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A MULTILEVEL SYSTEM OF ALGORITHMS FOR DETECTING AND ISOLATING SIGNALS IN A BACKGROUND OF NOISE

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A MULTILEVEL SYSTEM OF ALGORITHMS FOR DETECTING AND ISOLATING SIGNALS IN A BACKGROUND OF NOISE

L. S. Gurin, K. A. Tsoy

ABSTRACT. In a number of cases (in particular when conducting a controlled space experiment), the problem of detecting and isolating signals in a background of noise should be solved in several stages. To start with, the information is processed with the help of less laborious algorithms, and then on the basis of such processing, a part of the information is subjected to further processing with the help of more precise algorithms. In this paper, such a system of algorithms is studied, a comparative evaluation of a series of lower level algorithms is given, and the corresponding algorithms of higher level are characterized.

1. Statement of the Problem

An experiment in outer space, in particular a controlled experiment, is a complex system with many levels of hierarchies from many aspects. As is well-known, three types of hierarchies are distinguished [1]: strata (levels of description and abstraction), layers (levels of complexity of the given solutions), and echelons (organizational levels). Let us consider one stratum — the statistical processing of information. It consists of several layers and echelons; moreover, the levels are not independent with regard to these aspects. We are only interested in the layers.

*Numbers in the margin indicate pagination in the foreign text.

Assume that in the spacecraft, results from the measurement of a random process are obtained which are an additive mixture of signal and noise. These results are processed onboard by means of a specific algorithm, and then the processed materials are transmitted for subsequent processing on the ground by means of other algorithms. The purpose of the processing is to detect and isolate the signal with maximum accuracy and reliability. Here it is necessary to take into account the limitations on the high-speed response of the electronic computers onboard and on the ground and the carrying capacity of the telemetric channels. The development of a multilevel system of algorithms for the solution of this problem is required.

The establishment of such a system of algorithms is subdivided into several stages.

1) Various models of the signals and noise are considered. To work out the methodology, the simplest model is chosen.

2) Various algorithms for working out the lower level are considered. Their characteristics (reliability of detection, the time for realization on an electronic computer) in the assumed models of the signals and the noise are studied. It turns out that even in the simplest cases such an investigation can be only partially completed analytically. Therefore, the universal method of statistical modeling is applied (even though it is very laborious). /4

The characteristics obtained depend on certain selected parameters of the tuning (threshold values). These parameters are selected during the solution of the system problem, i.e., at one of the later stages; therefore, at the first stages it is desirable for the characteristics to achieve independence from the parameters. This increases the volume of the calculations investigated, as well as the volume of fixed results.

3) All this is repeated for the algorithms of higher level — only more precisely and more laboriously.

4) A concrete physical problem is considered. Based on the processing of the first results of the measurements, conclusions are drawn concerning the adequacy of the models of the signal and the noise which were considered. If necessary, stages 1) - 3) are repeated for models which are in better agreement with the physical problem.

5) On the basis of all of the preceding, a concrete system of algorithms for the given physical problem is developed, including a determination of the tuning parameters, etc.

In the present paper, we discuss the first results obtained during the completion of stages 1), 2), and in part 3).

2. Models of the Signal and the Noise

In connection with the discrete character of the measurements and the finite length of the realization, the following form of representation is convenient. Given the values:

$$y_i = f(x_i) + \xi_i = s_i + \xi_i \quad (1)$$

on the segment $i = 1, \ldots, N$, where $s_i = f(x_i)$ is the signal, and ξ_i is the noise. For various models of the signal and the noise, it is convenient to introduce provisional notation. The model $\underline{A_1}.\, f(x)$ is a triangular splash, i.e.,

$$\begin{aligned} s_i &= 0 \text{ when } |i - i_0| > k; \\ s_i &= C\left(1 - \frac{|i - i_0|}{k}\right) \text{ when } |i - i_0| < k, \end{aligned} \quad (2)$$

$\xi_i = \mathcal{N}(a_i, \sigma_i^2)$ are independent, normally-distributed random quantities, where:

$$\begin{aligned} a_i &= a_0 x^2 + a_1 x + a_2; \\ \sigma_i^2 &= b_0 x^2 + b_1 x + b_2. \end{aligned} \quad (3)$$

Let us consider the following particular cases of the model A_1.

$A_{11}-a_i=0$; $\sigma_i^2=\sigma^2$; k are known, i_o is unknown but determinate.

$A_{12}-a_i=a_0x^2+a_1x+a_2$, the rest similarly. Here a_0, a_1, a_2 are determinate but unknown.

$A_{13}-a_i=0$; $\sigma_i^2=\sigma^2$; k is known, C and i are unknown, but determinate.

A_{14} — the same as in A_{13}; only k is also unknown. A'_{11}, A'_{13}, A'_{14} — the same as A_{11}, A_{13}, A_{14}, respectively, but the unknowns (i_o; i_o and C; k, i_o and C) have given laws of distribution and unknown parameters.

The set of particular cases of the model A_1 may be increased. The general characteristic feature is the uncorrelatedness of the noise. For the present we shall consider σ^2 to be known

The model A_2. In contrast to the model A_1, the noise ξ_i is a normally distributed random vector with mathematical expectation $A=(a_1,\dots,a_N)$ and covariant matrix:

$$B=\|\sigma_{ij}^2\|_{N,N} \quad . \tag{4}$$

Here, a_i can be determined in accordance with (3).

In the particular case (stationary random process), we obtain the model A_2:

$$a_i=0; \quad \sigma_{ij}^2=\sigma^2(j-i) \quad . \tag{5}$$

For the present we shall assume that the function (5) is known. The presence of a prime or a second subscript in the designation of the model A_2 (e.g., A'_{23}) has the same meaning as in the case of the model A_1.

3. Algorithms of the Lower Level

A) The algorithm of moving summation. Let us introduce here the average:

$$\bar{y}_i = \frac{1}{2k+1} \sum_{j=i-k}^{i+k} y_j \quad . \tag{6}$$

for the models $A'_{11}, A_{12}, A_{13}, A'_{11}, \ldots$ as k assumes its known value. The question as to the models A_{14}, A_{14}, ... we shall not consider at present for the given algorithm. It is clear that $\bar{y}_i$ is determined for $k+1 \leqslant i \leqslant N-k$.

To determine the acceptance of the solution concerning the presence of the signal or its absence, one of the following decision rules can be applied.

Decision rule A_1. Let

$$\bar{y}_{max} = \max_{k+1 \leqslant i \leqslant N-k} \bar{y}_i \quad . \tag{7}$$

Let us introduce two decision functions (for the models A_1 and $\bar{A}_2$):

$$\begin{aligned} F_1 &= \bar{y}_{max} - \frac{c_1 \sigma}{\sqrt{2k+1}} \quad ; \\ F_2 &= \bar{y}_{max} - \frac{c_2 \sigma}{\sqrt{2k+1}} \quad , \end{aligned} \tag{8}$$

where $c_2 > c_1$, i.e., $F_1 > F_2$, and the cases $F_2 \geqslant 0, F_1 \leqslant 0$ are impossible.

The decision rule has the form:

$$\begin{array}{ll} F_1 \leqslant 0 & \text{— there is no signal} \\ F_2 \geqslant 0 & \text{— there is a signal} \\ F_1 > 0, F_2 < 0 & \text{— the situation is indeterminate} \end{array} \tag{9}$$

It is possible to write analytic expressions for the characteristic of this algorithm. Henceforth we shall consider groups of $N_1 < N$ points, i.e., in formula (7) the limits of variation of i will be still narrower. For the probability of a false alarm, we have: /7

$$\alpha_1 = 1 - \frac{1}{(\sqrt{2\pi}\sigma)^{N_1+2k}} \int_\Omega exp\left(-\sum_{j=1}^{N_1+2k} \frac{t_j^2}{2\sigma^2}\right) dt_1, \dots dt_{N_1+2k}, \quad (10)$$

where the region Ω in the space $t_1, \dots, t_{N_1+2k}$ is defined by the system of inequalities:

$$\sum_{j=\mu}^{\mu+2k} t_j \leq c_2 \sqrt{2k+1}\ \sigma\ ; \quad \mu = 1, 2, \dots, N_1 . \quad (11)$$

The formulas for the probability of the admission of the signal and for the situation of indeterminacy are obtained by algorithms. Since the computation of integrals of high multiplicity reduces, one way or another, to statistical modeling, it is more convenient to apply statistical modeling to the direct calculation of the characteristics of interest to us. The results of such calculations will be discussed below.

Decision rule A_2. In place of (8), we introduce two functions of the number of the point j:

$$\left. \begin{aligned} F_1(j) &= \bar{y}_j - \frac{c_1 \sigma}{\sqrt{2k+1}} ; \\ F_2(j) &= \bar{y}_j - \frac{c_2 \sigma}{\sqrt{2k+1}} . \end{aligned} \right\} \quad (12)$$

Let us designate two further whole numbers n_1 and n_2. The decision rule has the following form:

- if the number ν_1 of points of the group for which $F_1(j) \geqslant 0$ is less than or equal to n_1, there is no signal;

- if the number ν_2 of points of the group for which is greater than or equal to n_2, there is a signal;

- in all the remaining cases, the situation is indeterminate.

For this rule it is difficult to obtain analytically the probability of a false alarm or the admission of a signal. However, we shall present some estimate, or more precisely, the mathematical expectation of the numbers ν_1 and ν_2 for various groups of points (containing or not containing the signal). Let μ denote a number which equals 1 or 2. Then in the case of the model A_{11} where there is no signal, the probability of the event $F_\mu(j) \geq 0$ is:

$$P_\mu(j) = \frac{1}{\sqrt{2\pi}} \int_{C_\mu}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(C_\mu) . \tag{13}$$

In connection with the fact that the theorem about the mathematical expectation of the sum is also valid for the independent random variable, we have:

$$\mathcal{M} \nu_\mu = \mathcal{N}_1 \left(1 - \Phi(C_\mu)\right). \tag{14}$$

If there is a signal and it is (has a maximum) at the point numbered i_o, then we have for $\bar{y}$ $(j \geq 0)$:

$$\bar{y}_{i_o \pm j} = \begin{cases} \frac{kC}{2k+1} - \frac{j(j-1)}{2} \cdot \frac{C}{k(2k+1)}; & j = 0, 1, \ldots k; \\ \frac{(2k+1-j)(2k+2-j)}{2} \cdot \frac{C}{k(2k+1)}; & j = k+1, \ldots, 2k \\ 0 & ; \; j > 2k. \end{cases} \tag{15}$$

Next,

$$P_\mu(i_o \pm j) = 1 - \Phi(C_\mu - \bar{y}_{i_o \pm j}) . \tag{16}$$

Let Ω_1 denote the set of value $\pm j$ for which the point numbered $i_0 \pm j$ lies in the group of N_1 points under consideration, and let ω_1 be the number of points of Ω_1. Then for the model A_{11} in case there is a signal we have:

$$M\nu_\mu = (N_1 - \omega_1)(1 - \Phi(C_\mu)) + \sum_{\pm j \in \Omega} [1 - \Phi(C_\mu - \bar{y}_{i_0 \pm j})] \quad . \tag{17}$$

Decision rule A_3. This rule differs from rule A_2 in that the numbers χ_1 and χ_2 replace ν_1 and ν_2, respectively, where χ_μ is the maximum length of the series of events $F_\mu(j) \geqslant 0$ for the group of points N_1. It is natural that for rule A_3 the values c_μ and n_μ must be selected differently than for rule A_2. /9

For decision rule A_3 also, it is possible to obtain analytic expressions for some estimate of the characteristics (for the characteristics themselves, this is difficult). We shall present some of them.

In order to estimate the mathematical expectation of a number of series (in the present case we also include a series consisting of a single point), note that each series begins with a jump in the corresponding level, i.e., for example, $F_\mu(j-1) < 0;\ F_\mu(j) \geqslant 0$.

Let us find the probability of a jump at the given point j in case there is no signal. When the value C_μ is given, we have for the desired probability:

$$P_j = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \Phi\left[\sqrt{2k}\left(z - C_\mu \sqrt{\frac{2k+1}{2k}}\right)\right] \left\{1 - \Phi\left[\sqrt{2k}\left(z - C_\mu \sqrt{\frac{2k+1}{2k}}\right)\right]\right\} dz, \tag{18}$$

where, as usual,

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{z^2}{2}} dz \quad . \tag{19}$$

For the mathematical expectation of the number of series in the group of N_1 points, we obtain the value $N_1 P_j$. Since the total number of points above the given level in the absence of a signal is furnished by the formula (14), then according to a familiar theorem of the theory of restoration, for the provisional mathematical expectation m of the length of a series (provided the series occurs) we have: /10

$$m = \frac{1 - \Phi(C_\mu)}{P_j} . \tag{20}$$

Thus, when $C_\mu = 2.0$, calculation gives $P_j = 0.0208$, i.e., $\bar{m} = 1.095$. Hence the provisional probability of obtaining a series of length greater than unity is very small. This is confirmed by experimental results. Note, however, that formula (20) is imprecise. The reason for the imprecision of formula (20) lies in the fact that the random variables which appear in the application of the theorem from the theory of restoration do not satisfy the condition of independence.

The probability of obtaining a series of length χ, beginning at the point numbered j, is given by the following formula:

$$P_j^{\chi} = \frac{1}{(\sqrt{2\pi}\,\sigma)^{2k+\chi+2}} \int\limits_{\Omega_\chi} \exp\Big(-\frac{1}{2\sigma^2} \sum_{\nu=1}^{2k+\chi+2} t_\nu^2\Big) dt_1 \ldots dt_{2k+\chi+2} \tag{21}$$

where the region Ω_χ is defined by the inequalities:

$$\left.\begin{aligned} &\sum_{\nu=\nu_0}^{\nu_0+2k} t_\nu \geq C_\mu \sigma \sqrt{2k+1} ; \quad \nu_0 = 2 \ldots \chi+1 ; \\ &\sum_{\nu=\nu_0}^{\nu_0+2k} t_\nu < C_\mu \sigma \sqrt{2k+1} ; \quad \nu_0 = 1 \ \text{и} \ \nu_0 = \chi+2 . \end{aligned}\right\} \tag{22}$$

As regards formula (21), we may repeat the same reasoning which was stated above regarding formula (10).

Using the decision rules introduced, it is possible to draw the following conclusion relative to a data array N: if there is a signal and a situation of indeterminacy arises in, say, even one of the groups of N_1 — information is transmitted to a higher level. So far we have considered the model A_{11}. If it is necessary to evaluate the quantity C (in the model A_{13}), then, depending on which rule is applied, we proceed in the appropriate manner. For example, for rule A_1 we may assume that:

$$\hat{C} = \frac{2k+1}{k} \bar{y}_{max} \quad (23)$$

In the case of the model A_{12}, before applying the algorithm /11
described, it is necessary to eliminate the trend (3) by the usual methods, after which we obtain the model without trend, e.g., A_{11}. Let us call such a procedure the <u>algorithm of moving summation for a model with a trend</u>. By contrast, we shall call the following algorithm <u>the more precise algorithm of moving summation for a model with a trend.</u>

a) Let us apply the usual algorithm of moving summation for a model with a trend: i.e., as a preliminary measure we exclude the trend.

b) If we have obtained the solution "no signal" or "indeterminate", the calculation ends.

c) If we have obtained the solution "there is a signal" — we estimate the parameters of the signal and subtract the signal from the initial process with the trend and repeat the calculation. The new solution is definitive.

B) The algorithm "forward-backward". This algorithm was worked out and used to solve another problem concerning the subject, and it is discussed in [2]. It consists of the following. For each value of i from $k+1 \leq i \leq N-k$, let us draw two straight lines of regression, forward and backward, through the $k+1$ points:

$$\left.\begin{aligned} y_j^{(1)} &= a_i^{(1)} + b_i^{(1)}(j-i); \quad j=i,\dots i+k; \\ y_j^{(2)} &= a_i^{(2)} - b_i^{(2)}(i-j); \quad j=i,\dots i+k. \end{aligned}\right\} \tag{24}$$

As a statistic, on the basis of which the decision rule is constructed, let us take:

$$\Delta(i) = b_i^{(2)} - b_i^{(1)} \; . \tag{25}$$

For the dispersion σ_Δ^2 of the random quantity $\Delta(i)$, we have [cf. [2]]: /12

$$\sigma_\Delta^2 = \frac{24\,\sigma^2}{k(k+1)(k+2)} \; . \tag{26}$$

We shall construct the decision rule by analogy with the algorithm A), only we shall replace $\bar{y}_i$ by $\Delta(i)$ and $\frac{\sigma^2}{2k+1}$ by the expression (26).

<u>Decision rule B_1</u>. Let:

$$\Delta_{max} = \max_{k+1 \leq i \leq N-k} \Delta(i) \; . \tag{27}$$

Let us introduce two decision functions $(\mu = 1,2)$:

$$F_\mu = \Delta_{max} - C_\mu \sigma \sqrt{\frac{24}{k(k+1)(k+2)}} \tag{28}$$

and apply the decision rule (9).

<u>Decision rule B_2</u>. The same as A_2, except that in place of (12), we write:

$$F_\mu(j) = \Delta(j) - C_\mu \sigma \sqrt{\frac{24}{k(k+1)(k+2)}} \; . \tag{29}$$

<u>Decision rule B_3</u>. Analogous to A_3, except that the function (29) is used.

The advantage of the algorithm B is that in the case of the model A_{12}, the preliminary elimination of a trend is not required.

C) An optimal filter. It is easy to see that the algorithms discussed above are linear filters. If k is known exactly, then it is possible to use an optimal filter, giving the maximum signal to noise ratio at the output. It is easy to show that it has the form:

$$\bar{y}_i = \sum_{j=i-k+1}^{i+k-1} c_j y_j \quad , \tag{30}$$

where

$$c_j = \frac{1}{k}\left(1 - \frac{|i-j|}{k}\right) . \tag{31}$$

Here in formulas (8) and (12) we must write $C_\mu \sigma \sqrt{\frac{2k^2+1}{3k^3}}$ instead of $\frac{C_\mu \sigma}{\sqrt{2k+1}}$.

Formula (15) is replaced by the following:

$$\bar{y}_{i\pm j} = \begin{cases} \frac{C(2k^2+1)}{3k^2} + \frac{C_j}{6k^3}\left[6k^2 - 4k - 2kj + 3(j^2-1)\right], & j=0,1\ldots,k-1 \\ \frac{C}{6k^3}(2k-j)(2k-j-1)(2k-j+1) ; & j=k,\ldots,2k-2 \\ 0 ; & j > 2k-2 . \end{cases} \tag{32}$$

For the algorithm A) the signal to noise ratio is $\frac{C}{\sigma}\frac{k}{\sqrt{2k+1}}$, and for /13
the algorithm C) it is $\frac{C}{\sigma}\sqrt{\frac{2k^2+1}{3k}}$, i.e., we obtain an increase of $\sqrt{\frac{(2k^2+1)(2k+1)}{3k^3}}$ times, which equals 1.21 times when k = 6. This insignificant increase does not compensate for the substantial increase in the labor of calculating with C) in comparison to A). Moreover, for C) it is mandatory to know the parameter k. Therefore we did not consider the algorithm C) in more detail.

Note that for the algorithm B), the signal to noise ratio equals:

$$\frac{C}{\sigma}\sqrt{\frac{(k+1)(2k+1)}{6k}}.$$

When k = 6, this is 4.5% less than for the algorithm A). However, we consider this algorithm, since when k increases, it improves in comparison to the algorithm A) (when k = 2, it is already 1.25 times better). Moreover, as was pointed out above, it does not require the exclusion of a trend. The final concludions will be presented below. Finally, note that some increase in the signal to noise ratio for the smoothing algorithm can be obtained without an increase in the labor involved, due to the proper choice of the width of the smoothing interval. In formula (6) we shall take k_1 in place of k (assuming it is known); it also replaces k in formulas (8) - (12), etc. In formula (15), when j = 0, we obtain:

$$\bar{y}_{i_0}=\frac{C}{2k_1+1}\left[k_1+(k_1+1)\left(1-\frac{k_1}{k}\right)\right]. \tag{33}$$

Here the signal to noise ratio equals: /14

$$\frac{C}{\sigma}\,\frac{k_1+(k_1+1)\left(1-\frac{k_1}{k}\right)}{\sqrt{2k_1+1}} \tag{34}$$

Let us find the maximum of expression (34), assuming that k_1 is continuous and ranging over the interval (1, k). Thus, we obtain:

$$k_{1.opt}=\frac{2k-5+\sqrt{(2k+4)^2-3}}{6}\approx\frac{2}{3}k. \tag{35}$$

In our example when k = 6, we must take k_1 = 4. This improves the signal to noise indicator by 10%.

4. Upper Level Algorithms

Various realizations of methods for detecting and separating signals from the noise (with precision with respect to optimization, the number of iterations, etc.) based on the study of probability functions, which are familiar from the literature (cf., 3.g., [3, 4]) may be taken as upper level algorithms. For the model A_1, we have the following expressions for the probability functions ℓ_0 in the case of a null hypothesis (there is no signal), and ℓ_1 in the case a signal is present:

$$\ell_0 = \frac{1}{(2\pi)^{N/2}\sigma_1 \ldots \sigma_N} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} \frac{(y_i - a_i)^2}{\sigma_i^2}\right\}, \quad (36)$$

$$\ell_1 = \frac{1}{(2\pi)^{N/2}\sigma_1 \ldots \sigma_N} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} \frac{(y_i - S_i - a_i)^2}{\sigma_i^2}\right\}. \quad (37)$$

Note that these formulas are not applicable to the model A_1'. In the case A_1' and equally for the case A_2, it is possible to obtain other expressions for ℓ_0 and ℓ_1, in which event the following discussion is to be changed somewhat.

First of all, by whatever means the problem of detection is solved, it is necessary to estimate the parameters, i.e., in essence, to solve the problem of isolating the signal. This has led to a large set of algorithms. Let us consider the method of maximum plausibility only, which reduces for constant σ to the method of least squares. The maximum plausibility ℓ_1 corresponds to the maximum of the quantity: /15

$$S = \sum_{i=1}^{N} (y_i - S_i - a_i)^2 . \quad (38)$$

Depending on the model assumed, the quantity S is a function of the following parameters:

- in the model A_{11} $S = S(i_0)$;
- in the model A_{12} $S = S(i_0, a_0, a_1, a_2)$;
- in the model A_{13} $S = S(C, i_0)$;
- in the model A_{14} $S = S(k, C, i_0)$:

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and also possible is the variant $A_{12.4}$, where $S = S(k, C, i_0, a_0, a_1, a_2)$.

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To minimize S it is possible to apply arbitrary algorithms. The choice of the best of them depends on the number of unknown parameters: for one parameter (case A_{11}) the "golden section" method may be used; for two parameters, the Gauss-Seidel method may be used, etc. We shall make only two comments.

In the first place, the problem of minimizing S may turn out to be multi-extremal, in which case one of the methods of global optimization which is described in [5] may be employed.

Secondly, in connection with the fact that estimating the parameters is subordinate to the general problem of detection, the requisite precision of the estimation algorithm depends on the results of comparing the hypotheses. Here we have our own hierarchy of algorithms: after having estimated the unknown parameters in the first approximation, we solve the detection problem with the help of two /16
decision functions F_0 and F_1 as this was done at the lower level, and only in the event of indeterminacy do we proceed to a more precise estimate of the parameters. When performing the last-mentioned estimate it was possible to have used only one decision rule (excluding the possibility of indeterminacy), provided we did not have in view a further type of hierarchy (e.g., with respect to the number of realizations to be used).

Let us proceed to the solution of the detection problem, assuming that the parameters are known. As is well known, in the case of a single decision function, this problem is solved as follows: let us find the plausibility ratio:

$$\ell = \frac{\ell_1}{\ell_c} \tag{39}$$

and let us assign a certain threshold c. A solution concerning the presence of a signal is assumed if:

$$\ell \geqslant c\ . \tag{40}$$

In the contrary case, we shall assume that there is no signal. As regards the choice of the threshold c, this depends on the solution of a system of problems as a whole: the criteria, the limitations, and the available information.

In the case of two decision functions (at the lower or intermediate stages of the hierarchy) let us assign two thresholds c_1 and c_2 $(c_2 > c_1)$, and let us define:

$$\left.\begin{aligned} F_0 &= \ell - c_1\ ; \\ F_1 &= \ell - c_2\ . \end{aligned}\right\} \tag{41}$$

Everything that follows corresponds to what was discussed above.

For systematic calculations, it is necessary to know the probability characteristics of the algorithms obtained. In the simplest cases they can be written out in explicit form in terms of the quantity ℓ. Taking into account, however, the fact that the true values of these characteristics still depend on the accuracy of the preliminary parameter estimates, we shall not write down these formulas. The method of statistical modeling is successful in the case of various models and concrete algorithms in obtaining the characteristics, taking into account all the decisive circumstances. /17

5. The Results of Statistical Modeling

Statistical modeling was carried out on the model A_{12} with the number of points N = 100. All together, there were 100 realizations with a constant value for σ^2 and a random trend (3), the coefficients a_0, a_1 and a_2 in the expression

$$a_i = a_0 i^2 + a_1 i + a_2$$

being chosen uniformly distributed on the intervals:

$$a_0 - (-0.0005;\ 0.0005); \quad a_1 - (-0.05;\ 0.05); \quad a_2 - (0;5).$$

The signal was formed in accordance with (2) for $i_0 = 50$, $k = 6$; the coefficient C assumed three values, depending on the version of the calculation: $C = 3\sigma$; $C = \sigma$ and $C = 0.33\sigma$.

Next, when using the moving summation algorithm, the trend was excluded in advance with the aid of the least squares method. The trend was not excluded when the "forward-backward" method was used.

Then portions of length $N_1 = 25$ were processed, using various decision rules. Here when there was a signal each of the versions represented 100 realizations, and when there was no signal — 200 /18
realizations (since in the general interval $N = 100$, a central portion with a signal, and two extremes without signal, were selected). On each portion in each realization, the values of $\bar{y}_{max}$ (corresponding to Δ_{max} for the "forward-backward" algorithm), the number of rejections ν_μ and the maximum length of the series χ_μ of the criteria $\bar{y}$ (or, correspondingly, Δ) for the levels C equal to the following set of values: $2\sigma'$; $2.5\sigma'$; $3.0\sigma'$; $3.5\sigma'$ and $4\sigma'$, were fixed, where by σ' is meant $\sigma_{\bar{y}}$ for moving summation, and σ_Δ for the "forward-backward" algorithm.

In Table 1 - 14 below are given the estimates of the probabilities:

$$P(\bar{y}_{max} \geq \ell_y);\ P(\Delta_{max} \geq \ell_\Delta);\ P(\nu_\mu \geq \ell_\nu);\ P(\chi_\mu \geq \ell_\chi)$$

for various versions. Versions which have no interest from the point of view of obtaining conclusions are not presented.

Since the smaller values of C_μ turned out to be more effective, the versions $C = 0.33\sigma$ and $C = 3\sigma$ were calculated for $\bar{y}$ when $C_\mu = \sigma'$ and $C_\mu = 1.5\sigma'$. The corresponding results are presented in Tables 15 - 18.

The following experiment was performed regarding the upper level algorithm. In the model with a trend which was previously described, the trend was eliminated. Next, by the method of maximum plausibility i_o and C were estimated and the detection problem was solved according to the criterion of the plausibility ratio (40). The distribution of the plausibility ratio ℓ for the cases when the signal was present or absent with respect to the 100 realizations (as well as for the signals 0.33 σ, σ and 3σ) was outstanding for the choice of the threshold C.

The results obtained are presented in Tables 19 - 20.

The distribution $|i_o - 50|$ (the true value of i_o is 50) is presented in Table 19 when there is a signal for the versions $C = \sigma$ /19
and $C = 3\sigma$.

In Table 20, the distribution of ℓ is given in the absence of a signal and for three versions when it is present.

Besides the results presented in the tables, the required computing time was estimated. An analysis of the results obtained and some preliminary conclusions are presented in section 6.

6. Analysis of the Results Obtained and Conclusions

Before analyzing the results obtained, it is necessary to comment on accuracy. As is well known, when estimating some probability p concerning an empirical frequency which was obtained as the result of N trials, 5% of the confidence interval equals

$$\pm 1{,}96\sqrt{\frac{p(1-p)}{N}}. \qquad (42)$$

In our case, when there is no signal, i.e., when N = 200, for probabilities of a false alarm which are nearly zero, the estimate turns out quite crude; thus, for example, for p = 0.10 we obtain an error of $\pm$ 0.025. However, account must be taken of the fact that when two estimates obtained by different algorithms are compared, the situation turns out more favorable because the trials, on the basis of which these estimates are obtained, are dependent, since all the algorithms are subjected to exactly the same realizations. The basis for this position is in [6] (and in earlier works of Yu. G. Polyak).

Further, if the results obtained by statistical modeling are compared with certain theoretical results presented in section 3, it must be kept in mind that the former are obtained on a model with a trend, and the latter — on a model without a trend. Therefore, the degree of their coincidence makes it possible to judge the quality and the influence of the trend exclusion.

Finally, it must not be forgotten that the results obtained pertain to a case of uncorrelated noise and a concrete value of the number k. Here we are limited by the preliminary analysis, making it possible to obtain some conclusions with regard to the investigation of a concrete system. /20

The difficulty of the preliminary analysis consists in the fact that the comparison must take a number of criteria into account: the time required to realize the algorithms on an electronic computer, the probability of a flase alarm α_1, and the probability of the admission of a signal α_2. However, in individual cases, the realization times for versions being compared are practically identical. In the first place, realization occurs during the comparison of various decision rules and various threshold values for one and the same lower level algorithm. Moreover, it occurs with sufficient accuracy in practice even for different algorithms of lower level, since the operation of trend exclusion turns out not to effect calculation time significantly. Therefore, the preliminary analysis of lower level algorithms can be performed without taking the time

criterion into account. As regards the criteria α_1 and α_2, for the preliminary analysis, one of them could be fixed and compared with the other. Unfortunately, this approach is impractical in view of the fact that usually the values of α_1 and α_2 do not agree for the versions to be compared (cf. the tables). This difficulty may be avoided as follows. Suppose for one of the versions we have the values α_1 and α_2, and for another, the values α'_1 and α'_2. Let us consider an hypothetical system such that for the first version m independent detection channels are used, and the signal is assumed to be detected if it is detected in at least one of the channels; for the second version, let the corresponding number of channels be /21
m'. Then the probability of a false alarm is $m\alpha_1$ for the first system, and $m'\alpha'_1$ for the second. Let us select m and m' from the condition:

$$m\alpha_1 = m'\alpha'_1 . \qquad (43)$$

Now it is possible to compare the systems with respect to the general probability of the admission of a signal: the first system is preferable if:

$$\alpha_2^{m} < \alpha_2'^{\,m'},$$

i.e.,

$$\alpha_2^{m} < \alpha_2'^{\,\frac{m\alpha_1}{\alpha'_1}}$$

or

$$\alpha_2^{\frac{1}{\alpha_1}} < \alpha_2'^{\,\frac{1}{\alpha'_1}} \qquad (44)$$

Thus, for example, for the decision rule A_2, let us compare the threshold levels $C_\mu = \sigma_{\bar{y}}$ (Table 15) and $C_\mu = 2\sigma_{\bar{y}}$ (Table 2) when a weak signal $C = 0.33\sigma$ is detected. In the first case (for $\ell_\nu = 7$) we obtain $\alpha'_1 = 0.08$, and in the second case (for $\ell_\nu = 1$)

$\alpha_1 = 0.05$. The corresponding values of α_2 are $1 = 0.26 = 0.74$ and $1 - 0.14 = 0.86$. Comparison according to formula (44) shows that the first version is to be preferred.

As the result of a similar analysis, the following conclusions are obtained.

1. The best of the lower level algorithms is the algorithm A) with decision rule A_2 and $C_\mu = \sigma_{\bar{y}}$.

Here the choice of the number ℓ_ν is determined by the solution of the network problem.

2. The previous conclusion may be changed in favor of algorithm C) in case: /22

a) the noise is correlated;
b) of smaller values of the number k;
c) a trend has been eliminated in advance.

In each of these cases additional investigation is required, analogous to that discussed above.

3. According to all the indicators, except for the calculation time, the upper level algorithm is better than the best of the lower level algorithms. In terms of calculation time, it is only half as good.

4. Except for the preliminary conclusions mentioned, when a physical problem is to be solved, the results presented in Tables 1 - 20 may be used.

TABLE 1. DISTRIBUTION /23

$P(\bar{y}_{max} > \ell_y)$ *

ℓ_y	No signal	$C = 0{,}33\sigma$	$C = \sigma$	$C = 3\sigma$
-0,2	I,00	I,00	I,00	I,00
-0,I	0,99	I,00	I,00	I,00
0,0	0,98	I,00	I,00	I,00
0,I	0,9I	0,96	I,00	I,00
0,2	0,7I	0,82	0,97	I,00
0,3	0,45	0,59	0,88	I,00
0,4	0,23	0,36	0,70	I,00
0,5	0,I0	0,I8	0,47	I,00
0,6	0,006	0,09	0,26	I,00
0,7	0,0	0,0	0,I2	0,99
0,8	-	-	0,05	0,9I
0,9	-	-	0,0	0,78
I,0	-	-	-	0,63
I,I	-	-	-	0,5I
I,2	-	-	-	0,24
I,3	-	-	-	0,I4
I,4	-	-	-	0,07
I,5	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 2. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 2\sigma_\xi$ *

l_ν	No signal	C = 0,33σ	C = σ	C = 3σ
0	I,00	I,00	I,00	I,00
I	0,05	0,I4	0,33	I,00
2	0,0	0,08	0,26	0,99
3	-	0,05	0,20	0,99
4	-	0,02	0,I4	0,99
5	-	0,0I	0,II	0,99
6	-	0,0I	0,08	0,99
7	-	0,0I	0,06	0,94
8	-	0,0I	0,05	0,92
9	-	0,0I	0,04	0,8I
I0	-	0,0I	0,02	0,68
II	-	0,0	0,0	0,4I
I2	-	-	-	0,24
I3	-	-	-	0,II
I4	-	-	-	0,0I
I5	-	-	-	0,0I
I6	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 3. DISTRIBUTION $P(\varkappa_\mu \geq \ell_\varkappa)$ FOR $C_\mu = 2\sigma_{\bar{y}}$ *

$\ell_\varkappa$	No signal	C = 0,33σ	C = σ	C = 3σ
0	I,00	I,00	I,00	I,00
I	0,05	0,I4	0,33	I,00
2	0,0	0,06	0,24	0,99
3	-	0,03	0,I8	0,99
4	-	0,02	0,I2	0,99
5	-	0,0I	0,09	0,99
6	-	0,0I	0,07	0,96
7	-	0,0I	0,06	0,92
8	-	0,0I	0,05	0,89
9	-	0,0I	0,02	0,77
I0	-	0,0	0,02	0,68
II	-	-	0,0	0,4I
I2	-	-	-	0,24
I3	-	-	-	0,II
I4	-	-	-	0,0I
I5	-	-	-	0,0I
I6	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 4. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 2{,}5\sigma_y$ * /26

l_ν	No signal	$C = 0{,}33\sigma$	$C = \sigma$	$C = 3\sigma$
0	I,00	I,00	I,00	I,00
I	0,0	0,0I	0,I2	I,00
2	-	0,0	0,09	0,98
3	-	-	0,06	0,96
4	-	-	0,04	0,93
5	-	-	0,03	0,87
6	-	-	0,02	0,82
7	-	-	0,0I	0,78
8	-	-	0,0I	0,66
9	-	-	0,0I	0,54
I0	-	-	0,0	0,36
II	-	-	-	0,I7
I2	-	-	-	0,07
I3	-	-	-	0,0I
I4	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 5. DISTRIBUTION $P(\psi_\mu \geq l_\psi)$ FOR $C_\mu = 2.5\,\sigma_\psi$ *

l_ψ	No signal	$C = 0,33\sigma$	$C = \sigma$	$C = 3\sigma$
0	I,00	I,00	I,00	I,00
I	0,0	0,0I	0,I2	0,99
2	-	0,0	0,09	0,97
3	-	-	0,05	0,93
4	-	-	0,02	0,88
5	-	-	0,02	0,8I
6	-	-	0,02	0,75
7	-	-	0,0I	0,73
8	-	-	0,0I	0,65
9	-	-	0,0I	0,5I
I0	-	-	0,0	0,36
II	-	-	-	0,I7
I2	-	-	-	0,07
I3	-	-	-	0,0I
I4	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 6. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 3\sigma_y$ *

l_ν	No signal	$C=0{,}33\sigma$	$C=\sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,0	0,0	0,04	0,90
2	-	-	0,0I	0,85
3	-	-	0,0I	0,78
4	-	-	0,0	0,70
5	-	-	-	0,64
6	-	-	-	0,58
7	-	-	-	0,48
8	-	-		0,29
9	-	-	-	0,I6
I0	-	-	-	0,07
II	-	-	-	0,04
I2	-	-	-	0,0I
I3	-	-	-	0,0I
I4	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 7. DISTRIBUTION $P(\gamma_\mu \geq l_\gamma)$ FOR $C_\mu = 3,6\gamma$ * /29

l_γ	No signal	C= 0,336	C = 6	C = 35
0	I,00	I,00	I,00	I,00
I	0,0	0,0	0,04	0,90
2	-	-	0,0I	0,83
3	-	-	0,0I	0,76
4	-	-	0,0	0,67
5	-	-	-	0,60
6	-	-	-	0,54
7		-	-	0,42
8		-	-	0,26
9		-	-	0,I5
I0		-	-	0,07
II		-	-	0,04
I2		-	-	0,0I
I3	-	-	-	0,0I
I4	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 8. DISTRIBUTION $P(\Delta_{max} > \ell_{\Delta})$ * /30

ℓ_{Δ}	No signal	C=0,33 σ	C = σ	C =3 σ
0,2	I,00	I,00	I,00	I,00
0,3	0,98	0,97	0,98	I,00
0,4	0,88	0,86	0,94	I,00
0,5	0,66	0,6I	0,77	I,00
0,6	0,47	0,44	0,60	0,87
0,7	0,22	0,23	0,35	0,94
0,8	0,09	0,II	0,2I	0,89
0,9	0,03	0,07	0,09	0,7I
I,0	0,0I	0,03	0,04	0,6I
I,I	0,0	0,0	0,02	0,49
I,2	-	-	0,0I	0,34
I,3	-	-	0,0	0,22
I,4	-	-	-	0,I3
I,5	-	-	-	0,06
I,6	-	-	-	0,02
I,7	-	-	-	0,02
I,8	-	-	-	0,0I
I,9	-	-	-	0,0I
2,0	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 9. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 2\sigma_\Delta$* /31

l_ν	No signal	$C=0,33\sigma$	$C=\sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,55	0,59	0,68	0,99
2	0,29	0,35	0,4I	0,97
3	0,II	0,03	0,2I	0,89
4	0,25	0,06	0,I0	0,79
5	0,0I	0,02	0,04	0,46
6	0,05	0,00	0,0	0,I4
7	0,0	-	-	0,0I
8	-	-	-	0,0

TABLE 10. DISTRIBUTION $P(\eta_\mu \geq l_\eta)$ FOR $C_\mu = 2\sigma_\Delta$*

l_η	No signal	$C=0,33\sigma$	$C=\sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,55	0,55	0,68	0,99
2	0,I9	0,25	0,35	0,94
3	0,03	0,07	0,I4	0,85
4	0,05	0,0I	0,04	0,75
5	0,0	0,0	0,0I	0,42
6	-	-	0,0	0,02
7	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 11. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 2,5\sigma$.* /32

l_ν	No signal	$C = 0,33\sigma$	$C = \sigma$	$C = 3\sigma$
0	I,00	I,00	I,00	I,00
I	0,28	0,30	0,47	0,97
2	0,08	0,I4	0,I9	0,90
3	0,03	0,03	0,07	0,73
4	0,05	0,0I	0,02	0,53
5	0,005	0,0	0,0	0,20
6	0,0	-	-	0,20
7	-	-	-	0,0

TABLE 12. DISTRIBUTION $P(\gamma_\mu \geq l_\gamma)$ FOR $C_\mu = 2,5\sigma$.*

l_γ	No signal	$C=0,33\sigma$	$C=\sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,28	0,29	0,47	0,97
2	0,05	0,06	0,I6	0,87
3	0,0I	0,0I	0,06	0,73
4	0,0	0,0	0,0I	0,50
5	-	-	0,0	0,I6
6	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 13. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 3\sigma_\Delta$ *

l_ν	No signal	$C = 0,33\sigma$	$C = \sigma$	$C = 3\sigma$
0	I,00	I,00	I,00	I,00
I	0,095	0,II	0,20	0,86
2	0,0I5	0,0I	0,07	0,7I
3	0,05	0,0I	0,02	0,47
4	0,0	0,0	0,0	0,25
5				0,02
6				0,0

TABLE 14. DISTRIBUTION $P(\gamma_\mu \geq l_\gamma)$ FOR $C_\mu = 3\sigma_\Delta$ *

l_γ	No signal	$C=0,33\sigma$	$C = \sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,095	0,II	0,20	0,86
2	0,0I	0,0I	0,07	0,68
3	0,005	0,0I	0,0I	0,46
4	0,0	0,0	0,0	0,24
5	-	-	-	0,02
6	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 15. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = \sigma_{\bar{y}}$ * /34

l_ν	No signal	$= 0{,}33\sigma$	$C = \sigma$	$C = 3\sigma$
0	I,00	I,00	I,00	I,00
I	0,52	0,60	0,86	I,00
2	0,38	0,54	0,8I	I,00
3	0,30	0,44	0,75	I,00
4	0,2I	0,4I	0,66	I,00
5	0,I7	0,37	0,59	I,00
6	0,I3	0,3I	0,57	I,00
7	0,08	0,26	0,53	I,00
8	0,06	0,20	0,46	I,00
9	0,04	0,I8	0,35	I,00
I0	0,03	0,I2	0,29	0,99
II	0,02	0,09	0,24	0,98
I2	0,0I	0,06	0,70	0,94
I3	0,005	0,03	0,I6	0,72
I4	0,0	0,0	0,I0	0,40
I5	-	-	0,05	0,24
I6	-	-	0,02	0,II
I7	-	-	0,07	0,02
I8	-	-	0	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 16. DISTRIBUTION $P(\nu_\mu > l_\nu)$ FOR $C_\mu = 1{,}5\sigma_y$ *

l_ν	No signal	C=0,33б	C= б	C = 3б
0	I,00	I,00	I,00	I,00
I	0,52	0,60	0,86	I,00
2	0,36	0,48	0,79	I,00
3	0,24	0,4I	0,70	I,00
4	0,I6	0,36	0,59	I,00
5	0,I2	0,27	0,54	I,00
6	0,I0	0,22	0,50	I,00
7	0.07	0,I7	0,44	I,00
8	0.04	0,I0	0,36	I,00
9	0.02	0,08	0,29	I,00
I0	0.005	0,06	0,22	0,99
II	0,005	0,05	0,I8	0,97
I2	0,005	0,04	0,I6	0,94
I3	0,005	0,03	0,I4	0,72
I4	0,0	0,0	0,07	0,39
I5	-	-	0,04	0,23
I6	-	-	0,02	0,I0
I7	-	-	0,0I	0,02
I8	-	-	0,0	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 17. DISTRIBUTION $P(\nu_\mu \geq l_\nu)$ FOR $C_\mu = 1{,}5\sigma_{\bar{y}}$ * /36

l_ν	No signal	$C=0{,}33\sigma$	$C=\sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,I9	0,35	0,67	I,00
2	0,I3	0,23	0,52	I,00
3	0,08	0,I5	0,45	I,00
4	0,05	0,II	0,39	I,00
5	0,04	0,I0	0,34	I,00
6	0,02	0,08	0,3I	I,00
7	0,0I	0,08	0,23	0,99
8	0,005	0,06	0,I6	0,99
9	0,0	0,04	0,I2	0,97
I0	-	0,02	0,08	0,94
II	-	0,02	0,08	0,83
I2	-	0,0	0,03	0,58
I3	-	-	0,02	0,32
I4	-	-	0,0I	0,I7
I5	-	-	0,0	0,03
I6	-	-		0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 18. DISTRIBUTION $P(\aleph_\mu \geq l_\aleph)$ FOR $C_\mu = 1{,}5\sigma_y$ *

$l_\aleph$	No signal	$C=0{,}33\sigma$	$C=\sigma$	$C=3\sigma$
0	I,00	I,00	I,00	I,00
I	0,I9	0,25	0,67	I,00
2	0,I2	0,20	0,5I	I,00
3	0,06	0,I2	0,37	I,00
4	0,02	0,08	0,35	0,99
5	0,02	0,30	0,30	0,99
6	0,0I	0,04	0,26	0,99
7	0,005	0,03	0,I7	0,99
8	0,0	0,02	0,I2	0,99
9		0,02	0,09	0,97
I0	-	0,0I	0,08	0,9I
II	-	0,0I	0,07	0,83
I2	-	0,0	0,03	0,58
I3	-	-	0,02	0,32
I4	-	-	0,0I	0,I7
I5	-	-	0,0	0,03
I6	-	-	-	0,0

*Translator's note. Commas in numbers represent decimal points.

TABLE 19. DISTRIBUTION $p(|i_0-50| \geq l_i)$ * /38

l_i	C = 6	C = 36
0	I,0	I,0
I	0,80	0,54
2	0,50	0,I2
3	0,29	0,0I
4	0,I6	0,0
5	0,I0	
6	0,05	
7	0,0	

*Translator's note. Commas in numbers represent decimal points.

TABLE 20. DISTRIBUTION $P(l \geq l_l)$ * /39

l_l	No signal	$C = 0{,}33\sigma$	$C = \sigma$	$C = 3\sigma$
0	I,00	I,00	I,00	I,00
2	0,92	0,94	0,99	I,00
4	0,66	0,76	0,97	I,00
6	0,46	0,6I	0,93	I,00
8	0,30	0,45	0,8I	I,00
I0	0,I5	0,36	0,66	I,00
I2	0,I0	0,26	0,53	I,00
I4	0,07	0,I7	0,43	I,00
I6	0,02	0,I3	0,36	I,00
I8	0,0I	0,07	0,23	0,98
20	0,0I	0,04	0,I4	0,97
22	0,0	0,0I	0,06	0,79
24		0,0	0,04	0,59
26			0,0I	0,40
28			0,0	0,22
30				0,I5
32				0,0

*Translator's note. Commas in numbers represent decimal points.