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LOCALIZATION OF NONSTATIONARY SOURCES OF ELECTROMAGNETIC RADIATION WITH THE AID OF PHASOMETRY

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Introduction

In the description of sources of electromagnetic radiation it is customary to regard them as located either at a /3* finite or at an infinitely large distance("celestial" sources). An actual source is regarded as celestial if the parallax of the source is close to the precision of its measurement. It is proposed that phasometry of the radiation permits the determination of the moments of passage of the radiation at various points of cosmic space at which are located stellite observatories (SO) with recording instruments. This method is used to localize the source. The assumption of motion is necessary to resolve the problems of ambiguity of phasometry, leading to ambiguity in the localization. Such a model is applicable, for example, to bursts of gamma-radiation, which occur as arae and short-lived increases in the intensity of gamma radiation[1]. The methods of localization of such sources is similar to the methods used in hyperbolic navigation (the LORAN system) [2].

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LOCALIZATION OF NON-STATIONARY SOURCES OF ELECTROMAGNETIC RADIATION WITH THE AID OF PHASOMETRY

1. The Problem of Localization and Fundamental Relationships

By localization is meant the determination of the three /4*
coordinates of a source or the determination of the direction to a celestial source. It may be assumed that the surface of the wave of radiation (that is, the geometric multiplicty of points in space at the same phase of radiation) appears either as a plane, normal to the direction to a celestial source, or as a sphere, at the center of which the source is located. For the localization we employ the measurements of the moments t_i of the passage of the wave at the points in space $\bar{r}_i$, given in a certain coordinate system OXYZ. The possibilities of localization are defined by the collection n of the employed base points $\bar{r}_i$(that is, by the number of SOs used in the experiment).

In the case of the recording of a plane wave the quantities $\bar{r}_i$ and t_i are related by the expressions:

$$\bar{r}_i * \bar{l}_\gamma + c(t_i - t_\gamma) = 0, \quad i = \overline{1, n-1}, \tag{I.I}$$

$$\bar{l}_\gamma^* \bar{l}_\gamma = 1, \tag{I.2}$$

where $\bar{l}_\gamma$ is the direction to the source in the coordinate system OXYZ;

t_γ is the moment of passage of the wave at the origin of OXYZ: and

c is the speed of transmission of the radiation (an asterisk denotes the inner product)

From (1.1)-(1.2) the measured t_i and the given $\bar{r}_i$ define the direction $\bar{l}_\gamma$. The quantity t_γ appears only formally and need not be determined.

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*Numbers in the margin indicate pagination in the foreign text

$$|\bar{r}_i - \bar{r}_\gamma| = c(t_i - t_\gamma), \quad i = \overline{1, n-1}, \tag{I.3}$$

where $\bar{r}_\gamma$ is the radius-vector of the source in the system OXYZ;

t_γ is the moment of the burst of radiation.

Here the t_γ does not appear ~~simply formally~~ and must be defined together with $\bar{r}_\gamma$. /5

The system (1.3) includes within itself the solutions of the systems (1.1) and (1.2) as asymptotes. However, if it is known beforehand that the source may be regarded as celestial, it is preferable to employ (1.1)-(1.2). Therefore for the construction of an algorithm for localization we will employ both systems.

Let us take the origin of the coordinate system OXYZ as at the point $\bar{r}_o$, designate it as $O\xi\eta\zeta$, and employ the radius-vectors, defined by it, as $\bar{\rho}_i$, and pass to the time τ, calculated from the moment t_o:

$$\bar{\rho}_i = \bar{r}_i - \bar{r}_o, \quad \tau_i = t_i - t_o. \quad i = \overline{1, n-1}, \tag{I.4}$$

$$\bar{\rho}_\gamma = \bar{r}_\gamma - \bar{r}_o, \quad \tau_\gamma = t_\gamma - t_o. \tag{I.5}$$

We introduce the matrix R, the i-th column of which is the vector $\bar{\rho}_i$, and the vector $\bar{\tau}$, the i-th element of which is τ_i, and we rewrite (1.1) in the form:

$$R^* \bar{l}_\gamma + c\bar{\tau} = 0, \tag{I.6}$$

and (1.3) in the form:

$$|\bar{\rho}_\gamma| = -c\tau_\gamma, \tag{I.7}$$

$$|\bar{\rho}_i - \bar{\rho}_\gamma| - |\bar{\rho}_\gamma| = c\tau_i, \quad i = \overline{1, n-1}. \tag{I.8}$$

Since (1.8) may be regarded as independent of (1.7), in the following we will regard only (1.8) as defining $\bar{\rho}_\gamma$.

In order to evaluate the precision of the localization we will take into account the error $\Delta\bar{\tau}$ of the measurement of $\bar{\tau}$ and the errors $\Delta\bar{\rho}_i$ of the given vectors $\bar{\rho}_i$. Since the $\bar{\rho}_i$ are defined in the coordinate system $O\xi\eta\zeta$, the origin of which is located at the point $\bar{r}_o$, the error in the designation of $\bar{r}_o$ is excluded from the analysis. It is only necessary to regard this error as systematic in the case of a spherical wave. The importance of the errors will be evaluated in /6
the linear approximations, with the following relationships, obtained by differentiation of (1.6) and (1.8), employed in the linerization:

$$\bar{\rho}_i^* \frac{\partial \bar{\ell}_\gamma}{\partial \tau_j} = -c\,\delta_{ij}\,, \tag{I.9}$$

$$\bar{\rho}_i^* \frac{\partial \bar{\ell}_\gamma}{\partial \bar{\rho}_j} = -\delta_{ij}\,\bar{\ell}_\gamma^*\,, \tag{I.10}$$

$$(\bar{\ell}_i - \bar{\ell}_o)^* \frac{\partial \bar{\rho}_\gamma}{\partial \tau_j} = c\,\delta_{ij}\,, \tag{I.11}$$

$$(\bar{\ell}_i - \bar{\ell}_o)^* \frac{\partial \bar{\rho}_\gamma}{\partial \bar{\rho}_j} = \delta_{ij}\,\bar{\ell}_i^*\,, \tag{I.12}$$

where

$$\bar{\ell}_i = (\bar{\rho}_\gamma - \bar{\rho}_i)/|\bar{\rho}_\gamma - \bar{\rho}_i|\,, \qquad i = \overline{1, n-1}\,, \tag{I.12a}$$

$$\bar{\ell}_o = \bar{\rho}_\gamma / \bar{\rho}_\gamma\,. \tag{I.12б}$$

and where δ_{ij} is the Kroenecker delta function.
However, besides (1.9)-(1.10) it is necessary to consider in linearization the differential relationship impled by (1.2):

$$\bar{\ell}_\gamma^*\, d\bar{\ell}_\gamma = 0. \tag{I.13}$$

It follows from (1.10) and (1.12) that the precision of the localization of a celestial source in the linear approximation is only influenced by the projection of the errors $\Delta\bar{\rho}_i$ on $\bar{\ell}_\gamma$:

$$\Delta \rho_i^{\ell_\gamma} = \bar{\ell}_\gamma^{*} \Delta \bar{\rho}_i , \qquad i = \overline{1, n-1} , \tag{I.I4}$$

and the precision of the localization of the source of a spherical wave by the projection of the errors on $\bar{\ell}_i$:

$$\Delta \rho_i^{\ell_i} = \bar{\ell}_i^{*} \Delta \bar{\rho}_i , \qquad i = \overline{1, n-1} . \tag{I.I5}$$

As a measure of the quality of the statistical character of the errors of measurement we will employ the deviation $\sigma^2_{\tau i}$ /7
of the errors $\Delta\tau_i$ and the deviations $\sigma^{\ell_\gamma^2}_{\rho_i}$ and $\sigma^{\ell_i^2}_{\rho_i}$ of the errors $\Delta\rho_i^{\ell_\gamma}$ and $\Delta\rho_i^{\ell_i}$,assuming these errors to be irremovable. We will assume the errors of measurement of the time of passage of the wave $\Delta\tau_i$ to be uncorrelated to the errors of the coordinates of the base points $\Delta\bar{\rho}_i$. The correlation among the $\Delta\tau_i$, and among the $\Delta\rho_i^{\ell_\gamma}$ and the $\Delta\rho_i^{\ell_i}$, we will assume to be the "worst" possible case, that is, maximizing the size of the elements $\mu_i(V)$ of the covariant matrices $V_{\ell/\tau}$, $V_{\ell/\rho}$ of the errors $\Delta\bar{\ell}_{\gamma/\tau}$,$\Delta\bar{\ell}_{\gamma/\rho}$ of the localization of celestial sources and the covariant matrices $V_{\rho/\tau}$, $V_{\rho/\rho}$ of the errors $\Delta\bar{\rho}_{\gamma/\tau}$, $\Delta\bar{\rho}_{\gamma/\rho}$ of the localization of the source of a spherical wave, denoting the size of the errors of measurement of the time and the given coordinates of the SOs, respectively.

2. Case of n=2

The solution to (1.6) in this case will be the simple vector $\bar{\ell}_\gamma$, equal to:

$$\bar{\ell}_\gamma = \bar{\rho}_1/\rho_1 \cos\alpha + \bar{\varkappa}/\varkappa \sin\alpha , \tag{2.I}$$

where

$$\cos\alpha = -c\tau_1/\rho_1 . \quad 0 \leqslant \alpha \leqslant \pi , \tag{2.2}$$

and where κ is an arbitrary vector orthogonal to $\bar{\rho}_1$:

$$\bar{\varkappa}^* \rho_1 = 0. \tag{2.3}$$

In th celestial sphere the collection of possible solutions forms a circle of radius α with its center at the point $\bar{\rho}_{1/\rho}$. It follows from (1.9), (1.10), and (1.13), that only the projections $\Delta\ell^S_{\gamma/\tau}$ and $\Delta\ell^S_{\rho/\tau}$ of the errors $\Delta\bar{\ell}_{\gamma/\tau}$ and $\Delta\bar{\ell}_{\gamma/\rho}$ may be defoined in the direction $\bar{S}$, orthogonal to $\bar{\ell}_\gamma$ and lying in the plane passing through $\bar{\ell}_\gamma$ and $\bar{\rho}_1$:

$$\bar{S} = \bar{\ell}_\gamma \times (\bar{\varkappa} \times \bar{\rho}_1)/|\bar{\ell}_\gamma \times (\bar{\varkappa} \times \bar{\rho}_1)| . \tag{2.4}$$

The size of these projections will be equal to:

$$\Delta\ell^s_{\gamma/\tau} = c\Delta\tau_1/\rho_1 \sin\alpha , \tag{2.5}$$

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$$\Delta\ell^s_{\gamma/\rho_1} = \dot{\bar{\ell}}^*_\gamma \Delta\bar{\rho}_1/\rho_1 \sin\alpha , \tag{2.6}$$

and their mean-square deviation:

$$\sigma^s_{\ell/\tau_1} = c\,\sigma_{\tau_1}/\rho_1 \sin\alpha \quad , \tag{2.7}$$

$$\sigma^s_{\ell/\rho_1} = \sigma^{\ell_\gamma}_\rho / \rho_1 \sin\alpha . \tag{2.8}$$

The sum of the mean-square deviations σ_ℓ will equal:

$$\sigma_\ell = \sigma_\Sigma / \rho_1 \sin\alpha \ , \qquad (2.8a)$$

where

$$\sigma_\Sigma = (\sigma_{\rho_1}^{\ell_{r^2}} + c^2 \sigma_{\tau_1}^2)^{1/2} \ . \qquad (2.8б)$$

With the help of (2.8a) we may define the half-width of the ring in the celestial sphere (take, for example, triple σ_ℓ) within the limits of which the source is located. Clearly, the error of localization grows without limit upon the approach of the direction to the source to the line of action of the vector $\bar{\rho}_1$.

Let us now look at the localization of the source of a spherical wave. The multiplicity of solutions of the system (1.8) forms the surface of one of the branches of a hyperboloid of revolution. One focus of the hyperboloid is located at the origin of the coordinate system $O\xi\eta\zeta$, and the other at the point $\bar{\rho}_1$. The major axis of the hyperboloid is equal to $|c\tau|/2$. For $\tau>0$ the surface of solutions is that branch of the hyperboloid which encloses the focus $\bar{\rho}_1$, and for $\tau<0$ the one which envelopes the focus located at the origin of the doordinates. If $\tau=0$ the hyperboloid is transformed into a plane surface normal to $\bar{\rho}_1$ and passing through $\bar{\rho}_1/2$. The asymptotic surface of the hyperboloid is the circular cone, formed by the direction of (2.1), with its vertex lying at the point $\bar{\rho}_1/2$. In this way the asymptotic solution to (1.8) becomes the solution of (1.6).

The consequences of the continuous solutions of (1.8) appear practically more suited to the solution of the problem /9
which is the reverse of the problem of localization : the confirmation by means of system (1.8) of the coordinates of possible astronomical objects (for example, a portion of the region of the surface of the Sun) which might be the source of the radiation. From this follows the consideration, that

the source may be located not only on the surface of the hyperboloid, but also in the neighborhood defined by the errors of measurement. For a definition of the possible statistical dispersion of the source from the surface of the hyperboloid, we turn our attention to the errors of measurement. According to (1.11), (1.12), the errors of localization $\Delta\bar{\rho}_{\gamma/\tau}$ and $\Delta\bar{\rho}_{\gamma/\rho}$ are related to $\Delta\tau_1$ and $\Delta\rho_1^{\ell_1}$ in the following expressions:

$$(\bar{\ell}_1-\bar{\ell}_0)^* \Delta\bar{\rho}_{\gamma/\tau} = c\Delta\tau_1, \tag{2.9}$$

$$(\bar{\ell}_1-\bar{\ell}_0)^* \Delta\bar{\rho}_{\gamma/\rho} = \Delta\rho_1^{\ell_1}. \tag{2.I0}$$

These relationships only define the projections of the errors, $\Delta\rho^S_{\gamma/\tau}$ and $\Delta\rho^S_{\gamma/\rho}$, on the perpendicular S_1 normal to the surface of the hyperboloid (1.8) at the point $\bar{\rho}_\gamma$:

$$\bar{S}_1 = (\bar{\ell}_i-\bar{\ell}_0)/|\bar{\ell}_i-\bar{\ell}_0|, \qquad i=\overline{1,n-1}. \tag{2.I1}$$

The mean-square deviations of these projections are equal to:

$$\sigma^s_{\rho/\tau} = c\,\sigma_{\tau_1}/2\sin(p_1/2), \tag{2.I2}$$

$$\sigma^s_{\rho/\rho} = \sigma^{\ell_1}_{\rho_1}/2\sin(p_1/2). \tag{2.I3}$$

where p_1 is the angle between $\bar{\ell}_1$ and $\bar{\ell}_0$ (that is, the parallactic displacement of the source arising from the transfer of the point of observation from the origin of the coordinate system to the point $\bar{\rho}_1$:

$$p_i = 2\arcsin(|\bar{\ell}_i-\bar{\ell}_0|/2), \qquad i=\overline{1,n-1}. \tag{2.I4}$$

The sum of the mean-square deviations σ_ρ will be equal to:

$$\sigma_\rho = \sigma_\Sigma/2\sin p_{1/2}, \tag{2.I5}$$

where

$$\sigma_{\Sigma} = \left(\sigma_{\rho_1}^{\ell_1^2} + c^2 \sigma_{\tau_1}^2\right)^{1/2} . \tag{2.16}$$

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<u>3. Case n=3</u>

We will assume, that

$$\text{rank } (R) = 2 \tag{3.1}$$

since for a rank of R equal to 1 the problem is reduced to that of part 2. We introduce the vector $\bar{\kappa}$, equal to

$$\bar{\varkappa} = \bar{\rho}_1 \times \bar{\rho}_2 , \tag{3.2}$$

with the help of which we rewrite (1.6) in the form

$$[R\,\bar{\varkappa}]^* \,\bar{\ell}_\gamma + \begin{bmatrix} c\bar{\tau} \\ \lambda \end{bmatrix} = 0 , \tag{3.3}$$

where $[R\bar{\kappa}]$ is a compound matrix, and $\begin{bmatrix} c\bar{\tau} \\ \lambda \end{bmatrix}$ is a compound vector, with the unknown element λ. On the basis of (3.1) we may write:

$$\bar{\ell}_\gamma = -[R\,\bar{\varkappa}]^{*-1} \begin{bmatrix} c\bar{\tau} \\ \lambda \end{bmatrix} . \tag{3.4}$$

Having inverted the matrix (see Appendix, p.1) we obtain:

$$\bar{\ell}_\gamma = \left([\bar{\varkappa} \times \bar{\rho}_2] c\tau_1 + [\bar{\rho}_1 \times \bar{\varkappa}] c\tau_2 + \bar{\varkappa}\lambda\right) / \varkappa^2 . \tag{3.5}$$

We now find the value of λ fromthe condition (1.2):

$$\lambda = \pm \varkappa \sqrt{1 - c^2 \bar{\tau}^* D^{-1} \bar{\tau}} , \tag{3.6}$$

where

$$D = R^* R . \tag{3.7}$$

Thus, for

$$c^2 \bar{\tau}^* D^{-1} \bar{\tau} < 1 \tag{3.8}$$

the system (1.6) has two solutions symmetrically placed about the base plane passing through the points 0, $\bar{\rho}_1$,and $\bar{\rho}_2$.

For

$$c^2 \bar{\tau}^* D^{-1} \bar{\tau} = 1 \qquad (3.9)$$

the system (1.6) has one solution, lying in the base plane. The ellipse (3.9) bounds the possible area of measurement of $\bar{\tau}$. This ellipse is inscribed in the rectangle $|c\tau_1| \leq \rho_1$, $|c\tau_2| \leq \rho_2$.

Because of errors the measure quantity $\bar{\tau}$ may pass beyond the permissible region specific to the localized source near the base plane. In such a case it is necessary to place $\bar{\tau}$ within the boundary of the region corresponding to the statistical fluctuations of the errors.

It follows from (1.9), (1.10), and (1.13) that errors of localization are related to the errors of measurement in the following manner:

$$\Delta \bar{\ell}_{\gamma/\tau_1} = \frac{c\Delta\tau_1}{\bar{\ell}_\gamma^* \bar{\varkappa}}\left[\bar{\ell}_\gamma \times \bar{\rho}_2\right]. \quad \Delta \bar{\ell}_{\gamma/\tau_2} = \frac{c\Delta\tau_2}{\bar{\ell}_\gamma^* \bar{\varkappa}}\left[\bar{\rho}_1 \times \bar{\ell}_\gamma\right]. \qquad (3.10)$$

$$\Delta \bar{\ell}_{\gamma/\rho_1} = \frac{\Delta\rho_1^{\ell_\gamma}}{\bar{\ell}_\gamma^* \bar{\varkappa}}\left[\bar{\ell}_\gamma \times \bar{\rho}_2\right]. \quad \Delta \bar{\ell}_{\gamma/\rho_2} = \frac{\Delta\rho_2^{\ell_\gamma}}{\bar{\ell}_\gamma^* \bar{\varkappa}}\left[\bar{\rho}_1 \times \bar{\ell}_\gamma\right]. \qquad (3.11)$$

Obviously, errors of localization grow without limit as $\bar{\ell}_{\gamma} * \bar{\kappa} \to \ell$. The quantity $\bar{\ell}_{\gamma} * \bar{\kappa}$ is equal to zero where $\bar{\kappa} = 0$, that is, when the vectors $\bar{\rho}_1$and $\bar{\rho}_2$ are colinear, or when $\bar{\ell}_\gamma$ is orthogonal to $\bar{\kappa}$. In the first case the entire system of measurement degenerates, since the rank of R is equal to 1. In the second case $\lambda = 0$ and the system has a single solution, lying in the base plane, but its precision in the direction orthogonal to the base plane is undefined.

It is possible to show (see the Appendix p.2-p.6) that for the worst possible corrleation between $\Delta\tau_1$ and $\Delta\tau_2$ and between $\Delta\rho_1^{\ell_\gamma}$ and $\Delta\rho_2^{\ell_\gamma}$ the mean-square deviation of the errors (3.10), (3.11) will be:

$$\sigma_{\ell/\tau} = \frac{c}{\bar{\ell}_\gamma^* \bar{\varkappa}} \max\left|\bar{\ell}_\gamma \times (\sigma_{\tau_1}\bar{\rho}_2 \pm \sigma_{\tau_2}\bar{\rho}_1)\right| ; \qquad (3.12)$$

$$\sigma_{\ell/\tau} = \frac{1}{\bar{\ell}_\gamma^* \bar{\varkappa}} \max |\bar{\ell}_\gamma \times (\sigma_{\rho_1}^{\ell_\gamma} \bar{\rho}_2 \pm \sigma_{\rho_2}^{\ell_\gamma} \bar{\rho}_1)|. \tag{3.13}$$ /12

With the help of the triangular inequality it is possible to obtain the more satisfactory but larger values for $\sigma_{\ell/\tau}$ and $\sigma_{\ell/\rho}$:

$$\sigma_{\ell/\tau} \leq c(\sigma_{\tau_1}/\rho_1 + \sigma_{\tau_2}/\rho_2)/\sin\beta \sin\varphi\,, \tag{3.14}$$

$$\sigma_{\ell/\rho} \leq c(\sigma_{\rho_1}^{\ell_\gamma}/\rho_1 + \sigma_{\rho_2}^{\ell_\gamma}/\rho_2)/\sin\beta \sin\varphi, \tag{3.15}$$

where β is the angle between $\bar{\rho}_1$ and $\bar{\rho}_2$

ϕ is the angle between $\bar{\ell}_\gamma$ and the base plane.

Designating the smaller value between $\bar{\rho}_1$ and $\bar{\rho}_2$ as ρ_{min}, we obtain the following final value for the sum of the mean-square deviations of the errors of localization:

$$\sigma_\ell \leq \sigma_\Sigma/\rho_{min} \sin\beta \sin\varphi\,, \tag{3.15a}$$

$$\sigma_\Sigma = \left((c\sigma_{\tau_1} + c\sigma_{\tau_2})^2 + (\sigma_{\rho_1}^{\ell_1} + \sigma_{\rho_2}^{\ell_2})^2\right)^{1/2} \leq 2(c^2\sigma_\tau^2 + \sigma_\rho^2)^{1/2}\,, \tag{3.15б}$$

where σ_τ and σ_ρ are the larger of $\sigma_{\tau 1}$, $\sigma_{\tau 2}$ and $\sigma_{\rho 1}$ and $\sigma_{\rho 2}$, respectively.

We move now to the localization of the source of a spherical wave. According to (1.8), the multiplicity of possible solutions forms a curve defined by the line of intersection of two hyperboloids. This curve has two asymptotes, colinear to (3.5). As with n=2, it is approrpiate here to solve the /13 reverse problem. For this, in order to define the area within the limits of which the source may be found, we turn our attention to the errors of measurement.

The expressions (1.11), (1.12) relate the errors of localization in the perpendicular (2.11) and the curve (1.8) at the point $\bar{\rho}_\gamma$ solely to the projections of the errors of measurement $\Delta\bar{\rho}_{\gamma/\tau}$ and $\Delta\bar{\rho}_{\gamma/\rho}$. If the vectors (2.11) are not linearly independent, that is if the source is not located along one side or the extension of a side of the triangle formed by the base points $0, \rho_1, \rho_2$, then it is possible to define the projection $\Delta\rho_\gamma^p$ of the errors of the localization of the source on the plane passing through the normal vectors (2.11). Assuming, as before, the worst possible correlation between $\Delta\tau_1$ and $\Delta\tau_2$ and between $\Delta\rho_1^{\ell 1}$ and $\Delta\rho_2^{\ell 2}$, it is possible to obtain the following values for the mean-square deviation of this projection $\Delta\rho_\gamma^p$:

$$\sigma^p_{\rho/\tau} \leq c(\sigma_{\tau_1}/2\sin(p_1/2)+\sigma_{\tau_2}/2\sin(p_2/2))/\sin\gamma\,, \quad (3.16)$$

$$\sigma^p_{\rho/\rho} \leqslant (\sigma^{\ell_1}_{\rho_1}/2\sin(p_1/2)+\sigma^{\ell_2}_{\rho_2}/2\sin(p_2/2))/\sin\gamma\,, \quad (3.17)$$

where γ is the angle between $\bar{\ell}_1-\bar{\ell}_o$ and $\bar{\ell}_2-\bar{\ell}_o$ and p_i is defined in correspondence with (2.14).

Analogous to (3.15a) and (3.15b) we obtain the following values for the sum of maen-square deviations of the errors of localization:

$$\sigma^p_\rho \leqslant \sigma_\Sigma/2\sin(p_{min}/2)\sin\gamma\,, \quad (3.18)$$

and for small parallax:

$$\sigma^p_\rho \leqslant \sigma_\Sigma/p_{min}\sin\gamma\,. \quad (3.19)$$

4. Case n=4

We will assume that:

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$$\text{rank}\ (R) = 3 \tag{4.1}$$

that is, the base points do not lie in one plane. For the localization of celestial sources it is possible to employ algorithm (3.5)-(3.7), selecting the three of the four points which provide the most precision of localization, while using the remaining point to resolve ambiguity. However, it is more satisfactory to immediately obtain the sole solution to the system (1.6), using condition (4.1). We introduce the vector $\bar{S}$, equal to: /14

$$\bar{S} = -R^{*-1} c\bar{\tau}. \tag{4.2}$$

In the absence of errors of measurement the vector $\bar{S}$ would be the solution to (1.6). The influence of errors makes apparent the consequences of the redundance of (1.6) for n=4 and introduces a violation of the condition (1.2). Therefore the solution to (1.6) will be the normalized vector $\bar{S}$:

$$\bar{\ell}_\gamma = \bar{S}/S. \tag{4.3}$$

Expression (4.3), essentially, averages the measurements, including the redundance of the fourth measurement. Therefore the precision of the localization in this case can be greater than with the use of (3.5)-(3.7). However, this greater precision does not qualitatively change the preceding analysis, and the values obtained there may be emplayed in this case.

Let us look at a more general algorithm, which includes the localization of both a celestial source and the source of a spherical wave. We square both sides of equation (1.8) and rewrite it in the form:

$$\bar{q} - \rho_\gamma c\bar{\tau} = R^* \bar{\rho}_\gamma. \tag{4.4}$$

where the elements of the vector $\bar{q}$ are:

$$q_i = (\rho_i^2 - c^2\tau_i^2)/2 , \qquad i = \overline{1, n-1} . \tag{4.5}$$

Since $|c\tau_i| \leq \rho_i$, then:

$$\rho_i^2/2 \geqslant q_i \geqslant 0 , \qquad i = \overline{1, n-1} \tag{4.6}$$

As a consequence of (4.1) we may solve (4.4) with respect /15
to ρ_γ:

$$\bar{\rho}_\gamma = \bar{p} + \rho_\gamma \bar{S} , \tag{4.7}$$

Where $\bar{S}$ is as defined in (4.2), but :

$$\bar{p} = R^{*-1} \bar{q} . \tag{4.8}$$

If all q_i equal zero, then:

$$\bar{\rho}_\gamma = \rho_\gamma \bar{S} . \tag{4.9}$$

and since according to this $S \neq 1$, the only solution to equation (4.9) would be $\bar{\rho}_\gamma = \bar{0}$. On the other hand, from $\bar{\rho}_\gamma = \bar{0}$ it follows that $\bar{q} = \bar{0}$. Excluding from further analysis the case $\bar{\rho}_\gamma = \bar{0}$ let us assume that:

$$|c\tau_i| < \rho_i , \quad \rho_i^2/2 \geqslant q_i > 0 , \quad \rho_\gamma > 0 . \tag{4.10}$$

and let us introduce the variable:

$$\mu_\gamma = 1/\rho_\gamma . \tag{4.11}$$

with the help of which we rewrite (4.7) in the form:

$$\bar{l}_0 = \mu_\gamma \bar{p} + \bar{S} . \tag{4.12}$$

where $\bar{l}_0$ is the single vector of the direction to the source from the origin of the coordinate system $O\xi\eta\zeta$ (1.12b).

If $\bar{S} = \bar{0}$, that is if $\bar{\tau} = \bar{0}$, then it follows from (4.10) that $\bar{\rho}_0$ is equal to the vector $\bar{p}_0$:

$$\bar{p}_0 = \frac{1}{2} R^{*-1} \begin{bmatrix} \rho_1^2 \\ \rho_2^2 \\ \rho_3^2 \end{bmatrix} . \tag{4.13}$$

Since for $c\bar{\tau} = 0$ the source is located at an equal distance from all base points, the $\bar{p}_0$ is the radius-vector of the center of the sphere which passes through the base points $\bar{0}$, $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{\rho}_3$.

In the general case we obtain from (4.12) the following equation: /16

$$p^2 \mu_\gamma^2 + 2\bar{p}^* \bar{S} \mu_\gamma + S^2 - 1 = 0, \quad \mu \geq 0, \; p \neq 0 . \tag{4.14}$$

For S<1 (4.2) has the single positive solution:

$$\mu = (-\bar{p}^* \bar{S} + (p^2 - |\bar{p} \times \bar{S}|^2)^{1/2}) / p^2 . \tag{4.15}$$

In conjunction with (4.2) the condition S<1 satisfies the collection of vectors $c\bar{\tau}$ lying within the region of the ellipsoid:

$$c^2 \bar{\tau}^* D^{-1} \bar{\tau} = 1. \qquad D = R^* R . \tag{4.16}$$

For $S \geq 1$ both solutions of (4.14) may be positive. In particular, when S = 1 (that is, when $c\bar{\tau}$ lies on the surface (4.16)), one of the solutions corresponds to a celestial source, the direction to which is :

$$\bar{\ell}_\gamma = \bar{S} . \tag{4.17}$$

and the second, to a source located at the distance:

$$\rho_\gamma = -p^2/2\bar{\rho}^*\bar{s}\,. \qquad (4.18)$$

For S>1 the solution lies on the various sides of the three-sided figure with its vertex at the point $\bar{0}$ and edges $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{\rho}_3$. For the solution lying on the interior side (that is, on the side on which is located the center of the sphere described by (4.13)), the quantity μ_γ is equal to (4.15) and $0<\rho_\gamma \leq p_o$. For rhe solution lying on the exterior side of the figure, the value of μ_γ is equal to

$$\mu_\gamma = (-\bar{\rho}^*\bar{s} - (p^2 - |\bar{\rho}\times\bar{s}|^2)^{1/2})/p^2 \qquad (4.19)$$

and $0 < \rho_\gamma < \infty$.

We turn now to the precision of the localization of the source of a spherical wave. In accordance with (1.11) and (1.12) the errors $\Delta\bar{\rho}_{\gamma/\tau}$ and $\Delta\bar{\rho}_{\gamma/\rho}$ will be equal to: /17

$$\Delta\bar{\rho}_{\gamma/\tau} = L^{*-1} c\Delta\bar{\tau}, \qquad (4.20)$$

$$\Delta\bar{\rho}_{\gamma/\rho} = L^{*-1}\Delta\bar{\rho}^{\ell}, \qquad (4.21)$$

wher L is the matrix whose i-th column is $\bar{\ell}_i - \bar{\ell}_o$ and $\Delta\bar{\rho}^{\ell}$ is the vector whose i-th element is $\Delta\rho_i^{\ell_i}$ (1.15). Thus, the system of measurement degenerates for $|L| = 0$.

The determinant $|L|$ is equal to zero when the ends of the vectors $\bar{\ell}_o$, $\bar{\ell}_1$, $\bar{\ell}_2$, $\bar{\ell}_3$, constructed from one axis, all lie in the same plane. In particular, $|L| = 0$ if the source is situated along one of the edges or the extension of an edge of the pyramid whose vertices are the base points. In the general case only the quantities of the parallaxes p_i (2.14) influence the precision of the localization, since the matrix L, and consequently the characteristics of the precision of localization as well, remain unchanged under the arbitrary rearrangement of the base points from any original arrangement along a straight line passing through the original base points and the source.

In the case where the distance to the source is comparable to the distances between the base points, there is an optimal distribution of base points which maximizes the precision of the localization, where the columns of the matrix L are orthogonal and the parallaxes p_i (2.14) are equal to 2 arcsin $(1/\sqrt{3})$. In this case, assuming the worst possible correlation (p.2-p.6), the mean-square deviation of the errors of localization $\Delta\bar{\rho}_{\gamma/\tau}$ and $\Delta\bar{\rho}_{\gamma/\rho}$ are bounded by the following quantities:

$$\sigma_{\rho/\tau} \leq \frac{3\sqrt{3}}{2} c\, \sigma_\tau \,, \qquad (4.22)$$

$$\sigma_{\rho/\rho} \leq \frac{3\sqrt{3}}{2} \sigma_\rho \,. \qquad (4.23)$$

In the case where the distance to the source is significantly greater than the distances between base points, the precision of localization is reduced proportionate to the square of the smallest parallax of the p_i (2.14) (let it be p_1): /18

$$\sigma_{\rho/\tau} \leq 2 c\, \sigma_{\tau_1} / p_1^2 \,, \qquad (4.24)$$

$$\sigma_{\rho/\rho} \leq 2\, \sigma_{\rho_1}^{\ell_1} / p_1^2 \,. \qquad (4.25)$$

The sum of the mean-square deviations of the errors of localization will be bounded by the following quantities:

$$\sigma_\rho \leq 2\sigma_\Sigma / p_1^2 \,, \qquad (4.26)$$

where

$$\sigma_\Sigma = \left(c^2 \sigma_{\tau_1}^2 + \sigma_{\rho_1}^{\ell_1 2}\right)^{1/2} \,. \qquad (4.27)$$

Since $\rho_\gamma \simeq \rho_1 / p_1$, then for the relative error of localization we obtain the following expression:

$$\tilde{\sigma}_\rho \equiv \sigma_\rho / \rho_\gamma = \frac{2}{p_1}\left(\frac{\sigma_\Sigma}{\rho_1}\right) . \quad (4.28)$$

Assuming that the relative error may not, in any case exceed unity, we may conclude that:

$$\frac{2}{p_1} \frac{\sigma_\Sigma}{\rho_1} \leqslant \frac{1}{K} \quad , \; K \geqslant 1 . \quad (4.29)$$

Thus, for a given precision σ_Σ there is a limiting distance $\rho_{\gamma max}$, for which we may obtain the statistically probable value:

$$\rho_{\gamma max} = \rho_1^2 / 2 K \sigma_\Sigma . \quad (4.30)$$

5. Conclusion

For the localization of a celestial source it is sufficient to register the burst at three SOs. Ambiguities which arise in this case may be removed by the use of directional detectors. With the use of detectors with an angle of directional reception equal to 2Π steradians, for unambiguous localization there must be in one SO a plane surface, coinciding with the base plane, dividing it into two directional detectors. Another possible means to resolve ambiguity is to take steps leading to a fourth SO. /19

Using three SOs it is impossible to distinguish a plane wave from a spherical one (this is only possible if the source is located on the base plane). Thus the treatment of the detection of a pulse from a nearby source defines a direction to a celestial source which may be altogether different form the direction to the true source. Therefore, in the use of three SOs it is necessary to verify (the reverse problem) the possibility of radiation pulses from "suspicious" areas. If these regions are sufficiently small in one measurement (for example, a thin layer of the surface of the Sun), then in the process of solving the reverse problem one may define all three coordinates of the source.

Identification of the form of the wave and the resulting definition of the direction to a celestial source or the three coordinates of a nearby source is possible with the use of four SOs.

In the general case with the use of four SOs the coordinates of a nearby source are defined ambiguously.

In all cases localization suffers a degeneration of the system of measurement if the source is located on a line extending through any two of the base points. In the treatment one of these points must be discarded, and localization based on the remaining points. /20

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APPENDIX

We introduce several relationships used in the fundamental parts of the work.

a.) For the nondegenerate matrix A, its inverse may be represented as:

$$A^{-1} = (\bar{a}, \bar{b}, \bar{c})^{-1} \left[\bar{b} \times \bar{c} \quad \bar{c} \times \bar{a} \quad \bar{a} \times \bar{b}\right], \tag{p.1}$$

where $\bar{a}$, $\bar{b}$, $\bar{c}$ are columns in the matrix A*.

b.) Consider matrices of the type V = AKA*, where A is the given matrix and K is an arbitrary correlation matrix:

$$K_{ii} = 1, \quad K_{ij} = K_{ji}. \quad |K_{ij}| \leq 1, \quad i,j = \overline{1, n}.$$

The largest result S of the matrix V is reached when $K = K_{max}$, where

$$K_{max\,ij} = \mathrm{sign}(\bar{a}_i^* \bar{a}_j), \quad i,j = \overline{1, n}, \tag{p.2}$$

and equal to

$$S_{max} \equiv \sup_K S(V) = \sum_{i,j}^{n} |\bar{a}_i^* \bar{a}_j| \leq \left(\sum_i^n |\bar{a}_i|^2\right), \tag{p.3}$$

where a_i is the i-th column of A.

The maximum of the values of the matrix V is bounded by S_{max}:

$$\mu_{max} \equiv \sup_K \mu_1 (AKA^*) \leq S_{max}. \tag{p.4}$$

In particular, if the rank of V_{max} is equal to unity, then:

$$\mu_{max} = S_{max}. \tag{p.5}$$

For n=2 the rank of K_{max} is always equal to one. For n>2 /21
μ_{max} and S_{max} are constructed of various K_{max}. For example, if n=3 there exist four different K_{max}, dependent on A, from which S_{max} may be constructed:

$$K_1=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix},\ K_2=\begin{bmatrix}1&-1&1\\-1&1&1\\1&1&1\end{bmatrix},\ K_3=\begin{bmatrix}1&-1&-1\\-1&1&1\\-1&1&1\end{bmatrix},\ K_4=\begin{bmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{bmatrix} \quad \text{(p.6)}$$

The rank of K_1 and K_3 is equal to 1, but the rank of K_2 and K_4 is equal to 3.