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EARTH ORIENTATION FROM LUNAR LASER RANGE-DIFFERENCING

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PREFACE

This project is under the supervision of Professor Ivan I. Mueller, Department of Geodetic Science, The Ohio State University, and under the technical direction of Dr. David E. Smith, Code 921, NASA, Goddard Space Flight Center, Greenbelt, Maryland 20771.

This report together with Department of Geodetic Science Report No. 262 was submitted to the Graduate School of The Ohio State University as partial fulfillment of the requirements for the Ph.D. degree.

ABSTRACT

For the optimal use of high precision Lunar Laser Ranging (LLR), an investigation regarding a clear definition of the underlying coordinate systems, identification of estimable quantities, favorable station geometry and optimal observation schedule is given.

In Section 2, the least squares adjustment formulation for range-differencing is presented. Taking advantage of the earth-moon geometry, this procedure determines the coordinate differences of the stations particularly well. The body-fixed motions of the celestial pole (polar motion) and the earth rotation parameter are derived from an orthogonal transformation relative to a standard epoch. This is accomplished by a second least squares solution which utilizes the estimable parameters of the first adjustment as new "observations. A separation between earth rotation variations and ephemeris errors in lunar right ascension is not possible. Various station distributions are analyzed. A station geometry consisting of two north-south lines, being separated in longitude by 90°, and one east-west line determine the three orientation parameters virtually independent of ephemeris errors in declination. However, they include the common motions of the stations due to crustal motions.

The third section presents various analyses of variance models and numerical results. The simplifications consist of neglecting the earth rotation during the travel time of the pulse. In some models, the terms of the characteristic order of 1/60, i.e., those terms depending on the ratio of geocentric station distance to geocentric reflector distance, are neglected and the declination is taken constant during one interval. The analysis shows that for the given station distribution and an adequate observation schedule, the orientation parameters can be given daily with at least the measurement accuracy.

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LIST OF SYMBOLS

This list of symbols contains only those symbols which are used throughout the text.

τ	Return travel time of the laser pulse.
ω	Angular velocity of the earth rotation.
Θ	Greenwich apparent sidereal time.
x, y	Polar motion coordinates, y is positive westward.
C	Celestial pole.
$(U) = (U, V, W)$	Geocentric coordinate system whose body-fixed motion is due only to the common crustal motions of the participating stations.
ρ, Φ, Λ	Spherical coordinates in the system (U); ρ is the geocentric station distance; Φ and Λ are the station latitude and longitude.
$(U') = (U', V', W')$	Geocentric coordinate system whose third axis coincides with the celestial pole (C); U' axis is along the Greenwich mean astronomic meridian.
ρ, Φ', Λ'	Spherical coordinates in the system (U'); $\Lambda' = 0$ is in the Greenwich mean astronomic meridian.
$(U'') = (U'', V'', W'')$	Geocentric coordinate system whose third axis coincides with the celestial pole (C). The U''-axis is nearly body-fixed. Its body-fixed position is a function of an error in Greenwich apparent sidereal time Θ and right ascension of the lunar reflector as used in the computations.

$(X) = (X, Y, Z)$	Geocentric coordinate system whose third axis coincides with the celestial pole (C). The X-axis is along the direction of the true vernal equinox.
Δ	Geocentric reflector distance.
α, δ	Right ascension and declination of the reflector.
[X]	Set of estimable parameter combinations.
A	Design matrix for estimable parameter.
$\tilde{A}$	Design matrix in analysis of variance models.
[Y]	Set of all parameters (estimable and non-estimable).
$\bar{A}$	Design matrix corresponding to parameter set [Y].
H	Matrix which transforms non-estimable parameters to estimable ones.
T_o	Epoch of the initial interval.

1. INTRODUCTION

Lunar laser ranging (LLR) is on the verge of becoming a widely used tool in high precision geodesy. Observations have been successfully carried out at the McDonald Observatory at Austin, Texas for about six years. Ranging occurs regularly to the reflectors placed on the moon by Apollo 11, 14 and 15 and to the reflector on Lunakhod II. The Apollo 15 reflector is the most favorable to acquire since it is the largest among the lunar reflectors and it is located in close vicinity to distinctive surface features which make easy guiding of the laser possible. The accuracy of ranging has been increased already during its time of operation. A typical accuracy at this time is about 15 cm and better [Mulholland, 1975]. This measure is based on several returns which form a normal point [Abbot et al., 1973]. It is basically the laser pulse width, the electronic calibration, and the finite size of the reflectors which limit the ranging accuracy. A large number of parameters have reportedly been improved using lunar laser ranging data. Because the presently available observations are being made at the same observatory, one cannot solve yet for all parameters which are of scientific interest. In the near future, stations in Hawaii, Australia, Japan, France and possibly West Germany are expected to start a lunar laser ranging program.

Besides the stations mentioned above, which are all of the observatory type, i.e., fixed to their respective locations, mobile laser stations maybecome available shortly. In order to ensure the optimal use of range observations, it is necessary to design an optimal station distribution for the mobile lasers to fulfill the objective of the observation campaign. The analysis presented by Silverberg et al.

[1976] already indicates a possible function of the transportable laser station as a means for densification of accurately determined geocentric station positions. The positions of the fixed observatories are regarded as fundamental points and the mobile stations can be set up in geophysically interesting areas. This procedure requires good earth orientation parameters (polar motion and earth rotation variation). This information is supposed to be provided by the fixed observatories. However, the fixed observatories are also likely to be engaged in programs designed to improve the models about the lunar orbital motion and lunar rotation (libration). Such an observational program includes systematic ranging to all reflectors and is not necessarily the best program to give the earth orientation parameters frequently. It is, therefore, desirable to establish an earth orientation service independently of the fixed observatories. It is the subject of this study to investigate the feasibility of mobile laser stations to provide such an earth orientation service.

The method to be investigated is that of range-differencing whereby a new observable is formed by differencing the range measurements of two widely separated stations. These two co-observing stations constitute one observational unit. As for terminology, the two stations form the end points of a "line." In this study, the influence of the length, location and orientation of a line on the recoverable accuracy of the orientation parameters will be investigated. For the ideal case of simultaneous observations (same reflection time of the two pulses at the reflector) the range difference is very insensitive to changes in the geocentric reflector distance because of the small angle subtended by the paths of the two pulses. A change in declination or right ascension, on the contrary, has a strong effect on the range difference depending on the orientation of the line. It is assumed throughout this study that a good lunar ephemeris is available.

It may be a numerically integrated ephemeris or an analytical ephemeris. Since the primary objective of this study is the design of a method to determine the earth orientation and not the motion characteristic of the moon, no attempt will be made to separate the lunar orbital motion from the rotational motion (libration). It is therefore sufficient to carry out the range observations to only one reflector, say, the Apollo 15 reflector. The spatial position of the reflector will be parametrised by its geocentric distance, its declination and right ascension. The approximate values of these positional elements are computed from the available ephemeris and libration model. Constant corrections to the declination and right ascension of the reflector will be solved for daily. For nearly simultaneous observations, range-differencing occurs between those two observations whose reflection times are closest. In such cases, the requirements on the lunar ephemeris are more stringent as is explained in Section 2.3.1.

The coordinates of the stations will be transformed to their differences and sums. Range differences allow for the accurate determination of coordinate differences. However, coordinate differences completely determine the orientation of the earth. It is pointed out that lunar laser ranging gives the orientation of the earth only, relative to the lunar motion. If the results of lunar laser ranging are to be related to a frame defined by quasars, complementary observations such as differential VLBI (very long base line interferometry) observations of the Apollo lunar surface experiment packages (ALSEP s) and quasars are needed.

The earth orientation parameters will be solved for each interval. Since the progressive Chandler motion is smaller or equal to 10 cm per day, it is natural to limit the length of the interval to one day for which the orientation parameters can be taken as a constant. Actually, the length of one interval depends on the station distribution.

Each interval includes only one lunar transit. The interval begins with the epoch of the first observation at the most eastern station pair and ends with the last observation at the most western station pair. The shorter the interval, the simpler is the mathematical model of the adjustment and the number of solution parameters is minimized. However, a short interval requires a favorable station distribution and an adequate observation schedule in order to provide sufficient geometrical strength and enough observations for the parameters to be determined accurately in each interval.

In the subsequent section, the observation equations for range-differencing are set up assuming that the range observations are already corrected for influences of atmosphere and solid body earth tides. Fortunately, the atmospheric correction can be computed accurately to ± 1 cm for the wave length of the lunar lasers [Mulholland, 1975]. The solid body earth tides can reach an amplitude for the vertical displacement of 30 to 40 cm. It will be necessary to compute these displacements as accurately as possible using elaborate geophysical models. The complete description of the observation equations also requires relativistic considerations. Without taking recourse in great detail to relativistic theory, the pertinent procedure is given for computing the exact time delay. The mathematical formulation used in this study is such that not only chords and angles between the stations are estimated but also the orientation angles relative to an epoch T_0 which is the zero point of counting. This procedure is distinctly different from what is usually referred to as Inner Constraint in which case the resulting "orientation parameters" do not allow a simple interpretation. Due to the lack of real observations, an analysis of variance is made to investigate the capabilities of the proposed method as a function of the number of stations, their distribution and the observation schedule. The analysis of variance model is derived from the rigorous model by making certain simplifications. One of the simplifications is to neglect the earth rotation during the travel time of the pulse, or equivalently, an infinitely large velocity of light is assumed so that the instants of transmission, reflection and reception are the same. In other analysis models, the terms of the characteristic order 1/60, i.e., those terms which are a function of the ratio of the geocentric station distance to the geocentric reflector distance are neglected, and the declination is taken constant during one interval. Special attention is given to the minimum number of stations required in order to uniquely determine the earth orientation and to the difficulties introduced by crustal motions. Any data loss due to weather conditions is not taken into account. If for a particular interval no determination of the orientation parameters can be made because of loss of data due to adverse weather conditions, the parameters can be interpolated from their values in the adjacent intervals.

2. MODELLING RANGE OBSERVATIONS

The direction of the celestial pole (C) is best suited as a reference direction for the third axis of the coordinate system in which range equations are formulated. In fact, the body motions of the celestial pole (C) are directly estimable from laser ranges. In Leick [1978], an extensive discussion on the implications of the differences between the instantaneous rotation axis and the celestial pole is given. The principal property of the celestial pole is that its direction has neither body-fixed nor space-fixed periodic diurnal motions. In this section, the rigorous least squares formulation for range-differencing is given. The estimable parameters are identified, and the earth orientation parameters are specified on the basis of an orthogonal transformation (over determined case).

2.1 Time Delay Computation

Processing range data requires the precise computation of time delay, i.e., computing the elapsed round trip time for the pulse from the approximate parameters. Such a computation makes relativistic corrections necessary. The corrections to be considered here are undisputed in relativity theory. Although there are several competing relativities theories, each gives the same corrections, at least to the accuracy which is needed for laser ranging within the solar system. The following time scales need to be distinguished:

a) coordinate time t

b) proper time τ

Coordinate time is the time argument in the ephemerides of the planets, the earth-moon bary-center and the moon in the heliocentric coordinate system. These ephemerides are, of course, based on the relativistic equations of motion which are obtained from the Newtonian equations by amending relativistic perturbative accelerations. For the purpose of this discussion, one can consider the time scale of the atomic clock on the earth as a realization of the proper time scale.

In relativity the ratio of the intervals of the coordinate and proper time scale is not constant. The proper time scale is a function of the total potential U at the position of the clock and of the solar system barycentric velocity $\dot{s}$ of the clock. The relativistic corrections for laser ranging in the solar system are developed in great detail in Moyer [1976] from where the following differential equation is taken which relates proper time and coordinate time:

$$\frac{d\tau}{dt} = 1 - \frac{U}{c^2} - \frac{1}{2}\left(\frac{\dot{s}}{c}\right)^2 \qquad (2.1\text{-}1)$$

This is an approximate expression in which terms of order $1/c^4$ are ignored. c is the light velocity. Since U is positive, it is seen that for a fixed atomic clock on earth $d\tau < dt$, proper time τ falls behind coordinate time. Since

$$d\tau = \frac{dN}{n}$$

where dN is the number of observed cycles and n is the number of cycles per second atomic time (conversion factor), one can make the interval of proper time agree on the average with coordinate time by selecting the appropriate n. This procedure was, in fact, followed when the TAI (International Atomic Time) second was defined. The remaining periodic variation between proper and coordinate time is derived by integrating equation (2.1-1). Moyer [1976] gives the following expression:

$$
\begin{aligned}
t - \tau = {} & \Delta\tau \\
& + 1.658 \times 10^{-3} \sin E \\
& \mp 3.18 \times 10^{-10} \rho \cos \Phi \sin (UT1 + \Lambda) \\
& + \ldots
\end{aligned} \qquad (2.1\text{-}2)
$$

The first term is the constant of integration. The IAU (1977) adopted

$$\Delta\tau = 32.184 \text{ s}$$

It basically represents the constant shift between atomic time TAI and ephemeris time ET at the initial epoch of atomic time. The second term is the largest periodic term. Its magnitude is about 2. ms. E is the eccentric anomaly of the heliocentric orbit of the earth-moon barycenter. The third term has a magnitude of about 2 μs. $\rho \cos \Phi$ denotes the spin axis distance of the clock in kilometers, UT1 stands for universal time, and Λ for the station longitude. The complete expression contains many more terms including terms which are a function of the position of Jupiter and Saturn.

Given the $(t - \tau)$ correction of equation (2.1-2) coordinate time can be transformed to proper time at any epoch and vice versa. The computation of the travel time of the pulse is complicated by another relativistic phenomenon, which is called "radar time delay." This delay is a function of the light propagation characteristics as the light travels through the potential field of the sun. The theoretical background of this phenomenon is given in [Misner et al., 1973, p. 1103]. Computationally, the radar time delay is expressed by the so-called "Light-time equation (LTEq.)." Moyer [1976] gives

$$T_1 - T_k = \frac{r_{k1}}{c} + \frac{2GS}{c^3} \ln \frac{r_k + r_1 + r_{1k}}{r_k + r_1 - r_{1k}} \qquad (2.1\text{-}3)$$

with

$$
\begin{aligned}
r_{k1} &= \left\| \vec{r}_1^{\,s}(T_1) - \vec{r}_k^{\,s}(T_k) \right\| \\
r_k &= \left\| \vec{r}_k^{\,s}(T_k) \right\| \\
r_1 &= \left\| \bar{r}_1^{\,s}(T_1) \right\|
\end{aligned}
$$

The superscript s denotes heliocentric position. GS is the heliocentric gravitational constant. The light time equation thus relates the coördinates of two points, k and l, to the coordinate time t for light to travel from one of the points to the other. This equation has to be solved by iteration. In case of LLR, the up and down legs have to be solved separately.

In summary, the computed range observable τ_o is arrived at according to the following scheme:

$$\tau_3 \xrightarrow{(t - \tau)_{t_3}} T_3 \xrightarrow{\text{LTEq}} T_2 \xrightarrow{\text{LTEq}} T_1$$

and $\tau_o = T_3 - T_1 - (t - \tau)_{t_3} + (t - \tau)_{t_1}$ (2.1-4)

The subscripts 1, 2, and 3 denote the instants of transmission, reflection and reception of the laser pulse. Thus, τ_3 is the observed reception time of the pulse. It is converted via equation (2.1-2) to coordinate time T_3, which is used together with an estimate of T_2 in order to solve the light time equation (2.1-3). The solution gives an improved value for T_2 so that the solution of (2.1-3) can be repeated until no significant change in T_2 occurs. The same iteration is done for the downward leg. The light time equation has to be solved twice because the upward and downward path of the laser pulse is at a different position within the gravitational field of the sun. The third and fourth terms in equation (2.1-4) convert the round-trip light time from an interval of coordinate time to an interval of proper time at the epoch of observation. Finally, the computed time delay, τ_o, and observed time delay on the atomic clock, $\tau_b = \tau_3 - \tau_1$, can be compared.

2.2 Basic Observation Equation

2.2.1 Single Station Observations

Two solutions of the light time equation (2.1-3) and the relativistic corrections (2.1-2) have resulted in heliocentric positions of the reflector, the geocenter, the laser station and in the computed time delay τ_0. This computation is, of course, based upon the approximate values of reflector position, earth orientation parameters and laser station position. On the basis of the observed time delay, some of the parameters, particularly those for earth orientation, can be improved.

In Figure 2.1 the symbols C and R denote the celestial pole and the lunar reflector, respectively. The numbers 1, 2, and 3 denote the instants of transmission, reflection, and reception of the laser pulse. The observation equation will be expressed in a geocentric frame whose third axis coincides with the celestial pole (C). It is readily derivable from the light time equation as applied to the up and down leg. Denoting the second term on the right-hand side of equation (2.1-3), which is the contribution to the light time from general relativity, by Δ_{k1} the observation equation becomes

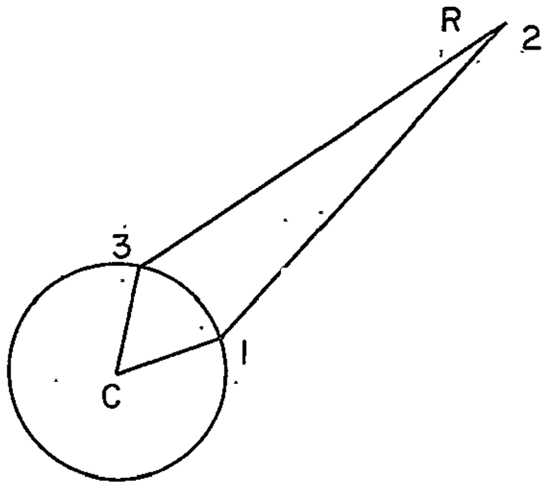


Figure 2.1 Earth-Moon Geometry

$$\|\vec{X}_{R2} - \vec{X}_1\| + \|\vec{X}_{R2} - \vec{X}_3\| - c\tau - cw = 0 \qquad (2.2\text{-}1)$$

where

$$w = -\Delta_{12} - \Delta_{23} + (t - \tau)_{t_3} - (t - \tau)_{t_1}$$

is a small but computable term. τ is the time delay. The coordinates refer to the system (X), the first axis being located along the true vernal equinox. The light velocity c determines the scale of the configuration. By dividing equation (2.2-1) by c, each coordinate is expressed in units of light velocity, so that any changes in the adopted light velocity result in a computable change in the coordinate length (scaling).

The station position at the instants of transmission and reception are related by a rotation around the third axis as follows:

$$\vec{X}_3 = R_3(\omega\tau)\vec{X}_1$$

where ω is the angular velocity of the earth rotation. The rotational position of the earth is introduced by

$$\vec{X}_1 = R_3(-\Theta_1)\vec{U}'$$

where Θ is the apparent Greenwich sidereal time at the instant of transmission and (U') is the coordinate system whose third axis still coincides with the celestial pole but whose first axis is fixed to the Greenwich mean astronomic meridian. Substituting these equations in equation (2.2-1) gives the basic mathematical model:

$$F(L_a, X_a) \equiv \|\vec{X}_{R2} - R_3(-\Theta_1)\vec{U}'\| + \|\vec{X}_{R2} - R_3(\omega\tau - \Theta_1)\vec{U}'\| - c\tau - cw = 0 \qquad (2.2\text{-}2)$$

This is the standard model in adjustment theory for the case where observed quantities and unknown parameters are in the same equation. With the notation of Uotila [1967], the linearized form of (2.2-2) is

$$ {}_{r}B_{n} V + {}_{r}A_{u} X + {}_{r}W_{1} = 0 \tag{2.2-3} $$

where

$$ B = \frac{\partial F}{\partial L_a}, \quad A = \frac{\partial F}{\partial X_a}, \quad \text{and } W = F(L_b, X_o) $$

r denotes the number of equations and u is the number of parameters. The weight matrix is the inverse of the variance-covariance matrix of observations multiplied by the variance of unit weight,

$$ P = \sigma_o^2 \, \Sigma_{L_b}^{-1} $$

As usual, the subscripts a, b, and o refer to adjusted, observed and approximate values, respectively. The partial derivatives in B and A are evaluated for the approximate and observed values. V denotes the residuals. X is the set of solution parameters; they are corrections to the approximate values. In order to avoid confusion, the parameters will sometimes be denoted by [X]. The usual minimization of

$$ V' \, PV $$

gives the least squares estimate for the parameters [X]

$$ \hat{X} = -(A^T M^{-1} A)^{-1} A^T M^{-1} W \tag{2.2-4} $$

with

$$ M = B P^{-1} B^T $$

The adjusted variance of unit weight is

$$ \hat{\sigma}_o^2 = \frac{V' PV}{r - u} \tag{2.2-5} $$

Finally, the variance-covariance matrix of the adjusted parameter is

$$ \Sigma_{\hat{X}} = \hat{\sigma}_o^2 \, (A^T M^{-1} A)^{-1} \tag{2.2-6} $$

The partial derivatives of the function F will be given below for spherical and Cartesian coordinates. Both systems are, of course, strictly related, but in some cases of analysis of variance, one or the other system is more convenient to interpret.

The subscript "2" on the lunar positional elements is subsequently omitted. Unless otherwise stated, its position is always evaluated at reflection time. Similarly, the subscript "1" on the symbol Θ is deleted, remembering that the apparent Greenwich sidereal time has to be evaluated at transmission time. Denoting the spherical coordinates of the station in the (U') system by $(\rho,\ \Lambda',\ \Phi')$ the partial derivatives in the design matrix A are computed from equation (2.2-2) as follows:

$$
\left.
\begin{aligned}
A_{\Delta} &= \frac{1}{r_{12}}\left\{\Delta - \rho\cos\Phi'\cos\delta\cos(\Lambda' + \Theta - \alpha) - \rho\sin\Phi'\sin\delta\right\} \\
&\quad + \frac{1}{r_{23}}\left\{\Delta - \rho\cos\Phi'\cos\delta\cos(\Lambda' - \omega\tau + \Theta - \alpha) - \rho\sin\Phi'\sin\delta\right\} \\
A_{\alpha} &= -\frac{\rho\Delta}{r_{12}}\cos\Phi'\cos\delta\sin(\Lambda' + \Theta - \alpha) \\
&\quad - \frac{\rho\Delta}{r_{32}}\cos\Phi'\cos\delta\sin(\Lambda' - \omega\tau + \Theta - \alpha) \\
A_{\delta} &= \frac{\rho\Delta}{r_{12}}\left\{\cos\Phi'\sin\delta\cos(\Lambda' + \Theta - \alpha) - \sin\Phi'\cos\delta\right\} \\
&\quad + \frac{\rho\Delta}{r_{32}}\left\{\cos\Phi'\sin\delta\cos(\Lambda' - \omega\tau + \Theta - \alpha) - \sin\Phi'\cos\delta\right\} \\
A_{\rho} &= \frac{1}{r_{12}}\left\{\rho - \Delta\cos\Phi'\cos\delta\cos(\Lambda' + \Theta - \alpha)\right\} \\
&\quad + \frac{1}{r_{32}}\left\{\rho - \Delta\cos\Phi'\cos\delta\cos(\Lambda' - \omega\tau + \Theta - \alpha)\right\} \\
A_{\Phi'} &= \frac{\rho\Delta}{r_{12}}\left\{\sin\Phi'\cos\delta\cos(\Lambda' + \Theta - \alpha) - \sin\delta\cos\Phi'\right\} \\
&\quad + \frac{\rho\Delta}{r_{32}}\left\{\sin\Phi'\cos\delta\cos(\Lambda' - \omega\tau + \Theta - \alpha) - \sin\delta\cos\Phi\right\} \\
A_{\Lambda'+\Theta} &= \frac{\rho\Delta}{r_{12}}\cos\Phi'\cos\delta\sin(\Lambda' + \Theta - \alpha) \\
&\quad + \frac{\rho\Delta}{r_{32}}\cos\Phi'\cos\delta\sin(\Lambda' - \omega\tau + \Theta - \alpha)
\end{aligned}
\right\}
\tag{2.2-7}
$$

Note that $\Lambda' + \Theta - \alpha$ is the hour angle of reflector. The symbol r stands for the distance between the observation station and reflector, in particular

$$r_{12} = \| \vec{X}_{R2} - \vec{X}_1 \|$$

$$r_{32} = \| \vec{X}_{R2} - \vec{X}_3 \|$$

It is seen that the coefficients A_α and $A_{\Lambda'+\Theta}$ are linearly dependent. Thus, only the linear combination $\Lambda' + \Theta - \alpha$, i.e., the hour angle of the reflector, is estimable so that the list of parameters contains five estimable quantities

$$[X] = [d\Delta, d\delta, d\rho, d\Phi', d(\Lambda' + \Theta - \alpha)]. \qquad (2.2\text{-}8)$$

Note that parameters or linear combinations of them are called estimable if the corresponding design matrix A is non-singular.

The Matrix B is diagonal since each equation of (2.2-3) contains only one observation. The diagonal term is

$$B_\tau = \frac{-\rho\Delta}{r_{32}} \omega \cos\Phi' \cos\delta \sin(\Lambda' - \omega\tau + \Theta - \alpha) - c$$

Each observation from the same station adds one equation to the system (2.2-3) and three parameters, concerning the reflector position and the earth rotation, to the list of (2.2-8). For k observations, the complete parameter list is

$$[X] = [d\Delta^1, d\delta^1 \ldots d\Delta^k, d\delta^k, d\rho, d\Phi, d^1(\Lambda' + \Theta - \alpha), d^2(\Lambda' + \Theta -$$

$$\ldots d^k(\Lambda' + \Theta - \alpha)]$$

A least squares solution becomes possible by imposing the constraints

$$\begin{aligned} d\Delta^\eta &= d\Delta \\ d\delta^\eta &= d\delta \\ d\alpha^\eta &= d\alpha \\ d\Theta^\eta &= d\Theta \end{aligned} \qquad (2.2\text{-}9)$$

for $\eta = 1 \ldots k$. These constraints make it necessary that a good lunar ephemeris and station clock are available in order for such simple modelling to be permissible. A detailed discussion on these constraints follows in Section 2.3.1. If one incorporates the constraints, there remain five parameters which can be solved by least squares

$$[X] = [d\,\Delta,\ d\,\delta,\ d\rho,\ d\,\Phi',\ d(\Lambda' + \Theta - \alpha)\,] . \qquad (2.2\text{-}10)$$

The station positions in (2.2-10) can be interpreted as referring to a system (U'') which differs from (U') by a small rotation around the third axis. The rotation is due to errors in the clock, $d\,\Theta$, and lunar right ascension $d\alpha$. Since the station coordinates in the (U')-system are a function of time due to polar motion, a reformulation of the mathematical model is given in terms of station coordinates in a conventional geocentric terrestrial system (U), which coincides with the (U'')-system at some standard epoch,

$$(U) \equiv (U'') \text{ at epoch } T_0 .$$

The relation between these systems is given by the polar motion coordinates (x, y). The same representation as found in [Mueller, 1967, p. 82] is chosen; i.e., the origin of the polar motion coordinate system is at the pole of the (U)-system, the x-axis is along the direction of the zero longitude, and y points westward. The latitude and longitude in the (U)-system, Φ and Λ, are related to those in the (U')-system by [Mueller, 1967, p. 87]

$$d(\Phi - \Phi') = y \sin\Lambda - x\cos\Lambda$$

$$d(\Lambda - \Lambda') = -(x \sin\Lambda + y\cos\Lambda)\tan\Phi$$

The partial derivatives for latitude and longitude in the observation equation (2.2-3) now become

$$A_{\Phi'} d\Phi' + A_{\Lambda'+\Theta-\alpha} d\Lambda'$$

$$= A_{\Phi'}\,[d\Phi - y\sin\Lambda + x\cos\Lambda] + A_{\Lambda'+\Theta-\alpha}\,[d(\Lambda + \Theta - \alpha) +$$

$$+ (x\sin\Lambda + y\cos\Lambda)\tan\Phi]$$

$$= A_{\Phi'} d\Phi - A_{\Phi'}\, y \sin\Lambda + A_{\Phi'}\, x \cos\Lambda + A_{\Lambda'+\Theta-\alpha}\, d\Lambda$$
$$+ A_{\Lambda'+\Theta-\alpha}\, x \sin\Lambda \tan\Phi + A_{\Lambda'+\Theta-\alpha}\, y \cos\Lambda \tan\Phi \qquad (2.2\text{-}11)$$

The coefficients (2.2-7) are evaluated with Φ' and Λ' , i.e., with the approximate station coordinates in the (U')-system. If those latitudes and longitudes are replaced by Φ and Λ then one formally obtains the partials which one would have obtained if the station position in equations (2.2-2) had been expressed in the (U)-system, and the partials had been evaluated with $x = y = 0$. No further changes are needed in the coefficients (2.2-7).

The parameters of equation (2.2-11) together with those of (2.2-8) give the following list of estimable parameter combinations:

$$[X] = [d\Delta,\ d\delta,\ d\rho,\ \mu,\ \nu] \qquad (2.2\text{-}12)$$

with

$$\mu = d\Phi' = d\Phi - y \sin\Lambda + x \cos\Lambda$$

$$\nu = d(\Lambda' + \Theta - \alpha) = d(\Lambda + \Theta - \alpha) + x \sin\Lambda \tan\Phi + y \cos\Lambda \tan\Phi$$

The coefficients of the last two parameters are

$$A_\mu = A_\Phi$$

$$A_\nu = A_{\Lambda+\Theta-\alpha}$$

There exist an alternative set of estimable parameters. From equation (2.2-11) the coefficients for polar motion are

$$A_x = A_\Phi \cos\Lambda + A_{\Lambda+\Theta-\alpha} \sin\Lambda \tan\Phi$$
$$A_y = -A_\Phi \sin\Lambda + A_{\Lambda+\Theta-\alpha} \cos\Lambda \tan\Phi \qquad (2.2\text{-}13)$$

Incorporating the inverse relations of (2.2-13),

$$A_{\Phi} = A_x \cos\Lambda - A_y \sin\Lambda$$

$$A_{\Lambda+\Theta-\alpha} = \cot\Phi\,(A_x \sin\Lambda + A_y \cos\Lambda)$$

in the general observation equation gives the following set of estimable parameters:

$$[X] = [d\Delta,\ d\delta,\ d\rho,\ \mu',\ \nu'] \tag{2.2-14}$$

with

$$\mu' = x + \cos\Lambda\, d\Phi + \cot\Phi \sin\Lambda\, d(\Lambda + \Theta - \alpha)$$

$$\nu' = y - \sin\Lambda\, d\Phi + \cot\Phi \cos\Lambda\, d(\Lambda + \Theta - \alpha)$$

and

$$A_{\mu'} = A_x \quad A_{\nu'} = A_y$$

It is emphasized again that all coefficients have to be evaluated with the approximate station coordinates in the (U)-system.

Analogous expressions to (2.2-7) for Cartesian station coordinates are:

$$A_{\Delta} = \frac{1}{r_{12}}\{-U' \cos\delta \cos(\Theta-\alpha) + V' \cos\delta \sin(\Theta-\alpha) - W' \sin\delta + \Delta\}$$
$$+ \frac{1}{r_{32}}\{-U' \cos\delta \cos(-\omega\tau + \Theta - \alpha) + V' \cos\delta \sin(-\omega\tau + \Theta - \alpha)$$
$$- W' \sin\delta + \Delta\}$$

$$A_{\delta} = \frac{\Delta}{r_{12}}\{U' \sin\delta \cos(\Theta-\alpha) - V' \sin\delta \sin(\Theta-\alpha) - W' \cos\delta\}$$
$$+ \frac{\Delta}{r_{32}}\{U' \sin\delta \cos(-\omega\tau + \Theta - \alpha) - V' \sin\delta \sin(-\omega\tau + \Theta - \alpha)$$
$$- W' \cos\delta\}$$

$$
\left.
\begin{aligned}
A_{\alpha-\Theta} &= \frac{\Delta}{r_{12}}\{-U'\cos\delta\,\sin(\Theta-\alpha) - V'\cos\delta\cdot\cos(\Theta-\alpha)\}\\
&\quad + \frac{\Delta}{r_{32}}\{-U'\cos\delta\,\sin(-\omega\tau+\Theta-\alpha) - V'\cos\delta\cos(-\omega\tau+\Theta-\alpha)\}\\
A_{U'} &= \frac{1}{r_{12}}\{U' - \Delta\cos\delta\cos(\Theta-\alpha)\} + \frac{1}{r_{32}}\{U' - \Delta\cos\delta\cos(-\omega\tau+\Theta-\alpha)\}\\
A_{V'} &= \frac{1}{r_{12}}\{V' + \Delta\cos\delta\sin(\Theta-\alpha)\} + \frac{1}{r_{32}}\{V' + \Delta\cos\delta\sin(-\omega\tau+\Theta-\alpha)\}\\
A_{W'} &= \frac{1}{r_{12}}\{W' - \Delta\sin\delta\} + \frac{1}{r_{32}}\{W' - \Delta\sin\delta\}
\end{aligned}
\right\} \quad (2.2
$$

The diagonal element of the B-matrix is

$$
B_\tau = \frac{\Delta\omega}{r_{32}}\{-U'\cos\delta\,\sin(-\omega\tau+\Theta-\alpha) - V'\cos\delta\cos(-\omega\tau+\Theta-\alpha)\} - c
$$

Combining the linearly dependent coefficients $A_{\alpha-\Theta}$, $A_{U'}$, and $A_{V'}$

$$
A_{\alpha-\Theta} = -U' A_{V'} + V' A_{U'}
$$

one finds again a list of five estimable quantities

$$
[X] = [X_1 \vdots X_2] = [d\Delta,\ d\delta \vdots dU'',\ dV'',\ dW''] \qquad (2.2\text{-}16)
$$

with

$$
\begin{aligned}
dU'' &= dU' + V'\,d(\alpha-\Theta)\\
dV'' &= dV' - U'\,d(\alpha-\Theta)\\
dW'' &= dW'
\end{aligned}
$$

and the coefficients

$$
A_{U''} = A_{U'}, \quad A_{V''} = A_{V'}
$$

The constraints (2.2-9) have been incorporated in the list (2.2-16).

The expressions are transformed to those of the (U)-system by introducing polar motion coordinates as follows (only terms of first order are retained):

$$\begin{aligned} U' &= U - xW \\ V' &= V + yW \\ W' &= xU - yV + W \end{aligned} \qquad (2.2\text{-}17)$$

Differentiating equations (2.2-17) and combining them with the parameters (2.2-16) gives another set of estimable quantites:

$$[X] = [X_1 \vdots X_2] = [d\Delta,\ d\delta \vdots\ dU'',\ dV'',\ dW''] \qquad (2.2\text{-}18)$$

with

$$\begin{bmatrix} dU'' \\ dV'' \\ dW'' \end{bmatrix} = \begin{bmatrix} dU \\ dV \\ dW \end{bmatrix} + \begin{bmatrix} 0 & d(\alpha - \Theta) & -x \\ -d(\alpha - \Theta) & 0 & y \\ x & -y & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \qquad (2.2\text{-}19)$$

The coefficients are

$$A_{U''} = A_U,\quad A_{V''} = A_V,\quad A_{W''} = A_W$$

Note that the coordinates (U′) of the coefficients (2.2-15) have to be replaced by (U). The partials for polar motion, A_x, and A_y, are not needed explicitly although they can be derived in a similar manner as was done for the case of spherical coordinates (2.2-11). For reasons of abbreviation, the rotation parameter in (2.2-19) is denoted by $\bar{\alpha}$, thus

$$d\bar{\alpha} \equiv d(\alpha - \Theta)$$

The quantities dU'', dV'' and dW'' can be interpreted as coordinate corrections in the estimable frame of reference. The third axis of this frame coincides with the direction of the celestial pole (C), whereas the first axis deviates from that of the (U′)-system by an angle $d\bar{\alpha}$.

Equation (2.2-19) can equivalently be written as

$$d\vec{U}'' = d\vec{U} + [R_3 (d\bar{\alpha})\, R_1 (y)\, R_2 (x) - I]\, \vec{U}$$

The derivatives so far have resulted in a set of estimable parameters for laser observations at one station. The columns of the design matrix A were checked for linear dependencies and the corresponding parameters combined in order to yield estimable parameters. This concept is identical to that of rank factorization. For the sake of subsequent discussions, the identities between both approaches are pointed out. Let [Y] be the vector of all the m parameters which have entered the formulation

$$[Y] = [X_1 \vdots Y_2 \vdots Y_3] = [d\Delta, d\delta \vdots\; x,\; y, d\bar{\alpha} \vdots dU, dV, dW]$$

The corresponding singular design matrix $\bar{A}$ of size (r x m) is

$$_r\bar{A}_m \doteq (A_1 \vdots A_2 \vdots A_3)$$

The rank is

$$R(\bar{A}) = m - s = u$$

with rank deficiency s = 3 for the present case. $[Y_2]$, which denotes the parameters of the earth orientation, still contains the combination of right ascension and Greenwich sidereal time. Both could, of course, also be separated, and thus increase the rank deficiency to four. In linear algebra it is proven that the matrix $\bar{A}$ can be factorized such that

$$_r\bar{A}_m = {}_rD_u\; {}_uH_m \text{ with } r > u,\; m > u$$

and

$$R(D) = u$$

Thus,

$$\bar{A}Y = DX$$

and

$$X = H\,Y$$

gives the estimable parameters [X]. The following three identifications can be made:

1) The set of estimable parameters is the same as in (2.2-18),
2) The design matrix for the estimable parameters is $D = (A_1 \vdots A_3)$,
3) The non-estimable parameters are transformed to estimable quantities by the matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -W & 0 & V & 1 & 0 & 0 \\ 0 & 0 & 0 & W & -U & 0 & 1 & 0 \\ 0 & 0 & U & -V & 0 & 0 & 0 & 1 \end{bmatrix}$$

The H matrix is written for abbreviation as

$$H = \begin{bmatrix} I & O & O \\ O & F^T & I \end{bmatrix}$$

Evaluating

$$\begin{aligned} \bar{A} &= DH \\ &= (A_1 \;\; A_3) \begin{bmatrix} I & O & O \\ O & F^T & I \end{bmatrix} \\ &= (A_1 \vdots A_3 F^T \vdots A_3) \end{aligned}$$

yields the linear relation between the coefficients:

$$A_2 = A_3 F^T$$

2.2.2 Range Differencing

The adjustment model for range-differencing is readily derived from the basic model (2.2-3) and its set of estimable parameters of equation (2.2-18). Denoting the two co-observing stations by i and j, the linearized form of the adjustment model is

$$
\begin{aligned}
B_i V_i + A_i X_i + W_i &= 0 \\
B_j V_j + A_j X_j + W_j &= 0
\end{aligned}
\qquad (2.2\text{-}20)
$$

Since these equations included the constraints (2.2-9), there are 2r equations for u = 8 estimable parameters, i.e., the parameters $d\Delta$, $d\delta$, $d\bar{\alpha}$ are common to both sets. Subtracting the first equation from the second in (2.2-20) gives the new set of equations,

$$
(-B_i \; B_j) \begin{bmatrix} V_i \\ V_j \end{bmatrix} + (-A_i A_j) \begin{bmatrix} X_i \\ X_j \end{bmatrix} + W_{ij} = 0 \qquad (2.2\text{-}21)
$$

with

$$
W_{ij} = W_j - W_i
$$

It is understood that in (2.2-21) those terms are differenced which correspond to the observation equations for simultaneous or near simultaneous range observations at the stations i and j. This is automatically achieved if the equations in (2.2-20) are ordered successively in time. The parameters are conveniently transformed to their sums and differences as follows:

$$
(-A_i \; A_j) \begin{bmatrix} X_i \\ X_j \end{bmatrix} = (A_i + A_j) \left[\frac{-X_i + X_j}{2} \right] + (A_j - A_i) \left[\frac{X_j + X_i}{2} \right] .
$$

With the notation

$$
X_{j-i} = \frac{X_j - X_i}{2} , \quad X_{j+i} = \frac{X_j + X_i}{2} \qquad (2.2\text{-}22)
$$

the equation (2.2-21) now becomes

$$(-B_i\, B_j)\begin{bmatrix} V_i \\ V_j \end{bmatrix} + (A_i + A_j \,\vdots\, A_j - A_i)\begin{bmatrix} X_{j-i} \\ X_{j+i} \end{bmatrix} + W_{ij} = 0$$

The list of estimable parameters in the above equation is

$$\begin{aligned}[X] &= [X_1 \,\vdots\, X_2] \\ &= [d\Delta,\ d\delta \,\vdots\, dU''_{j-i},\ dV''_{j-i},\ dW''_{j-i},\ dU''_{j+i},\ dV''_{j+i},\ dW''_{j+i}]'\end{aligned} \qquad (2.2\text{-}23)$$

and the design matrix consists of the differences and sums of the original coefficients,

$$\begin{aligned}(A) &= (A_1 \,\vdots\, A_3) \\ &= (A_{j\Delta} - A_{i\Delta},\ A_{j\delta} - A_{i\delta} \,\vdots\, A_{jU} + A_{iU},\ A_{jV} + A_{iV},\ A_{jW} + A_{iW}, \\ &\qquad A_{jU} - A_{iU},\ A_{jV} - A_{iV},\ A_{jW} - A_{iW})\end{aligned} \qquad (2.2\text{-}24)$$

Note that the geocentric reflector distance and the declination in (2.2-24) are not split up into their sums and differences because $d\Delta_j - d\Delta_i = 0$ and $d\delta_j - d\delta_i = 0$ is valid throughout the interval according to constraints (2.2-9). The last six parameters in (2.2-23) are the corrections to the sums and differences of the station coordinates in the (U'')-system. They are related to the corresponding parameters in the conventional terrestrial system (U) by the three orientation angles x, y and $d\bar{\alpha}$. With equations (2.2-19) and (2.2-22) one gets the following expressions:

$$\begin{bmatrix} dU''_{j-i} \\ dV''_{j-i} \\ dW''_{j-i} \end{bmatrix} = \begin{bmatrix} dU_{j-i} \\ dV_{j-i} \\ dW_{j-i} \end{bmatrix} + \begin{bmatrix} 0 & d\bar{\alpha} & -x \\ -d\bar{\alpha} & 0 & y \\ x & -y & 0 \end{bmatrix} \begin{bmatrix} U_{j-i} \\ V_{j-i} \\ W_{j-i} \end{bmatrix}, \qquad (2.2\text{-}25)$$

and

$$\begin{bmatrix} dU_{j+1} \\ dV_{j+1} \\ dW_{j+1} \end{bmatrix} = \begin{bmatrix} dU_{j+1} \\ dV_{j+1} \\ dW_{j+1} \end{bmatrix} + \begin{bmatrix} 0 & d\bar{\alpha} & -x \\ -d\bar{\alpha} & 0 & y \\ x & -y & 0 \end{bmatrix} \begin{bmatrix} U_{j+1} \\ V_{j+1} \\ W_{j+1} \end{bmatrix} \qquad (2.2\text{-}26)$$

To the first order of the orientation parameters these equations can be written as:

$$d\vec{U}_{j-1} = d\vec{U}_{j-1} + [R_3(d\bar{\alpha})\ R_1(y) R_2(x) - I]\vec{U}_{j-1}$$

$$d\vec{U}_{j+1} = d\vec{U}_{j+1} + [R_3(d\bar{\alpha})\ R_1(y)\ R_2(x) - I]\ \vec{U}_{j+1}$$

I is the identity matrix. The least squares estimates according to the general equations (2.2-4) to (2.2-6) are, with

$$M = (B_i \Sigma_{Lbi} B_i^T + B_j \Sigma_{Lbj} B_j^T),$$

$$[\hat{X}]^T = [\hat{X}_1, \hat{X}_3]^T = \begin{bmatrix} A_1^T M^{-1} A_1 & A_1^T M^{-1} A_3 \\ A_3^T M^{-1} A_1 & A_3^T M^{-1} A_3 \end{bmatrix}^{-1} \begin{bmatrix} A_1^T M^{-1} \\ A_3^T M^{-1} \end{bmatrix} W_{ij}$$

Taking the inverse analytically yields

$$\hat{X}_1 = G_1 W_{ij} \qquad (2.2\text{-}27)$$

where

$$G_1 = -\{A_1^T M^{-1} A_1 - A_1^T M^{-1} A_3 (A_3^T M^{-1} A_3)^{-1} A_3^T M^{-1} A_1\}^{-1} A_1^T M^{-1}$$
$$+ (A_1^T M^{-1} A_1)^{-1} A_1^T M^{-1} A_3 Q_{x_3} A_3^T M^{-1}$$

$$\hat{X}_3 = Q_{x_3} \{A_3^T M^{-1} A_1 (A_1^T M^{-1} A_1)^{-1} A_1^T M^{-1} - A_3^T M^{-1}\} W_{ij} \qquad (2.2\text{-}28)$$

with

$$Q_{x_3} = \{A_3^T M^{-1} A_3 - A_3^T M^{-1} A_1 (A_1^T M^{-1} A_1)^{-1} A_1^T M^{-1} A_3\}^{-1}$$

and

$$\Sigma_{\hat{x}_3} = \hat{\sigma}_0^2 Q_{x_3} \qquad (2.2\text{-}29)$$

whereby $\hat{\sigma}_0$ is computed from (2.2-5).

As for rank factorization, the necessary identifications are readily available. Let [Y] include all the parameters

$$
\begin{aligned}
[Y] &= [X_1 \vdots Y_2 \vdots Y_3] \\
&= [d\Delta,\ d\delta \vdots\ x,\ y, d\bar{\alpha} \vdots dU_{j-1},\ dV_{j-1},\ dW_{j-1}, \\
&\quad dU_{j+1},\ dV_{j+1},\ dW_{j+1}]
\end{aligned}
\tag{2.2-30}
$$

with design matrix

$$\bar{A} = (A_1 \vdots A_2 \vdots A_3)$$

which has a rank deficiency of three, then the rank factorization is

$$
\begin{aligned}
\bar{A}Y = DHY &= DX \\
X &= HY.
\end{aligned}
$$

The design matrix D for the estimable parameters [X] is

$$D = (A_1 \vdots A_3),$$

where

$$R(D) = u.$$

The non-estimable parameters are transformed to a set of estimable parameters by

$$
H = \left[\begin{array}{cc|ccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -W_{j-1} & 0 & V_{j-1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & W_{j-1} & -U_{j-1} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & U_{j-1} & -V_{j-1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -W_{j+1} & 0 & V_{j+1} & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & W_{j+1} & -U_{j+1} & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & U_{j+1} & -V_{j+1} & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

For abbreviation H is written as

$$
H = \begin{bmatrix} I & O & O \\ O & F^T & I \end{bmatrix}
\tag{2.2-31}
$$

The estimable parameters [X] are those of equation (2.2-23). Further,

$$\bar{A} = DH$$

$$= (A_1\ A_3) \begin{bmatrix} I & O & O \\ O & F^T & I \end{bmatrix}$$

$$= (A_1 \vdots A_3 F^T \vdots A_3)$$

yield again a relation between the coefficients

$$A_2 = A_3 F^T.$$

2.3 Aspects of Implementation

2.3.1 Ephemeris Modelling

The adjustment formulation includes the constraints (2.2-9). The first three constraints imply that for the length of an interval the corrections to the reflector position are modelled by three constant parameters. Such a constraint makes simultaneous observations superfluous, at first sight. It requires that a good lunar ephemeris be used to compute the reflector positions at the instant of reflection of the laser pulse. This simplified modelling seems justified in view of the smooth motion of the moon due to its large moment of inertia. The shortest libration term is 13.6 days. Even the presence of short periodic terms does not invalidate the above constraints as long as the coefficients of these terms are correct. Since the lunar orbit is quite accurately derived from rigid body theory, the author is not aware of any frequencies, say, analogous to the critical frequencies in the nutations of the earth as a result of core motion, that are significantly wrong in the available conventional ephemerides.

The constraints appear even more reasonable if one accounts for the fact that a given range accuracy is capable of resolving the reflector position to only a certain level. Figure 2.2 displays the geometry of the reflector and stations i and j. The earth rotation during the travel time of the pulse is neglected for simplicity. Figure 2.2 is valid for either a north-south or an east-west line.

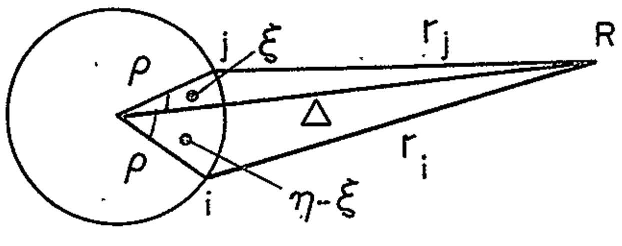


Figure 2.2 Resolution of Reflector Position

The expressions for the topocentric distances

$$r_j^2 = \rho^2 + \Delta^2 - 2\rho\Delta\cos\xi$$

$$r_i^2 = \rho^2 + \Delta^2 - 2\rho\,\Delta\cos(\eta - \xi)$$

can be expanded in terms of $\frac{\rho}{\Delta}$ in order to give the range difference as follows:

$$r_j - r_i \cong \rho\,[\cos\xi - \cos(\eta - \xi)]$$

Differentiating this equation with respect to the lunar position ξ results in the error estimate

$$|d(r_j - r_i)| \leq 2\,\rho\,d\,\xi.$$

In terms of the linear distance at the moon, this estimate becomes

$$|\,d(r_j - r_i)| \leq \frac{1}{30}\,d\xi_{Lin}$$

If a range difference accuracy of $3\sqrt{2}$ cm is assumed, then the linear reflector position can be determined at best with an accuracy of ±1.3 m. In actual computation, several parameters are solved simultaneously. The existing correlations between the parameters tend to increase the uncertainty even more as compared to the simple calculation above. From this point of view, the modelling of $d\delta$ and $d\alpha$ as a constant per interval seems adequate. The modelling of the geocentric reflector range error $d\Delta$ is more critical if strictly simultaneous observations are not possible. For near simultaneous observations, the condition $d\Delta_j = d\Delta_i = d\Delta$ requires that the change in geocentric reflector distance between the two reflection epochs is computable at least with measurement accuracy since the computed range difference observable will directly depend on such an error. Fortunately this error can be decreased, theoretically, even completely eliminated, by scheduling the observations "as simultaneous as practically possible." If the analysis of actual observations indicates that the modelling of the reflector position in the manner described here is not sufficient, one can attempt to create artificial simultaneous observations by interpolating single station ranges at certain epochs. In fact, the simplified interpolation method which leads to the construction of the "normal points" [Abbot et al. 1973] might still be sufficiently accurate. In any case, the investigation on the proper interpolation method should be carried out with real data and not with simulated observations.

2.3.2 Timing

The condition $d\Theta^{\eta} = d\Theta$ in (2.2-9) expresses a perfect synchronization between the two co-observing station clocks. Through frequent comparison with transportable clocks, this condition can be fulfilled quite accurately. If both stations are capable of utilizing LORAN C transmissions, it is possible to maintain a long term clock synchronization of $1\,\mu s$ (ground waves). This corresponds to an equatorial rotational motion of the earth of less than a millimeter. Time synchronization errors are, therefore, virtually negligible and the parameter $\mathrm{l}\bar{\alpha}$, indeed, contains only irregularities of the earth rotation and errors of the lunar right ascension.

2.3.3 Coordinate System Definition

The estimable quantities of (2.2-23) are the corrections to coordinates in the (U'')-system, whose third axis coincides with the celestial pole (C), and whose X''-axis differs from the Greenwich mean astronomical meridian by $d\bar{\alpha}$. Range observations give coordinates, whereas for range differences, the parameter set is preferably transformed to coordinate differences and sums. The origin of the (U'')-system is at the instantaneous center of mass. This is so because the earth rotates around its center of mass. Both requirements, i.e., origin at the instantaneous center of mass and alignment of the third axis with the celestial pole, are operationally achieved by expressing the station positions as follows:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_3\,(-\Theta)\,\rho \begin{bmatrix} \cos\Phi' \cos\Lambda' \\ \cos\Phi' \sin\Lambda' \\ \sin\Phi' \end{bmatrix}$$

where ρ is the geocentric distance and Φ', Λ' are the latitude and longitude of the station in the (U')-system. This formulation was used when setting up the mathematical model.

The problem of measuring the orientation in space can be regarded as solved as soon as the estimable parameters of the participating observatories become available, say, in the form of a table to be issued every day. The reference direction is the "mean" instantaneous north celestial pole for that particular interval; the word "mean" refers to the Chandler motion of approximately 10 cm or less per day. An equivalent way of representing the orientation of the earth is to issue daily a list of polar motion coordinates of the terrestrial position of the celestial pole relative to a conventional pole. This list should also contain the rotation parameter $d\bar{\alpha}$. It is emphasized that both methods of representation are strictly equivalent. Both are merely related by an orthogonal transformation (3 rotations) which leaves the angles and distances between stations invariant. The adjusted differences and sums of the coordinates in the (U'')-system and their variance-covariance matrix serve as input for a "second" adjustment (transformation) which determines the orientation parameters

$$[Y_2] = [\ x, \quad y, \ d\bar{\alpha}\ ] \tag{2.3-1}$$

The "observations" are according to (2.2-23)

$$L_{b_2} \equiv \hat{X}_3 = [dU''_{j-1},\ dV''_{j-1},\ dW''_{j-1},\ dU''_{j+1},\ dV''_{j+1},\ dW''_{j+1}\ ]$$

The variance-covariance matrix of observations is given by equation (2.2-29)

$$\Sigma_{L_{b_2}} = \Sigma_{\hat{X}_3}$$

We note that this covariance matrix is a submatrix of $\Sigma_{\hat{X}_1 \hat{X}_3}$; it is generally a full matrix. The mathematical model for the second adjustment is readily given by equations (2.2-25) and (2.2-26) which, with the help of submatrix F in (2.2-31) and equation (2.3-1) is written as:

$$Y_3 \doteq -F^T\, Y_2 + \hat{X}_3 \tag{2.3-2}$$

This is the linearized form of the adjustment model with observation equations. The residuals are $[Y_3]$, i.e., the corrections to the coordinate differences and sums in the (U)-system. The least squares estimate of the parameters $[Y_2]$ are obtained by minimizing

$$Y_3^T\ \Sigma^{-1}_{\hat{X}_3}\ Y_3$$

Note that the minimization is based on a full weight matrix. The least squares estimate is given by the standard formula. Substituting expression (2.2-28) for $\hat{X}_3$, one obtains

$$\hat{Y}_2 = G_2 W_{ij}$$

$$= [F\, Q^{-1}_{\hat{X}_3} F^T]^{-1} F[A_3^T M^{-1} A_1 (A_1^T M^{-1} A_1)^{-1} A_1^T M^{-1} - A_3^T M^{-1}] W_{ij} \qquad (2.3\text{-}3)$$

The variance-covariance matrix of the adjusted parameters $[\hat{Y}_2]$ is

$$\Sigma_{\hat{Y}_2} = \hat{\sigma}^2_{02} \; (F\, \Sigma^{-1}_{\hat{X}_3} F^T)^{-1}$$

the adjusted variance of unit weight being

$$\hat{\sigma}^2_{02} = Y_3^T \; \frac{\Sigma^{-1}_{\hat{X}_3}}{DF} \; Y_3 \quad .$$

DF denotes the degree of freedom. It is a function of the number of participating stations. The residuals, $[Y_3]$, which are the corrections to the coordinates in the (U)-system, are

$$\hat{Y}_3 = G_3 W_{ij}$$

$$= -F^T \hat{Y}_2 + \hat{X}_3 \qquad (2.3\text{-}4)$$

$$= \{-F^T [F\, Q^{-1}_{X_3} F^T]^{-1} F + Q_{X_3}\} \{A_3^T M^{-1} A_1 (A_1^T M^{-1} A_1)^{-1} A_1^T M^{-1} - A_3^T M^{-1}\} W_{ij}$$

The variance-covariance matrix of the adjusted residuals is given by the standard formula in least squares

$$\Sigma_{\hat{Y}_3} = \Sigma_{\hat{X}_3} - F^T \Sigma_{\hat{Y}_2} F$$

The implied condition of the second least squares solution is

$$F\, \Sigma^{-1}_{\hat{X}_3} Y_3 = 0$$

Combining (2.2-27), (2.2-28), (2.3-3), and (2.3-4), the least squares estimates of all parameters are

$$\begin{bmatrix} \hat{X}_1 \\ \hat{Y}_2 \\ \hat{Y}_3 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} W_{ij} = \begin{bmatrix} I & O \\ O & [F\Sigma_{\hat{X}_3}^{-1} F^T]^{-1} F\ \Sigma_{\hat{X}_3}^{-1} \\ O & I - F^T(F\Sigma_{\hat{X}_3}^{-1} F^T)^{-1} F\Sigma_{\hat{X}_3}^{-1} \end{bmatrix} \begin{bmatrix} \hat{X}_1 \\ \\ \hat{X}_3 \end{bmatrix} \quad (2.3\text{-}5)$$

It is a characteristic of the solution (2.3 - 5) that $[X_1]$ does not depend on any minimization which occurs on $[Y_3]$. The effects of the off-diagonal terms in $\Sigma_{\hat{X}_3}$ on the adjusted parameters, $[Y_2]$ and $[Y_3]$, depend on the magnitude of the correlations between the observations. Magness and McGuire [1962] derived the following limits:

$$\lambda_{min}\, \Sigma_{uc} \leq \Sigma_{Y_2} \leq \lambda_{max}\, \Sigma_{uc} \quad (2.3\text{-}6)$$

where Σ_{uc} is the variance-covariance matrix of the adjusted parameters $[Y_2]$ if only the diagonal elements (uncorrelated observations)

$$P_{uc} = (\Sigma_{\hat{X}_3\, ii})^{-1} \quad (2.3\text{-}7)$$

are used. λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the correlation matrix, S, of the observation noise,

$$S = P_{uc}^{\frac{1}{2}}\, \Sigma_{\hat{X}_3}\, P_{uc}^{\frac{1}{2}}$$

The procedure discussed above yields polar motion coordinates, i.e., the daily mean position of the celestial pole (C) with respect to the pole of the conventional terrestrial system (U) whose position coincided at the zero epoch, T_0, with the celestial pole. The definition of the system (U) depends initially on the coordinate differences in the (U'')-system of the participating stations during the interval T_0. Subsequently only crustal motion of the defining stations can change the terrestrial position of the coordinate axis. Since the residuals of the second adjustment, $[Y_3]$, are the coordinate corrections in the (U)-system, it is possible to monitor station motions by analyzing the residuals over a longer period of time. It is very important to note that <u>common station motions</u> will be absorbed in the orientation parameters (polar motion and earth rotation). This property is implied in the set-up of the second adjustment.

The station coordinates to be used for evaluating the partial derivatives in the first adjustment of each interval are always the same adopted coordinates of the system (U). Equations (2.3-3) and (2.3-4) express an orthogonal transformation between the systems (U) and (U$''$), whereby the covariance matrix of the "observed" coordinates in the system (U) is taken as zero (adopted coordinates without error). This is a special case of the more general transformation where a non-zero covariance matrix is assigned to each set of coordinates. One could, of course, consider assigning the covariance matrix of the adjusted coordinates at interval T_0 to the adopted set which defines the system (U). In that case the coordinates in both systems (U) and (U$''$) would receive residuals (corrections). However, in the event of crustal motion the adopted covariance matrix would become increasingly inaccurate leading to distorted residuals. It is, therefore, reasonable to proceed with a zero covariance matrix for the coordinates in the (U)-system, and thus consider the adopted coordinates as uncorrelated and having no error, and perform the computations as discussed above. Strictly speaking one should interpret $\hat{Y}_3$ in this case as a correction to the observed coordinates in the system (U$''$). But such an interpretation does not diminish the value of $\hat{Y}_3$ as an indicator for crustal motions.

3. ANALYSIS OF VARIANCE

A numerical study is carried out in order to investigate the expected accuracy obtainable from range-differencing. The basic time span is one interval (one day). The study is a classical covariance analysis using hypothetical observations to increment the normal matrix.

The normal matrix is formed for a simplified model (analysis of variance model). Arnold [1974] also discusses simplification leading to a somewhat different analysis of variance model. Numerical investigations were previously reported by Fajemirokun [1971], Kaula [1973], and Stolz and Larden [1977]. All three studies are concerned with single ranges. Their assumptions and parametrization vary widely. The first two investigations carry out the adjustment over a longer period of time. Kaula models polar motion by four frequencies having periods between half a month and one month. Stolz and Larden assume a perfectly known lunar ephemeris. They find that the orientation parameters are usually obtainable to better than measurement accuracy if the averaging interval is two days.

3.1 Model Simplification

As a first approximation, the rotation of the earth which occurs during the travel time of the pulse is neglected. The basic model (2.2-2) simplified as

$$r = \| \vec{X}_R - \vec{X} \| \qquad (3.1\text{-}1)$$

The simple adjustment model for observation equation

$$L_a = F(X_a)$$

can be used. The linearized form is

$$V = AX + L$$

V denotes again the residuals. L is the difference between the computed and "observed" time delay. In the present case, only the inverse of the normal matrix

$$N^{-1} = (A^T P A)^{-1} \qquad (3.1\text{-}2)$$

is analyzed due to lack of real observations. P is the weight matrix of the observations. The coefficients of the design matrix A are obtained from those of the rigorous model, (2.2-7), by setting

$$r_{12} = r_{32} = r$$

deleting $\omega\tau$ in the arguments, and by dividing the resultant coefficient by 2. With these approximations, the coefficients of the estimable parameters of (2.2-12) are (spherical station coordinates):

$$\left.\begin{aligned}
\tilde{A}_{\Delta} &= \frac{1}{r}\{\Delta - \rho\cos\Phi\cos\delta\cos(\Lambda + \Theta - \alpha) - \rho\sin\Phi\sin\delta\} \\
\tilde{A}_{\delta} &= \frac{\rho\Delta}{r}\{\cos\Phi\sin\delta\cos(\Lambda + \Theta - \alpha) - \sin\Phi\cos\delta\} \\
\tilde{A}_{\rho} &= \frac{1}{r}\{\rho - \Delta\cos\Phi\cos\delta\cos(\Lambda + \Theta - \alpha)\} \\
\tilde{A}_{\Phi} &= \frac{\rho\Delta}{r}\{\cos\delta\sin\Phi\cos(\Lambda + \Theta - \alpha) - \sin\delta\cos\Phi\} \\
\tilde{A}_{\Lambda+\Theta-\alpha} &= \frac{\rho\Delta}{r}\cos\delta\cos\Phi\sin(\Lambda + \Theta - \alpha)
\end{aligned}\right\} \qquad (3.1\text{-}3)$$

The polar motion coefficients are according to equation (2.2-13):

$$\tilde{A}_X = \frac{\Delta\rho}{r}\{\sin\Phi\cos\delta\cos(\Theta-\alpha) - \cos\Phi\cos\Lambda\sin\delta\}$$

$$\tilde{A}_Y = \frac{\Delta\rho}{r}\{\sin\Phi\cos\delta\sin(\Theta-\alpha) + \cos\Phi\sin\Lambda\sin\delta\} \qquad (3.1\text{-}4)$$

For the case of Cartesian station coordinates the coefficients of the estimable parameter (2.2-18) are

$$\left.\begin{aligned}
\tilde{A}_\Delta &= \frac{1}{r}\{-U\cos\delta\cos(\Theta-\alpha) + V\cos\delta\sin(\Theta-\alpha) - W\sin\delta + \Delta\} \\
\tilde{A}_\delta &= \frac{\Delta}{r}\{U\sin\delta\cos(\Theta-\alpha) - V\sin\delta\sin(\Theta-\alpha) - W\cos\delta\} \\
\tilde{A}_U &= \frac{1}{r}\{U - \Delta\cos\delta\cos(\Theta-\alpha)\} \\
\tilde{A}_V &= \frac{1}{r}\{V + \Delta\cos\delta\sin(\Theta-\alpha)\} \\
\tilde{A}_W &= \frac{1}{r}\{W - \Delta\sin\delta\}
\end{aligned}\right\} \qquad (3.1\text{-}5)$$

In the case of range-differencing, the coefficients of the parameters (2.2-23) are, according to (2.2-24) and (3.1-5), as follows:

$$\left.\begin{aligned}
\tilde{A}_\Delta &= \frac{2}{r}\{-U_{j-1}\cos\delta\cos(\Theta-\alpha) + V_{j-1}\cos\delta\sin(\Theta-\alpha) - W_{j-1}\sin\delta\} \\
&= \text{order } 1/60 \\
\tilde{A}_\delta &= \frac{2\Delta}{r}\{U_{j-1}\sin\delta\cos(\Theta-\alpha) - V_{j-1}\sin\delta\sin(\Theta-\alpha) - W_{j-1}\cos\delta\}
\end{aligned}\right\} \qquad (3.1\text{-}6)$$

$$\left.\begin{aligned}
\tilde{A}_{U_{j-1}} &= \frac{2}{r}\{U_{j+1} + \Delta\cos\delta\cos(\Theta-\alpha)\} \\
\tilde{A}_{V_{j-1}} &= \frac{2}{r}\{V_{j+1} + \Delta\cos\delta\sin(\Theta-\alpha)\} \\
\tilde{A}_{W_{j-1}} &= \frac{2}{r}\{W_{j+1} - \Delta\sin\delta\}
\end{aligned}\right\} \qquad (3.1\text{-}7)$$

$$\left.\begin{aligned}\widetilde{A}_{U_{j+1}} &= \frac{2}{r}U_{j-1} = \text{order } 1/60\\ \widetilde{A}_{V_{j+1}} &= \frac{2}{r}V_{j-1} = \text{order } 1/60\\ \widetilde{A}_{W_{j+1}} &= \frac{2}{r}W_{j-1} = \text{order } 1/60\end{aligned}\right\} \quad . \tag{3.1-8}$$

The expressions above are valid for simultaneous observations. The coefficients of the station coordinates are of either of two characteristic magnitudes, i.e., order 1 or order 1/60, because of the earth-moon geometry for which $r/\Delta \sim 1$ and $\rho/\Delta \sim 1/60$. The coefficients to the coordinate sums belong to the latter group as well as $\widetilde{A}_{\Delta}$. The respective parameters cannot be determined accurately from range difference observables. The coordinate difference along the third axis is also determined very weakly because its coefficient $\widetilde{A}_{W_{j-1}}$ is of order 1/60 for $\delta = 0$. Moreover, $\widetilde{A}_{W_{j-1}}$ is independent of the hour angle of the moon so that its change in magnitude during one interval is very small. Depending on the location of the two stations, there will be a more or less strong correlation between $d\delta$ and dW''_{j-1}. Therefore, range difference observables are capable of determining only <u>two</u> parameters accurately during the time of one interval; they are according to equation (2.2-25)

$$\begin{aligned} dU''_{j-1} &= -W_{j-1}\,x + V_{j-1}\,d\bar{\alpha} + dU_{j-1}\\ dV''_{j-1} &= W_{j-1}\,y - U_{j-1}\,d\bar{\alpha} + dV_{j-1}\end{aligned} \quad . \tag{3.1-9}$$

There is no disadvantage in having some coefficients only weakly determined. Their significance is found by statistical testing, and, if insignificant, they are deleted altogether or their approximate values can be weighted. In the latter case, there will be no numerical problems when inverting the normal matrix. Good approximate station coordinates are easily available. The accuracy of Doppler positioning, say ± 1 m, will be shown to be sufficient.

The coefficients (3. 1-8) change their order of magnitude if the observations are not simultaneous. For example,

$$\tilde{A}_{\substack{j\\j+i}} = \frac{1}{r_j}\{U_j - \Delta_j \cos\delta_j \cos(\Theta_j - \alpha_j)\}$$

$$-\frac{1}{r_i}\{U_i - \Delta_i \cos\delta_i \cos(\Theta_i - \alpha_i)\}$$

Using the approximations

$$r_j \cong r_i = r$$

$$\delta_j \cong \delta_i = \delta$$

$$\alpha_j \cong \alpha_i = \alpha$$

this coefficient becomes

$$\tilde{A}_{\substack{ij\\j+i}} = \frac{1}{r}\{U_{j-i} - \Delta\cos\delta\,[\cos(\Theta_j - \alpha) - \cos(\Theta_i - \alpha)]\}$$

$$\cong -2\cos\delta \sin\left(\frac{\Theta_i + \Theta_i}{2} - \alpha\right)\sin\frac{\Theta_j - \Theta_i}{2} + O\left(\frac{1}{60}\right).$$

The first term exceeds the order of 1/60 if the difference in the epochs of observations is

$$\Theta_j - \Theta_i > 4 \text{ min}.$$

The same limit is found for the coefficients of V_{j+i} and W_{j+i} in (3.1-8). Therefore, in order to keep the coefficients (3.1-8) at order 1/60 or less, the range observations, which form the range difference observable, should occur within 4 minutes.

Another analysis of variance model is arrived at if we neglect terms <u>of the order 1/60</u> and <u>assume a constant declination during one interval</u>. Such a model allows us to study the effect of changes in declination on the parameter separation. In case of spherical station coordinates, the coefficients (3.1-3) are

$$\left.\begin{aligned}
\widetilde{A}_{\Delta} &= 1 \\
\widetilde{A}_{\delta} &= \rho \cos\Phi \sin\delta \cos(\Lambda + \Theta - \alpha) - \rho \sin\Phi \cos\delta \\
\widetilde{A}_{\rho} &= -\cos\Phi \cos\delta \cos(\Lambda + \Theta - \alpha) \\
\widetilde{A}_{\Phi} &= \rho \sin\Phi \cos\delta \cos(\Lambda + \Theta - \alpha) - \rho \sin\delta \cos\Phi \\
\widetilde{A}_{\Lambda + \Theta - \alpha} &= \rho \cos\delta \cos\Phi \sin(\Lambda + \Theta - \alpha)
\end{aligned}\right\} \quad (3.1\text{-}10)$$

The coefficients for polar motion are, according to equations (3.1-4),

$$\left.\begin{aligned}
\widetilde{A}_{x} &= \rho \sin\Phi \cos\delta \cos(\Theta - \alpha) - \cos\Phi \cos\Lambda \sin\delta \\
\widetilde{A}_{y} &= \rho \sin\Phi \cos\delta \sin(\Theta - \alpha) + \cos\Phi \sin\Lambda \sin\delta
\end{aligned}\right. \quad (3.1\text{-}11)$$

In order to make interpretation easier, the polar motion coordinates (x, y) are transformed into along-meridian and across-meridian components (x', y') by

$$x' = x \cos\Lambda - y \sin\Lambda$$

$$y' = x \sin\Lambda + y \cos\Lambda$$

Their respective coefficients are derived with the help of (3.1-11) as

$$\left.\begin{aligned}
\widetilde{A}_{x'} &= \rho \sin\Phi \cos\delta \cos(\Lambda + \Theta - \alpha) - \rho \cos\Phi \sin\delta \\
\widetilde{A}_{y'} &= \rho \sin\Phi \cos\delta \sin(\Lambda + \Theta - \alpha)
\end{aligned}\right. \quad (3.1\text{-}12)$$

Since δ is assumed constant, equations (3.1-10) and (3.1-12) make some additional linear combinations possible. The part of the observation equation which pertains to the geocentric distance ρ and to the latitude Φ can be rewritten so as to contain the spin axis distance, $\rho \cos\Phi$, explicitly. Using the expressions of (3.1-10), one gets

$$\widetilde{A}_{\rho}\, d\rho + A_{\Phi}\, d\Phi = \widetilde{A}_{SA}\, d(\rho \cos\Phi) - \rho \sin\delta \cos\Phi\, d\Phi$$

where the coefficient $\widetilde{A}_{SA}$ of the spin axis distance is

$$\widetilde{A}_{SA} = -\cos\delta \cos(\Lambda + \Theta - \alpha) \qquad (3.1\text{-}13)$$

Combining equations (3.1-10) to (3.1-13) gives, together with equation (2.2-12), the following estimable parameters

$$[X] = [\zeta, \eta, \xi]$$

where

$$\begin{aligned}
\zeta &= \Delta - \rho \sin\delta \cos\Phi \, d\Phi - x'\rho \sin\delta \cos\Phi - \rho \sin\Phi \cos\delta \, d\delta \\
\eta &= d(\rho \cos\Phi) - \rho \cos\Phi \tan\delta \, d\delta - x'\rho \sin\Phi \qquad (3.1\text{-}14) \\
\xi &= d(\Lambda + \Theta - \alpha) + y' \tan\Phi
\end{aligned}$$

The respective coefficients are

$$(A) = (1, \widetilde{A}_{SA}, \widetilde{A}_{\Lambda + \Theta - \alpha}) \qquad (3.1\text{-}15)$$

The number of estimable parameters has been reduced to three as compared to five (equation (2.2-12)) when δ is varying.

The estimable parameters for the case of range differencing are readily obtained. For a <u>north-south line</u> with stations i and j symmetric to the equator and for which $\rho_i = \rho_j$, the coefficients of (3.1-15) are identical for both stations. The estimable parameters are, therefore, after differencing the observation equations:

$$[X] = [\zeta_j - \zeta_i, \eta_j - \eta_i, \xi_j - \xi_i]$$

with

$$\begin{aligned}
\zeta_j - \zeta_i &= -2\rho \sin\Phi \cos\delta \, d\delta \\
\eta_j - \eta_i &= d(\rho \cos\Phi)_j - d(\rho \cos\Phi)_i - 2x'\rho \sin\Phi \qquad (3.1\text{-}16) \\
\xi_j - \xi_i &= d(\Lambda_j - \Lambda_i) + 2y' \tan\Phi
\end{aligned}$$

Φ is the latitude of the station in the northern hemisphere. Analyzing the coefficients (3.1-15), it is seen that the separation of all three parameters is only possible if the observations cover a wide range of the lunar hour angle. For observations within a small range of the lunar hour angle one can set

$$\cos(\Lambda + \Theta - \alpha) \cong 1$$
$$\sin(\Lambda + \Theta - \alpha) \cong \Lambda + \Theta - \alpha$$

and the coefficient $\widetilde{A}_{SA}$ becomes a constant. Thus, the parameters $d\delta$ and x' cannot be separated. Because of the geometric observability conditions, the useable range of the lunar hour angle will be limited. It is, therefore, expected that the along-meridian polar motion component x' and the difference in spin axis distance (the latter parameter is equivalent to dU_{j-i} if the U-axis is located in the meridian of the two stations) will be more affected by ephemeris errors in lunar declination than the across-meridian polar motion component.

The estimable parameters of the <u>east-west line</u> are generally influenced by ephemeris errors in declination. Deleting terms of the order 1/60 and taking δ constant reduces the coefficients in (3.1-6) and (3.1-7) to

$$\widetilde{A}_{\delta} = 2\,U_{j-i} \sin\delta \cos(\Theta - \alpha) - 2\,V_{j-i} \sin\delta \sin(\Theta - \alpha)$$
$$\widetilde{A}_{\zeta_{j-i}} = 2\cos\delta \cos(\Theta - \alpha)$$
$$\widetilde{A}_{\eta_{j-i}} = 2\cos\delta \sin(\Theta - \alpha)$$

Combining these three coefficients and taking the relations (2.2-25) into account gives the parameters

$$[X] = [\zeta_{j-i}, \; \eta_{j-i}]$$

with

$$\begin{aligned} \zeta_{j-i} &= V_{j-i}\, d\bar{\alpha} + dU_{j-i} + U_{j-i} \tan\delta\, d\delta \\ \eta_{j-i} &= -U_{j-i}\, d\bar{\alpha} + dV_{j-i} - V_{j-i} \tan\delta\, d\delta. \end{aligned} \qquad (3.1\text{-}17)$$

The coefficients are

$$(A) = (\widetilde{A}_{\zeta_{j-i}}, \; \widetilde{A}_{\eta_{j-i}})$$

Equations (3.1-17) show that the estimates of ζ_{j-i} and η_{j-i} will strongly depend on ephemeris errors in declination. In actual computation, these two parameters correspond to dU''_{j-i} and to dV''_{j-i}. The expressions in (3.1-17) can be simplified by introducing a new coordinate system ($\bar{U}$),

$$\vec{\bar{U}} = R_3\left(\frac{\Lambda_i + \Lambda_j}{2}\right)\vec{U}$$

whose first axis lays in the "instantaneous" meridian which goes through the center of the points of the east-west line. Equations (3.1-17) then become

$$\begin{aligned} \bar{\zeta}_{j-i} &= \bar{V}_{j-i}\, d\bar{\alpha} + d\bar{U}_{j-i} \\ \bar{\eta}_{j-i} &= d\bar{V}_{j-i} - \bar{V}_{j-i} \tan\delta \; d\delta \end{aligned} \qquad (3.1\text{-}18)$$

According to the first equation in (3.1-18), the estimation of the earth rotation parameter does not depend significantly on the ephemeris uncertainty in declination. The estimates of the earth rotation parameter and the across-meridian component of polar motion are, therefore, expected to be of the same accuracy. However, this equation does not imply that the rotation parameter $d\bar{\alpha}$ is obtained independently of crustal motions. The common east-west crustal motions of the stations will be absorbed in $d\bar{\alpha}$. The second equation determines the chord length of the east-west line. Its estimate depends on that of the declination.

3.2 Station Geometry

From the previous analysis of estimable parameters, we see that each line, consisting of two co-observing stations, yields only two accurately determined station parameters. In order to determine all three orientation parameters at least two lines are needed. Inspecting the coefficients of equations (3.1-9), it is clear that two stations located along the meridian determine polar motion completely, assuming that there is no crustal motion between them. If they are, in addition, located symmetrically with respect to the equator, the polar motion estimates are virtually independent of an error in the earth rotation parameter $d\bar{\alpha}$. If the two stations are, however, located on the same parallel, the range difference determines the rotation parameter $d\bar{\alpha}$ and not polar motion. Lines of north-south and east-west directions give the most favorable station geometry since the normal matrix of the second adjustment is diagonal for such cases, and thus minimizes the estimated correlations between the parameters. In view of crustal motion, a clear definition of the terrestrial coordinate system requires that a third line be added if the other stations are located along meridians and parallels. One north-south and one east-west line give no over determination for polar motion. Therefore, a second north-south line is needed to make polar motion free from the effect of individual station motions. Of course, the common station motions are still inseparable from the orientation parameters.

For an ideal station distribution, approximate formulas can be given for the variance propagation. The configuration consisting of two north-south lines of equal lengths, symmetric with respect to the equator, and separated in longitude by 90 degrees, is considered first. For simplicity, a diagonal variance-covariance matrix of observations is assumed in the second adjustment. Under those conditions, the normal matrix in equation (2.3-3) is (compare also equations (3.1-9)):

$$F\Sigma^{-1}_{\ell_B}F^{T} = \begin{bmatrix} \dfrac{W^2_{j-i}}{\sigma^2_{U''_{j-i}}} + \dfrac{W^2_{l-k}}{\sigma^2_{U''_{l-k}}} & 0 \\ 0 & \dfrac{W^2_{j-i}}{\sigma^2_{U''_{j-i}}} + \dfrac{W^2_{l-k}}{\sigma^2_{V''_{l-k}}} \end{bmatrix} \qquad (3.2\text{-}1)$$

The parameters are the polar motion coordinates x and y. The station pairs i,j and k,l are co-observing respectively. Because of the symmetry with respect to the equator, the coordinate differences of the third coordinates are equal,

$$W_{j-i} = W_{l-k}$$

Since the two lines are 90 degrees apart, there is a symmetry in the coefficient $\widetilde{A}_{U_{j-i}}$ and $\widetilde{A}_{V_{j-i}}$ in (3.1-7) for the respective parameters of the two lines, which results in

$$\frac{1}{\sigma^2_{U''_{j-i}}} + \frac{1}{\sigma^2_{U''_{l-k}}} \cong \frac{1}{\sigma^2_{V''_{j-i}}} + \frac{1}{\sigma^2_{V''_{l-k}}}$$

Substituting the two equations above in (3.2-1) and taking the square root of the inverse elements gives the standard deviation

$$\sigma_x \cong \sigma_y \cong \frac{\rho}{W_{j-i}} \; \frac{\sigma_{U''_{j-i}}\,\sigma_{U''_{l-k}}}{\sqrt{\sigma^2_{U''_{j-i}} + \sigma^2_{U''_{l-k}}}} \qquad (3.2\text{-}2)$$

If the U-axis is chosen in the meridian of the north-south station pair i, j, then the variances in equation (3.2-2) are those of the along-meridian and across-meridian component of the estimable parameters. But these two variances are characteristically different because the estimate of the along-meridian component is strongly effected by ephemeris errors in declination. One may, therefore, neglect across-meridian variance in the denominator of (3.2-2). In doing so, and evaluating the factor ρ/W_{j-i}, the expressions for the error propagation become

$$\sigma_{x'} \cong \sigma_{y'} \cong \frac{\sigma_{U''_{i-k}}}{\sin \Phi} \qquad (3.2\text{-}3)$$

where Φ is the station latitude. Consequently, two north-south lines which are separated in longitude by 90 degrees yield the most accurate estimates of polar motion. The accuracy is the same in all directions (circular error ellipse).

For a single north-south line, symmetric with respect to the equator, the variances are largest and smallest for the along and the across-meridian component, respectively. With the U-axis again located in the meridian, their accuracies are

$$\begin{aligned} \sigma_{x'} &\cong \frac{\sigma_{U''_{j-i}}}{\sin \Phi} \\ \sigma_{y'} &\cong \frac{\sigma_{V''_{j-i}}}{\sin \Phi} \end{aligned} \qquad (3.2\text{-}4)$$

Similarly, an approximate relation for the variance of the rotation parameter can be obtained from equation (3.1-18) (east-west line):

$$\sigma_{\overline{\alpha}} = \frac{\sigma_{\overline{\zeta}_{j-i}}}{\cos \Phi \, \sin\left(\dfrac{\Lambda_j - \Lambda_i}{2}\right)} \qquad (3.2\text{-}5)$$

The latter relation holds because the estimate of $\overline{\zeta}_{j-i}$ is insensitive to errors in lunar declination.

It is emphasized that in the subsequent numerical experiment the variance of dU''_{j-i} and dV''_{j-i} (equations 3.1-9) will be given instead of $\overline{\zeta}_{j-i}$ and $\overline{\eta}_{j-i}$ (equations 3.1-18) since the numerical results are not transformed to the special coordinate system ($\overline{U}$) and the declination is varied.

All these approximate expressions depend on the variance of the estimable parameters $[X_3]$ of the first adjustment, which, in turn, are a function of the number of observations and their distribution over the hour angle of the moon. These dependencies are the subject of subsequent numerical calculation.

The chords and angles between stations can be computed from the estimable coordinate difference in the (U'')-system or from the transformed coordinates in the (U)-system. Both procedures give the same results since the orthogonal transformation leaves chords and angles invariant. Generally, the third coordinate (W-parameter) will set the limit to the achievable accuracy. As special cases, the azimuth of a north-south line and the chord length of an east-west line are determinable with high accuracy.

3.3 Numerical Analysis

3.3.1 Incrementing the Normal Matrix

The analysis consists of incrementing the normal matrix based on hypothetical observations. The normal matrix of (3.1-2)

$$N_X = \begin{bmatrix} A_1^T PA_1 & A_1^T PA_3 \\ A_3^T PA_1 & A_3^T PA_3 \end{bmatrix} \qquad (3.3\text{-}1)$$

is formed from the coefficients (3.1-6) to (3.1-8). The respective parameters are

$$[X] = [X_1 \vdots X_3] = [d\Delta, d\delta \vdots dU''_{j-1}, dV''_{j-1}, dW''_{j-1}, dU''_{j+1}, dV''_{j+1}, dW''_{j+1} \ldots]$$

whereby each new line consisting of two stations adds six parameters to the set $[X_3]$. The inverse of the normal matrix is the variance-covariance matrix of the adjusted parameters

$$Q_X = N^{-1}$$

Particularly, the variance-covariance matrix of the subset $[X_3]$ is

$$Q_{X_3} = (A_3^T PA_3 - A_3^T PA_1 \ (A_1^T PA_1)^{-1} A_1^T PA_3)^{-1} \qquad (3.3\text{-}2)$$

The a priori and a posteriori variance of unit weight is set equal to unity. P is the weight matrix of the range difference observables

$$P = \frac{1}{(3\sqrt{2})^2} \ I \qquad (3.3\text{-}3)$$

corresponding to a single range accuracy of 3 cm. I is the unit matrix. Next, the normal matrix (2.3-3) of the second adjustment

$$N_{Y_2} = F \ Q_{X_3} F^T \qquad (3.3\text{-}4)$$

is set up and inverted. The orientation parameters $[Y_2]$ are

$$[Y_2] = (x, \ y, \ d\alpha)$$

Finally, the variances of the residuals (2.3-2) are computed by

$$Q_{\hat{Y}_3} = Q_{X_3} - F^T Q_{Y_2} F \qquad (3.3\text{-}5)$$

3.3.2 Design Characteristics

Each of the experiments is distinguished by the following design characteristics:

a) number of lines

b) orientation of lines

c) length of the lines

d) observation schedule

In Table 3.1, column 2, the station configuration for the various

Table 3.1

Design Characteristics

Figure	Lines/Observation Schedule*	Station Position	Modification
3.5	NS - EW - NS/A (65 - 43 - 21)	NS: $\Lambda_{6-5} - \Lambda_{2-1} = 90°$ EW: $\Lambda_3 = 80°$, $\Lambda_4 = 20°$ $\Phi_{6-5} = \Phi_{2-1} = 80°$ $\Phi_3 = \Phi_4 = 0$	
3.6	as above	as above	observation density
3.7	NS - EW - NS/B (65 - 43 - 21)	as above	
3.10	as above	as above	$50° \leq \Lambda_{6-5} - \Lambda_{2-1} \leq 90°$
3.11	NS - EW/A (43 - 21)	NS: $\Phi_3 - \Phi_4 = 80°$ EW: $\Lambda_1 = 80°$, $\Lambda_2 = 20°$ $\Phi_1 = \Phi_2 = 0$	
3.12	NS - EW/B (43 - 21)	as above	
3.15	as above	as above	$(0.1\,m)^2 \leq \sigma^2_{W''_{j-1}} \leq (1.4\,m)^2$
3.16	as above	as above	$80° \geq \Phi_3 - \Phi_4 \geq 40°$
3.17	as above	as above	azimuth of NS $0 \leq$ azimuth $\leq 42°$
3.18	NS - EW - NS/B (65 - 43 - 21)	NS: $\Lambda_3 - \Lambda_4 = 90°$ EW: $\Lambda_3 = 80°$, $\Lambda_4 = 20°$ $\Phi_{6-5} = \Phi_{2-1} = 80°$ $\Phi_3 = \Phi_4 = 0$	$20° \leq \Lambda_4 \leq 60°$
3.19	as above	Table 3.2	Φ_3 and Φ_4
3.20	NS - EW/B (42 - 21)	NS: $\Phi_3 - \Phi_4 = 80°$ EW: $\Lambda_1 = 80°$, $\Lambda_2 = 20°$ $\Phi_1 - \Phi_2 = 0$	azimuth of EW $90° \geq$ azimuth $\geq 48°$
3.21	conceivable real station distribution (Table 3.3)		

* NS: Stations are located on the meridian.
EW: Stations are located on the parallel.

experiments and observation schedules is given. The letters NS and EW denote a north-south and east-west line, respectively. The first symbol in this notation denotes the most western line. It is always a north-south line and defines the zero longitude. An exception to this convention is the case of Figure 3.21 where the longitude is zero at Greenwich. Figure 3.1 shows the system of station numbering for the three- and two-line design. Thus, the lines 56 or 34 define zero longitude, respectively.

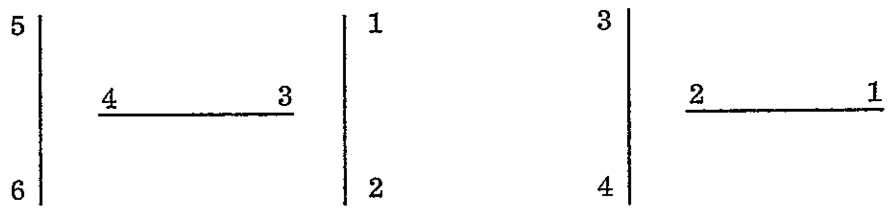


Figure 3.1 Station Numbering for Three-Line (NS-EW-NS) and Two-Line (NS-EW) Design

Two observation schedules, A and B, were used. In each schedule the observation spacing is ten minutes. This time limit was selected for practical reasons since it allows one to compute normal points of a certain epoch in case of near simultaneous observations. In schedule A, the observations are equally spaced throughout the lunar hour angle. The only additional limitation is that the altitude of the moon has to be larger than 20 degrees in order to avoid disturbing effects of the atmosphere. Schedule B used three hours of observations, whereby after the first and third hour there is an interruption of one hour. The observation time is arranged symmetrically with respect to the lunar transit, i.e., the observations start 2.5 hours before transit and finish 2.5 hours after transit. The moon, therefore, has to stay at least five hours above an altitude of 20 degrees.

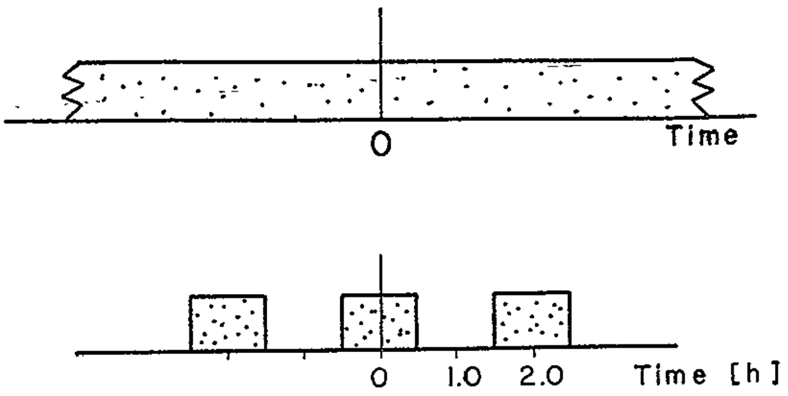


Figure 3.2 Observation Schedule
(The observation spacing is ten minutes within the dotted area.)

There are two modifications of the station geometry:

(a) Basic Design: The stations are located exactly along meridians (symmetric to equator) and parallels. In case of the three-line design, the two north-south lines are 90 degrees apart. In order to demonstrate the influence of the lunar declinational position on the variances, the computations are carried out for 14 successive intervals, each interval lasting one day. In interval 1, the declination is approximately -17.5 degrees, between the intervals 5 and 6 it passes zero, and at interval 13 the maximal declination is reached (Figure 3.3).

(b) Modified Design: One of the lines changes its position in azimuth, length, etc. In all variations the lunar declination of interval 1 (-17.5 degrees) is used.

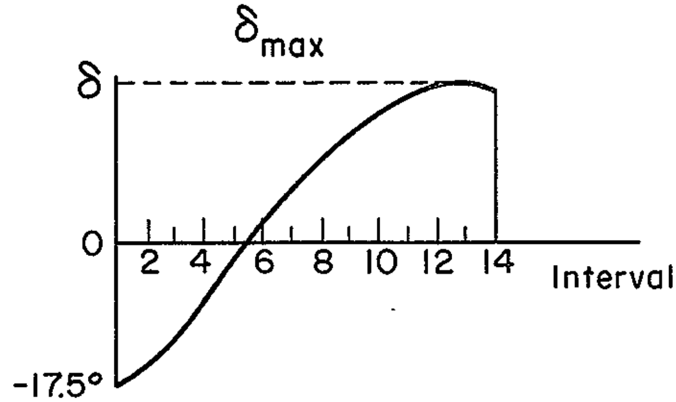


Figure 3.3 Variation in Declination

3.3.3 A Priori Weighting

Some consideration has to be given to the a priori weights of the parameters of the first adjustment. Since the coefficients of the parameters dU''_{j+1}, dV''_{j+1}, and dW''_{j+1} (see coefficients 3.1-8) are of order 1/60, these parameters are not expected to be of any significance in view of the good approximate coordinates which are available. An initial adjustment was carried out which included these parameters. The normal matrix was ill-conditioned. A numerical inversion with double precision arithmetic on the computer failed. As a next step, these parameters were weighted with

$$\sigma_{U''_{j+1}} = \sigma_{V''_{j+1}} = \sigma_{W''_{j+1}} = \pm\ 1\ \mathrm{m}$$

Also the geocentric reflector distance, Δ, and the third coordinate difference, dW''_{j-1}, were weighted constraint with $\sigma_\Delta = \pm$ 100 m and $\sigma_{W''_{j-1}} = \pm$ 1 m. This a priori weighting of the station parameters is in accordance with what can be expected from present day Doppler positioning. The variances, for the parameter dU''_{j+1}, dV''_{j+1}, and dW''_{j+1}, after inversion of the normal matrix, corresponded very closely to those in the a priori weights. This, of course, confirms that the range difference observables in LLR do not improve these parameters.

A final adjustment was made deleting these parameters entirely. The resultant variances of the remaining parameters, as well as the variance of unit weight, which was based on simulated observations with Gaussian noise superimposed, did not differ from the case which included the weighted parameters dU''_{j+1}, dV''_{j+1} and dW''_{j+1}, thus making any further statistical testing unnecessary.

The parameter dW''_{j-1} plays a special role. In all numerical studies the a priori weight is based on $\sigma_{W''_{j-1}} = 1$ m. The estimated accuracy of the declination depends very much on the knowledge of W''_{j-1}. The coefficients $\widetilde{A}_{W_{j-1}}$ and $\widetilde{A}_\delta$ for the north-south line are according to equations (3.1-6)

$$\widetilde{A}_\delta \quad \propto \quad \cos \delta$$

and

$$\widetilde{A}_{W_{j-1}} \quad \propto \quad \sin \delta$$

Since the declination does not change very much during one interval, both parameters will be strongly correlated. The numerical effect of the a priori weight of W''_{j-1} on the declination is shown in Figure 3.15. The weighting of W''_{j-1} creates another but minor problem. Since W''_{j-1} is a coordinate difference in the (U'')-system, it is a time varying quantity. The use of the same a priori weight and the <u>same</u> approximate coordinates, e.g., those of the initial epoch T_0, will introduce distortions to the adjustment (too optimistic weight). As polar motion increases, the once adapted approximate values become increasingly worse. This problem can be avoided by introducing the adjusted coordinates of the first adjustment of the previous interval as approximate coordinates in the subsequent interval.

3.3.4 Graphical Representations

All figures represent the square root of the diagonal element of the inverse of the normal matrix (standard deviations). According to expression (3.3-3), the accuracy for one range difference is $3\sqrt{2}$ cm. For other accuracies the graphs change proportionately. For the purpose of checking the programming, artificial observations were simulated with a Gaussian noise superimposed. The variance of unit weight a posteriori thus obtained fluctuated around one. But this value is of no real use since it only reflects the "randomness" of the random number generator on the computer. It is understood that with real data, the a posteriori variance of unit weight of the first adjustment becomes the a priori variance of unit weight of the second adjustment.

The accuracies of the polar motion coordinates as well as the station coordinates are given in centimeters, whereas the $\sigma_{\bar{\alpha}}$ and σ_{δ} are given in one one-thousandth of an arcsec. All stations are located on a sphere with radius

$$R = 6370 \text{ km}$$

The scale of the plots varies among the figures. For each experiment, the scale of (σ_X, σ_Y), $(\sigma_{U_{j-i}}, \sigma_{V_{j-i}}, \sigma_{U''_{j-i}}, \sigma_{V''_{j-i}})$, and $(\sigma_{W_{j-i}}, \sigma_{W''_{j-i}})$ are, respectively, the same. In each case, there is a strong resemblance between the standard deviations of the coordinate differences in the (U)-system and the (U″)-system. This is to be expected because of the small degree of freedom in the second adjustment. Figure 3.4 reviews the geometric meaning of various parameters and related coordinate systems.

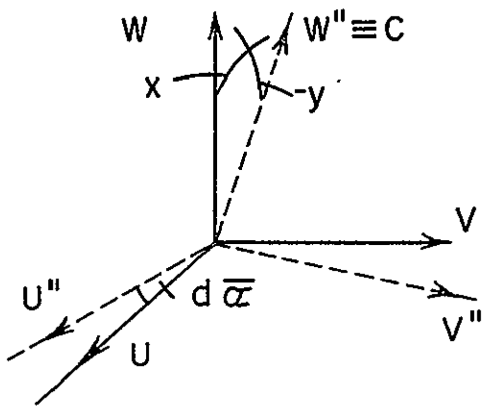
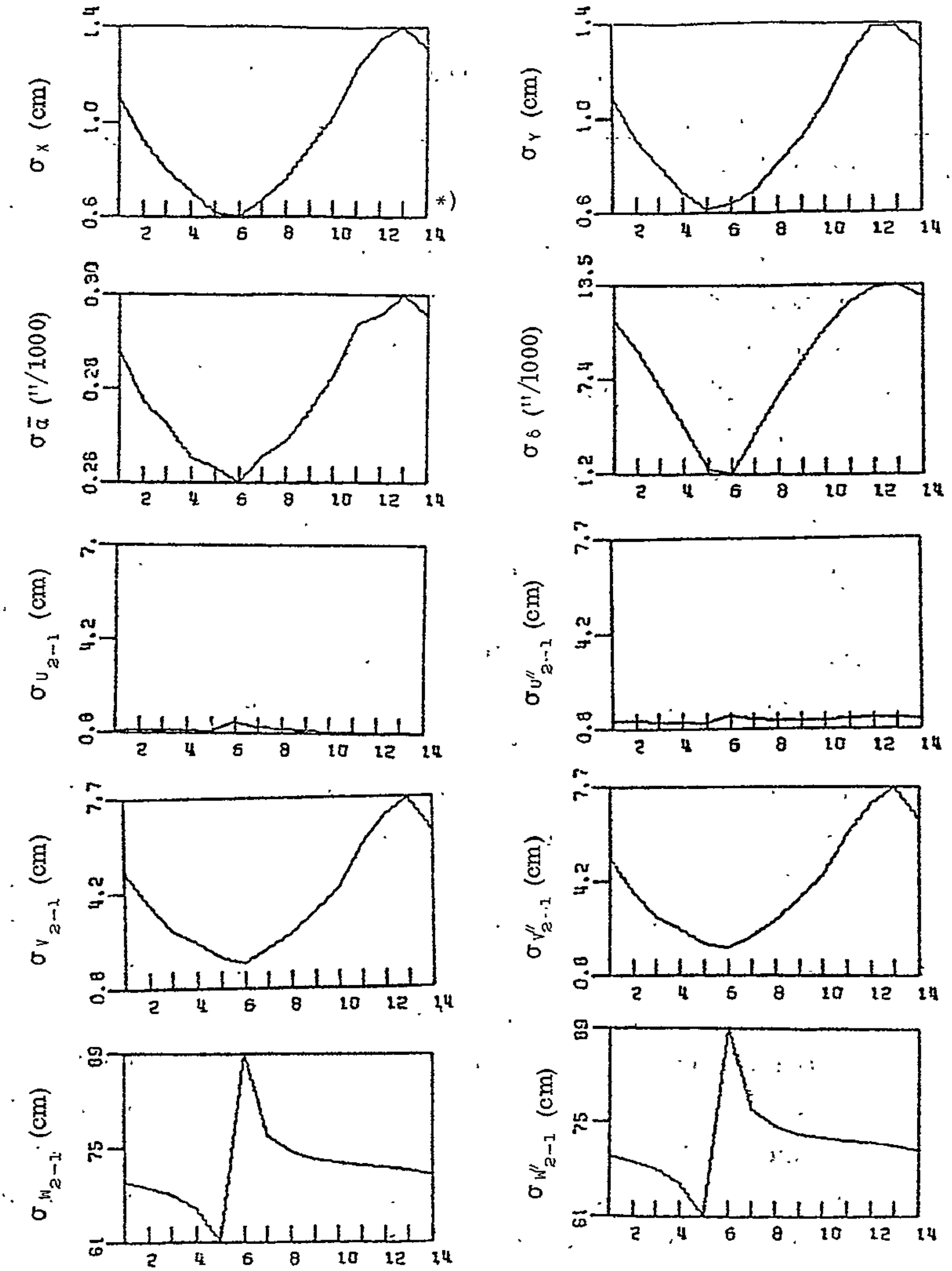


Figure 3.4 The Coordinate Systems (U) and (U″)

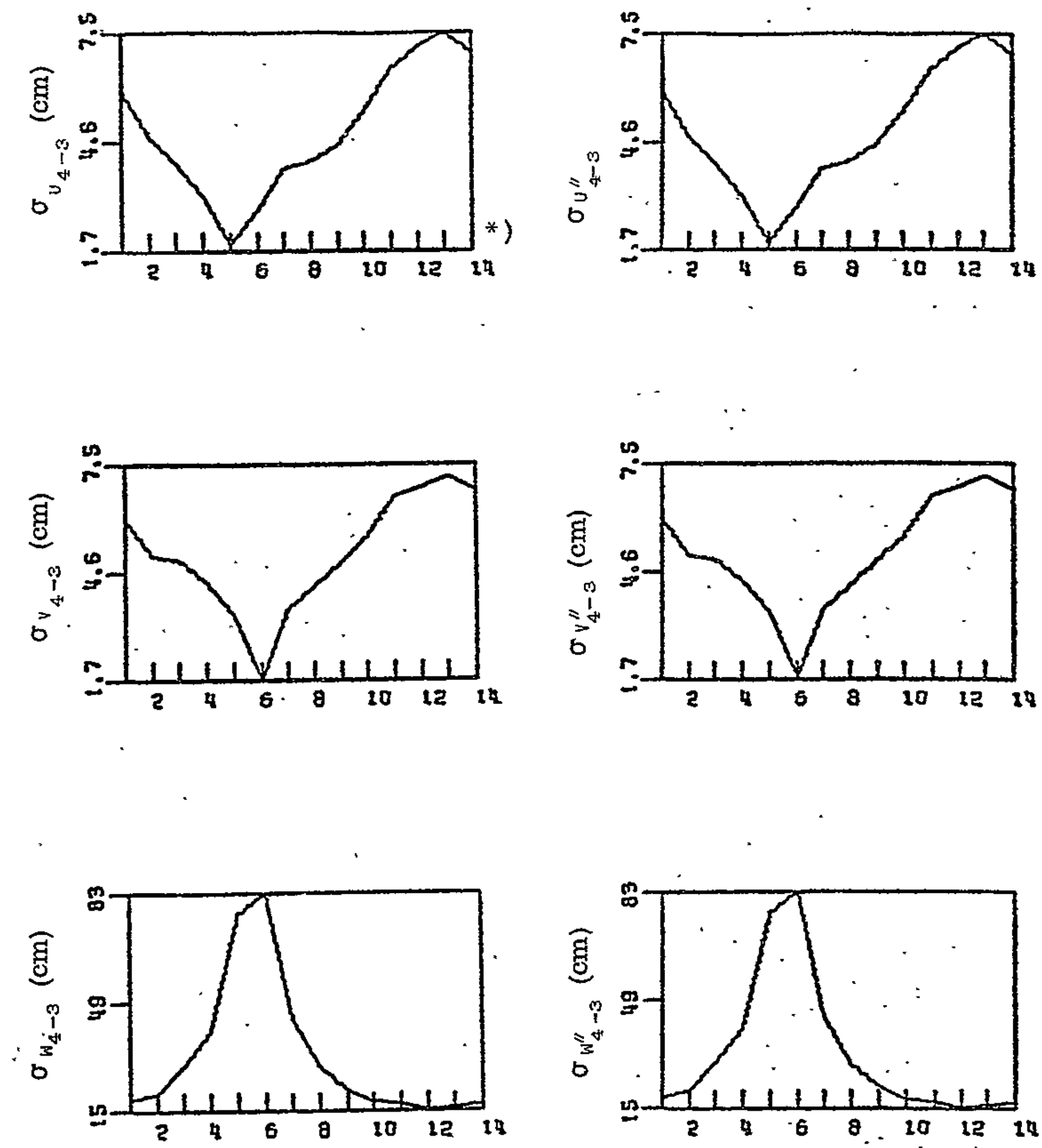
The common origin of the coordinate system (U) and (U″) is at the instantaneous center of mass. For the initial interval at T_0, the two coordinate systems coincide per definition. If there were no crustal motions, the (U)-system would be fixed to the crust. The body-fixed direction of the U-axis is determined by the Greenwich apparent sidereal time Θ and the reflector right ascension α which were used as approximate (adopted) values in the calculations during the interval T_0. Note that there is no second adjustment for the initial interval T_0. The system (U″) is not body-fixed. The third coordinate axis W″ coincides with the direction of the north celestial pole, and the position of the first axis U″is a function of the Θ and α which are used as approximate values in the calculations during a particular interval T_1. In case of crustal motions, the (U)-system is not body-fixed any more but rotates slowly by the amount of the common crustal motion component of all participating stations. It is exactly the same motion which, as mentioned earlier, is included in polar motion and earth rotation parameters. There is no way of separating this motion from laser ranging at the surface only.

Figure 3.5 shows the basic three-line configuration for observation schedule A. Generally, the best estimates are obtained for $\delta \cong 0$, which occurs between intervals 5 and 6. At that instant, the variation in δ is largest; therefore, best parameter separation is possible, and the number of observations is largest (longest visibility). The polar motion accuracy is the same in both directions. There is a dependence on the declination, but the total variation is very small. The standard deviations of the along-meridian components dV''_{2-1} and dU''_{6-5} show the expected strong dependence on the declinations, whereas the across-meridian components dU''_{2-1} and dV''_{6-5} do not show such a dependence. The same dependence is reflected in $\sigma_{V_{2-1}}$, $\sigma_{U_{6-5}}$ and $\sigma_{U_{2-1}}$, $\sigma_{V_{6-5}}$, respectively. The sudden jump in $\sigma_{W''_{2-1}}$ and $\sigma_{W''_{6-5}}$ results from a change in sign of the correlation coefficient $\sigma_{W'', \delta}$ as the moon passes zero declination. Remembering that the a priori variance of the W''_{j-i} parameters is 1 m^2, it is seen that only a minor improvement takes place. The accuracy estimates of the coordinate differences for the east-west line $\sigma_{U_{4-3}}$ and $\sigma_{V_{4-3}}$ expectedly depend very much on the accuracy of the declination. There is, of course, a strong correlation between dV_{4-3} and dU_{4-3} since, in the case of only one east-west line, they directly depend on the estimated rotation parameter $d\bar{\alpha}$. If the variance-covariance matrix of observations for the second adjustment were taken to be diagonal, the correlation between dV_{4-3} and dU_{4-3} would be identically 1. Another example of the effect of the full variance-covariance matrix of observation is reflected in the estimate $\sigma_{\bar{\alpha}}$. Based on the corresponding estimable quantities dU''_{4-3} and dV''_{4-3}, one could again set up an approximate error propagation for a diagonal variance-covariance matrix of observation. The geometry is given by equations (3.1-9). But such an approximate error estimation results in estimates of $\sigma_{\bar{\alpha}}$ which are too pessimistic since the correlation between the observations (estimable parameter of first adjustment) are neglected. A numerical example follows later.



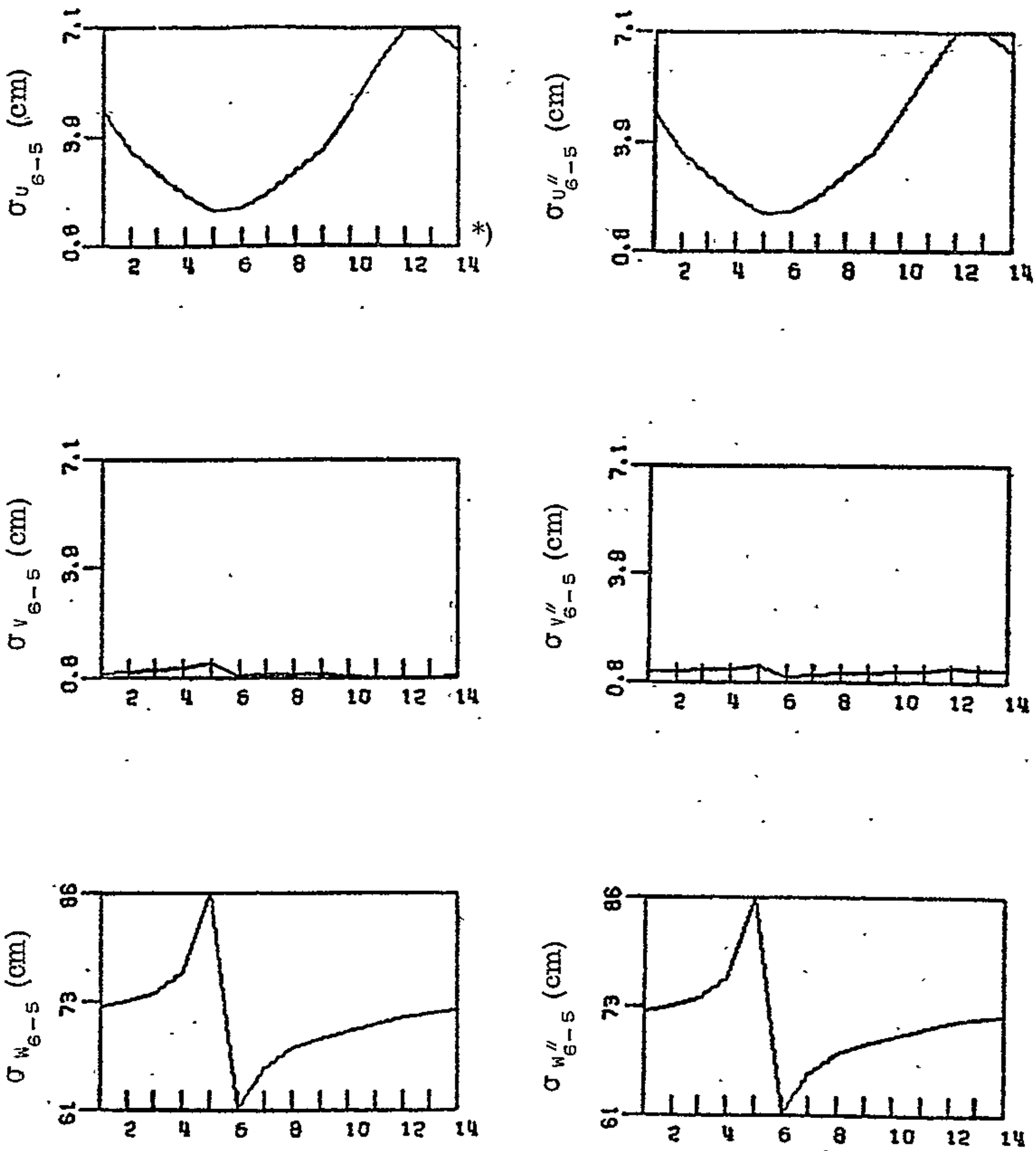
*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.5 Three-Line Design with Observation Schedule A



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.5 (cont'd) Three-Line Design with Observation Schedule A

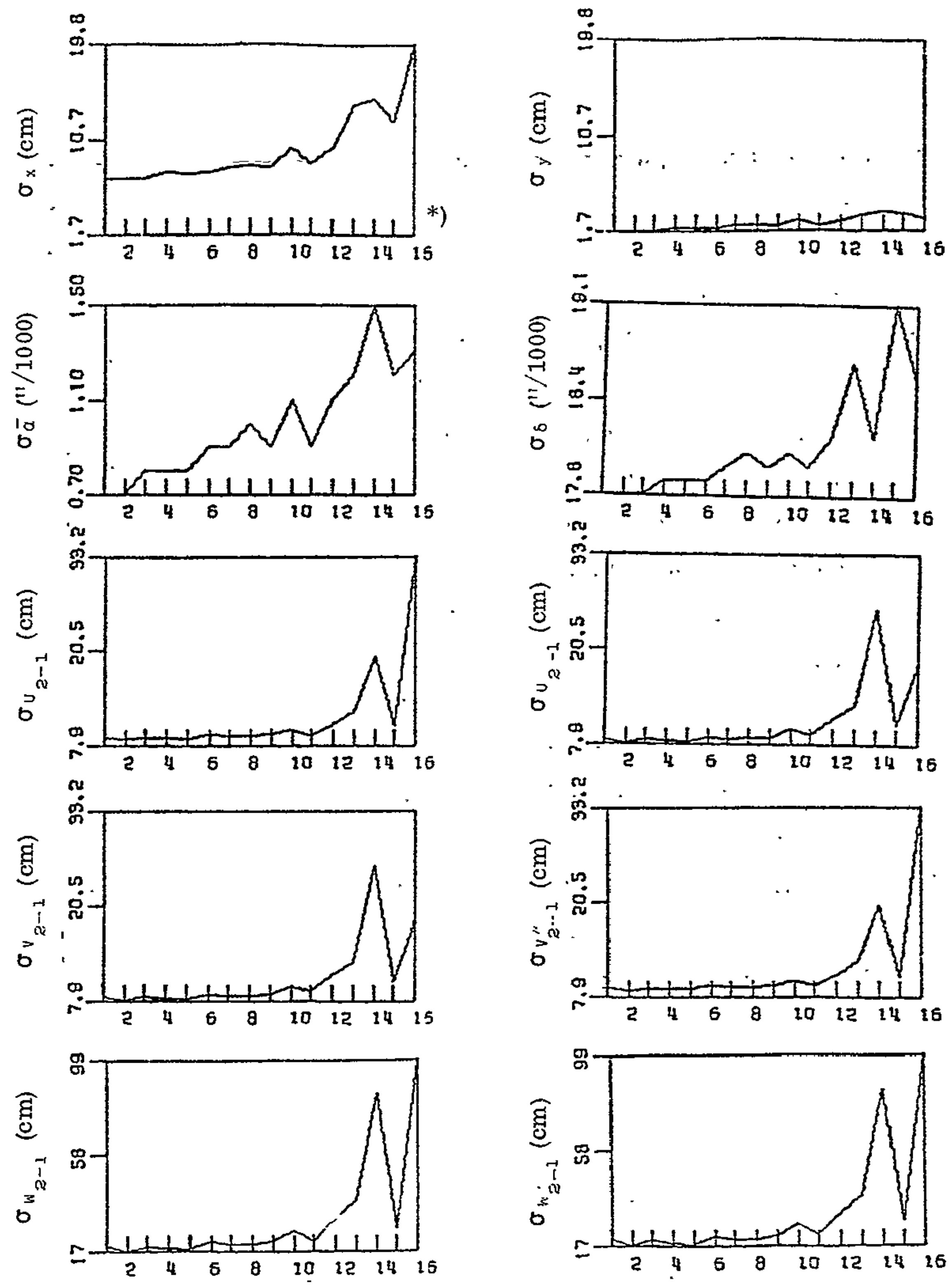


*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.5 (cont'd) Three-Line Design with Observation Schedule A

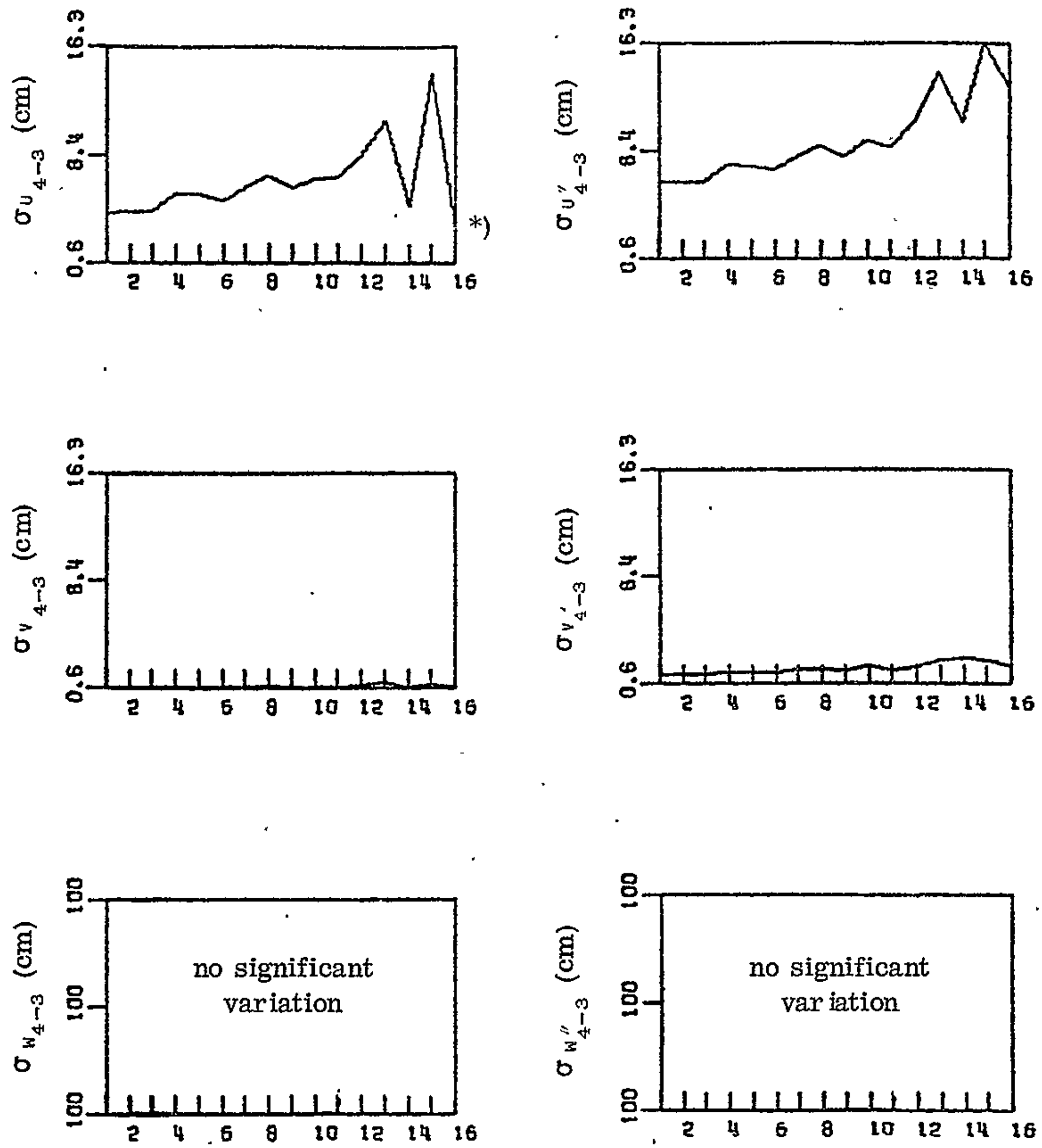
Figure 3.6 shows the estimated accuracies as a function of the number of observations. The basic observation schedule A, which included an observation every ten minutes as long as the zenith distance of the moon is smaller than 70 degrees, is modified by deleting up to 80% of the possible observations. Multiplying the number on the horizontal axis in the graphs by 5 gives the percentage of deleted observations from the basic schedule A. The deleted observations are selected randomly by use of the random number generator in order to eliminate the effect of observation geometry as much as possible. Each computation is made for the same lunar declination (-17.5 degrees). The graphs show that a significant deterioration in accuracy occurs after deleting 50%, and more, of the observations. The accuracy of the across-meridian polar motion component σ_y remains unchanged, even for a deletion of 80%. The results of this experiment justify the adoption of observation schedule B as the standard schedule.

Figure 3.7 shows the three-line configuration based on observation schedule B. There are no new features detectable. Generally, the accuracy still improves for zero declination, but not as much as in Figure 3.5 because of the deletion of observations at the extreme hour angle in schedule B.



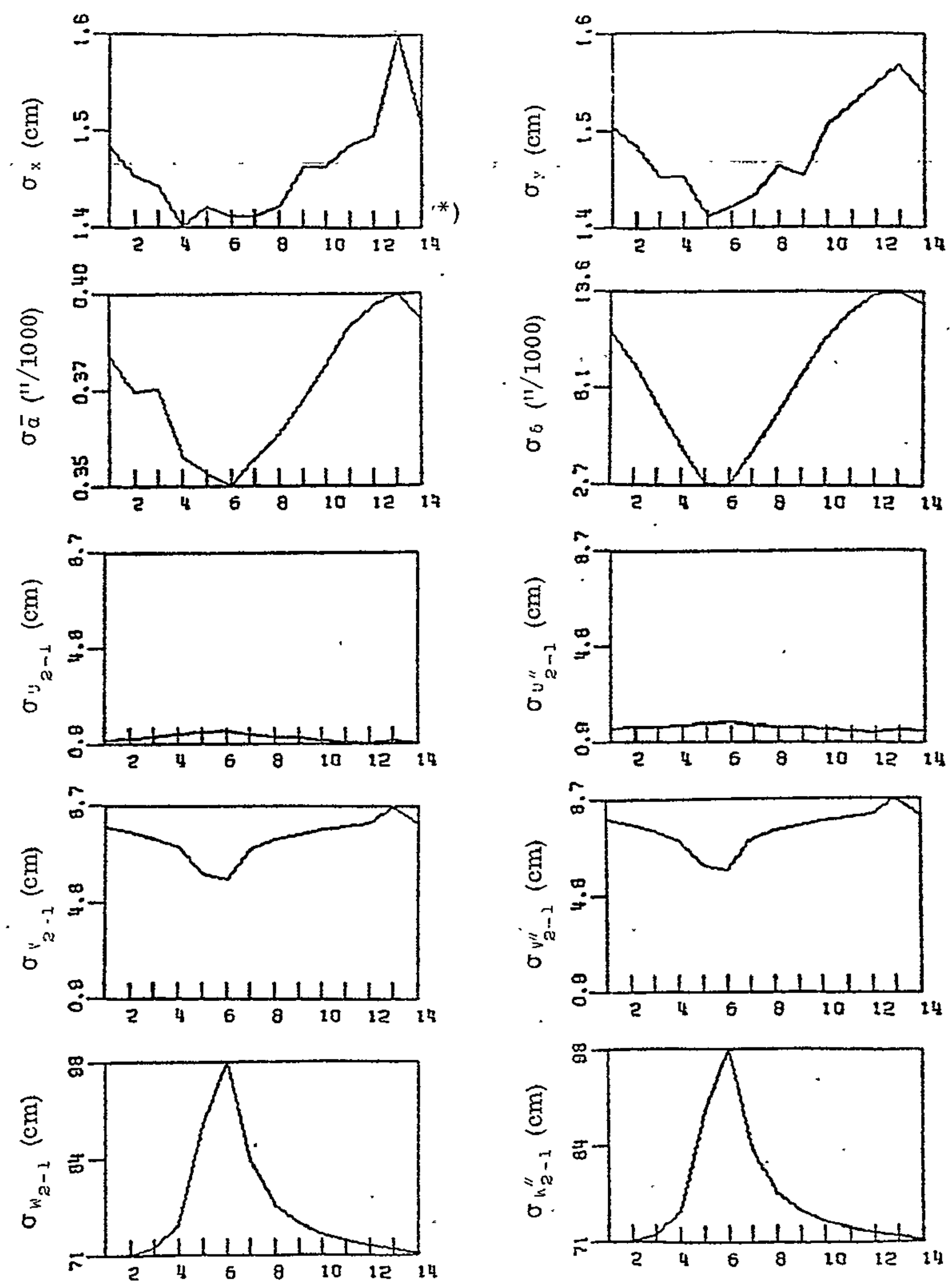
*) Multiplying the number in the abscissa by 5 gives the percentage of observations which are deleted from schedule A.

Figure 3.6 Observation Density (Two Lines)



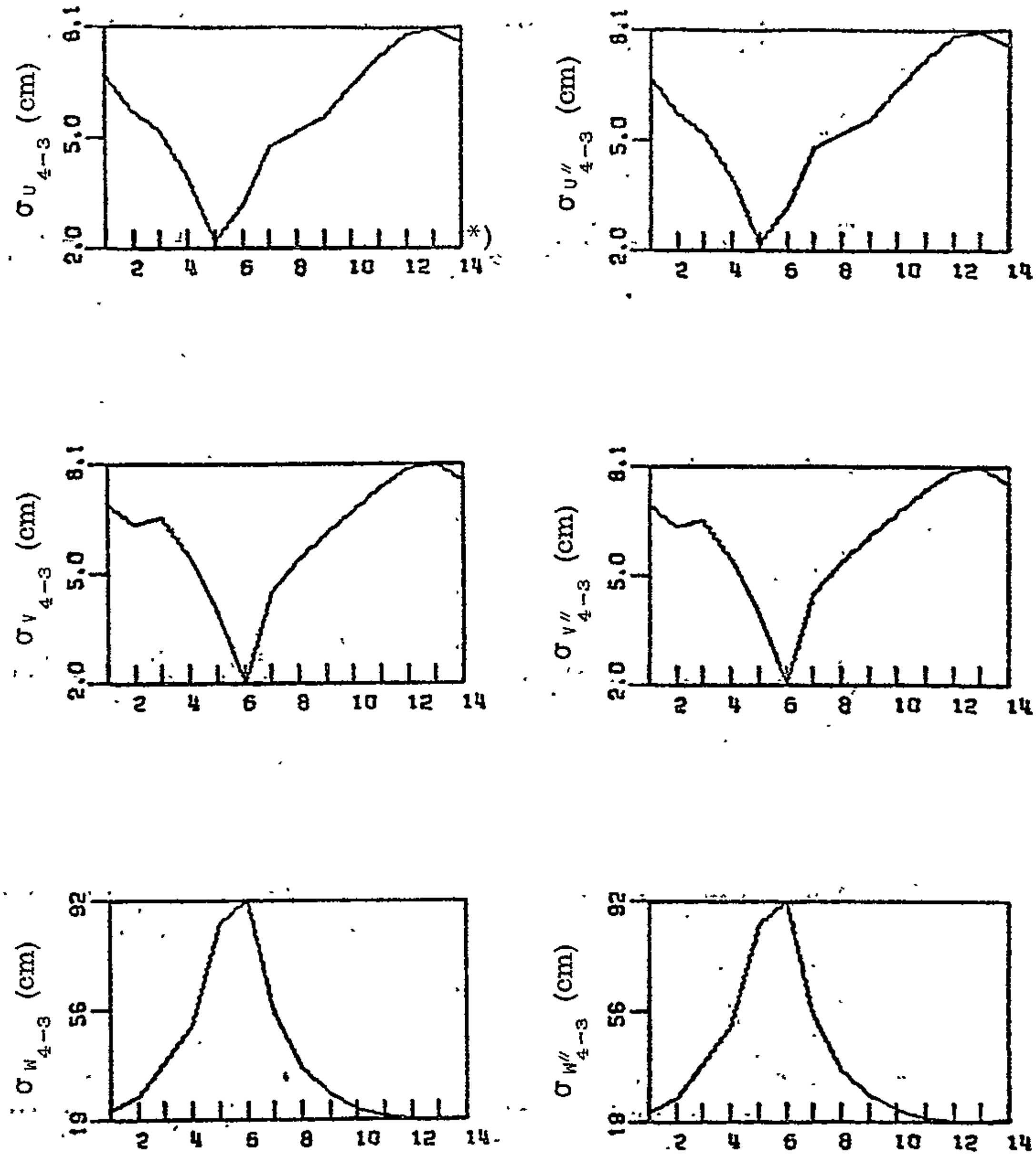
*) Multiplying the number in the abscissa by 5 gives the percentage of observations which are deleted from schedule A.

Figure 3.6 (cont'd) Observation Density (Two Lines)



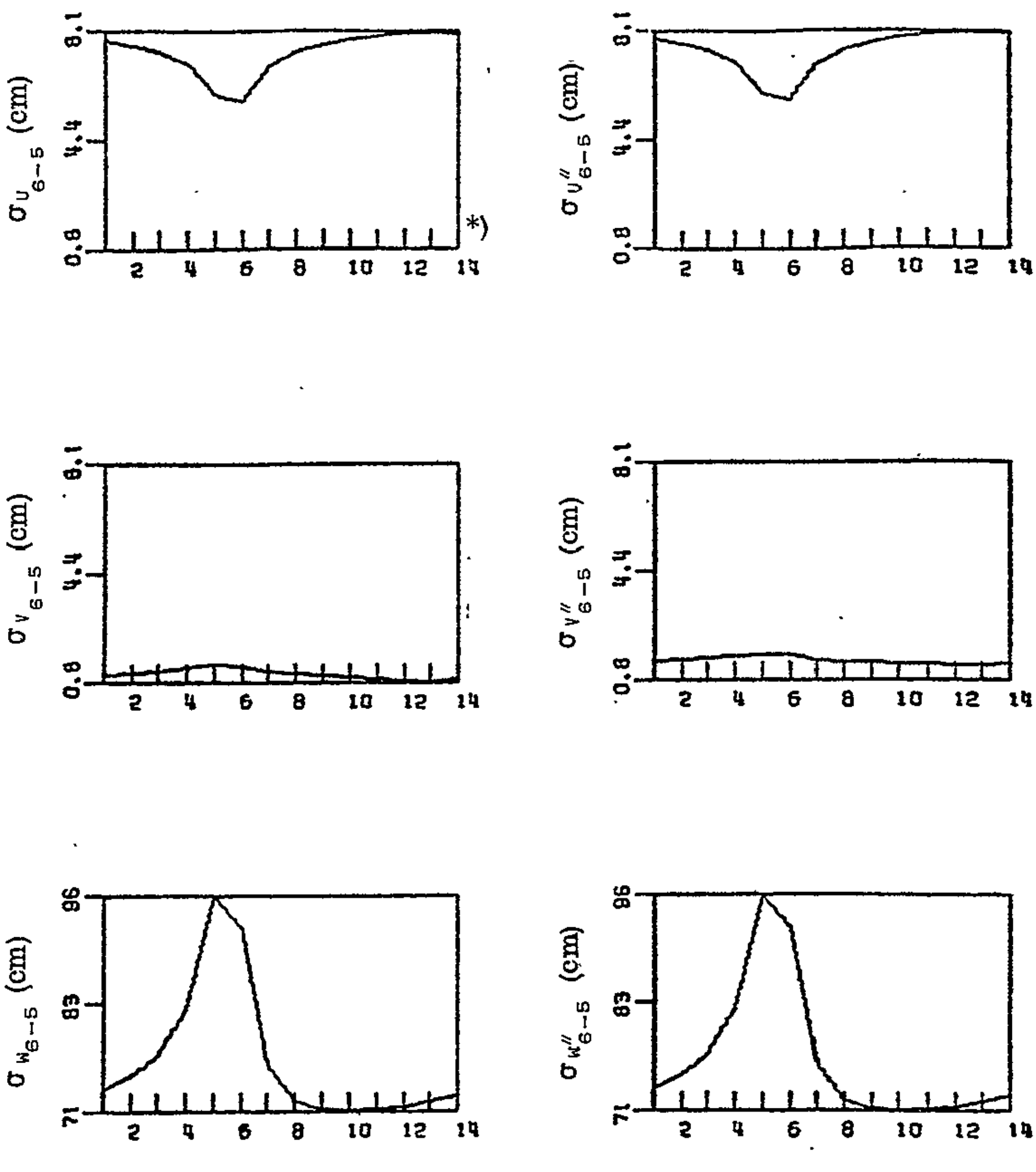
*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.7 Three-Line Design with Observation Schedule B



*). One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.7 (cont'd) Three-Line Design with Observation Schedule B



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.7 (cont'd) Three-Line Design with Observation Schedule B

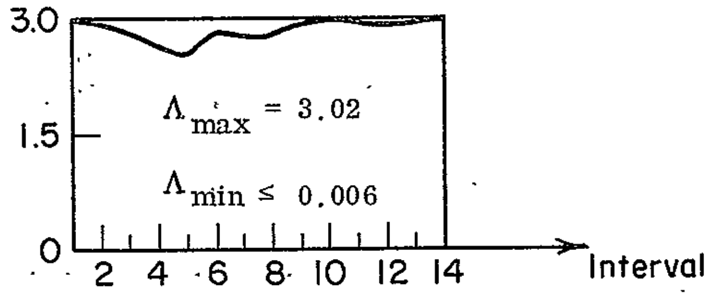
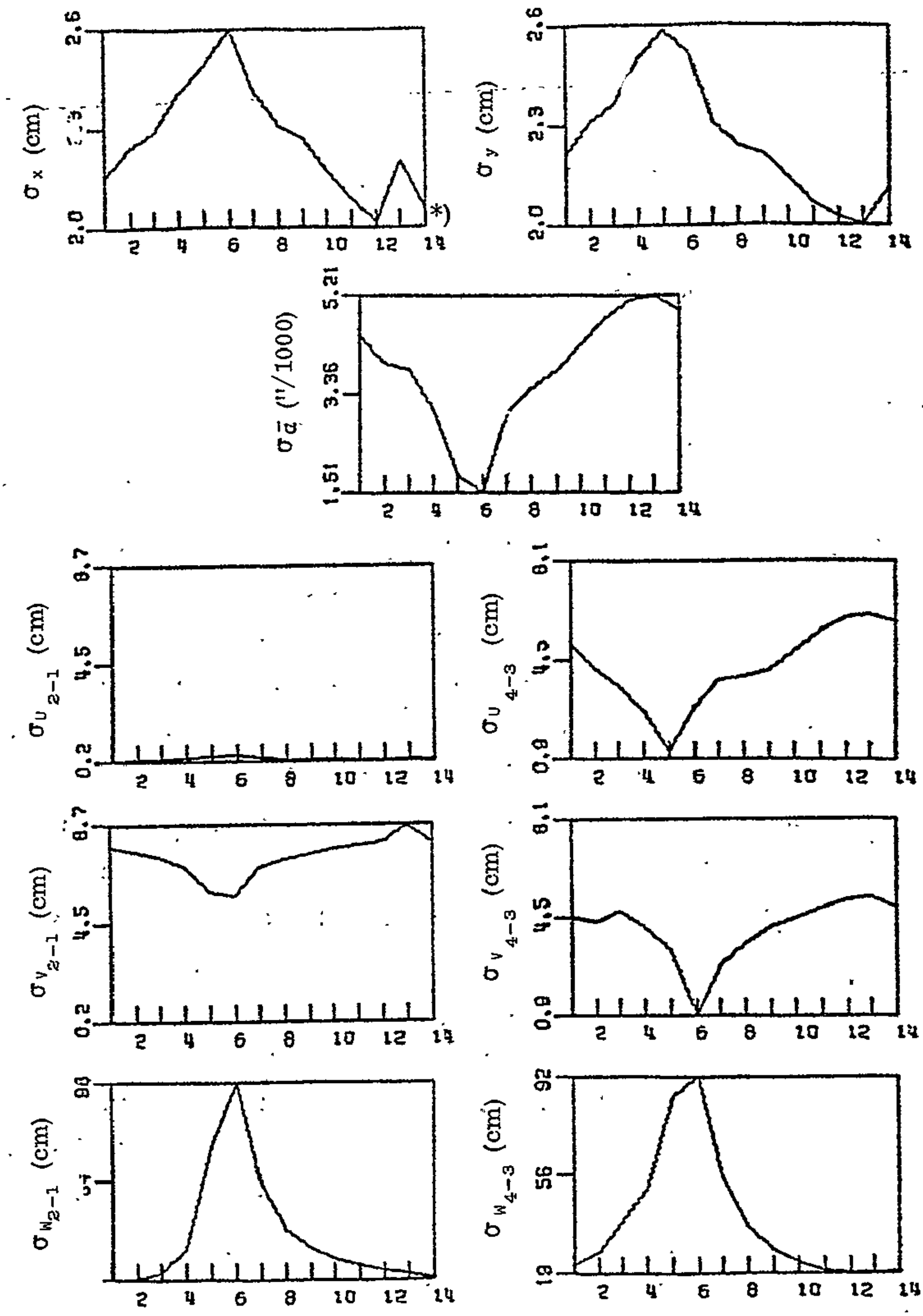


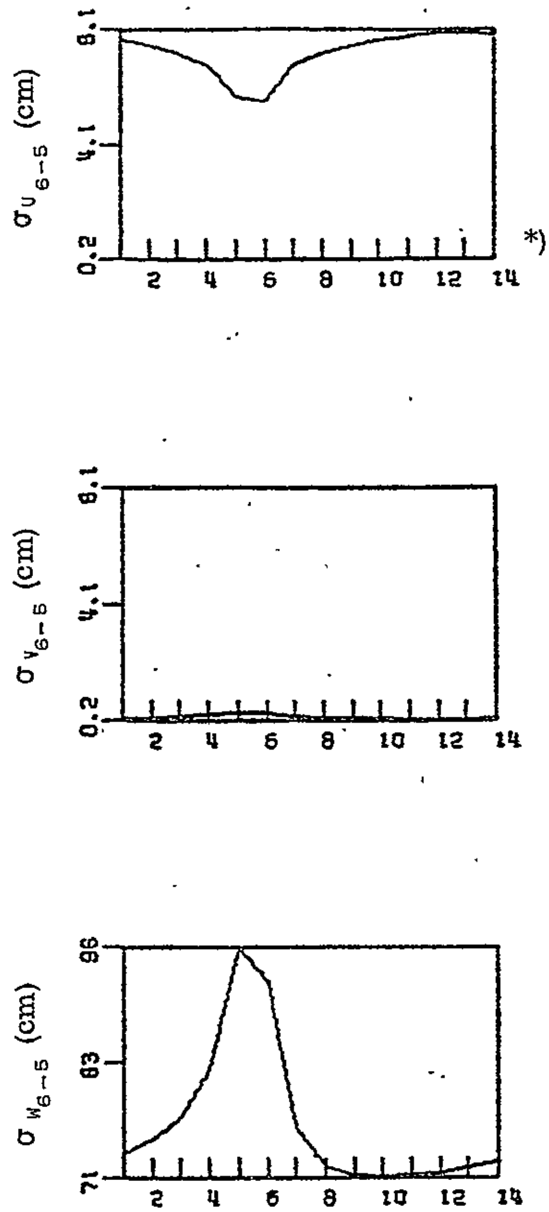
Figure 3.8 Eigenvalues for the Three-Line Design

Figure 3.8 shows the eigenvalues for the correlation matrix of the observations for the previous three-line configuration. Since the maximum and minimum eigenvalues differ appreciably from unity, one expects the correlations of the adjusted estimable parameters $[\hat{X}_3]$ to have a strong influence on the accuracy of the orientation parameters according to the inequality (2.3-6). The eigenvalues are slightly affected by the lunar declination. In Figure 3.9 the accuracy estimates of the parameters of the second adjustment are given for the three-line design using a diagonal variance-covariance matrix of observations. The σ_x, σ_y, and $\sigma_{\bar{\alpha}}$ have all increased. The figures do not include the estimable parameters of the first adjustment since they do not change by this operation. Moreover, the shape of the curves has changed significantly. In case of polar motion, one now gets a decreased accuracy for zero declination as opposed to an increase as in Figure 3.7. The largest <u>decrease</u> in accuracy occurs for $d\bar{\alpha}$. In fact, $\sigma_{\bar{\alpha}}$ is now solely determined from the estimable parameters dU''_{4-3} and dV''_{4-3}, which, according to equations (3.1-17), strongly depend on the ephemeris errors in declination. But this apparent dependence between σ_α and σ_δ is a fallacy. It appears only because the correlations between $[\hat{X}_3]$ were neglected. As was shown in equation (3.1-18), the rotation parameter depends only insignificantly on declination errors. It is therefore mandatory to base the second adjustment on the full variance-covariance matrix $\Sigma_{\hat{X}_3}$, not only because the variances increase but equally important because the adjusted orientation parameters lose their geometric meaning otherwise.



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.9 Orthogonal Transformation with Uncorrelated Observations (Three Lines)

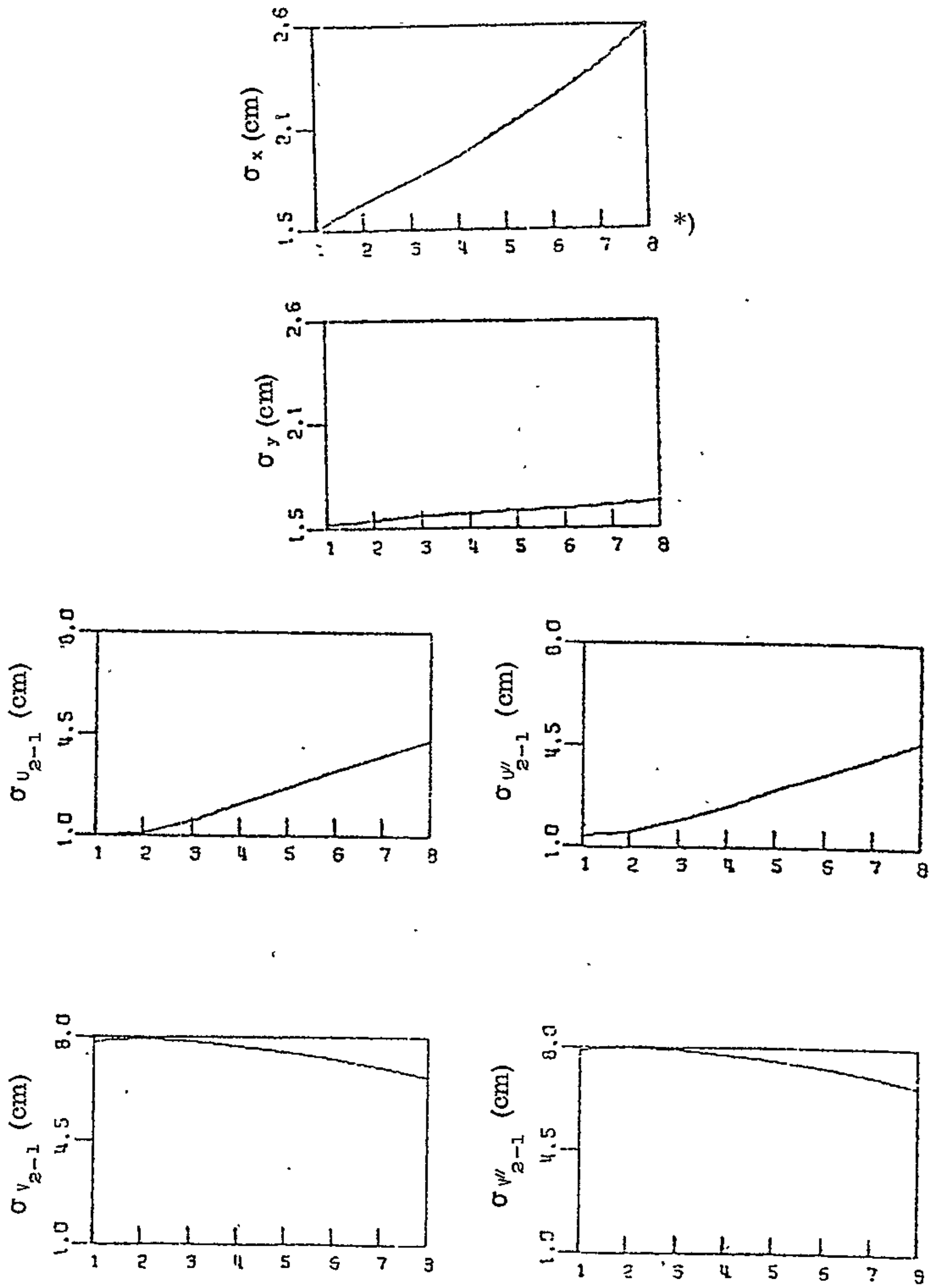


*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.9 (cont'd) Orthogonal Transformation with Uncorrelated Observations (Three Lines)

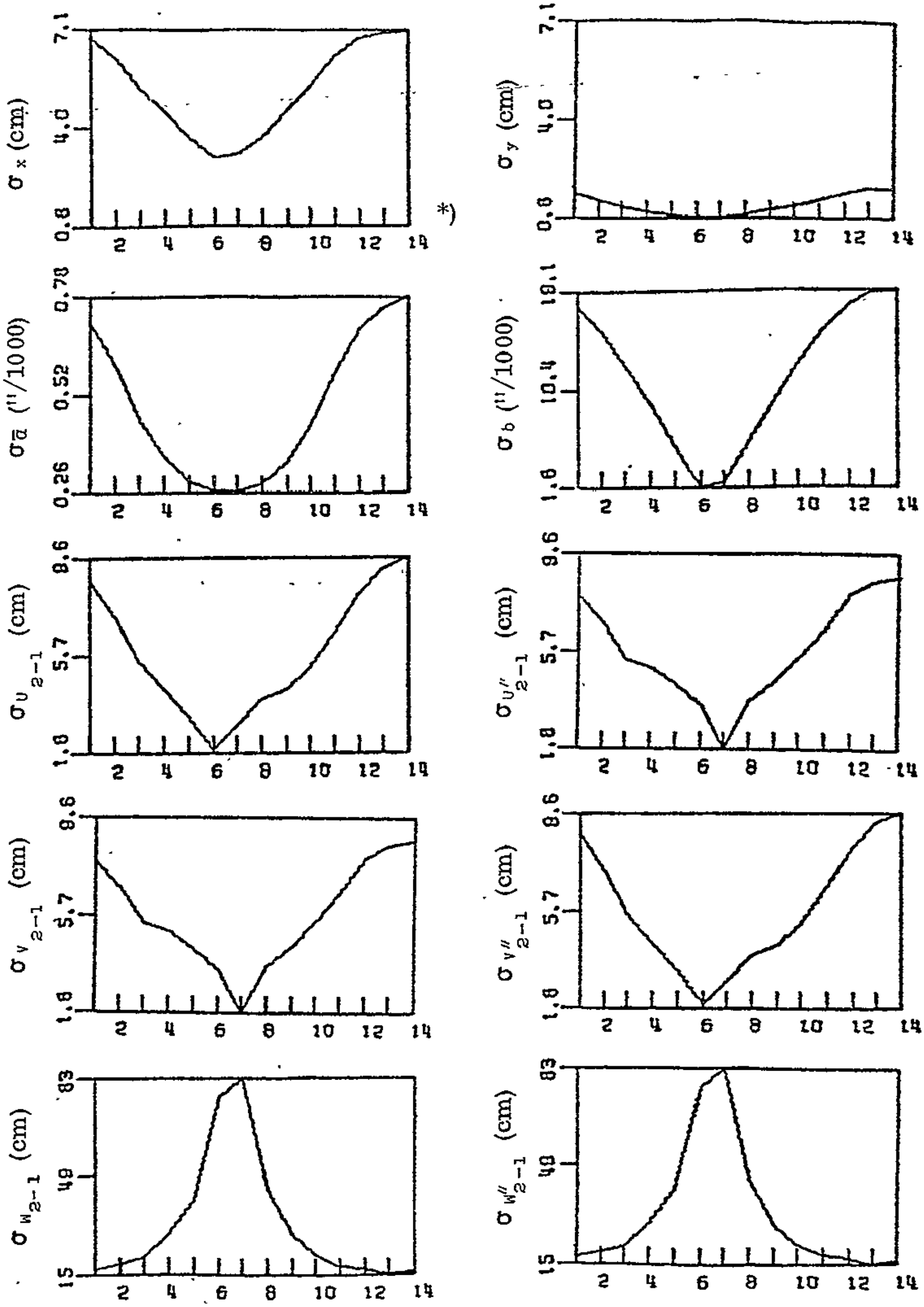
In Figure 3.10 the results of varying the separation in longitude between two north-south lines are shown. The basic three-line design is used. The line 6-5, which defines the zero meridian, remains fixed, whereas line 2-1 changes its longitudinal position from 90 degrees to 50 degrees in steps of five degrees. In the figure, only those parameters are given which are sensitive to such a change. The y-polar motion component, i.e., mainly the across-meridian component of the line 6-5 does not change significantly since this line is held fixed. The x-component, on the contrary, changes, although not significantly. Therefore, the two north-south lines have to be separated only approximately by 90 degrees.

Figures 3.11 and 3.12 show the result of the two-line design for observation schedules A and B, respectively. The only significant difference is in the along-meridian polar motion, where schedule B gives worse accuracy for $\delta \cong 0$ because of the neglect of observations at the extreme hour angle. The σ_y of the across-meridian component indicates only an insignificant variation.



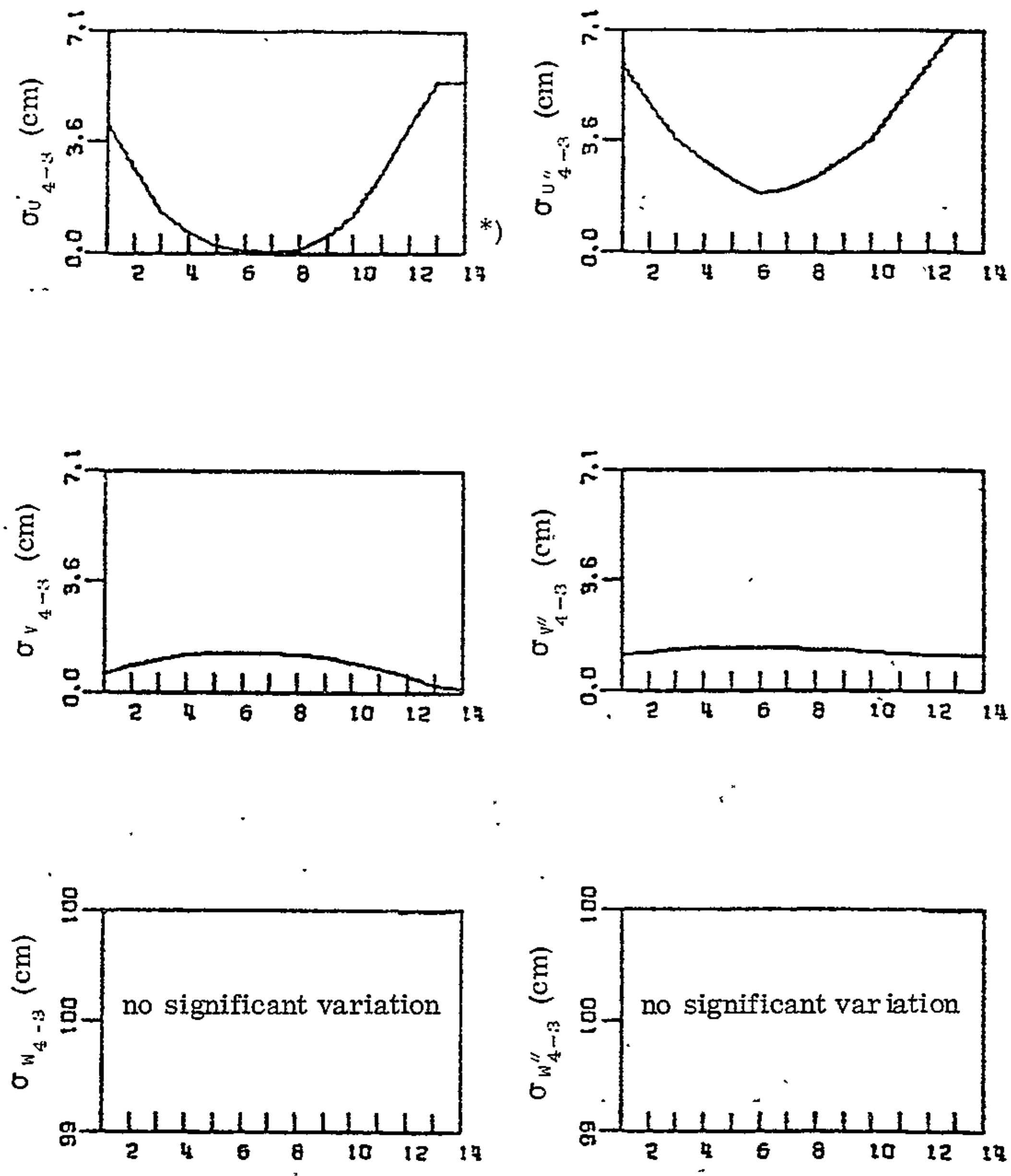
*) The separation in longitude is reduced by five degress in each step.
The initial separation in longitude is 90 degrees.

Figure 3.10 Separation in Longitude for Two North-South Lines



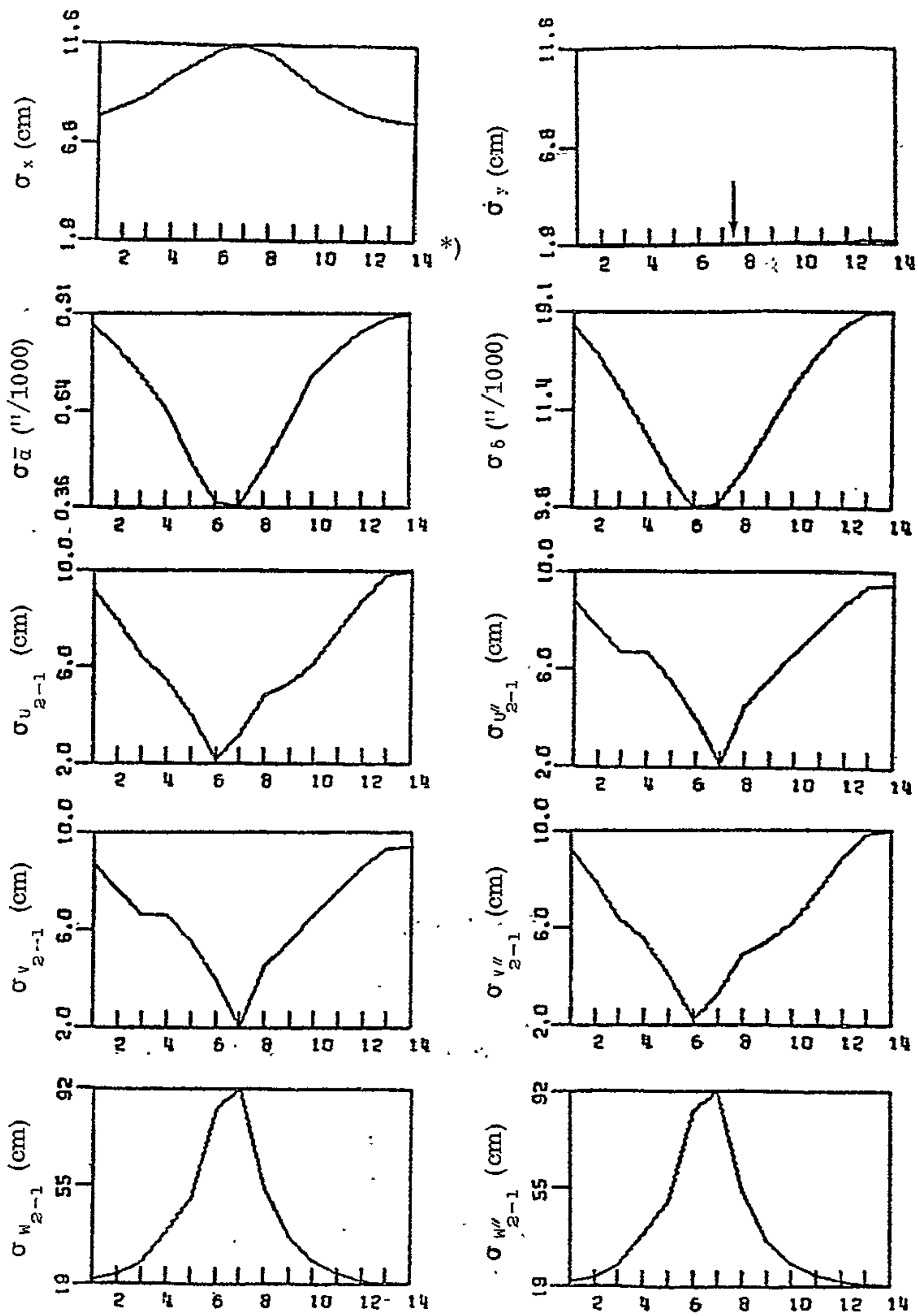
*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.11 Two-Line Design with Observation Schedule A



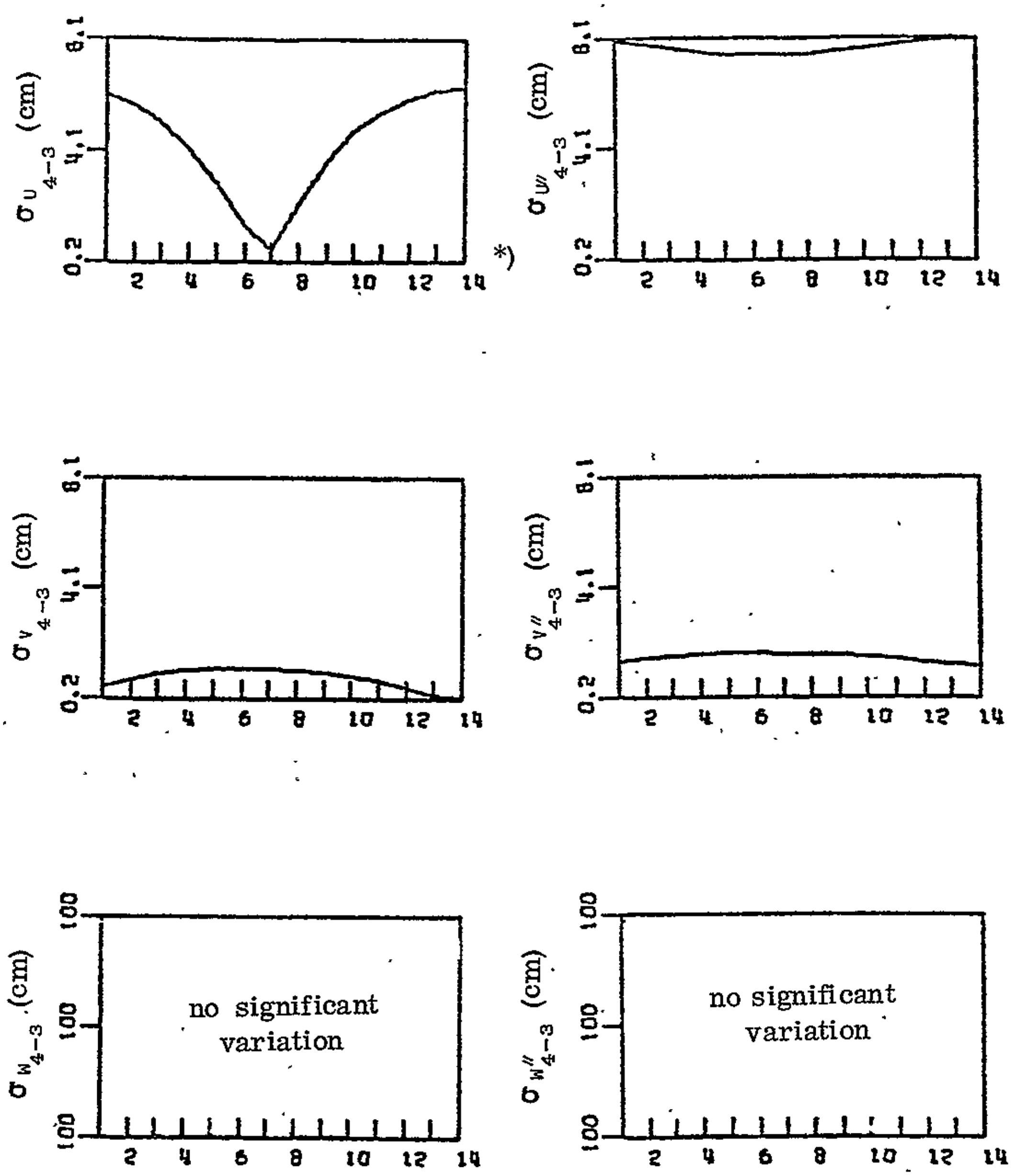
*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.11 (cont'd) Two-Line Design with Observation Schedule A



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.12 Two-Line Design with Observation Schedule B



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.12 (cont'd) Two-Line Design with Observation Schedule B.

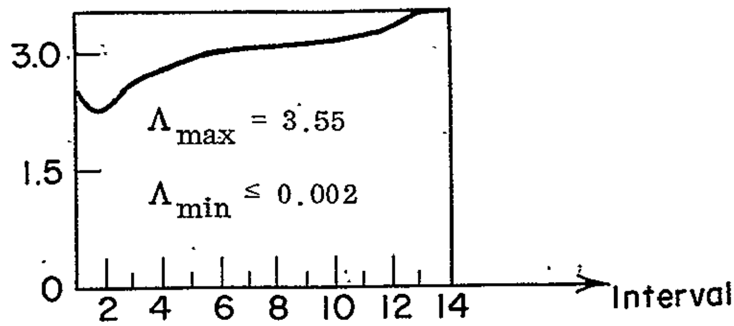


Figure 3.13 Eigenvalues for Two-Line Design

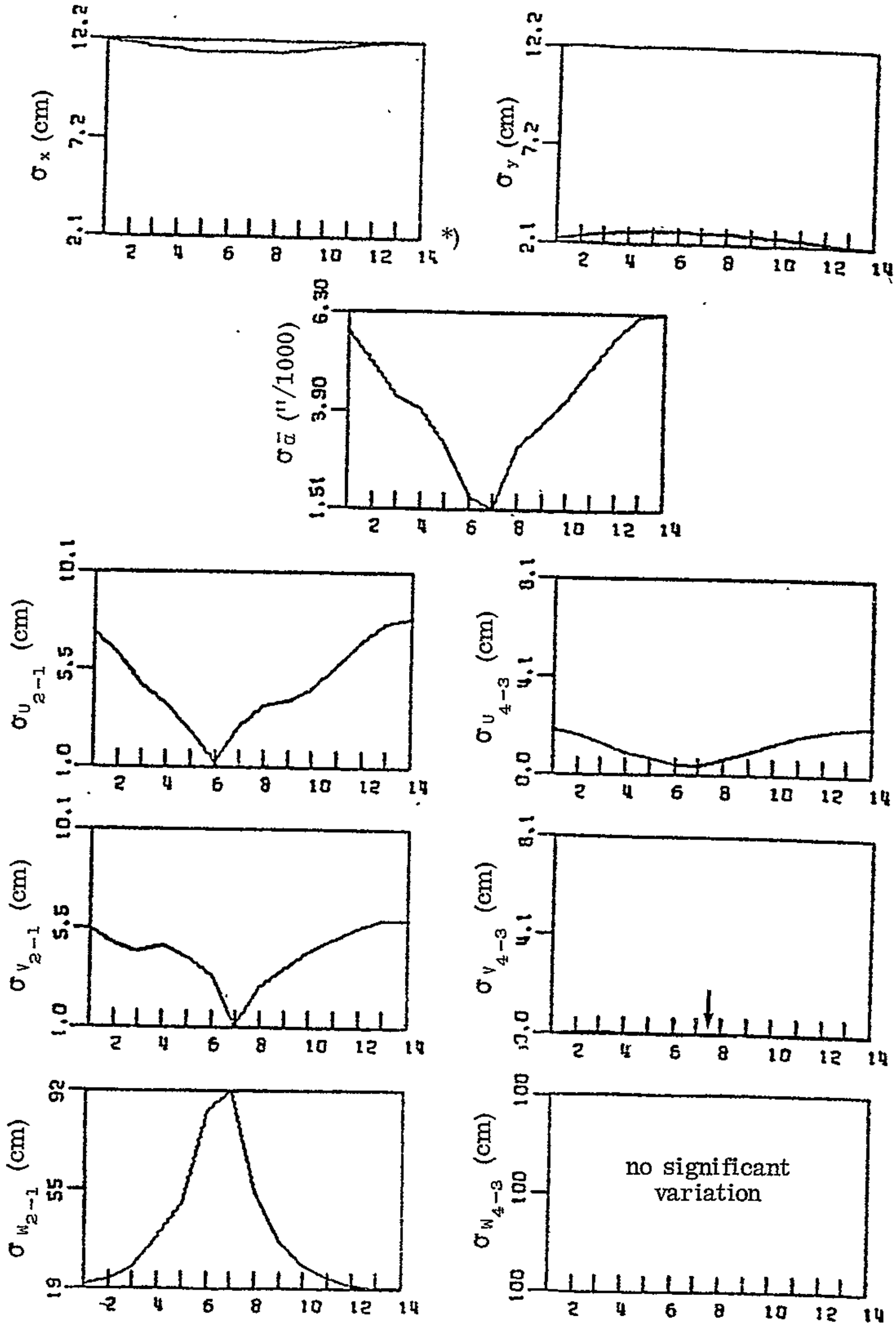
In Figure 3.13 the eigenvalues of the correlation matrix of $[\hat{X}_3]$ are shown for the two-line design. The maximum eigenvalue is increasing as the moon moves from minimum to maximum declinations. Figure 3.14 displays the two-line design if the correlation between the observations is neglected. All conclusions which were drawn for the three-line case are also valid here.

The effect of weighting the third coördinate difference W''_{j-1} on the obtainable accuracy in declination is demonstrated in Figure 3.15. The following a priori weights are used:

$$\frac{1}{(.1\text{m})^2} \geq \frac{1}{\sigma^2_{W''_{j-1}}} \geq \frac{1}{(1.4\ \text{m})^2}$$

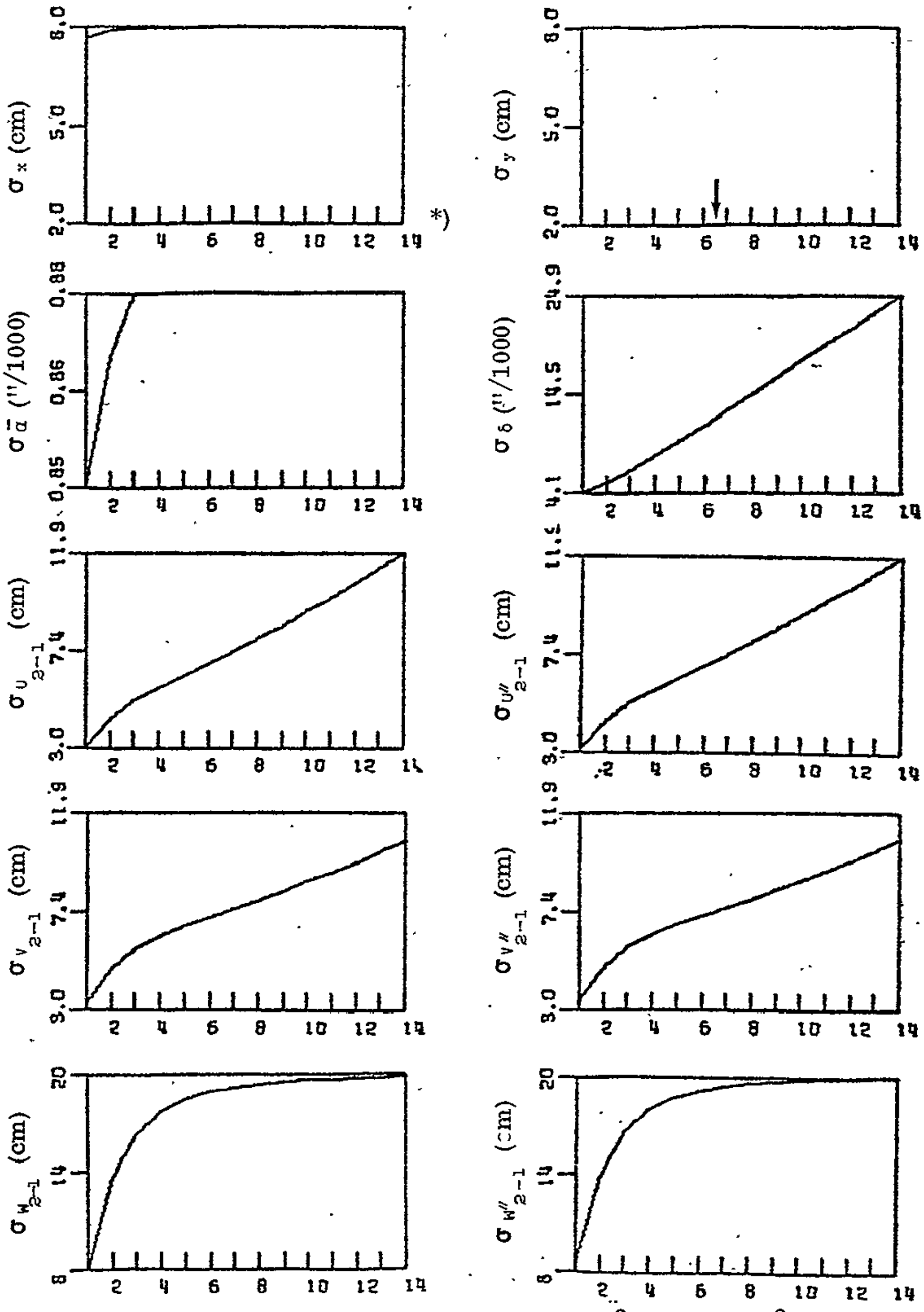
σ_δ is nearly directly proportional to the a priori $\sigma_{W''_{j-1}}$. Polar motion and rotation accuracies are essentially independent of the weighting. This is important since it confirms that the orientation parameters can be obtained accurately even if the third station coordinate is known to only ±1m (accuracy of Doppler positioning). Figure 3.16 shows the effect of variation in length for a north-south line symmetric with respect to the equator. The five experiments are based on the following latitude separation of the stations

$$80^\circ \geq \Phi_3 - \Phi_4 \geq 40^\circ$$



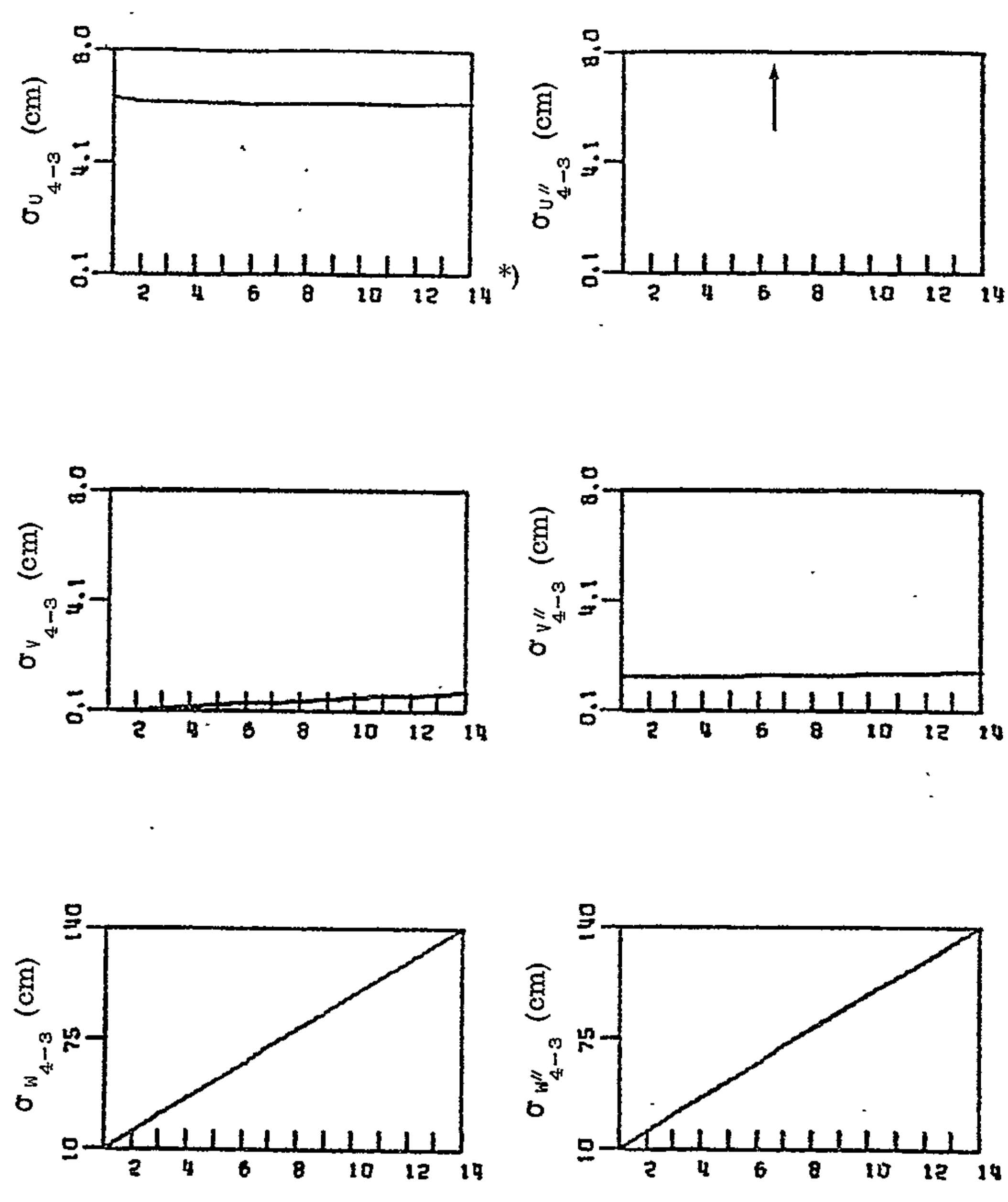
*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.14 Orthogonal Transformation with Uncorrelated Observations (Two-Line Design)



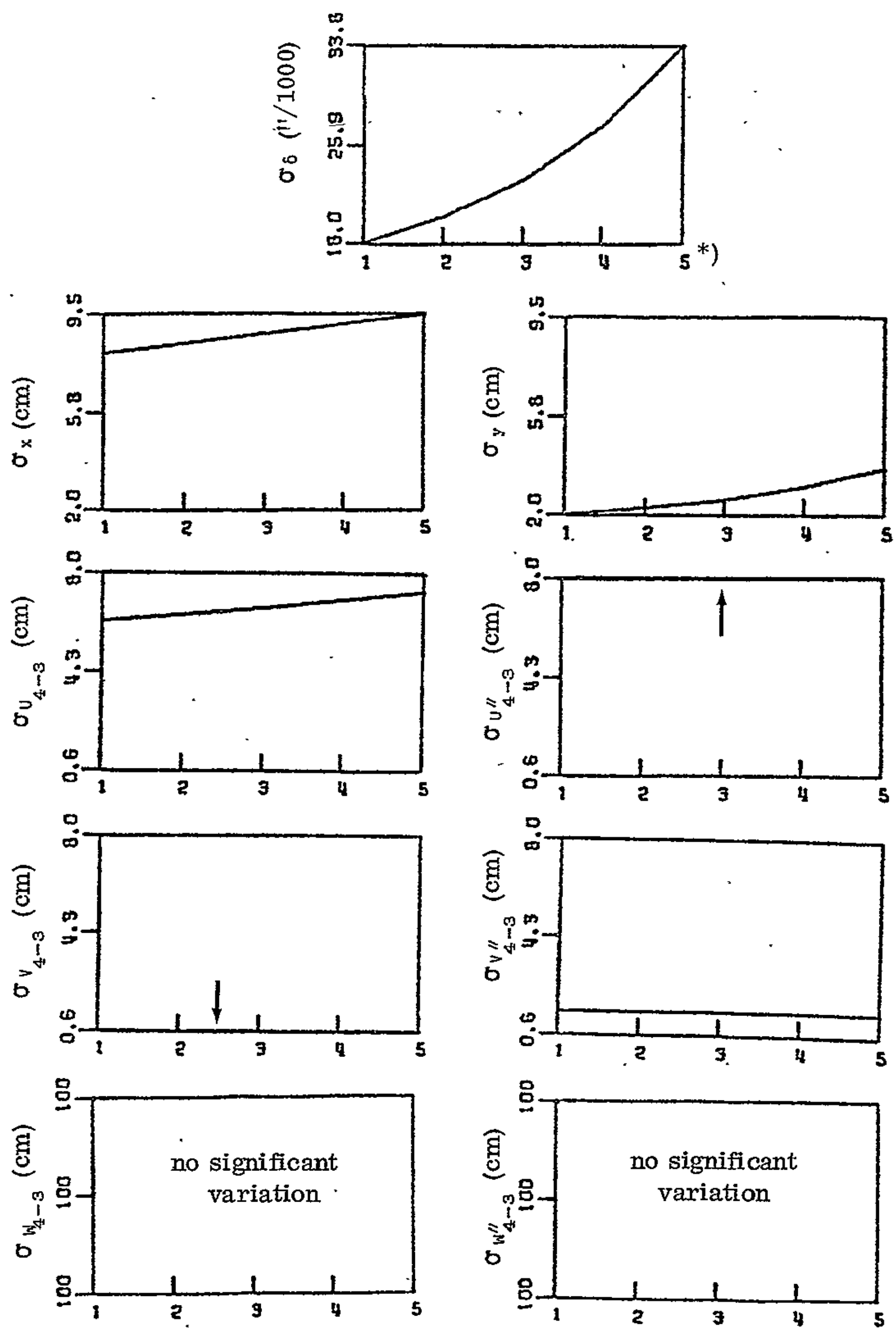
*) The a priori weight decreases from $(0.1\ \mathrm{m})^{-2}$ to $(1.4\ \mathrm{m})^{-2}$.

Figure 3.15 A priori Weighting of W''_{j-1}



*) The a priori weight decreases from $(0.1)^{-2}$ to $(1.4\ \mathrm{m})^{-2}$.

Figure 3.15 (cont'd) A priori Weighting of W''_{j-1}



*) One unit in abscissa denotes a shortening of the north-south line by 10°. Initial separation is 80°.

Figure 3.16 Variation in Length of North-South Line

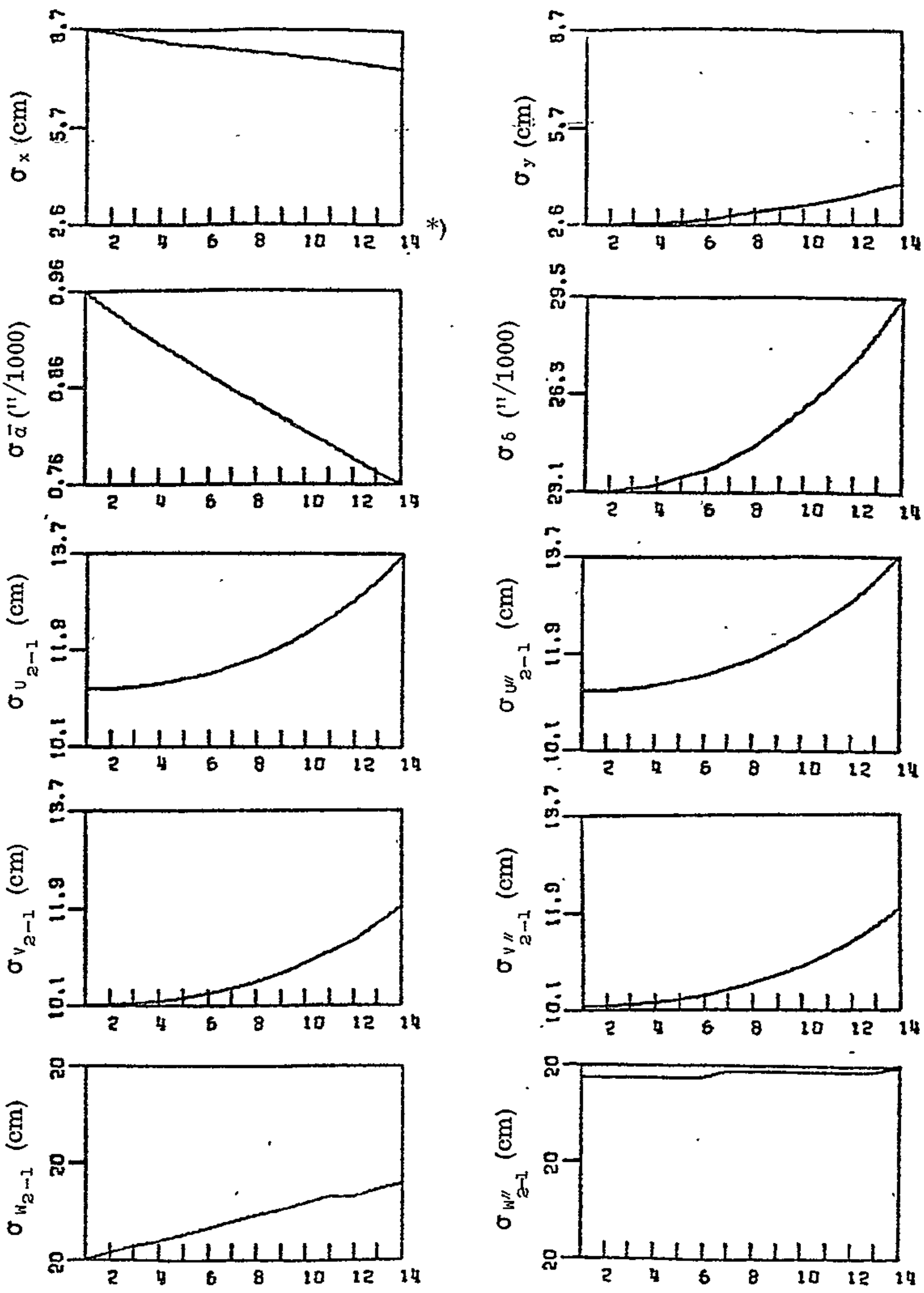
The length is decreased in steps of 10 degrees. A long north-south line gives obviously a better accuracy in declination and in polar motion. According to the approximate formula (3.2-4), $\sigma_{x'}$ and $\sigma_{y'}$ is inversely proportional to the sine of the station latitude. As a whole, the deterioriation in polar motion accuracy as a function of the latitude separation is not very critical. Any separation between 80 and 60 degrees in latitude is acceptable.

The effect of the variation in azimuth of the north–south line is shown in Figure 3.17. The azimuth is changed from 0 degrees to 42 degrees, in steps of 3 degrees. It is clear from the figures that the requirement for an exact north-south line is not very strong. Any line, running approximately north-south (±15 degrees) will be sufficient.

In the next three experiments, the variations of the east-west line regarding length, latitude and orientation are investigated. Figure 3.18 shows the results of an equatorial line with lengths

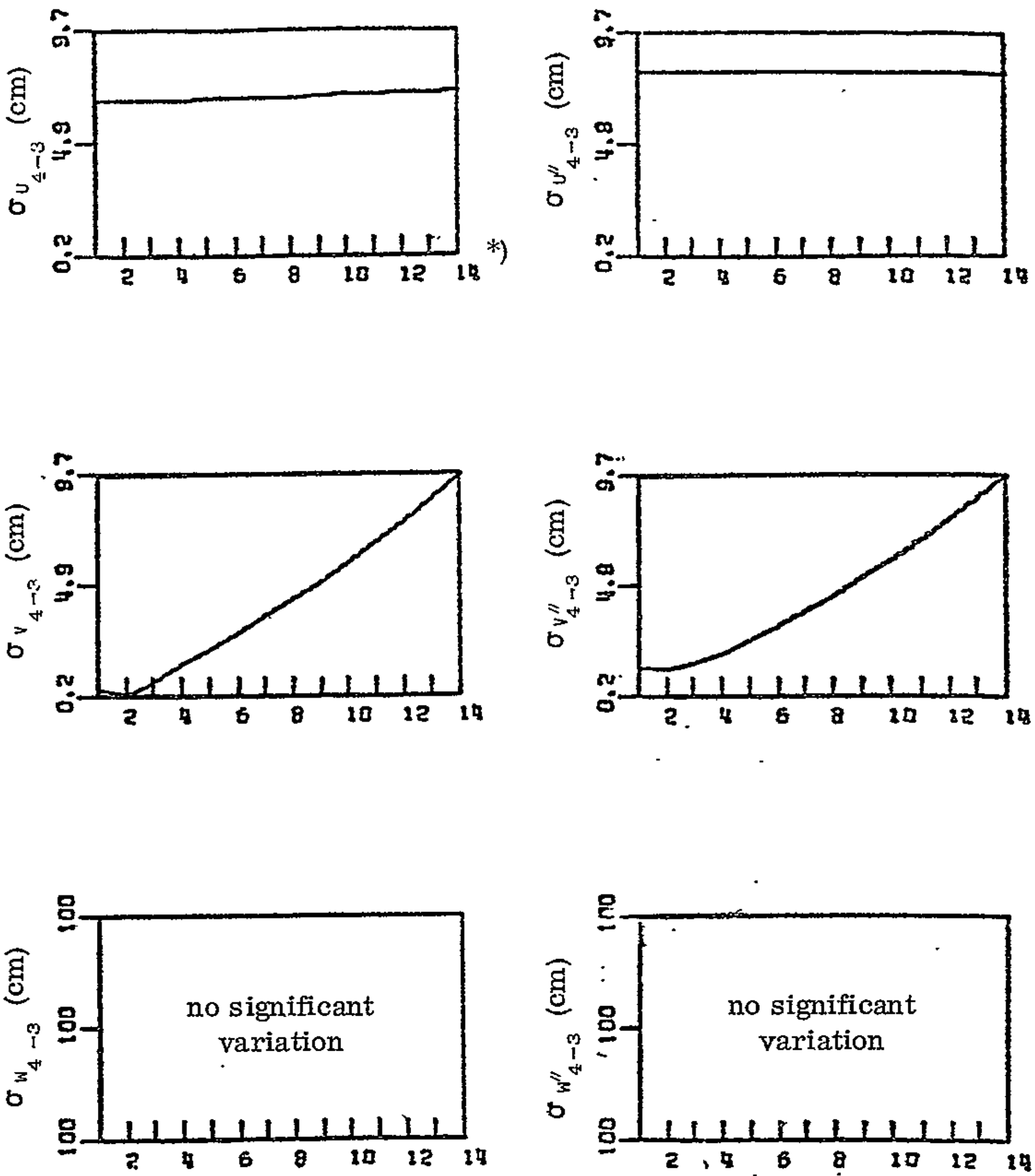
$$60^0 \geq \Lambda_3 - \Lambda_4 \geq 20^0$$

The variation occurs in steps of 10 degrees. Only the graphs for the east-west line and the rotation parameter are given. Station 3 is held fixed at $\Lambda_3 = 80^0$ and station 4 moves toward station 3. The accuracy of the rotation parameter decreases as expected from the approximate expression (3.2-5). Generally, a variation in length between the limits 60 degrees and 40 degrees is tolerable. In Figure 3.19 the latitude of the east-west line is varied. Because of the geometrical constraint imposed by schedule B, i.e., useful visibility of at least 5 hours daily throughout the month, the length of the line has to be decreased as the latitude increases. The following positions are used:



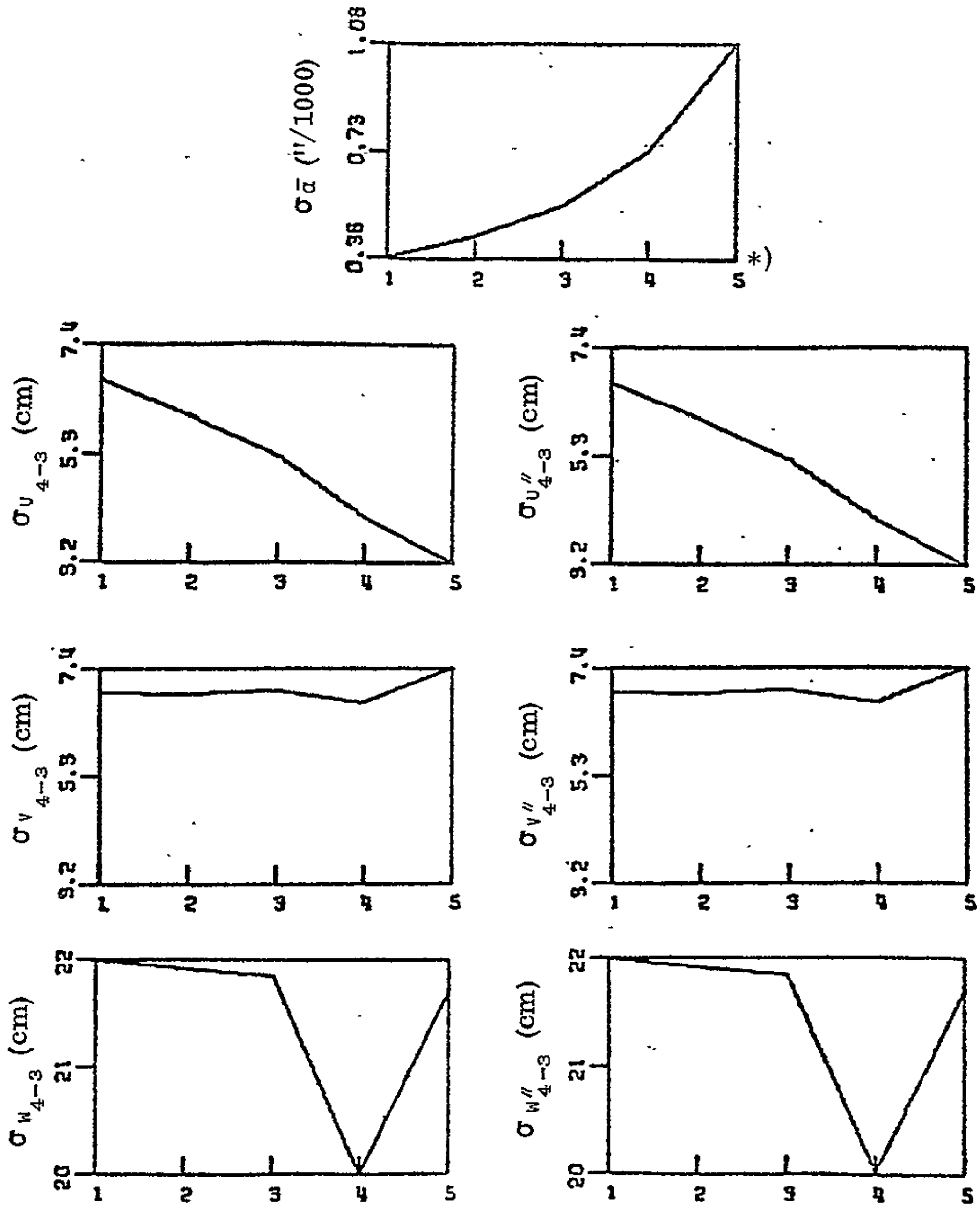
*) The azimuth changes from zero degree in steps of three degrees.

Figure 3.17 Variation in Azimuth of North-South Line



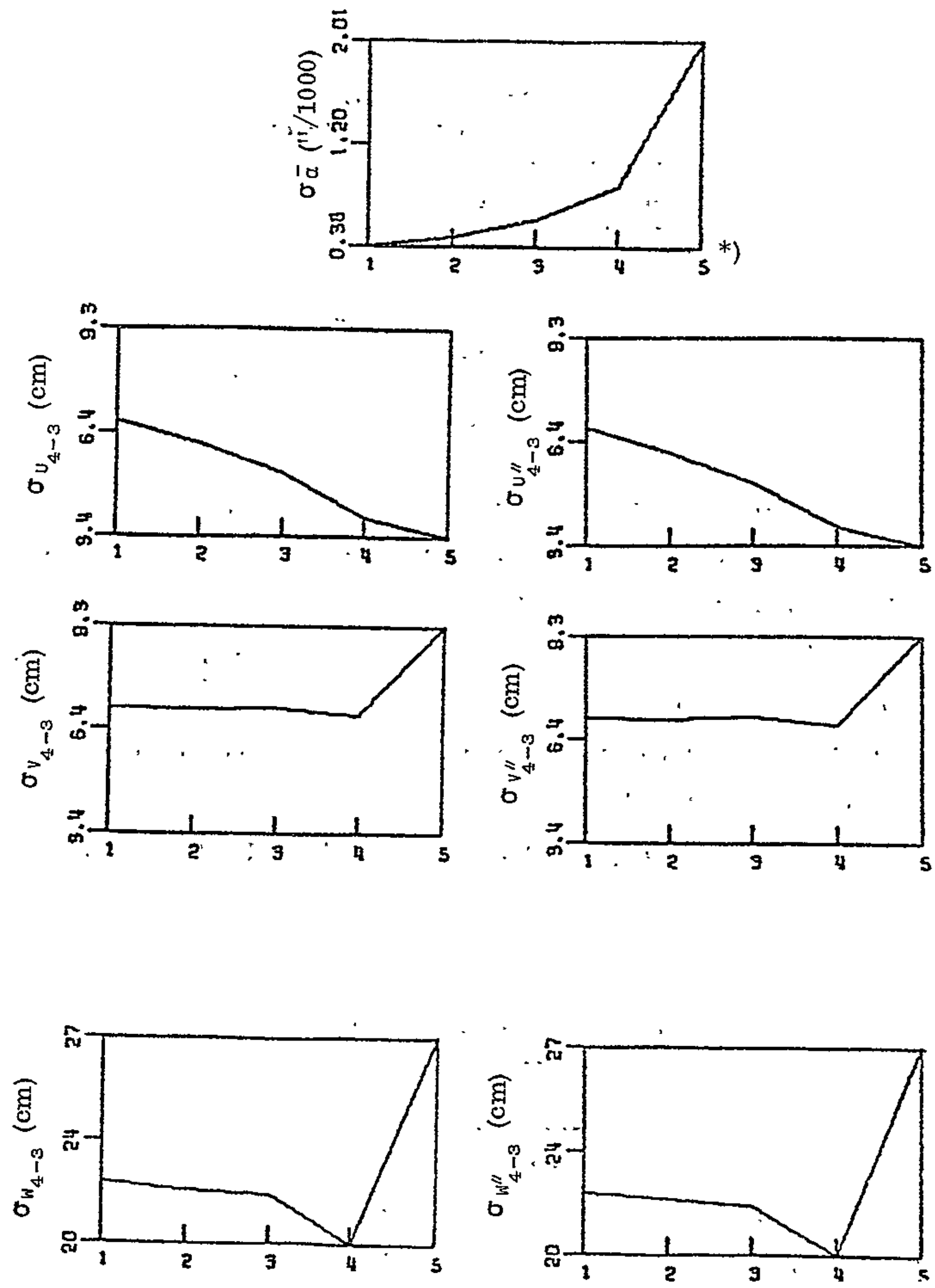
*) The azimuth changes from zero degree in steps of three degrees.

Figure 3.17 (cont'd) Variation in Azimuth of North-South Line



*) The length is reduced in steps of 10°. Initial station separation is 60°.

Figure 3.18 Variation in Length of Equatorial Line



*) The station location for each experiment is given in Table 3.1.

Figure 3.19 Variation in Latitude of East-West Line

Table 3.2

Station Variation for East-West Line
(Variation in Latitude)

Experiment	Φ	Λ_1	Λ_2
1	0	80	20
2	10	80	30
3	20	80	40
4	30	80	50
5	40	80	65

The variation σ_α agrees with that predicted by the approximate formula (3.2-5). A latitudinal position up to 20 degrees (-20 degrees) for the east-west line appears acceptable.

Finally, Figure 3.20 shows the variation in azimuth of the east-west line. The azimuth is varied from 90 degrees to 48 degrees in steps of three degrees. Any azimuth up to approximately 20 degrees has an insignificant effect on the rotational parameter.

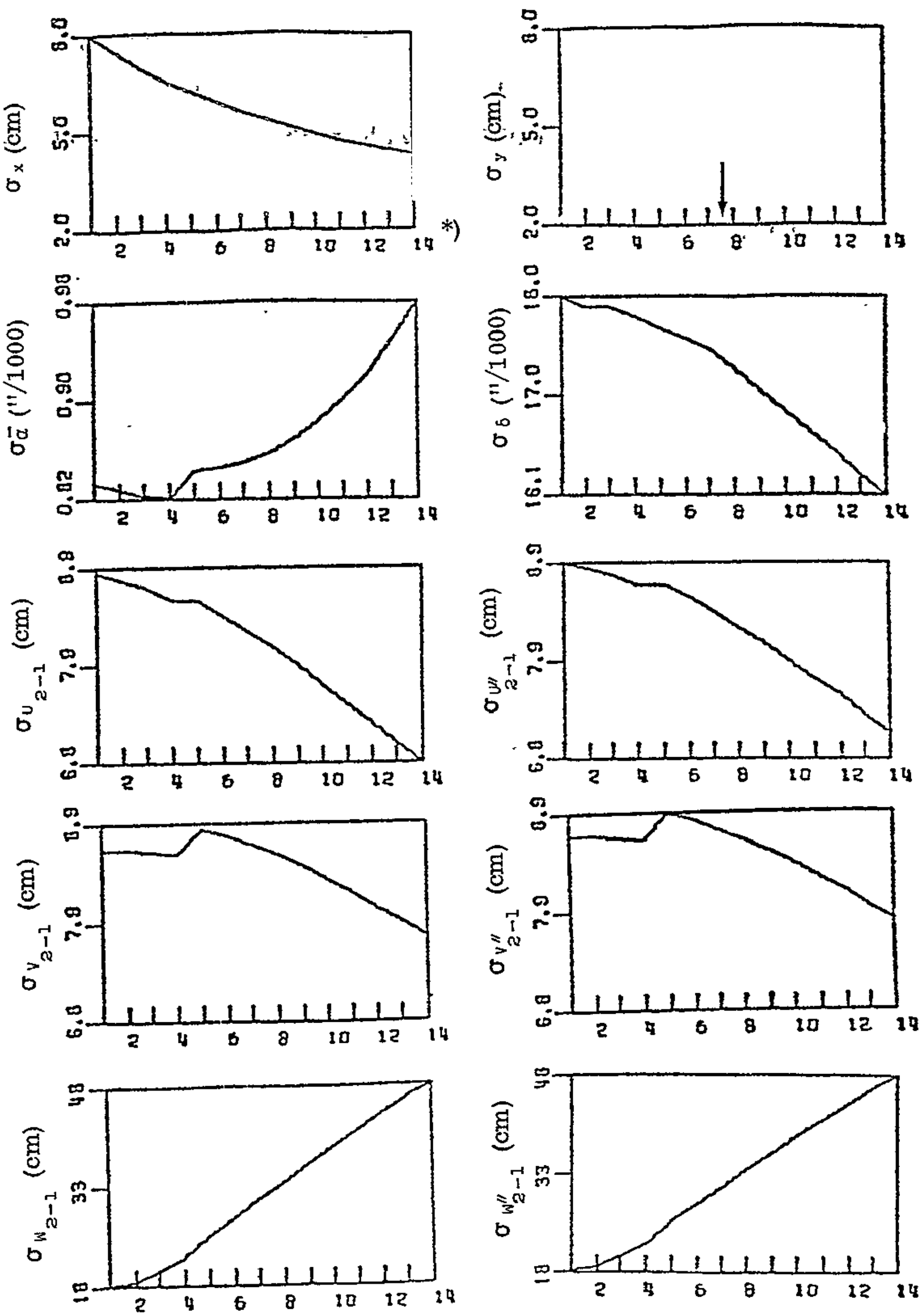
Figure 3.21 displays the accuracies obtainable for a conceivable real station distribution as a function of the lunar declination.

Table 3.3

Conceivable Real Station Distribution

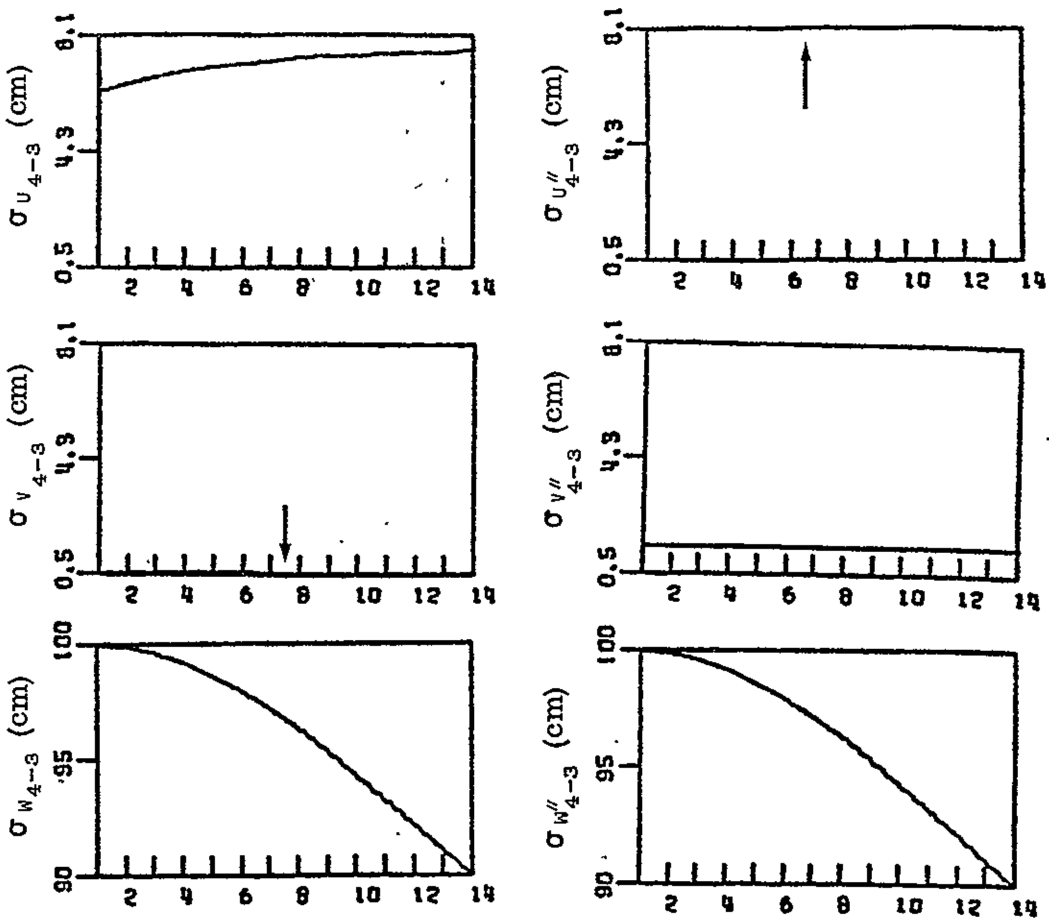
Station	Latitude	Longitude
(1) Texas (McDonald)	30°	256°
(2) Hawaii	20°	204°
(3) Japan	35°	138°
(4) Australia	-35°	149°
(5) Southern Europe	38°	15°
(6) Southern Africa	-34°	20°

The first four stations are expected to be in operation soon. In this experiment, the zero longitude is at Greenwich. The only new feature in these figures is that $\sigma_{\bar{a}}$ remains small for high lunar declination because the "effective east-west line," i.e., McDonald-Hawaii, is in the northern hemisphere.



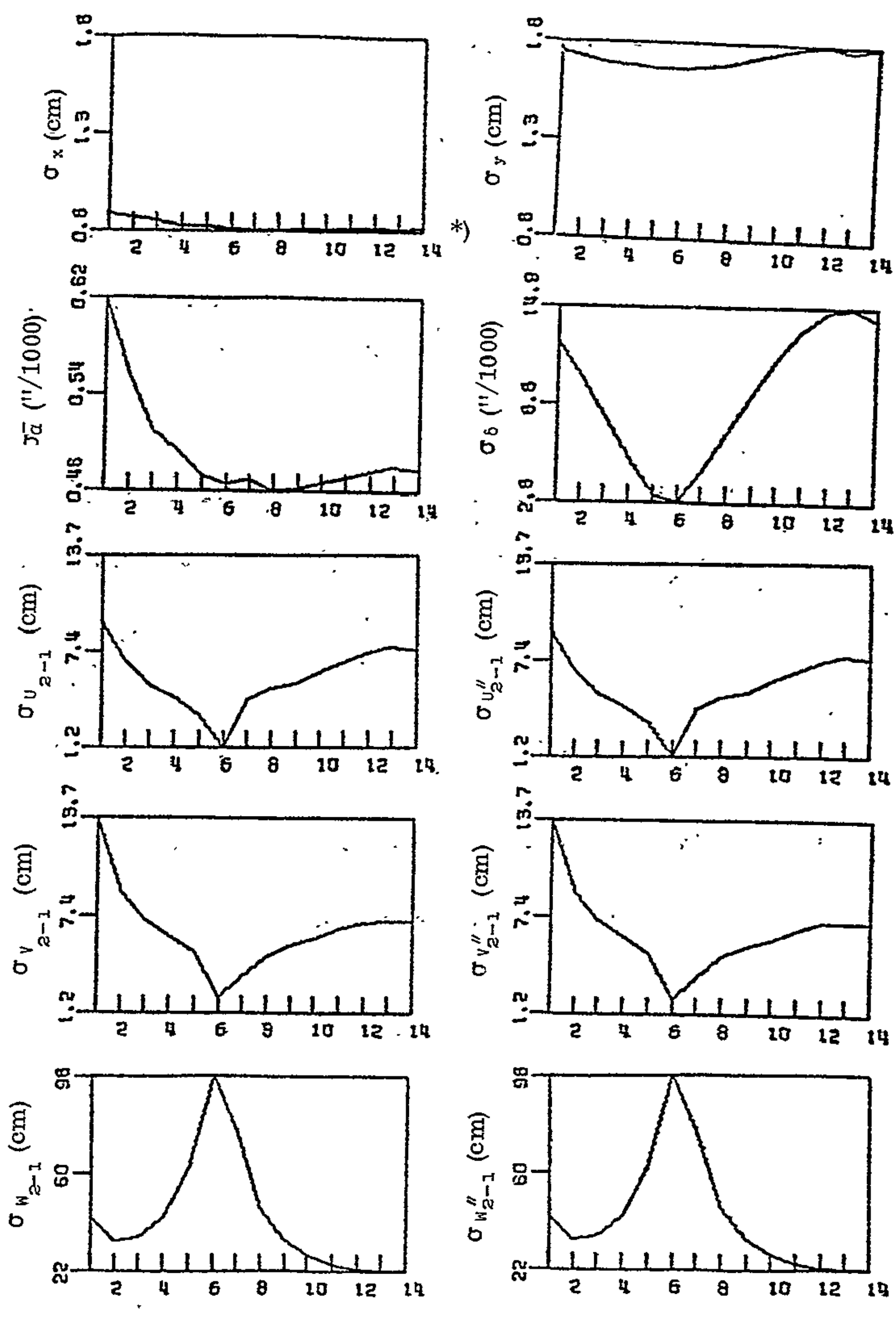
*) The azimuth varies in steps of 3°. The initial azimuth is 90°.

Figure 3.20 Variation in Azimuth of East-West Line



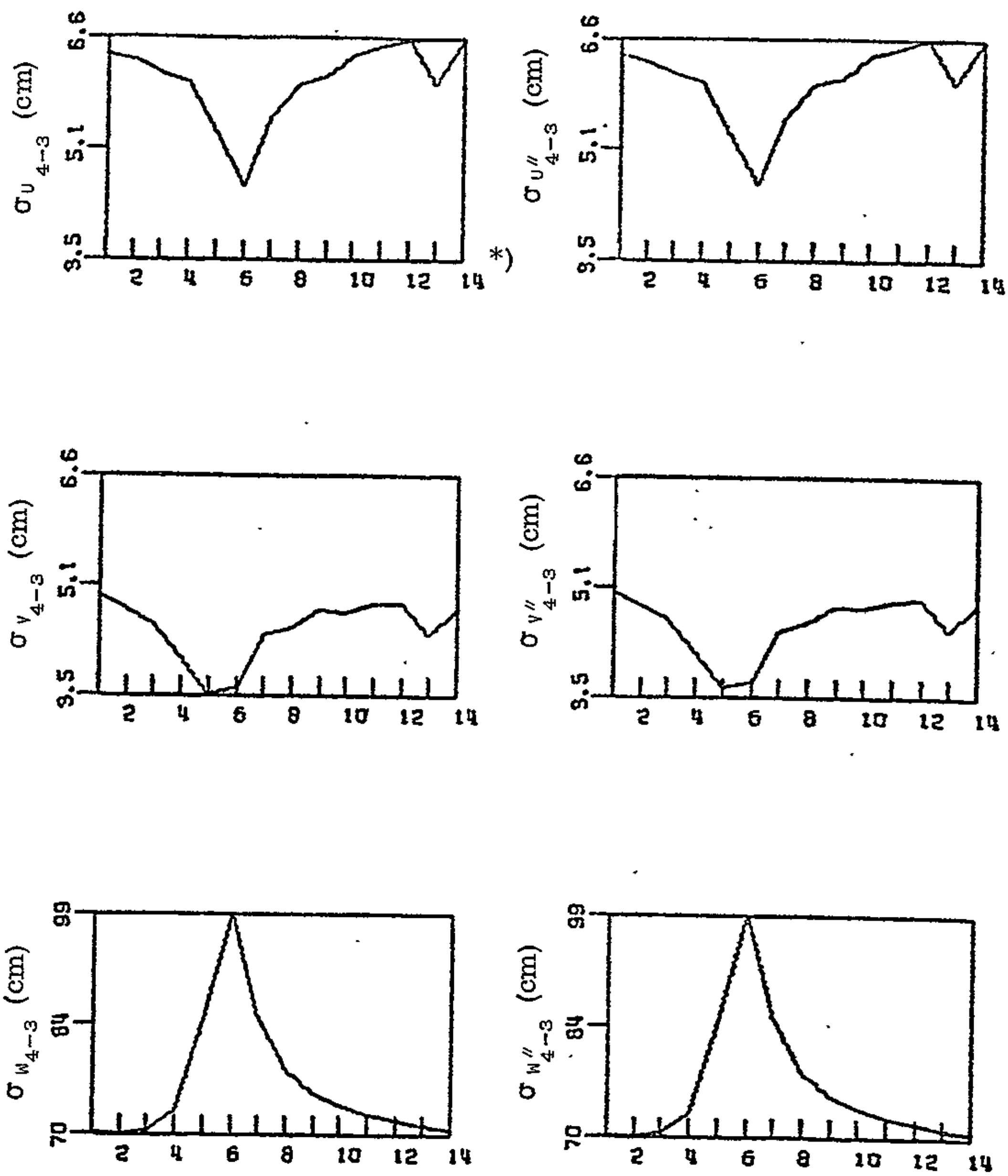
*) The azimuth varies in steps of three degrees. The initial azimuth is 90°.

Figure 3.20 (cont'd) Variation in Azimuth of East-West Line



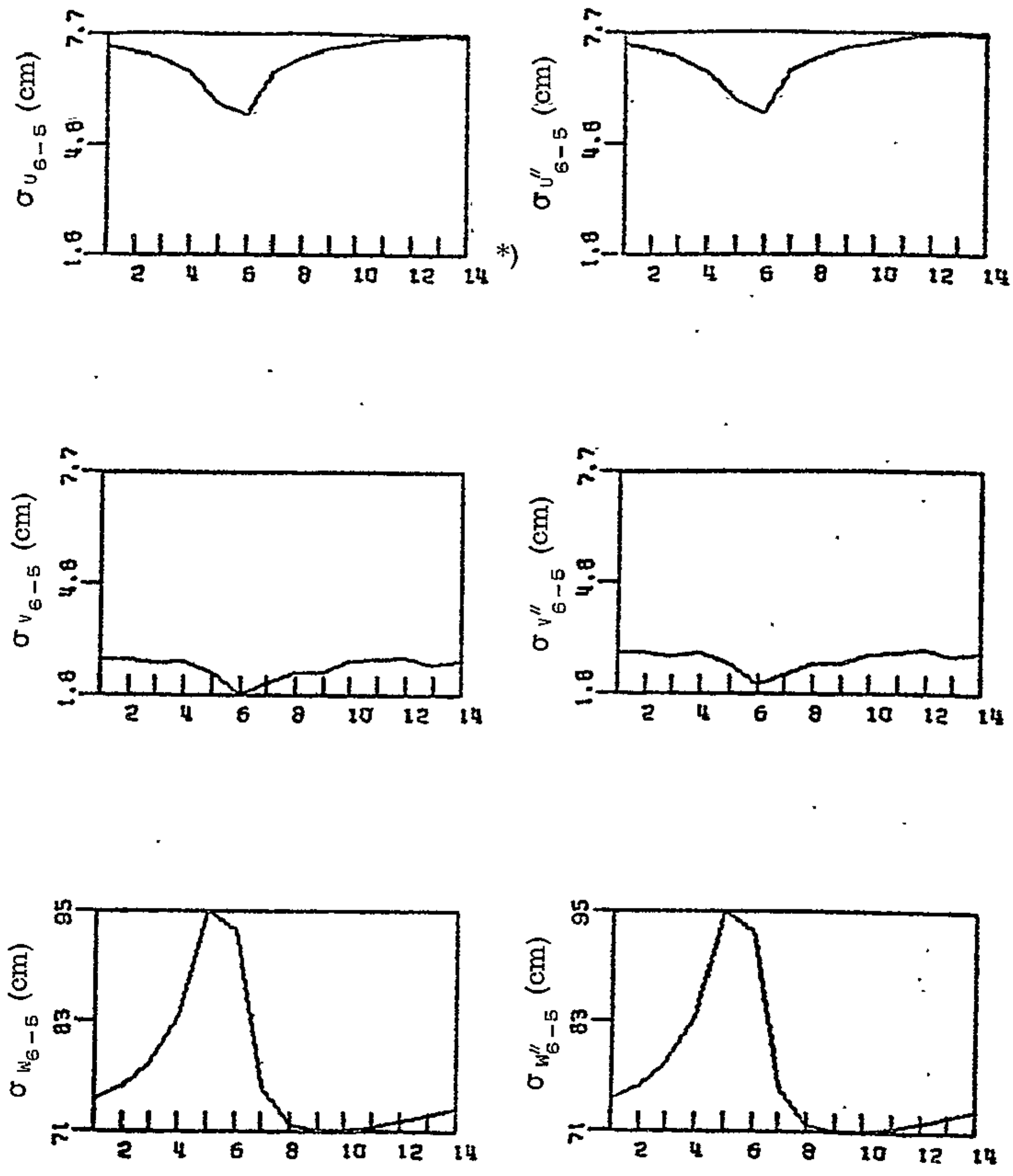
*) One unit in abscissa denotes one interval. The lunar declination is varied

Figure 3.21 Experiment with Realistic Station Distribution



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.21 (cont'd) Experiment with Realistic Station Distribution



*) One unit in abscissa denotes one interval. The lunar declination is varied.

Figure 3.21 (cont'd) Experiment with Realistic Station Distribution

4. SUMMARY AND RECOMMENDATIONS

In this study the least squares formulation is given for range difference observables. The necessary relativistic considerations as they relate to high precision lunar laser ranging are given. It is pointed out that the earth orientation with respect to the lunar position can be parametrized in two equivalent ways. The first method is based on the estimable parameters, which are obtained by forming parameter combinations (re-parametrization) such that the remaining design matrix is non-singular. They are shown to be the coordinates in a system (U'') whose third axis coincides with the celestial pole and whose origin is at the instantaneous center of mass. The first axis of this system is not strictly fixed to the crust but depends on errors of the station clock and the lunar ephemeris. No separate determination of the corrections to longitude, time, and right ascension of the reflector is possible. The second method uses the estimable quantities mentioned above and performs an orthogonal transformation (over determined case) so as to result in actual polar motion coordinates and in an earth rotation parameter which relate the system (U'') and (U). It is necessary to agree upon a standard epoch for which these two systems coincide. Although these orientation parameters are sometimes referred to as unestimable, they will in no way be inferior to the estimable quantities when a standard epoch is fixed. Still, the earth rotation parameter is linearly dependent on the correction to lunar right ascension. The parameters of the station coordinates appearing in the range difference observation equation have been transformed into their differences and sums. Because of the earth-moon geometry, the coefficients of the coordinate sums are characteristically of the order 1/60. They were found to be insignificant on the basis of the available approximate station coordinates. The range difference observable will, therefore, only determine the coordinate differences, which, however, determine the orientation parameters completely. Strictly simultaneous observations are the best since they reduce the significance of ephemeris errors. Processing near simultaneous observations requires a good ephemeris in order to compute the change in lunar position between the two epochs of reflection. It is possible to create artificially simultaneous observations using powerful interpolation tools. But such investigations should be carried out on real data.

An analysis of variance has been based on two types of approximate models. The numerical computations have been carried out with a model in which the earth rotation during the travel time of the pulse was neglected. The accuracy for the range difference was assumed to be $\pm 3\sqrt{2}$ cm. A second type of approximate model has been set up by neglecting terms of the order 1/60 and taking the declination constant for the time of one interval. To the approximation of this model it has been shown that the across-meridian polar motion component and the earth rotation parameter are independent of errors in declination whereas the along-meridian component strongly depends on such errors. The analysis was based on ideal station distributions in the form of north-south and east-west lines. Such a design reduces the correlations between the parameters. As a basic observation schedule, one observation every ten minutes to one and the same reflector was assumed for a period of three hours per day. The first and last observations were placed 2.5 hours before and after lunar transit. After one full hour of observations, an interruption of one hour was assumed so that the total observation span was five hours a day. This condition and the requirement that observations are only made at zenith distances smaller than 70 degrees put a limit on the station separation. Common visibility during the whole month is possible for the following station separations:

north-south line (symmetric to equator):		$\Delta\Phi \leq 80^\circ$
east-west line:	$\Phi = 0^\circ$	$\Delta\Lambda < 60^\circ$
	$\Phi = 10^\circ$	$\Delta\Lambda < 60^\circ$
	$\Phi = 20^\circ$	$\Delta\Lambda < 50^\circ$
	$\Phi = 30^\circ$	$\Delta\Lambda < 30^\circ$.

Each line gives only two estimable parameters accurate enough to be useful in determining the orientation parameters. Usually, the accuracy of the third coordinate is modestly accurate at best, because of the small change in declination during one day. Therefore, at least two lines are necessary. A three-line configuration is preferable because it allows the elimination of individual station motions due to crustal motion, although the common crustal motion component is absorbed by the orientation parameters. For a design which includes two north-south lines and one equatorial east-west line, the orientation parameters, i.e., polar motion and earth rotation variations, can be obtained at least with the measurement accuracy. The numerical analysis showed that the requirement for lines to run exactly north-south or east-west is not very stringent. It is quite sufficient if these two principal directions are approximately realized, say, within 10 or 15 degrees. Consequently, there is a large degree of freedom for the practical realization of such a network. Besides the station geometry, the weather conditions are very important indeed. The final selection of station sites should give due considerations to the local climate. Some of the stations, which are presently available, already fulfill the geometric requirements. The stations in Australia and Japan are located very ideally to form a north-south line, whereas the McDonald Observatory in Texas and the station in Hawaii can form the east-west line. It is suggested that the method of range-differencing be tested as soon as all of these stations become operational. The missing north-south line can be established by mobile laser stations.

Once accurate orientation parameters become available, they can be used as known parameters in long-term single station solutions which will provide geocentric coordinates and significant improvements in the lunar ephemeris. At that time, it will definitely be possible to reach an accuracy level in the ephemeris which allows the analysis of earth core motions [Leick, 1978]. Yet, an inseparability between the nutations and ephemeris corrections exists (Appendix B). This simply demonstrates that the orientation of the celestial pole in space can only be given relative to the motion of the moon in case of lunar laser ranging.

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APPENDIX A

Generalized Inverse Solutions and Estimability

The two step procedure to solve for polar motion and earth rotation described in the previous sections can also be formulated as a one step procedure. Such a re-formulation cannot effect the outcome of the adjustment, but it helps in demonstrating commonly used terminology.

Consider the equation (2.2-30) and the following equation which gives the general model for range differences as

$$BV + (A_1 \ A_2 \ A_3) \begin{bmatrix} X_1 \\ Y_2 \\ Y_3 \end{bmatrix} + W = 0$$

and which includes all the parameters explicitly. This adjustment model, in which each equation contains two range observations together with the parameters, had to be used because of the finite velocity of the light. For the purpose of this appendix, we can limit ourselves to the simplified model expressed by equations (3.1-6) to (3.1-8) where the earth rotation during the travel time of the pulse is neglected. These simplifications led to the model with observation equations

$$V = (\widetilde{A}_1 \ \widetilde{A}_2 \ \widetilde{A}_3) \begin{bmatrix} X_1 \\ Y_2 \\ Y_3 \end{bmatrix} + L \qquad (A.1)$$

where

$$\widetilde{A}_1 = (\widetilde{A}_\Delta, \ \widetilde{A}_\delta)$$

$$\widetilde{A}_2 = (\widetilde{A}_x, \ \widetilde{A}_y, \ \widetilde{A}_{\bar{\alpha}})$$

$$\widetilde{A}_3 = (\widetilde{A}_{U_{j-1}}, \widetilde{A}_{V_{j-1}}, \widetilde{A}_{W_{j-1}}, \widetilde{A}_{U_{j+1}}, \widetilde{A}_{V_{j+1}}, \widetilde{A}_{W_{j+1}})$$

V are the residuals and L are the observations.

The equation (A.1) is written as

$$V = AY + L \tag{A.2}$$

with

$$A = (\widetilde{A}_1 \; \widetilde{A}_2 \; \widetilde{A}_3). \tag{A.3}$$

The design matrix A has a rank deficiency of three. The weight matrix of the observations L is

$$P = \frac{1}{\sigma_0^2} I \tag{A.4}$$

The most general solution to this over determined system is the unique minimum norm least-squares solution [Rao and Mitra, 1971, p. 51] which satisfied the conditions

$$V^T PV = \text{minimum} \tag{A.5}$$

and

$$Y^T QY = \text{minimum} \tag{A.6}$$

The solution is

$$Y = A^+_{PQ} L. \tag{A.7}$$

A^+_{PQ} is called the minimum Q-norm P-least squares inverse. It fulfills the following four conditions [Rao and Mitra, 1971, p. 52] if P and Q are positive definite (p.d.) matrices:

$$\begin{aligned} AA^+_{PQ}A &= A \\ A^+_{PQ}AA^+_{PQ} &= A^+_{PQ} \\ (AA^+_{PQ})^T P &= PAA^+_{PQ} \\ (A^+_{PQ}A)^T Q &= QA^+_{PQ}A \end{aligned} \tag{A.8}$$

If Q is positive semi-definite (p.s.d.), the first two conditions in (A.8) are replaced by

$$
\begin{aligned}
P A A^{+}_{PQ} A &= P A \\
Q A^{+}_{PQ} A A^{+}_{PQ} &= Q A^{+}_{PQ}
\end{aligned}
\tag{A.9}
$$

The minimum norm least squares inverse A^{+}_{PQ} is given by the G-matrix of equation (2.3-5) if we replace the matrix M^{-1} by P of (A.4) in the expressions (2.2-27), (2.3-3) and (2.3-4). Denoting the thus obtained matrix by G' gives

$$
A^{+}_{PQ} = \begin{bmatrix} G_1' \\ G_2' \\ G_3' \end{bmatrix}
\tag{A.10}
$$

with

$$
Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma^{-1}_{\hat{X}_3} \end{bmatrix}
\tag{A.11}
$$

That the generalized inverse indeed fulfills all conditions in (A.8) and (A.9) can be verified by straightforward matrix multiplication. It is, therefore, formally established that the two step solution is identical to a minimum norm least squares solution.

In the main body of this study, we called the parameters $[X_1, X_3]$, which are the ephemeris corrections and the coordinates in the (U'')-system, estimable parameters. This was done in order to underline that the corresponding design matrix was of full rank. According to Rao [1965, p. 224], all linear parametric functions are estimable, if and only if the rank of the design matrix is full, i.e., if the rank is equal to the number of parameters to be solved. It is, therefore, always possible to find estimable parameters simply by inspecting the coefficients of the design matrix for linear dependencies and combining the corresponding parameters to form new but estimable parameter combinations. If one does not change the design matrix, i.e., leaves the linear dependent columns included, there is the following condition for unbiased estimation [Rao and Mitra, 1971, p. 139]: A parametric function $p^T Y$ is unbiasedly estimable by a linear function of L (the observations) if and only if

$$p^T A^- A = p^T \qquad (A.12)$$

or

$$p^T (A^T A)^- A^T A = p^T \qquad (A.13)$$

where A^- is any generalized inverse which fulfills

$$A A^- A = A \qquad (A.14)$$

From (A.12) and (A.14) it is clear that for

$$p^T = A^- A$$

the linear parametric function $p^T Y$ is unbiasedly estimable. It was shown that A^+_{PQ} fulfills the condition (A.14) which is a special case of the first condition in (A.8). With equations (A.3) and (A.10), one obtains the estimable parametric function as

$$p^T Y = A^+_{PQ} A Y = \begin{bmatrix} I & O & O \\ O & I & (F \Sigma^{-1}_{\hat{X}_3} F^T)^{-1} F \Sigma^{-1}_{\hat{X}_3} \\ O & O & I - F^T (F \Sigma^{-1}_{\hat{X}_3} F^T)^{-1} F \Sigma^{-1}_{\hat{X}_3} \end{bmatrix} \begin{bmatrix} X_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad (A.15)$$

Using the expression (2.3-2) for $[Y_3]$, i.e.,

$$Y_3 = -F^T Y_2 + X_3$$

the equations (A.15) can be rewritten as

$$p^T Y = \begin{bmatrix} I & O \\ O & (F \Sigma^{-1}_{\hat{X}_3} F^T)^{-1} F \Sigma^{-1}_{\hat{X}_3} \\ O & I - F^T (F \Sigma^{-1}_{\hat{X}_3} F^T)^{-1} F \Sigma^{-1}_{\hat{X}_3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \tag{A.16}$$

These equations express the parametric function $p^T Y$ in terms of the estimable parameters $[X_1, X_3]$. Comparing (A.16) with the least squares estimates in equation (2.3-5), the parametric function becomes

$$p^T Y = \begin{bmatrix} \hat{X}_1 \\ \hat{Y}_2 \\ \hat{Y}_3 \end{bmatrix}$$

It is recognized that both procedures, i.e., finding the estimable linear parameter combinations and performing a second adjustment, or using the formalism of generalized inverses, leads to the same result.

Finally, the present solution is compared with what is sometimes referred to as inner constraint or "pseudo inverse solution." At the outset, it is underlined that so far only the subset $[Y_3]$ takes part in the minimization of the second adjustment. The parameters $[X_1]$, i.e., the lunar declination and the geocentric reflector distance, are entirely independent of the definition of the coordinate system (U). The declination refers to the (U″)-system. Therefore the parameters $[X_1]$ are not a subject of any constraint whatsoever in determining polar motion.

Pope [1971] discusses the use of the Null space in solving singular geodetic systems. It is understood that polar motion is assumed to be known in that context. The singularity results from a lack of definition in shift, possibly in scale, and in a <u>single</u> rotation. The singularities are eliminated by incorporating a similarity transformation on the approximate station coordinates in the form of a constraint. He points out that any basis E of the design matrix is suitable in order to obtain an inner constraint solution. Adding the constraint

$$E^T Y = O \tag{A.17}$$

where E fulfills the condition

$$A E = O \tag{A.18}$$

to the normal equations gives an augmented non-singular normal matrix whose inverse is

$$\begin{bmatrix} N & E \\ E^T & O \end{bmatrix}^{-1} = \begin{bmatrix} N^+ & E(E^T E)^{-1} \\ (E^T E)^{-1} E^T & O \end{bmatrix} \tag{A.19}$$

N^+ is the pseudo inverse. It fulfills the conditions (A.8) with $P = I$ and $Q = I$. The parameters are

$$Y = N^+ A^T P L = (A^T P A)^+ A^T P L$$
$$= A^+_{PI} L$$

The latter equality is readily proven by use of the properties (A.8). Since the norm matrix, $Q = I$, is an identity matrix, <u>all</u> parameters participate in the minimization, i.e.,

$$Y^T Y = X_1^T X_1 + Y_2^T Y_2 + Y_3^T Y_3 = \text{minimum}$$

The matrix E is readily derived from our previous work. This is most easily seen by looking at the relations involved in the rank factorization theorem which was used repeatedly in Section 2. Graybill [1961, section 11.2.3], in what he refers to as re-parametrization, gives the following relations. Consider the model of observation equations as in (A.2)

$$V = A Y + L$$

with an m $\times$ m normal matrix

$$_mN_m = (A^T A)$$

of less than full rank

$$R(N) = m - s = u$$

The weight matrix of observations is assumed to be equal to the identity matrix for the present purpose, which can always be accomplished by transformation. According to well-known theorem in linear algebra, there exists non-singular matrix

$$_mM_m = ({}_mS_u \ {}_mE_s)$$

such that

$$M^T (A^T A) M = \begin{bmatrix} {}_uS^T (A^T A) S_u & O \\ O & O \end{bmatrix} \tag{A.20}$$

where the non-zero submatrix in (A . 20) is of size and rank u. The relation (A.20) implies

$$E^T (A^T A) E = O$$

which in turn implies

$$A E = O \tag{A.21}$$

Equation (A.20) also implies that $R(A S) = u$, i.e., the product is of full rank. E spans the Null space of the design matrix. The original observation equation (A.2) can be rewritten as

$$V = A Y + L = A M M^{-1} Y + L$$

Partitioning the inverse of M by

$$M^{-1} = \begin{bmatrix} H \\ \overline{H} \end{bmatrix}$$

leads to

$$V = ASHY + AE\bar{H}Y + L$$

$$= ASHY + L \qquad \text{(A.22)}$$

The last equality follows from (A-21). Denoting the product of A and S by D,

$$D = AS \qquad \text{(A.23)}$$

one obtains the rank factorization theorem by comparing (A.22) and (A.2) as

$$A = DH \qquad \text{(A.24)}$$

where D has full rank. The estimable parameters result from the non-estimable parameters by the transformation

$$X = HY \qquad \text{(A.25)}$$

According to (A.22), the matrix E is now readily obtained. The design matrix (D) of the estimable parameters [X] was given in Section 2.2.2 by

$$D = A'S = (A_1\, A_2\, A_3)\; S = (A_1 \;\; A_3) \qquad \text{(A.26)}$$

Equation (A.26) determines the matrix S as

$$S = \begin{bmatrix} I & O \\ O & O \\ O & I \end{bmatrix}$$

Since

$$MM^{-1} = SH + E\bar{H} = I$$

substitution of (2.2-31) for the matrix H gives

$$E\bar{H} = I - \begin{bmatrix} I & O & O \\ O & O & O \\ O & F^{T} & I \end{bmatrix} = \begin{bmatrix} O & O & O \\ O & I & O \\ O & -F^{T} & O \end{bmatrix}$$

which leads to the following identities:

$$\dot{E} \equiv \begin{bmatrix} O \\ I \\ -F^T \end{bmatrix} \tag{A.27}$$

and

$$\bar{H} \equiv (O \quad I \quad O)$$

The matrix E, which spans the basis of the Null space of the design matrix, can be easily found directly by solving the equation (A.21). The identity (A.27) can be verified using the coefficients (3.1-6) to (3.1-8).

It is concluded that the procedure of finding the basis of the design matrix and applying it straightforwardly as a constraint $\dot{E}Y = 0$ leads to an equally weighted minimization of the squares of all parameters. This is an undesirable procedure for the case of range differences to the moon. First, the $[X_1]$ parameters (geocentric reflector distance and declination) should not be included in the minimization. Second, the parameters $[Y_3]$, i.e., the station coordinate differences and sums, should not be minimized based on equal weights, because some of those parameters are weakly determined. Third, the inclusion of the orientation parameters, in particular polar motion, in the minimization adds a time varying component to the station coordinates. The resultant station coordinates are neither crustal fixed (even if no crustal motions occur) nor are they components in the celestial system. Since the primary concern of this study is not only the determination of chords and angles, but also of finding unique orientation parameters, the pseudo inverse solution is not pursued any further.

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APPENDIX B

Ephemeris Errors Vs. Corrections in Nutation

Polar motion modelling refers to the celestial pole (C). It was modelled as a constant per day, i.e., the mean of the progressive Chandler motion per day. The celestial pole, per definition, has no periodic diurnal motion relative to the crust. The adjusted parameters in declination and right ascension (the latter is linearly dependent on correction to time) will give the corrections to the lunar ephemeris in the celestial system (X). These corrections contain two types of errors which are not separable immediately. One error source is that the adopted set of nutations, whichever set one uses, is unlikely to describe the celestial pole (C) in space accurately, since any nutation set is derived from a hypothetical earth model. The adopted pole (adopted nutations) will have a nearly diurnal periodic body-fixed motion. The second type of error denotes actual ephemeris errors. Ephemeris errors can have several origins, such as errors due to truncation, errors in the constants, or even programming errors. The latter error source is particularly suitable to demonstrate the consequences of the non-separability of the two errors. Assume that during the programming of the ephemeris one term was forgotten. Not knowing this, one interprets the adjusted corrections in declination or right ascension as an error of the adopted set of nutations. If this set of nutations is compared with another experimental set as derived from, say, VLBI, there will be a discrepency just equal to the forgotten ephemeris term.

The polar motion which refers to the pole as defined by the adopted set of nutations is decomposed as

$$\begin{aligned} x &= x_c + x(t) \\ y &= y_c + y(t) \end{aligned} \tag{B.1}$$

x_c and y_c are constant for the time of one interval. They are the components of the celestial pole which have always been referred to in Section 2. The diurnal components are modelled with the same frequencies as have the nutations. It is assumed that the diurnal component can be totally accounted for by changing the coefficients of the adopted set of nutations.

Equation (2.3-13) in [Leick, 1978] gives for the diurnal polar motion terms the expression

$$\begin{aligned} x(t) + i\,y(t) &= -i \sum_j \overline{A}_j \; e^{-i(GMST + \Delta\alpha_j)} \\ &= \sum_j \left[-\overline{A}_j \sin(GMST + \Delta\alpha_j) - i\,\overline{A}_j \cos(GMST + \Delta\alpha_j)\right] \end{aligned}$$

where $\overline{A}_j$ is the coefficient, $\Delta\alpha_j$ the nutation argument for the nutation j, and GMST stands for Greenwich mean sidereal time. The y-axis is taken positive along $\Lambda = 90°$, whereas in the preceding part of this report y was taken positive along $\Lambda = 270°$. Changing the sign of y in the above equation, the residual diurnal polar motion is modelled as

$$x(t) + i\,y(t) = \sum_j \left[-d\overline{A}_j \sin(GMST + \Delta\alpha_j) + i\,d\overline{A}_j \cos(GMST + \Delta\alpha_j)\right) \tag{B.2}$$

In [Leick, 1978, Section 2.4], expressions for the change in declination and right ascension were given as a function of the nutation coefficients and frequencies. With α being the right ascension, the change in declination is

$$\delta_{adopted} - \delta_C = -\mathrm{Re}\ (i \sum_j \bar{A}_j\, e^{-i(\Delta\alpha_j + \alpha)})$$

$$= \sum_j -\bar{A}_j \sin(\Delta\alpha_j + \alpha)$$

Thus, the change in declination due to an error in the nutation coefficients is

$$d\,\delta(t) = \sum_j -d\bar{A}_j \sin(\Delta\alpha_j + \alpha) \qquad (B.3)$$

The change in right ascension is given by the same reference

$$\alpha_{adopted} - \alpha_C = \mathrm{Im}(i \sum_j \bar{A}_j\, e^{-i(\Delta\alpha_j + \alpha)})\ \tan\alpha = \sum_j \bar{A}_j \cos(\Delta\alpha + \alpha) \quad \tan\delta$$

Thus

$$d\alpha(t) = \sum_j d\bar{A}_j \cos(\Delta\alpha_j + \alpha) \tan\delta \qquad (B.4)$$

Note that the sign in (B.3) and (B.4) is always in the sense "adopted minus C."

The coefficients for polar motion, declination and right ascension are given by equations (3.1-3) and (3.1-4). After some lengthy algebraic manipulations, we find the following relations

$$\tilde{A}_x\ x(t) + \tilde{A}_y\ y(t) + \tilde{A}_\delta\ d\delta(t) + \tilde{A}_\alpha\ d\alpha(t) = 0, \qquad (B.5)$$

with

$$\tilde{A}_\alpha = \tilde{A}_{\Lambda + \Theta - \alpha}$$

The first two terms are identified as those terms in the observation equation which relate to diurnal polar motion. They are not included in the models discussed in the main body of this study. The linear relation (B.5) shows that errors in the nutations, i.e., the adopted set of nutations does not describe the direction of the celestial pole (C), will be absorbed in the daily adjusted declination and right ascension parameters. The nutation errors will be inseparable from any errors in lunar ephemeris. It is important to realize that the actual polar motion coordinates, that is, the constant part (neglecting the progressive

Chandler motion) is obtained uniquely. The adjustment station positions are independent of the relation in (B.5). If one were to compare the path of the celestial pole C with respect to the crust as obtained from LLR and, say, VLBI, there should be full agreement regardless of whether a "programming error" occurred during ephemeris implementation, or whether no such error occurred. The comparison can be made with the respective polar motion coordinates provided both parties select the same "zero point" for counting polar motion.

REFERENCES

Leick, A., 1978. "The Observability of the Celestial Pole and Its Nutations," Dept. of Geod. Sci. Report No.262, The Ohio State University, Columbus, Ohio.