induced velocity components.

$$
\begin{aligned}
& \text { Average component, } w_{i} \\
& \text { Longitudinal variation with pitching aerodynamic } \\
& \text { moment, } q_{i M R} \\
& \text { Lateral variation with roll aerodynamic moment, } p_{i M R}
\end{aligned}
$$

The downash velocity becomes

$$
\begin{equation*}
w_{i}=w_{i M R}+r q_{i M R} \cos \psi_{R}+r p_{i M R} \sin \psi_{R} \tag{423}
\end{equation*}
$$

The coefficients can be evaluated by equating the thrust and moment values for the main rotor equations to the integrals of the momentum expressions at hand. First consider the thrust expression. The evaluating task can be reduced by employing some boundary conditions. For rotor thrust only (no moment), $q_{i M R}$ and $p_{i M R}=0$. A convenient expression for the elementary area, $d A$, is shown in Figure 35 . While radial anulii would serve for thrust integration, the form selected is particularly suited for the moment expressions.


Figure 35. - Incremental area for shaft moment integration.

For the average rotor thrust,

$$
-F_{Z A_{N R, H}}=T=\int_{-B R}^{B R} \rho \sqrt{u_{H}^{2}+v_{H}^{2}+\left(w_{H}-w_{i M R}\right)^{2}}\left(2 \sqrt{(B R)^{2}-r^{2}} d r\right) 2 w_{i M R}
$$

A further assumption is required to solve the square root of this expression and the corollary momentum equations.

For forward flight

$$
\begin{equation*}
\left.w_{i} \ll v_{i M R}=\sqrt{u_{0}+v_{E}^{2}+\left(w_{v}-w_{i M R}\right)}=\text { (constant }\right) \tag{425}
\end{equation*}
$$

Completing the integration gives

$$
\begin{equation*}
-F_{\Sigma A_{M R, H}}=\rho \pi(B R)^{2} v_{i M R} 2 w_{i M R} \tag{426}
\end{equation*}
$$

Next consider the case of no rolling moment; i.e., only thrust and pitching moment. Figure 35 is used with the incremental strip considered to be rightleft oriented so that all equal values of $q_{i M R}$ are integrated at once.

Then,

$$
\begin{aligned}
M_{Y A_{M R, H}} & =\int_{-B R}^{B R} r d T \\
& =\int_{-B R}^{B R} r \dot{\rho} \sqrt{u_{H}^{2}+v_{H}^{2}+\left(w_{H}-W_{i M R}-r q_{i M R}\right)^{2}} \\
& \left(2 \sqrt{(B R)^{2}-r^{2}} d r\right) 2\left(w_{i M R}+r q_{i M R}\right)
\end{aligned}
$$

$$
\begin{equation*}
M_{Y A_{M R, H}}=\rho \frac{\pi(B R)^{4}}{4} v_{i M R} 2 q_{i M R} \tag{428}
\end{equation*}
$$

Likewise for rolling moment, and using fore-aft increment strips gives

$$
\begin{equation*}
M_{\mathrm{VA}_{V R, H}}=0 \frac{\Psi^{\prime}(\mathrm{BR})^{4}}{4} v_{i M R}=E_{i M R} \tag{429}
\end{equation*}
$$

Note the subscript $A$ on $F_{Z_{M R, H}}, M_{X A} A_{M R, H}$, and $M_{Y A}{ }_{M R, H}$ denotes the aerodynamic component only of main rotor loads in hub axes.

The foregoing expressions are now developed for hovering and low-speed fligat. In this sondition,

$$
\begin{equation*}
u_{\mathrm{H}}^{2}+{v_{H}}^{2} \ll\left(\mathrm{w}_{\mathrm{H}}-\mathrm{w}_{\mathrm{iMR}}\right)^{2} \tag{430}
\end{equation*}
$$

Integrating gives

$$
\begin{equation*}
-\mathrm{F}_{\mathrm{ZA}}^{\mathrm{MR}, \mathrm{H}},=\rho \pi(\mathrm{BR})^{2}\left(\mathrm{w}_{\mathrm{H}}-\mathrm{w}_{\mathrm{iMR}}\right) \quad 2 \mathrm{w}_{\mathrm{iMK}} \tag{431}
\end{equation*}
$$

and

$$
\begin{align*}
& M_{\mathrm{YA}, \mathrm{H}}=\left(\frac{\rho r(\mathrm{BR})^{4}}{4}\left(w_{H}-w_{i M R}\right)\right) 2 q_{i M R}\left(1-\frac{w_{i M R}\left(w_{H}-w_{i M R}\right)}{\left(w_{H}-w_{i M R}\right)^{2}}\right) \\
& M_{X A_{M R, i}}=\left(\frac{\rho \pi(\mathrm{BR})^{4}}{4}\left(w_{H}-w_{i M R}\right)\right) \quad 2 p_{i M R}\left(1-\frac{w_{i M R}\left(w_{H}-w_{i M R}\right)}{\left(w_{H}-w_{i M R}\right)^{2}}\right)
\end{align*}
$$

The consequence of cyclic, first-harmonic downash vas explored in Reference 9. Their conclusion, whirh parallels Lockheed's experience, is that the phase and magnitude of the flap response of a hingeless tlade to cyclic feathering is markediy affected by cyclic downash. The shaft moments variation vith feathering angle and the phase angle betveen flap and feathering are both reduced with cyclic downash, the effect being greater in hover than in forvard flight.

A physical interpretation can be rationalized for the formula above, at least in hover, in that the aerodymaic thrust and moment produces a flow of linear and angular mamentum. Lagine the flov as a continuous stack of disks having mass per unit thickness $p \|(B R)^{2}$ and dimetral inertia per unit thickness $\rho\left(\pi(B R)^{4} / 4\right)$. $2 v_{i N R}, 2 p_{i M R}$ and $2 q_{i N R}$ are the final, far downstream position, values of induced velocities obtain by these disks oriented with the flow. The terms $\rho \pi(B R)^{2} V_{i}$ anc $\rho \pi\left(B R ;{ }^{4} / 4 V_{i}\right.$ are the mass flow per unit time, and the moment of inertia flov per unit time through the actuator disk, which times $\mathbf{2} \mathbf{w}_{i M R}, \mathbf{2} p_{i M R}$ or $\mathbf{2 q}_{i M R}$ is the gain of momentum.

For programing purposes, an empirical blend of the forvard flight and hovering sets of expressions is used. The limiting cases of the empirical set give the derived ceses. The expressions used are:

$$
\begin{align*}
& -F_{Z A_{M R, H}}-\rho \pi(B R)^{2} V_{i M R} 2 w_{i M R}  \tag{434}\\
& -M_{Y_{M R, B}}=\frac{\rho \pi(B R)^{4}}{4} v_{i M R} 2 q_{i M R}\left[1-\frac{w_{i M R}\left(w_{H}-w_{i M R}\right)}{v_{i M R}{ }^{2}}\right]  \tag{435}\\
& { }^{-M_{X A}}{ }_{M R, H}=\frac{\rho \pi(B R)^{4}}{4} v_{i M R}{ }^{2} p_{i M R}\left[1-\frac{v_{i M R}\left(w_{H}-w_{i M R}\right)}{V_{i M R} ?}\right] \tag{436}
\end{align*}
$$

6.2.2.3 Variations in forward flight and is grounc effects. - The previous develcpment can be assembled and combined with linearized forward fiight distribution and ground effect factors.

$$
\begin{align*}
\left(\psi_{B L E}\right)_{D W, B L n}= & w_{i M R} f_{i M R}\left[1+K_{i M R} \frac{r}{R} \cos \left(\psi_{R}+\psi_{B L n}+\psi_{W}\right)\right] \\
& +r p_{i M R} \sin \left(\psi_{R}+\psi_{B L n}\right)+r q_{i M R} \cos \left(\psi_{R}+\psi_{B L n}\right) \tag{43i}
\end{align*}
$$

The ground effect factor, $f_{i N R}$, and the longitudinal linear gradient factor, $K_{i N R}$, accounts for forvard flight.

Note this Cormula is in rotating coordinates, and that the forvard flight distribution actually is applied along the line of the apparent airflow, $\psi_{W}$. In doing this the distribution is valid for forward flight, sidevard flight and sideslip conditions. The ansle $\psi_{W}$ is

$$
\begin{equation*}
\Psi_{W}=\operatorname{Tan}^{-1}\left(\frac{v_{H}}{u_{H}}\right) \tag{438}
\end{equation*}
$$

as shown in Figure 45.
The aerodynamic moment factors, $q_{i M R}$ and $p_{i N R}$, remain attached to the hub axis.

The downash factor $K_{i N R}$, as explained in assumption 3, is given as a function of the wake angle defined as

$$
\begin{equation*}
x_{i M R}=\operatorname{Tan}^{-1} \frac{\sqrt{u_{H}^{2}+v_{H}^{2}}}{w_{i M R}-w_{H}} \tag{439}
\end{equation*}
$$

which is zero in hover and rear 90 degrees in high-speed flight. The function can be constrained by a number of factors. In hover, the value is zero. A 90 -degree value of about 1.6 can be read from Figure 34 . Also from this rigure a set of linearized distributions is read, and plotted as Figure 36.

The ground effect factor,

$$
\begin{equation*}
f_{i M R}=1-\frac{1}{16}\left(\frac{R}{h}\right)^{2} \frac{1}{1 \div \frac{u_{H}^{2}+v_{H}^{2}}{w_{i M R^{2}}^{2}}} \tag{440}
\end{equation*}
$$

is taken from Reference 10 , where $h=-\left(20_{H}\right)_{E}$ is from Section 4.5.1.
6.2.2.4 Downwash transients. - Downwash transients exist due to an apparent mass associated with the induced flow field. Work by Peters, et al. derive


Figure 36. - Typical shape of longitudinal factor curve.
the expressions for collective and cyclic dornvash including unsteady components. A good summary of this work is Reference 11.

Converting the referenced vork to dimensional form gives equations comparable to ( $434,435,436$ ).

$$
\begin{align*}
& w_{i N R}+\frac{4 B R}{3^{\pi}} w_{i M R}=-\frac{F_{2 A} M_{R, H}}{2 \rho V_{i N R}} T(B R)^{2}  \tag{441}\\
& p_{i M R}+\frac{32}{45 \pi} \frac{B R}{V_{i M R}} \frac{\dot{p}_{i N R}}{1-w_{i M R}}\left(v_{H}-u_{i M R}\right) / V_{i M R} \\
& =\frac{-2 M_{X A_{N R, H}}}{\rho V_{i M R}(B R)^{4} \quad\left[1-w_{i N R}\left(w_{H}-w_{i N R}\right) / V_{i M R}^{2}\right)}  \tag{442}\\
& q_{i M R}+\frac{32}{45 \pi} \frac{B R}{v_{i M R}} \quad \frac{8_{i M R}}{\left[1-w_{i M R}\left(w_{H}-w_{i M R}\right) / V_{i M R}\right]}
\end{align*}
$$

These differential equations are solved for $w_{i M R}, P_{i M R}$ and $q_{i M R}$ using numerical (Euler) integration.
6.2.2.5 Iteration of domwash solution. - As is the aase with any rotary-wing loading calculetion, there is an interplay between the downash variance from calculating the loading and a variance in the loading from recomputing the downash. A common practice is to solve an iterative loop to satisfy both equations (i.e., lift and momentum). In REXOR II the iteratisn does not take place independently, but proceeds stepwise with the rotor azimuthal advance. With the normal, rapid convergence of the iteration the solvtion will essentially be complete with the step advance. However, large step sizes will incur an additional downash time lag.
6.2.3 Blade element velocity components. - In the following subsections the blade aerodynamic loading is categorized and developed along two lines. They are:

- Transient phencmena consisting of quasi-stendy aerodynamics and dynanic stall.
6.2.3.1 Sources and resolution from blade motion. - The steady aerodynanics are based on the air velocities while the quasi-steady aerodynamics (from flutter theory) and dynasic stall depend on accelerations.

The air velocity is the blade mechanical velocities sumed vith a component due to downaash. In a similar manner, the air acceleration is taken to be the mechanical blade accelerations minus the downash accelerations. The downash formalation as developed in Section 6.2 .2 allows for lags, and it is these lags that result in downash acceleration terns.
6.2.3.2 Steady aerodyamics. - The air velocities relative to blade section are desired for an axis systen vith origin at the quarter chord to match the airfoil table data. Fro Section 4.5.5, the mechanical blade velocities relative to the free strean or earth axes are available as $\left\{\dot{X}_{B L E}, \dot{\mathrm{Y}}_{\mathrm{BLE}}, \dot{\mathrm{Z}}_{\mathrm{BLE}}\right\}^{I}$. The desired relative air velocities at the quarter chord (or blade BIn reference axis) are.

Where the second vector on the right is the downmash velocity developed in Section 6.2.2, and the third transfers the velocity from the BLE reference point at the blade center of gravity back to the quarter chord. The distance ${ }^{\mathbf{Y}}{ }_{C G}$ is positive with the center of gravity ahead of the quarter chord. For zotational convenience,

$$
\left\{\begin{array}{c}
u_{S}  \tag{445}\\
u_{c} \\
u_{N}
\end{array}\right\}=\left\{\begin{array}{lll}
-\dot{x}_{1 / 4} & c \\
\dot{y}_{1 / 4} & c \\
\dot{z}_{1 / 4} & c
\end{array}\right\}
$$

$$
\alpha_{1 / 4 c}=\sin ^{-1}\left(U_{N} / \sqrt{U_{c}^{2}+U_{X N}^{2}}\right)
$$

Airflow aspects of quasi-steady aerodynamic formulation are developed at this point for convenience. The quasi-steady aerodynamic contribution is conceived as composed of circulatory and noncirculatory components. The circulatory components are taken to be equivalent to finding the aerodynamic force, and moment coefficients are based on an angle of attack at the three-quarter chord:

$$
\begin{equation*}
\alpha_{3 / 1} c=\alpha_{1 / 4 c}+\frac{p_{\mathrm{BL} \mathrm{E}} \frac{c}{2}}{\sqrt{U_{\mathrm{H}}^{2}+U_{C}^{2}}} \tag{447}
\end{equation*}
$$

As such, the effect of angular rates is included in deriving the steady aerodynamic coefficients. The formulation above does not attempt to account for local downwash rotation or curvature and its chorduise variation. The net result is that aeroaynamic coefficients determined in Section 6.2.4 are computed with a $3 / 4 e^{\text {. }}$

A number of quantities used in the dynamic stall computations, Section 6.2.3.4, are also available from the previous mechanical development. They are also defined here for convenience. First, the angle of sideslip appears only in the dynamic stall formulation. For this purpose it is defined as

$$
\begin{equation*}
\Lambda=\operatorname{Tan}^{-1}\left(\frac{U_{S}}{U_{C}}\right) \tag{448}
\end{equation*}
$$

Also, dynamic stall is based on the time derivative of the angle of attack at the three-quarter chord:

$$
\begin{align*}
\dot{a}_{3 / 4 \mathrm{c}}= & \frac{1}{1+\left(U_{N} U_{C}\right)^{2}}\left[\frac{\dot{U}_{N}}{U_{C}}-\frac{U_{N}}{U_{C}} \frac{\dot{U}_{C}}{U_{C}}\right] \\
& +\frac{c}{2}\left[\frac{\dot{p}_{B L E}}{\sqrt{U_{C}{ }^{2}+U_{N}{ }^{2}}}-p_{B L E}\left(\frac{U_{C} \dot{U}_{C}+U_{N} \dot{U}_{N}}{\left(U_{C}{ }^{2}+U_{N}{ }^{2}\right) \frac{3}{2}}\right)\right] \tag{449}
\end{align*}
$$

The above equation requires $\dot{U}_{C}$ and $\dot{U}_{\mathbf{I}}$.

$$
+\left\{\begin{array}{l}
-(\dot{r}-p q) x_{C G} \\
0 \\
(j+q r) Y_{C G}
\end{array}\right\}_{B L n}
$$

The inclusion of the gravity term places these accelerations in a true inertial axis systen, not earth inertial axes, as appropriate for aerodynamic calculations. See Section 4.5.1. Gravity does cause buoyancy forces, but these can be ignored. The surning acceleration components are also subtracted to produce linear accelerations which correctly model the blade element incident airflow. The gravity vector can be obtained from hub values as:

$$
\left\{\begin{array}{l}
\delta_{\mathrm{X}}  \tag{451}\\
\mathrm{~g}_{\mathrm{Y}} \\
g_{\mathrm{Z}}
\end{array}\right\}_{\mathrm{BLE}}=\left[\frac{\partial \zeta_{\mathrm{BLL}}}{\partial \zeta_{\mathrm{R}}}\right]\left[\frac{\partial 5_{\mathrm{R}}}{\partial 5_{\mathrm{H}}}\right]\left\{\begin{array}{l}
\mathrm{g}_{\mathrm{X}} \\
\mathrm{~g}_{\mathrm{Y}} \\
\mathrm{~g}_{\mathrm{Z}}
\end{array}\right\}_{\mathrm{H}}
$$

By differentiating the downash velocities, the downash acceleration is obtained:

$$
\begin{align*}
& \left(\dot{u}_{B L E}^{D W, B L n}\right)=\dot{u}_{i M R} \frac{\left(v_{3 L E}\right)_{D K, B L n}}{w_{i!R}} \\
& -w_{i M R} f_{i M R} \kappa_{i M R} \frac{X_{B L n}}{R} \sin \left(\psi_{R}+\psi_{B L n}+\psi_{h}\right) \dot{\psi}_{R} \\
& +r \dot{p}_{i M R} \sin \left(\psi_{R}+\psi_{B L n}\right)+r \dot{q}_{i M R} \cos \left(\psi_{R}+\psi_{B L n}\right) \\
& +r\left[p_{i M R} \cos \left(\psi_{R}+\psi_{B L n}\right)-q_{i M R} \sin \left(\psi_{R}+\psi_{B L n}\right)\right] \dot{\psi}_{R} \tag{452}
\end{align*}
$$

6.2.3.3 Quasi-steady aerodynamics. - Quasi-steady serodynamics is accounted for in REXOR II by incorporating the terms from the two-dimensional flutter theory of Theodorsen (reference 12). In the REYCR II analysis, Theodorsen's lift deficiency function $C(k)$ is taken as unity. This means that the flutter theory presently incorporated neglects shed wake effects, or in physical terms does not account for the phase change between blade element lift (or pitching moment) and angle of attack, due to shed vorticity, or the assumption of quasi-steady aerodynamics is expressed by $C(k)=1$.

Referring to a classic text on aeroelasticity by Bisplinghoff, Ashley, and Hoffman (Reference 13), the expressions for lift and pitching moment are given as:

$$
\begin{equation*}
L=\pi \rho b^{2}[\ddot{h}+U \dot{a}-b a \ddot{a}]+2 \pi \rho U b C(k)\left[\dot{h}+U a+b\left(\frac{1}{2}-a\right) \dot{a}\right] \tag{453}
\end{equation*}
$$

and

$$
\begin{align*}
M= & \pi \rho b^{2}\left[b a \ddot{h}-U b\left(\frac{1}{2}-a\right) \dot{a}-b^{2}\left(\frac{1}{8}+a^{2}\right) \ddot{a}\right] \\
& +2 \pi \rho U b^{2}\left(a+\frac{1}{2}\right) \quad C(k)\left[\dot{h}+U a+b\left(\frac{1}{2}-a\right) \dot{a}\right] \tag{454}
\end{align*}
$$

In REXOR II, the blade aerodyamics and quasi-steady aerodynamics are referenced to the local section quarter-chord properties. This is done because the majority of available airfoil data uses this reference. Hote that the final aerodynamic loads are translated to the local BLE axis (c.g. location) for use in the equations of metion.

Reviewing the above expressions, and referencing the rotation point to the quarter chord gives $a=-1 / 2$. If ve take $C(k)$ as unity, replace $2 \pi$ for circulatory lift by ( $\mathrm{dC}_{\mathrm{LA}} / \mathrm{da}$ ), and substitute $c / 2$ for the semichord $b$, these equations become

$$
\begin{equation*}
L=\frac{\pi \rho c^{2}}{4}\left[\ddot{\mathrm{~h}}+U \dot{\alpha}+\frac{c}{4} \ddot{a}\right]+\left(\frac{d C_{L R}}{d \alpha}\right) \frac{\rho c U}{2}\left[\dot{b}+U \alpha+\frac{c \dot{\alpha}}{2}\right] \tag{455}
\end{equation*}
$$

and:

$$
\begin{equation*}
n=\frac{\pi \rho c^{2}}{4}\left[-\frac{c}{4} \ddot{h}-\frac{U c}{2} \dot{\alpha}-\frac{c^{2}}{4}\left(\frac{3}{8} \quad \ddot{a}\right)\right] \tag{456}
\end{equation*}
$$

Hote that the entire last term in the moment equation vanishes with a $=-1 / 2$. Referring to the lift expression, noncirculatory aerodynamic lift is accounted for in REXOR II by the first term in wich $\ddot{\mathbf{h}}+\mathrm{U} \dot{\alpha}$ are combined into $\dot{U}_{N}$ in blade element coordinates. The second term results from table lookup where

$$
\begin{equation*}
\Delta L=\frac{1}{2} \rho U^{2} c\left(\frac{d C_{L R}}{d \alpha}\right) \alpha=\frac{1}{2} \rho U^{2} c C_{L R} \tag{457}
\end{equation*}
$$

in which the angle of attack is previously computed from

$$
\begin{equation*}
\alpha_{3 / 4 c}=\left[\frac{\dot{h}}{U}+\alpha+\frac{c \dot{a}}{2 \dot{U}}\right] \tag{458}
\end{equation*}
$$

The $\alpha$ within the brackets is identified as $\theta$, the actual physical angle of the blade with respect to the freestream direction. The $\alpha$ on the left hand is that due to the air velocities which include the plunging velocity $\dot{\mathrm{h}}$ and rotation component $c / 2 \dot{\alpha}$. Herce $p_{B L E}=\dot{\alpha}$ and $\dot{p}_{B L E}=\ddot{\alpha}$.

The total aerodynamic pitching moment is the sum of the quasi-steady loads computed above and the table lookup blade section properties (Section 6.2.4).
6.2.3.4 Dymamic stall. - Dynamic stall is included in REXOR II based upon the Boeing-Vertol formulation set forth in References 14,15 , and 16 . It is similar to the treatment of dynamic stiall in the Bell C-81 program. A comparison of REXOR II with the C-8l program is given in Appendix IV, pages 393404, of Reference 3. Dynamic stall is specifically addressed with respect to. the two programs beginning on page 395 of that report. A significant point of difference between the treatment of dynamic stall in the two programs is that C-81 puts a 20 -percent limit on the angle-of-attack overshoot in obtaining the dynamic maximum lift coefficient, whereas REXOR II has no limit. The correctness of the treatment of dynamic stall in either program is difficult to assess since the concensus of researchers in this area is that current methods are empirical at best, and much research still remains to be done in this area.

Reference 14 notes that, "The trends show that compressibility effects reduce dynamic-stall delay, and at about $M=0.6$ no dynamic-stall delay is evident." For this reason an upper Mach number limit of 0.6 was implemented in the dynamic stall calculations for REXOR II. The test data obtained by Boeing Vertol and given in the references cited was for the Mach number range 0.2 to 0.6 . As implemented in REXOR II if $M<0.25$, the value $M=0.25$ is used in the analytic expression for developing the stall hysteresis loop.

Reference 15 notes that it was found that, "airfoils used currently by the helicopter industry had stailing dominated by leading edge stall. For this type of stalling process, the dynamic $C_{L}$ extension was proportional to the time rate of change of the angle of attack."

In that reference, so as to use static airfoil data as much as possible, static stall and dynamic stall are empirically related by developing a reference angle of attack given by

$$
\begin{equation*}
a_{R E F}=\alpha-\left(r \sqrt{\left|\frac{\alpha c \dot{\alpha}}{2 V}\right|} \operatorname{sign}(\dot{\alpha})\right) \tag{459}
\end{equation*}
$$

in which,

$$
\begin{equation*}
\gamma=\log _{e} \frac{0.601}{M} \tag{460}
\end{equation*}
$$

and is physically related to dynamic stall delay. $\alpha$ is identified as $\alpha_{3 / 4}$ and

$$
\begin{equation*}
M=\alpha_{s} / \sqrt{U_{N}^{2}+U_{C}^{2}} \tag{461}
\end{equation*}
$$

As noted in Reference 16 in regard to dyamic stall..." as a blade element reaches and exceeds the static angle of attack, stall does not occur as long as a sufficient, positive time rate of chage of the airfoil angle of attack, $\dot{a}$, is present." The experimentally derived equation for dyamic stall delay is given in the reference as

$$
\begin{equation*}
\text { dynamic stall delay }=\gamma \sqrt{\frac{c Q}{2 V}} \tag{462}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{c \dot{a}}{2 V}=k \tag{463}
\end{equation*}
$$

the blade element reduced frequency.
Referring to the gamma expression, we note that $\gamma \rightarrow 0$ as $M \rightarrow 0.601$, which is the upper limit for Mach number values for dynamic stall calculations. Also, note that $\gamma \rightarrow 1$ as $M \rightarrow 0.2211$, which is approximated by the value of $M=0.25$, the lower limit in REXOR II for dynamic stall simulation.

The term $\alpha_{\text {REF }}$ given above is also called the dynamic angle of attack (Reference 15 ) and given by the notation $\alpha_{D r a}$.
6.2.3.4.1 Lift accounting for dynamic stall. - Using the reference or dynamic angle of attack computed from $a_{\text {REF }}$, the REXOR II program implements the "Fast Aerodynamic Table", Section 6.2.4, subroutine and determines the lift coesficient, $C_{L}$, corresponding to $\alpha_{\text {REF }}$ and the freestream Mach number for the specified blade element and blade azimuth position. Also computed at the given Mach number are the $C_{L}$ for zero angie of attack and the $C_{L}$ for a small increment $\Delta \alpha$ with respect to zero. Yawed or radial flow is accounted for by computing the yaw angle of the flow given by:

$$
\begin{equation*}
\Lambda=\operatorname{Tan}^{-1}\left(\frac{U_{S}}{U_{C}}\right) \tag{464}
\end{equation*}
$$

where $U_{S}$ and $U_{C}$ represent blade spanwise and chordwise componenus of flow respectively.

The slope of the lift curve is then found from:

$$
\begin{equation*}
\left(\frac{\partial C_{L}}{\partial \alpha}\right)_{D Y N}=\frac{C_{L}\left(\alpha_{R E F}, M\right)-C_{L}(0, M)}{\alpha_{R E F} \cos \Lambda} \tag{465}
\end{equation*}
$$

It can be argued from physical reasonings that the dynamic lift-curve slope cannot exceed the static life-curve slope. As a check, REXOR II also calculates:

$$
\begin{equation*}
\left(\frac{\partial C_{L}}{\partial \alpha}\right)_{0, M}=\frac{C_{L}(\Delta \alpha, M)-C_{L}(0, M)}{\Delta \alpha} \tag{466}
\end{equation*}
$$

Only in the event $\left(\partial C_{L} / \partial \alpha\right)_{D Y N}$ is greater than $\left(\partial C_{L} / \partial \alpha\right)_{O, M}$ is the iatter value used to calculate $C_{L}$. Otherwise $C_{L}$ is calculated by

$$
\begin{equation*}
C_{L}=\left(\frac{\partial \tau_{L}}{\partial \alpha}\right)_{D Y N} \alpha+C_{L}(0, M) \tag{467}
\end{equation*}
$$

The ability of this approximation to describe mathematically the lift hysteresis characterized by dynamic stall is shown in Figure 37, which compares analytical results with experimental two-dimensional airfoil data. (From Reference 16.)

The component of the lift force per unit span acting normal to the blade chord axis and including dynamic stall effects is then calculated from

$$
\begin{equation*}
\Delta F_{N O}=\delta_{L} \varepsilon \frac{\rho V^{2}}{2} \cos \alpha \tag{460}
\end{equation*}
$$

The total normal force is determined by adding to this term the drag component, $C_{D} \subset \rho V^{2} / 2 \sin \alpha$, and the unsteady aerodynamic terms discussed in the previous section. To account for dynamic stall effects on drag, twodimensional drag coefficient data are used, but as determined at $\alpha_{\text {REF }}$, not $\alpha$. This is consistent with Reference 15.


Figure 37. - Dynamic atall-lift coefficient vs angle-of-attack hysteresis loop.

The component of the lift force per unit span parallel to the blade chord axis is found correspondingly from:

$$
\begin{equation*}
\Delta F_{C}=\frac{C_{L} c \rho v^{2}}{2} \sin \alpha \tag{469}
\end{equation*}
$$

The total chordvise force is then obtained by auding the corresponding drag coefficient term multiplied by $\cos \alpha$.
6.2.3.4.2 Pitching moment accounting for dynamic stall. - For determining pitching moments due to dynamic stall (see Refererce 13), the reference or dynamic engle of attack given by ${ }^{\text {REF }}$ must be modified. In REXOR II, this is accomplished by multiplying the second term by en empirical constant, $K$. Hence,

$$
\begin{equation*}
\alpha_{\mathrm{REF}}^{\prime}=\alpha_{\mathrm{DYN}}^{\prime}=\alpha-K\left(\gamma \sqrt{\left|\frac{c \dot{\alpha}}{2 V}\right|} \operatorname{sign}(\alpha)\right) \tag{470}
\end{equation*}
$$

$K$ is selected based upon the dynamic stall characteristics of the airfoil. In general it has been found for conventional rotor blade airfoils that $K$ should be selected so that

$$
\alpha_{\mathrm{REF}}^{\prime}=\alpha_{\mathrm{REF}}+\Delta \alpha
$$

where $\Delta \alpha$ is of the order of 2.5 degreas. With $\alpha_{\text {REF }}$ calculated from the above equation, the moment coefficient is determined from tables such that,

$$
\begin{equation*}
C_{M}=C_{M}\left(\alpha_{R E F}^{\prime}, M\right) \tag{472}
\end{equation*}
$$

A comparison of test and theoretical dynamic $C_{M}$ from Reference 15 is shown in Figure 3d.

The total pitching moment acting per unit span on a blade element is then given by:

$$
T(i)=-C_{M} c^{2} \frac{p V^{2}}{2}-F_{N O} S_{V}(i)+\left[\begin{array}{c}
\text { quasi-steady }  \tag{473}\\
\text { aero } \\
\text { terms }
\end{array}\right]
$$

(Section 6.2.3.3)


Figure 38. - Dynamic stall - mement coefficient vs angle-oi-attack hysteresis loop.
where $S_{Y}(x)$ represents the distance from the aerodynamic center to the blade elastic axis, and the quasi-steady aerodyanic terns are included as described in Section 6.2.3.3.
6.2.4 Coefficient zable lookup - overview. - In cataloging blade section aerodyamic data, $C_{I}, C_{D}$ and $C_{M}$, there are two procedures available.

- Curve fit tre aerodynamic data to the specific airfoil geometry being investigated for the range of Mach number and angle of attack to be considered.
- Tabulate the data as a function of performance and geometric parameters, and interpolate to the exact conditions at hand.

REXOR II uses the second procedure. The data consists of $C_{L}, C_{D}$ and $C_{M}$ tables. Each table is tabulated as a function of angle of attack and Mach number. The table format is organized identically to the Army C-8i program. Thus C-81 airfoil decks may be directly used in REXOR II.

A table set of RACA 0012 section characteristics is included as part of REXOR II. Two external tables may be used; the first of which overrides the resident 0012 data. Changeover of external tables occurs at a preselected blade radial station.
6.2.4.1 Inputs and outputs. - Each table ( $C_{L}, C_{D}, C_{M}$ ) has a separate angle of attack entry and a common Mach entry. The separate entries are used for dyamic stall calculations. The outputs in addition to $C_{L}, C_{D}$ and $C_{M}$ are the zerc angle-of-attack $C_{L}$ and $C_{L}$ vs angle of attack slope.
6.2.5 Blade element and rotor aercdymanic loads sumary. - The required loads for use in the equations of motion are in BLn axis. Development to this form from BLE axis about this quarter chord point is covered in Section 5.6.4. The BLE axis form is:

$$
\begin{align*}
& +\left\{\begin{array}{l}
0 \\
0 \\
I_{I_{A}(i)} \\
M_{A}(i) \\
0 \\
0
\end{array}\right\}_{\text {Unsteazy, BLE }} \tag{474}
\end{align*}
$$

where,

$$
\left[F_{\alpha_{-a L E}}\right]=\left[\begin{array}{lll}
1 & 0 & 0  \tag{475}\\
0 & \cos \left(\alpha_{3 / 4} c\right) & -\sin \left(\alpha_{3 / 4} c\right) \\
0 & \sin \left(\alpha_{3 / 4}\right) & \cos \left(\alpha_{3 / 4}\right)
\end{array}\right]
$$

### 6.3 Interference Terms

6.3.1 Nature of the Phenomenon. - In the process of producing lift, the various parts of the rotorcraft impart a net momentum change to the air mass opposite to the direction of the force produced. This induced air velocity from the momentum change impinges upon other elements of the rotorcraft changing their aerodynamic behavior.

The sources of interest are the main rotor and wing (or iffing body characteristics of the fuselage). The surfaces being affected are the wing plus fuselage and the empennage. The impinging velocity is expressed in $Z_{F}$ (fuselage vertical) axis as a percentafe of the source flow and a function of the vake angle of this flow.

A second interference velocity source is to consider the circulation part of the Theodorsen functiun. Here the wing or wing equivalent of the fuselafe is producing iift at the quarter-chord point accoraing to the air velocity at the $3 / 4$-chord jocation. Accordingly, the vertical component of air yeiocity at the wing inciudes a comonent.

$$
\begin{equation*}
\frac{1}{2} c_{\text {wing }} \dot{q}_{F} \tag{476}
\end{equation*}
$$

Here the wing quarter chord is assumed to lie on the $y_{g}$ axis. This component is also effective at the horizontal tail via the wing to horizontal tail downasi factor.
6.3.2 Rotor to ving/fuselage. - The downash function (percentage of source flow) used in REXOR II is a lookup table of downash factor, $F(X)_{M R-W}$, and idealized main rotor wake angle $X_{M R}$
where,

$$
\begin{equation*}
x_{M R}=\tan ^{-1}\left(\frac{u_{H}}{u_{H}-u_{i N R}}\right) \tag{477}
\end{equation*}
$$

The table data is linearly interpolated to the required wake angle value.
The fuselage reference downash velocity at the wing (or equivalent) then is

$$
\begin{equation*}
W_{W I N G}=w_{F}-w_{i N R} F(x)_{M R-W}+\frac{1}{2} \delta_{W I M G} q_{F} \tag{478}
\end{equation*}
$$

and taking time derivatives,

$$
\begin{equation*}
\dot{w}_{W I M G}=\dot{w}_{F}-\dot{w}_{i M R} F(x)_{M R-W}+\frac{1}{2} c_{W I M G} \dot{G}_{F} \tag{479}
\end{equation*}
$$

The total air velocity to the wing/fuselage is

$$
\begin{equation*}
\left.v_{T_{W I N G}}=\left(u_{F}^{2}+v_{F}^{2}+w_{W I N G}\right)^{2}\right)^{1 / 2} \tag{480}
\end{equation*}
$$

and the angle of attack is

$$
\begin{equation*}
a_{\text {WIHG }}=\tan ^{-1}\left(\frac{\mathbf{W}_{\text {WING }}}{u_{F}}\right) \tag{481}
\end{equation*}
$$

The total velocity in the fuselage $X Z$ plane is used in the erpennage computations:

$$
\begin{equation*}
v_{X_{z}}=\left(u_{F}^{2}+w_{W I N G}^{2}\right)^{1 / 2} \tag{482}
\end{equation*}
$$

6.3.3 Rotor to horizontal tail. - A downash factor $F(x)_{\text {MR-HT }}$ between the main rotor and horizontal tail is computed in the same manner as $F(X)_{M R-W}$ from the main rotor wake angle $X_{M R}$. This data in conjunction with the wing to horizontal tail downash factor is used to compute incremental air velocities at the horizontal tail.

Evaluating the main rotor increment,

$$
\begin{equation*}
w_{i M R_{I}}=w_{i M R}\left(F(x)_{M R-H T}-F(x)_{M R-X}\right) \tag{483}
\end{equation*}
$$

A increment for an upper horizontal taii is likewise generated:

$$
\begin{equation*}
w_{i M R_{I U}}=w_{i M R}\left(F(x)_{M R-H T U}-F(x)_{M R-W}\right) \tag{434}
\end{equation*}
$$

6.3.4 Data sources. - The theoretical downwash factor ranges from 0 at $x=0$ and 180 degrees to 2 at $X=90$ in the fully contracted rotor wake. Several sources of measured data are available to construct a distribution for a given configuration. Reference 12 gives isolated rotor data for field distances and
 good data set for typical ving locations.
6.3.5 Empennage velocity components. - REXOR II models the emperange assembly as either part of the fuselage-ving aerodynanic table (tail on) or as a separate set of aerodynanic loads (tail off). In either case a set of perturbation velocities is used.

The wing to horizontal tail downash factor appears explicitly as a quasiunsteady aerodynamic term. An airflow time delay from the ving to horizontal tail is computed as

$$
\begin{equation*}
\Delta t=\frac{\ell_{H T}}{V_{X_{Z}}} \tag{485}
\end{equation*}
$$

Using the downash factor $\partial \varepsilon_{i} \partial \alpha$ the vertical airflow component at the horizontal tail is

$$
\begin{equation*}
\Delta t \dot{w}_{W I M G} \frac{\partial \varepsilon}{\partial a}=\frac{l_{\mathrm{HI}}}{V_{X_{Z}}} \dot{w}_{W I N G} \frac{\partial \varepsilon}{\partial \alpha} \tag{486}
\end{equation*}
$$

In a like manner the delay in sidewash gives rise to the term $\ell_{V T} / V_{X_{Z}} \dot{r}_{F}{ }^{\partial \sigma_{V I}} / \partial \beta$ on the vertical tail.

The vertical incremental horizontal tail velocity is then:

$$
\begin{align*}
\Delta w_{Y T}= & -w_{i N R_{I}}+\ell_{H T}\left(q_{F}+\frac{\partial \varepsilon}{\partial \alpha} \dot{u}_{W I N G} / v_{X_{Z}}\right) \\
& +\left(\tau_{E L} \delta_{E L}+\tau_{H T} \delta_{i}\right)_{H T} u_{H T} \tag{487}
\end{align*}
$$

The terms $\tau_{E L}$ and $\tau_{\text {HT }}$ intrcduce equivalent velocities due to elevator and horizontal tail incidence deflections respectively.

Similarly for the upper horizontal tail:

$$
\begin{equation*}
\Delta w_{H T U}=-w_{i M R}+\ell_{H T U}\left(q_{F}+\frac{\partial \varepsilon}{\partial \alpha} \dot{w}_{W I N G} / v_{X_{Z}}\right) \tag{488}
\end{equation*}
$$

Assembling the vertical tail lateral incremental relocity:

$$
\begin{equation*}
\Delta v_{V T}=-\ell{ }_{V T}\left(r_{F}-\frac{\partial \sigma_{V T}}{\partial \beta} \frac{\dot{r}_{F}}{V_{T_{W I N G}}}\right)+h_{V T} p_{F}+r_{R V D} \delta_{R V D} u_{V T} \tag{489}
\end{equation*}
$$

Where $r_{\text {RUD }}$ is used to introduce an equivalent velocity due to rudder deflection. A horizontal and vertical tail longitudinal total velocities are simply developed from a vake deficiency factor

$$
\begin{align*}
& u_{\mathrm{HIT}}=u_{F} \eta_{\mathrm{HI}},  \tag{490}\\
& u_{\mathrm{HHU}}=u_{F} n_{\mathrm{HIU}} \tag{491}
\end{align*}
$$

and

$$
\begin{equation*}
u_{V T}=u_{F}{ }^{\eta} V T \tag{492}
\end{equation*}
$$

Then the total vertical velocity at the horizontal tail is:

$$
\begin{equation*}
w_{H T}=w_{F}-w_{i M R} F(x)_{M R-W}-u_{F} \varepsilon_{F T}+\Delta w_{H T} \tag{493}
\end{equation*}
$$

and

$$
\begin{equation*}
w_{H T U}=w_{F}-w_{i M R} F(x)_{M R-W}-i_{F} \varepsilon_{H T U}+\Delta w_{H T U} \tag{494}
\end{equation*}
$$

The induced flow field angles $\epsilon_{\text {HT }}$ and $\varepsilon_{\text {HTV }}$ are a function of wing angle of attack, flap deflection and wing incidence change.

$$
\begin{gather*}
\varepsilon_{\mathrm{HT}}=\varepsilon_{O_{H T}}+\frac{\partial \varepsilon}{\partial \alpha} \sin a_{w}+\frac{\partial \varepsilon_{\mathrm{HT}}}{\partial \alpha_{\mathrm{FL}}} \delta_{\mathrm{FL}}+\frac{\partial \varepsilon_{\mathrm{HT}}}{\partial i_{w}} \delta_{i_{w}}  \tag{495}\\
\varepsilon_{\mathrm{HTU}}=\varepsilon_{\mathrm{O}_{\mathrm{HTU}}}+\frac{\partial \varepsilon_{\mathrm{HTU}}}{\partial \alpha} \sin \alpha_{w}+\frac{\partial \varepsilon_{\mathrm{HTU}}}{\partial \delta_{\mathrm{FL}}} \delta_{\mathrm{FL}}+\frac{\partial \varepsilon_{\mathrm{HTU}}}{\partial \mathrm{i}_{\mathrm{W}}} \delta_{\mathrm{i}_{W}} \tag{496}
\end{gather*}
$$

The total lateral velocity at the vertical tail is

$$
\begin{equation*}
v_{V T}=v_{F}-u_{F} \sigma_{V I}+\Delta v_{V I} \tag{497}
\end{equation*}
$$

Where $\sigma_{V T}$ is a sidevash coefficient from the fuselage.

$$
\begin{equation*}
\sigma_{V I}=\frac{\partial \sigma_{V T}}{\partial B} \sin _{B_{W}} \tag{498}
\end{equation*}
$$

### 6.4 Body Loads

In this section the aerodynamic contributions from the fuselage, wing and empennage are developed. These components are adivd together vith the tail rotor loads, Section 6.5, and auxiliary thrustors, Section 6.6. A transformation to fuselage axes is made from wind axes.

The fuselage, wing and empennage data is composed of SIATIC, DERIV, and CONTROL elemeats. The SRETIC data are the steady state load components as would be reasured in a wind tunnel. These data may te tail on or off. The DERIV data give additional loads due to velocity component variations from trim for tail on STATIC data as well as steady offsets (unequal wing tyist, etc.). Tail off DERIV loads use full tail velocity components rather than variations to generate the empennage forces und moments.

The CONTROL loads account for flap, dive brake, wing incidence, aileron deflection inputs via the cuntrol syster. The rudder, upper and lover torizontal tail incidence, lower horizontal tail ele ator inputs affect the empennage air velocity components, and are developed in Section 6.3.5.
6.4.1 Nonrotating airframe airloads. - The required lads are computed in REXOR II as the sum of steady-state forces and moments plus loads arising from stability derivative iype terms and controi surface inputs. The steady-state data are rormed in terms of overall $C_{L}, C_{D}$, and $C_{M}$ for the flias?age, wing, and empennage assembly.

where:

$$
\begin{equation*}
Q_{A}=\frac{1}{2} \rho S_{W I N G} V_{X_{Z}}^{2} \tag{500}
\end{equation*}
$$

The wing area, $S_{\text {WI }}$, and chord, $C$, ${ }^{\text {WING }}$, are actual or the equivalent of the lifting fuselage. Alternately, they may be the reference length and area used for the available wind tannel data. $C_{D_{I}}, C_{L_{T}}$ and $C_{M_{T}}$ are linearly interpolated from input data tables of $C_{L},{ }^{C} D_{D}, C_{M}$, versus añele of attack, $\alpha,{ }^{\text {. }}$ The data are interpolated on $a_{\text {WING }}$ from Section 6.3.2. The loads developed are in wind axis.

The stability derivative lcad contributions are computed as a 6 by 7 derivative matrix postmultiplied by a velocity component vector. For tail on fiselage aerodynamic data:

The components of the [ $F_{M N}$ ] matrix are discussed in Volume III. Two horizontal tails and one vertical are assumed. The upper tail is considered optional. Tine matrix also provides for asymmetric effects of wing incidence differential, for linear and quadratic sidesiip variations and for wing roll damping.

For tail off data the velocity vector is replaced by


These terms also produce forces and moments in wind axe3.
REXOR II includes the effects of flaps, ailerons and dive brakes ia the nonrotating aerodynamic loads. The flap deflections are modeled as inear stability derivatives of $C_{L}, C_{D}$ and $C_{M}$. The aileron load is the variation of aileron moment volume (rolling moment coefficient times wing area times wing span) with aileron deflection. The input is for one aileron. Dive brakes are represented as a variation of drag area with brake extension. The brake panels are assumed to be on the fuselage vertical axis and a distance - $h_{D B}$ below the fuselage reference.

The desired loads are:

The static and derivative terms are added to form the total body loads and transformed into fuselage axes.

where,

$$
\left[r_{\alpha_{W}}\right]=\left[\begin{array}{lll}
\cos \left(a_{W}\right) & 0 & -\sin \left(\alpha_{W}\right)  \tag{505}\\
0 & 1 & 0 \\
\sin \left(a_{W}\right) & 0 & \cos \left(\alpha_{W)}\right.
\end{array}\right]
$$

$$
\left[T_{\beta_{W}}\right]=\left[\begin{array}{lll}
\cos \left(\beta_{W}\right) & -\sin \left(\beta_{W}\right) & 0  \tag{506}\\
\sin \left(\beta_{W}\right) & \cos \left(\beta_{W}\right) & 0 \\
0 & 0 & 1
\end{array}\right]
$$

and

$$
\begin{gather*}
\cos \left(\alpha_{W}\right)=u_{F} / v_{X_{Z}}  \tag{507}\\
\sin \left(\alpha_{W}\right)=w_{W I N G} / v_{X_{Z}}  \tag{508}\\
\cos \left(\beta_{W}\right)=v_{X_{Z}} / v_{T_{W I N G}}  \tag{509}\\
\sin \left(\beta_{W}\right)=v_{F} / v_{T_{W I N G}} \tag{510}
\end{gather*}
$$

The air velocities $W_{W I N G}, V_{X_{Y}}, V_{T_{W I N G}}$, are defined in Section 6.3.2.
6.4.2 Component additional airloads. - A total array, \{QLOADS\}, of non main rotor air loads is computed in fuselage axes.

$$
\left\{\begin{array}{l}
F_{X}  \tag{511}\\
F_{Y} \\
F_{Z} \\
F_{\phi} \\
F_{\theta} \\
F_{4}
\end{array}\right\}=\left\{{ }_{F_{A}}=\left\{\begin{array}{c}
F_{X_{B}} \\
F_{Y_{B}} \\
F_{Z_{B}} \\
M_{X_{B}} \\
M_{Y_{B}} \\
M_{Z_{B}}
\end{array}\right\}+\left\{\begin{array}{c}
F_{X_{T R}} \\
F_{Y_{T R}} \\
F_{Z_{T R}} \\
M_{X_{T R}} \\
M_{Y_{T R}} \\
M_{Z_{T R}}
\end{array}\right\}+\left\{\begin{array}{c}
F_{F} \\
F_{X_{P}} \\
F_{Y_{P}} \\
{ }_{Z_{P}} \\
M_{X_{P}} \\
M_{Y_{P}} \\
M_{Z_{P}}
\end{array}\right\}\right.
$$

The first component is described above. The tail lotor load vector auxiliary thrustor load vector are developed in the following secticns.

### 6.5 Tail Rotor

A number of different levels of aerodynamic presentation accuracy and axis of representation may be used for tail rotor computations. In line with the stated objectives of REXOR II, a linear aerodynamic approach 1 s used. A shaft axis reference is used for the analysis, In this system, the air velocity quantities involved are easy to visualize. Also, the flapping ard feathering motions are the true, measureable quantities.
6.5.1 Formulations. - First, consider the airflow quantities available in fuselage axis, Figure 39. Note the tail rotor axes alignment with respect to the fuselage axis system. Formulating the components with respect to the fuselage axes gives:

$$
\left\{\begin{array}{l}
u_{T R}  \tag{512}\\
v_{T R} \\
w_{T R}
\end{array}\right\}_{F}=\left\{\begin{array}{l}
u_{F} n_{T R} \\
v_{F}-u_{F} \sigma_{T R}-\ell_{T R}\left(r_{F}-\frac{\partial \sigma_{T R}}{\partial B} \frac{\dot{v}_{F}}{\omega_{W}}\right)+h_{T R} P_{F} \\
\omega_{H T}
\end{array}\right\}
$$

where

$$
\begin{equation*}
\sigma_{T R}=\frac{\partial \sigma_{T R}}{\partial \beta} \sin \beta_{W} \tag{513}
\end{equation*}
$$

The vertical component is approximated by the horizontal tail value. Terms subscripted w refer to velocity, $w$, and sideslip angle, $\beta$, at the wing. The flow effects induced by the wing-fuselage bombination are described by the wake velocity deficiency fact or $\eta_{T R}$, the side wash angle, ${ }^{T} T R$, and its variation with angle of sideslip.

The velocity vector is then rotated through the tail rotor shaft lateral tilt, $\phi_{O_{T R}}$.

$$
\left\{\begin{array}{l}
u \\
v \\
w
\end{array}\right\}_{T R}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi_{0} & \sin \phi_{0} \\
0 & -\sin \phi_{0} & \cos \phi_{0}
\end{array}\right]_{T R} \quad\left\{\begin{array}{c}
u_{T R} \\
v_{T R} \\
w_{T R}
\end{array}\right\}_{F}
$$

Constructing the blade element tangential ( $U_{T}$ ) and perpendicular ( $U_{P}$ ) components, as shown in Figure 40 gives:

$$
\begin{gather*}
U_{T}=(r \Omega)_{T R}+u_{T H} \sin \psi_{T R}  \tag{515}\\
U_{P}=-v_{T R}-u_{i T R}-r \dot{B}-u_{T R} \in \cos \psi_{T R} \tag{516}
\end{gather*}
$$

where,

$$
w_{i r R} \text { is the tail rotor induced velocity }
$$

and

$$
\begin{equation*}
v_{T R}=-v_{T R}-w_{i T R} \tag{517}
\end{equation*}
$$

Expressing the blade element angle of attack as a small angle of approximation,

$$
\begin{equation*}
\alpha_{T R}=0+\frac{U_{P}}{U_{T}} \tag{518}
\end{equation*}
$$

shere,

$$
\begin{equation*}
\theta=\theta_{T R}-A_{1} \cos \psi_{T R}-B_{1} \sin \psi_{T R} \tag{519}
\end{equation*}
$$



$$
\begin{equation*}
\theta=\theta_{T R}+\delta_{-T R} a_{I_{T R}} \cos \psi_{T R}+\delta_{3 m R} b_{1_{T R}} \sin \phi_{T R} \tag{520}
\end{equation*}
$$

The tail rotor analysis assimes no coning.
The blace flapping, $B$, is then

$$
\begin{equation*}
\beta=-a_{1_{T R}} \cos \psi_{T R}-b_{I_{T R}} \sin \psi_{T R} \tag{5,21}
\end{equation*}
$$

240


Figure 39. - Overall tail rotor geometry.


Figure 40. - Tail rotor blade element detail.

The tail rotor expressions of interest are the prime forces added to the fuselage system. First looking at the tail rotor thrust. For a blade element re have

$$
\begin{equation*}
d T_{T R}=\left[\frac{1}{2} \rho a b c a U_{T}^{2} d r\right]_{T R} \tag{522}
\end{equation*}
$$

Yhere $a$ is the iift curve slope, $b$ is the number of tail rotor blades, and $c$ is the blade chord (assumed constant).

Substituting,

$$
\begin{equation*}
\left.d T_{T R}=\left[\frac{1}{2} \text { pabce } \theta U_{T}^{2}+U_{P} U_{T}\right) d r\right]_{T R} \tag{523}
\end{equation*}
$$

Integrating for the entire rotor,

$$
\begin{equation*}
T_{T R}=\left[\left.\frac{1}{2 \pi} \frac{1}{2} \rho a b c \int_{0}^{2 \pi} \int_{0}^{\mathrm{BR}}\left(\theta \mathrm{U}_{\mathrm{T}}^{2}+\mathrm{L}_{\mathrm{P}} \mathrm{U}_{\mathrm{F}}\right) \mathrm{dr} \mathrm{~d} \right\rvert\,\right]_{\mathrm{TR}} \tag{524}
\end{equation*}
$$

where $B$ is the finite airfoil lift factor expressed as a so-called tip loss factor.

Noting only even functions contribute to the integrand.

$$
\begin{align*}
T_{T R}= & \frac{1}{2} \rho a(b \subset R)_{T R}\left(\theta_{T R} \frac{B^{3}}{3}(\Omega R)_{T R}^{2}+\theta_{T R} \frac{B}{2} u_{T R}^{2}\right. \\
& +\frac{B^{2}}{2}(\Omega R)_{T R} V_{T R}+\left(\Omega R i_{T R} u_{T R} b_{1_{T R}} \frac{B^{2}}{2} \delta_{3 T R}\right) \tag{525}
\end{align*}
$$

Note the thrust is independent of the longitudinal flapping, but is a function of lateral cyclic shom as lateral flapping times delta 3.

The required lateral flapping angle is obtained by equating the lateral flapping moment equal to zero.

$$
\begin{equation*}
0=\left[\frac{1}{2^{\pi}} \frac{1}{2} \rho a b c \int_{0}^{2 \pi} \int_{0}^{\mathrm{BR}}\left(\theta \mathrm{U}_{\mathrm{T}}{ }^{2}+U_{\mathrm{P}} U_{\mathrm{T}}\right) \cos \psi r d r d \psi\right]_{T R} \tag{526}
\end{equation*}
$$

gives

$$
\begin{equation*}
b_{1_{T R}}=-a_{1} \delta^{6} 3 T R \tag{527}
\end{equation*}
$$

To obtain the longitudinal flapping angle, the longitudinal rotor moment is formed and set equal to zero.

$$
\begin{equation*}
0=\left[\frac{1}{2 \pi} \frac{1}{2} \rho a b c \int_{0}^{2 \pi} \int_{0}^{B R}\left(\theta U_{T}^{2}+U_{P} U_{T}\right) \sin \phi r d r d \dot{\psi}\right]_{T R} \tag{528}
\end{equation*}
$$

gives

$$
\begin{equation*}
a_{1 R R}=\frac{u_{R R}\left(\theta_{T R}(\Omega R)_{T R} \frac{8 B}{3}+2 V_{T R}\right)}{\delta_{3 T R}\left(E^{2}(\Omega R)_{T R}^{2}+\frac{3}{2} u_{T R}^{2}\right)+B^{2}(\Omega R)_{T R}^{2}-\frac{i}{2} u_{T R}^{2}} \tag{529}
\end{equation*}
$$

In formuating the tail rotor arive torque, the blade profile drag is expressed as

$$
\begin{equation*}
\bar{C}_{D}=C_{D_{0}}+k \bar{C}_{L}^{4} \tag{530}
\end{equation*}
$$

where $\bar{C}_{L}$ is the average lift coefficient. Reviewing the thrust equation with a constant (average) lift coefficient gives

$$
\begin{equation*}
T_{T R}=\left[\frac{1}{2 \pi} \frac{1}{2} \quad \rho b \in \bar{c}_{L} \int_{0}^{2 \pi} \int_{0}^{B R} U_{T}^{2} d r d \nabla\right]_{T R} \tag{531}
\end{equation*}
$$

gives

$$
\begin{equation*}
c_{L}=6 T_{T R} /\left(\rho \sigma_{T R} A_{T R}\left(B^{3}(\Omega R)_{T R}^{2}+\frac{3}{2} B u_{F}^{2}\right)\right) \tag{532}
\end{equation*}
$$

The drive torque is expressed as the reaction to turning the tail rotor shaft. The pius sign is associated with a clcckwise sense of rotation when -acing a left-hand mounted tail rotor.

$$
\begin{equation*}
d Q_{T R}= \pm\left[-\frac{1}{2} \rho c \quad b r U_{T}^{2} \bar{C}_{D}+\frac{1}{2} \text { pabcca\&r } U_{T}^{2}\right]_{T R} d r \tag{533}
\end{equation*}
$$

where

$$
\begin{equation*}
\phi=\frac{U_{P}}{U_{T}} \tag{534}
\end{equation*}
$$

## intesrating

$$
\begin{aligned}
Q_{R R}= & \pm\left[\frac { 1 } { 2 \pi } \frac { 1 } { 2 } \rho b c \int _ { 0 } ^ { 2 \pi } \left(\int_{0}^{R}-U_{T}^{2} \bar{C}_{D} r d r\right.\right. \\
& \left.\left.+\int_{0}^{B R}\left(a U_{T} U_{P} \theta-a U_{P}^{2}\right) r d r\right) d \psi\right] \\
= & \pm \frac{1}{2} \rho(b c c R)_{T R} R_{T R}\left(-\frac{\bar{C}_{D}}{4}(\Omega R)^{2}-\frac{\bar{C}_{D}}{4} u_{T R}{ }^{2}\right.
\end{aligned}
$$

(continued on next page)

$$
\begin{gather*}
+a(\Omega R)_{T R} \frac{B^{3}}{3} v \theta+a v_{T R}{ }^{2} \frac{\mathrm{~s}^{2}}{2} \\
+a_{1 T R}\left(1-\delta a_{3 T R}^{2}\right) v_{T R} u_{T R} \frac{B^{2}}{2}+(\Omega R)_{T R}{ }^{2} \frac{B^{2}}{8} a_{1 T R}{ }^{2}\left(1+\delta_{3 T R}^{2}\right) \\
\left.+a a_{1 T R}{ }^{2} u_{T R}{ }^{2} \mathrm{~B}^{2}\left(3-\delta \frac{2}{8 T R}\right)\right) \tag{535}
\end{gather*}
$$

The remaining load term in $X_{T R}$. Using the same formulation methodology,

$$
\begin{aligned}
d X_{T R}= & -\left[\frac{1}{2} \rho b c \bar{r}_{D} U_{T}^{2} \sin \psi\right. \\
& \left.-\frac{1}{2} \rho a b c a J_{T}^{2}(\sin \phi \sin \psi+\sin 6 \cos \psi)\right]_{T R} d r
\end{aligned}
$$

(536)

Making small angle approximations,

$$
\begin{align*}
d x_{T R}= & \frac{1}{2} \rho(b c)_{T R}\left[-\bar{C}_{D} U_{T}^{2} \sin \psi\right. \\
& +a\left(\theta T_{T} U_{P}+U_{D}^{2}\right) \sin \psi \\
& \left.-a\left(\theta U_{T}^{2}+U_{D} U_{T}\right)\left(a_{1} \cos ^{2} \psi+b_{1} \sin \psi \cos \psi\right)\right]_{T R} d r \tag{537}
\end{align*}
$$

## integrating

$$
\begin{align*}
& X_{T R}=\frac{1}{2 \pi} \frac{1}{2} \rho(b c) T \int_{T R}^{2 \pi}\left(\int_{0}^{R}-\bar{C}_{D} U_{T}^{2} \sin \phi d r\right. \\
& +\int_{0}^{\mathrm{BR}}\left(a\left(\theta U_{T} U_{P}+U_{P}^{2}\right) \sin \phi\right. \\
& -a\left(\theta U_{T}{ }^{2}+U_{P} U_{T}\right)\left(a_{1} \cos ^{2} \psi\right. \\
& \left.\left.\left.+b_{1} \sin (\cos \phi)\right) d r\right) d \psi\right] \\
& \text { TR } \\
& =\frac{1}{2} \rho a(b \text { c } R)_{T R}\left(-a_{1} \theta(\Omega R)^{2} \frac{B^{3}}{3}-a_{1} \frac{B^{2}}{2} V_{T R}(\Omega R)_{T R}\right. \\
& +V_{T R} u_{T R} \theta \frac{B}{2}-a_{1}{ }^{2}\left(1+3 \delta_{3 T R}^{2}\right) u_{T R} \frac{B^{2}}{4}(\Omega R)_{T R} \\
& \left.-a_{1}\left(1+\delta_{3 T R}^{2}\right)(\Omega R)_{T R} v_{T R} \frac{B^{2}}{4}-\frac{c_{D}}{2 a}(\Omega R)_{T R} u_{T R}\right) \tag{538}
\end{align*}
$$

The induced velocity is calculated from simplified momentum balance.

$$
\begin{equation*}
v_{i T R}=T_{T R} /\left(\left(2 \rho \pi R^{2} B^{2}\right)_{T R}\left(u_{T R}^{2}+v_{T R}^{2}+u_{T R}^{2}\right)^{1 / 2}\right) \tag{539}
\end{equation*}
$$

Hormally, the thrust, flapping, and induced velocity equations are solved as an iterative set. In REXOR II, these equations are solved for every pass (azimuth step) of the main rotor, and the tail rotor set convergence is assumed a priori.

Note that the pitch-flap coupling does not appear in the expressions developed. This is due to the equivalence of flapping and feathering, coupled with the absence of lateral flapping.
6.5.2 Airloads - control settings. - The force and moment terms are assembled for use in the oyerall fuselage loads, Section 6.4 The pilot controi is the rudder pedals $\theta_{T R}$.

$$
\begin{align*}
& +\left[\begin{array}{ccc}
0 & h & y \\
-h & 0 & l \\
-y & -l & 0
\end{array}\right]_{T R}\left\{\begin{array}{l}
F_{A T R} \\
F_{A T R} \\
\mathbf{F}_{A T R}
\end{array}\right\}_{F} \tag{541}
\end{align*}
$$

$S_{T R}$ is a factor to account for in blockage on the tail rotor thrust. The equations for $X_{T R}, T_{T R}$ and $Q_{T R}$ are based on $\rho_{0}$, the sea level density. The sign of $G$ TR allows for a tail rotating in a negative direction; i.e., upper top moving forward.

### 6.6 Auxiliary Thrustors

REXOR II models the compound helicopter configuration by the inclusion of an auxiliary source of forward thrust. A perturbation bypass jet math model is used. It is assumed that all the thrust units are at the same setting. Furthermore the dynamics of the engine rotating mass are ignored.

### 6.6.1 Formulations and airloads. - Based on a perturbation model the thrust

 for all units instailed is:$$
\begin{equation*}
T_{P}=\left(\frac{\partial T}{\partial \delta_{p}^{2}} \delta_{p}^{2}+\frac{\partial T}{\partial M_{p}} M_{p}+\frac{\partial T}{\partial\left(M_{p} \delta_{p}\right)} M_{p} \delta_{p}\right) \quad \frac{p}{P_{0}} \tag{542}
\end{equation*}
$$

where $M_{p}$ is the freestream Mach number and $\delta_{p}$ represents the total engine control paraweter.

The engines are located at height $h_{p}$ and distance $\ell_{p}$ aft of the fuselage axes. A thrust angle $\theta_{0}$ is also assumed. The engine contributions to the fuselage aerodynamic loads ${ }^{\text {p }}$ are then:

$$
\begin{align*}
& \left\{\begin{array}{l}
F_{X_{A P}} \\
F_{Y_{A P}} \\
F_{Z_{A P}}
\end{array}\right\}_{F}=\left[\begin{array}{ccc}
\cos \theta_{0} & 0 & \sin \theta_{0} \\
0 & 1 & 0 \\
-\sin \theta_{0} & 0 & \cos \theta_{0}
\end{array}\right] \quad\left\{\begin{array}{l}
T_{\mathrm{P}} \\
0 \\
0
\end{array}\right\}  \tag{543}\\
& \left\{\begin{array}{l}
M_{X_{A P}} \\
M_{Y_{A P}} \\
M_{Z_{A P}}
\end{array}\right\}_{F}=\left[\begin{array}{lll}
0 & h_{p} & 0 \\
-h_{p} & 0 & \ell_{p} \\
0 & -l_{p} & 0
\end{array}\right]\left\{\begin{array}{l}
F_{X_{A P}} \\
F_{Y_{A P}} \\
F_{Z_{A P}}
\end{array}\right\} \tag{544}
\end{align*}
$$

## 7. CONTROL SYSTEM

### 7.1 Overviey

REXOR II models vehicles ranging from pure helicopters to winged helicopters to compound helicopters with conventional airplane control surfaces. The control system is modeled as a set of pilot controls (stick, rudder pedals, coilective, etc.) which are coupled to the helicopter and airplane aerodynamic surfaces through a set of overall linkage factors (gains). These gains are slaved to a master control (phasing unit) which can be varied from the extremes of pure helicopter to pure airplane type of controls.

### 7.2 Pilot Controls

To simplify the operation of REXOR II the control inputs are mostly expressed as a percent of full scale (maximum input). The pilot inputs are:

| \% $X_{c, p}$ | Longitudinal stick |
| :--- | :--- |
| \% $Y_{c, p}$ | Lateral stick |
| \% $r_{c, p}$ | Rudder pedals |
| \% $\delta_{p, p}$ | Propulsion setting |
| \% $\theta_{o, p}$ | Collective blade angle |
| $\psi_{R, p}$ | Rotor speed setting |
| \% $\delta_{D B, p}$ | Dive brake extension angle |
| \% $\delta_{F L, p}$ | Flap extension angle |
| \% $\delta_{i w, p}$ | Command wing incidence change |
| \% $\delta_{i H P, p}$ | Command horizontal tail incidence change. |

Pilot controls are combined with trim ( $T$ ), initial condition (IC), rigging offset (subscript 0), and stability augmentation inputs (SAS). These combined inputs then operate the rotor and fixed aerodynamic surfaces. Scaling factors ( $K$ ) convert the percentage inputs into angular and linear deflections. $X_{c}^{\prime}=K_{X C F S} G_{c}\left(\$ X_{c, T}+\$ X_{c, p}\right)+X_{c, S A S}+\frac{\partial X_{c}}{\partial \theta_{0}} \theta_{0}$
$\delta_{E L}=K_{E L F S} G_{E L}\left(\delta X_{c, T}+\delta X_{c, p}\right)+\delta_{E L, 0}+\delta_{E L, S A S}$
$Y_{c}=K_{Y C R S} G_{c}\left(\$ Y_{c, T}+\Pi Y_{c, p}\right)+Y_{c, S A S}+\frac{\partial Y_{c}}{\partial \theta_{0}} \theta_{0}$
$\delta_{A I L}=Y_{A I L P S} G_{A I L}\left(\$ Y_{c, T}+\not Y_{c, p}\right)+\delta_{A I L, 0}+\delta_{A I L, S A S}$
$\theta_{O T R}=K_{T R F S} G_{T R}\left(\psi_{r_{c, T}}+\phi r_{c, P}\right)+\theta_{O T R, 0}+\dot{\theta}_{O T R, S A S}+\frac{\partial \theta_{O T R}}{\partial \theta_{0}} \theta_{0}$
$\delta_{R U D}=K_{\text {RUDFS }} G_{R U D}\left(\phi_{\dot{r}_{C, T}}+\phi_{r_{c, P}}\right)+\delta_{R U D, 0}+\delta_{R U D, S A S}$
$\theta_{0}=K_{\theta_{F S}}\left(\psi_{\theta_{O, P}}+\delta_{\theta_{O, T}}\right)-\left(Z_{s p}-z_{s p, T}\right) / e$
where $Z_{\text {sp }}=\left.Z_{\text {sp, }}\right|_{t=t_{\text {TRIM }}}=t_{\text {FLY }}$
$\delta_{p}=K_{P F S}\left(\delta_{p, T}+\delta_{p, p}\right)$
$\delta_{\mathrm{DB}}=\kappa_{\mathrm{DBFS}}\left(\$ \delta_{\mathrm{DB}, \mathrm{D}}+\$ \delta_{\mathrm{DB}, \mathrm{IC}}\right)$
$\delta_{\mathrm{FL}}=K_{\mathrm{FLFS}}\left(\$ \delta_{\mathrm{FL}, \mathrm{p}}+\boldsymbol{\alpha} \delta_{\mathrm{FL}, \mathrm{IC}}\right)$
$\delta_{i w}=K_{i w P S}\left(\% \delta_{i w, p}+\$ \delta_{i w, I C}\right)$
$\delta_{i H T}=K_{i H M F S}\left(\delta_{i H T, p}+\phi \delta_{i H T, I C}\right)$
$\delta_{i H T U}=K_{i H T F S}\left(\delta_{i H T U, I C}\right)$
The factors $G_{c}, G_{E L}, G_{A I L}, G_{T R}, G_{\text {RUD }}$ are the slaved gains controlled by the phasing unit.
The quantities $X^{\prime}{ }_{c}, Y^{\prime}{ }_{c}$ are processed through a first order lag and rate limiting prior to being applied as swashplate input commands, $X_{c}$ anc. $Y_{c}$.

$$
\left|\begin{array}{l}
X_{c}  \tag{559}\\
Y_{c}
\end{array}\right|=\left(\frac{1}{\tau_{A^{\prime}} \leq+1}\right)\left\{\begin{array}{l}
X_{c}^{\prime} \\
Y_{c}^{\prime}
\end{array}\right\}
$$

$$
\left\{\begin{array}{l}
\dot{X}_{c}  \tag{560}\\
\dot{X}_{c}
\end{array}\right\} \leq \dot{\delta}_{\mathrm{MAX}}
$$

The fixed aerodynamic surface motions are shown in Figure 41. The pilot inputs are depicted in Figure 42.

### 7.3 Stability Augmentation Systems

REXOR II incorporates stability augmentation inputs to the lateral and longitudinal cyclic inputs, elevator, aileron, tail rotor collective, and rudder. These SAS inputs are all derived from fuselage axis angular rate information. Signal processing consists of a low frequency washout and limiter applied to all throughput. A first order lag is also used on some signals.

The SAS coefficients are also computed on the basis of percentage of full scale deflection of the pilot control they are connected to. The same scaling conversion factors as used for the pilot inputs are applied to the SAS outputs.

The six SAS channels are shown below, Figures 43 through 48 .


Figure 41. - Fi.in aerodynamic surfaces.



Figure 43. - Longitudinal cycilc stability augmentation.



Figure 45. - Elevator atability augmentation.



Figure lit. - Tall rotor stability aumentation.



[^0]:    Figure 8. - Coordinate systems - freestream (earth) to principal reference uxis.

