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EXTENSION OF THE TRIDIAGONAL REDUCTION (FEER) METHOD FOR COMPLEX EIGENVALUE PROBLEMS IN NASTRAN

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SUMMARY

An extension of the Tridiagonal Reduction (FEER) method in Level 17 of NASTRAN for complex eigenvalue analysis is described. As in the case of real eigenvalue analysis, the eigensolutions closest to a selected point in the eigenspectrum are extracted from a reduced, symmetric, tridiagonal eigenmatrix whose order is much lower than that of the full-size problem. The reduction process is effected automatically, and thus avoids the arbitrary lumping of masses and other physical quantities at selected grid points. The statement of the algebraic eigenvalue problem admits mass, damping and stiffness matrices which are unrestricted in character, i.e., they may be real, symmetric or unsymmetric, singular or nonsingular.

The basic concepts underlying the method are summarized and special features, such as the estimation of errors and default modes of operation are discussed. In addition, the new user-information and error messages, and optional diagnostic output relating to the complex Tridiagonal Reduction method, are presented.

Some numerical results and initial experiences relating to usage in the NASTRAN environment are provided, including comparisons with other existing NASTRAN methods for complex eigenvalue extraction.

INTRODUCTION

The complex Tridiagonal Reduction method is an extension of the FEER algorithm (Fast Eigenvalue Extraction Routine) for real eigenvalue analysis to

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complex, algebraic eigenproblem formulations. A specified number of eigenvalues lying closest to a selected point in the complex plane are sought, as well as the associated eigenvectors. As in the case of real eigenvalue analysis (ref. 1), these eigensolutions are extracted from a symmetric, tridiagonal eigenmatrix whose order is much lower than that of the full-size problem. In fact, the size of this canonical, reduced matrix is of the same order of magnitude as the number of desired roots, even if the discretized system model possesses thousands of degrees of freedom. The reduction process is carried out via an automatic algorithm requiring a finite number of steps. Thus, a basic weakness of methods requiring the lumping of masses and other physical quantities at arbitrarily selected degrees of freedom (refs. 2-4) is avoided in reducing the problem size.

With regard to computational speed, the complex Tridiagonal Reduction method is somewhat slower than the Hessenberg method (refs. 5 and 6) for small problems (on the order of one hundred or less degrees of freedom), if all the existing eigensolutions are to be calculated. However, it becomes more efficient than the Hessenberg method when the number of requested eigensolutions is much less than the full problem size. Moreover, for much larger problems, the central memory requirement of the Hessenberg method exceeds the capabilities of most large computers, so that it becomes unavailable as a solution option. This limitation does not exist in the case of the Tridiagonal Reduction Method.

The complex Tridiagonal Reduction method employs a single initial shift point, and hence only one matrix decomposition is required for each neighborhood chosen in the complex plane. It therefore is more efficient than the complex Inverse Power method, which typically performs many shifts and decompositions for each region selected. In addition, both the complex Inverse Power method and the complex Determinant method require that the user supply the length and width of rectangular regions in the complex plane, within which the eigenvalues are desired, as well as the number of estimated roots in each region. This can be burdensome to the user, who usually does not have enough advance insight to select these parameters intelligently. An improper choice (e.g., a strip too wide, or too small an estimate on the number of roots within the strip) can lead to an inordinately large number of computations or failure to extract any roots at all within the allotted machine time. These disadvantages are eliminated in the complex FEER method, where the user is only required to select points, closest to which a specified number of eigensolutions are desired.

The theory and computational procedures for complex analysis depart from those of real analysis in the following major respects:

1. Both left and right bi-orthogonal vectors must be created in the process of constructing the reduced tridiagonal matrix.

2. The reduced tridiagonal matrix, while symmetric in form, is, in general, complex rather than real.

3. The calculated theoretical errors in the computed eigenvalues are estimates rather than upper bounds.

4. Eigensolutions closest to one or more specified points (shift points) in the complex plane are found. All eigensolutions obtained for previous shift points are swept out of the problem to prevent their regeneration when dealing with the current shift point.

AN OVERVIEW OF THE COMPLEX TRIDIAGONAL REDUCTION METHOD

A detailed development of the analytical and computational procedures, including programming aspects and flow charts can be found in the NASTRAN Level 17 Theoretical and Programmer's Manuals. The following is a summary of the basic features of the complex Tridiagonal Reduction Scheme.

The general complex eigenvalue problem is stated in the form

$$[Mp^2 + Bp + K]\{u\} = 0 \ , \tag{1}$$

where [M], [B], and [K] may be real, complex, symmetric or unsymmetric, singular or non-singular. A specified number of eigenvalues, p , lying closest to a specified point, λ_o , (called a shift point) in the complex plane are to be found, as well as the associated eigenvectors {u} . The eigenvalues may include multiplicities. By a suitable transformation, the above can be expressed in the standard inverse form,

$$[A]\{x\} = \Lambda\{x\} \ , \tag{2}$$

where [A] is double the size of the stiffness, mass and damping matrices, and

$$\Lambda = \frac{1}{p-\lambda_o} \ . \tag{3}$$

In the special case where [B] is null (e.g., no damping), the double-size eigenvalue problem can be avoided by considering the mathematical eigenvalues to be p^2 and defining

$$\Lambda = \frac{1}{p^2-\lambda_o^2} \tag{4}$$

in equation (2).

Since the eigenmatrix, [A] , is, in general, unsymmetric, the eigenvectors, {x} , are orthogonal to the eigenvectors, $\{\bar{x}\}$, of the transpose eigenproblem

$$[A]^T\{\bar{x}\} = \Lambda\{\bar{x}\} \ , \tag{5}$$

so that for $\Lambda_i \neq \Lambda_j$,

$$\{\bar{x}_j\}^T\{x_i\} = 0; \ i \neq j \ . \tag{6}$$

The above relationship is a biorthogonality condition and the associated eigenvectors, $\{x_i\}$ and $\{\bar{x}_j\}$, are called right and left eigenvectors, respectively.

A reduction of the order of the eigenvalue problem, equation (2), is effected through the transformation

$$\underset{nx1}{\{\hat{x}\}} = \underset{nxm}{[V]} \underset{mx1}{\{y\}} , \tag{7a}$$

and

$$\underset{nx1}{\{\hat{\bar{x}}\}} = \underset{nxm}{[\bar{\bar{V}}]} \underset{mx1}{\{\bar{\bar{y}}\}} , \tag{7b}$$

where $\{\hat{x}\}$ and $\{\hat{\bar{x}}\}$ are approximations of $\{x\}$ and $\{\bar{\bar{x}}\}$, respectively, n is the order of the unreduced problem, and $m \leq n$. The above transformation matrices are chosen to be biorthonormal, so that

$$[\bar{\bar{V}}]^T[V] = [I] . \tag{8}$$

From equations (2), (7), and (8), it is seen that

$$[H]\{y\} = \bar{\Lambda}\{y\} , \tag{9}$$

where

$$\underset{mxm}{[H]} = [\bar{\bar{V}}]^T[A][V] , \tag{10}$$

and $\bar{\Lambda}$ is an approximation of the eigenvalue, Λ.

Thus, equation (9) is an mth order eigenvalue problem, where $m \leq n$. The value of m is established according to the criteria given later.

As in the case of real eigenvalue analysis (ref. 1), the Lanczos algorithm is used to construct the transformation matrices vector by vector, i.e.,

$$\underset{nxm}{[V]} = [\{v_1\}, \{v_2\}, \ldots, \{v_m\}] , \tag{11a}$$

$$\underset{nxm}{[\bar{\bar{V}}]} = [\{\bar{\bar{v}}_1\}, \{\bar{\bar{v}}_2\}, \ldots, \{\bar{\bar{v}}_m\}] , \tag{11b}$$

such that the reduced mxm matrix, $[H]$, is tridiagonal and its eigenvalues accurately approximate the roots of equation (2) having the largest magnitude (or, equivalently, the physical roots, p, closest to the specified point of interest, λ_o, in the complex plane). Using symmetry arguments similar to those employed for real eigenvalue analysis, it can be shown that the transformed, reduced eigenmatrix in equation (9) is tridiagonal and symmetric, having the form,

$$[H] = [\bar{\bar{V}}]^T[A][V] = \begin{bmatrix} a_{11} & d_2 & & & \\ d_2 & a_{22} & d_3 & & \\ & d_3 & a_{33} & d_4 & \\ & & \ddots & \ddots & d_m \\ & & & d_m & a_{mm} \end{bmatrix} \tag{12}$$

The matrix coefficients are theoretically given by the simplified recurrence formulas

$$\left.\begin{aligned} a_{i,i} &= \{\bar{\bar{v}}_i\}^T[A]\{v_i\} \\ \{w_{i+1}\} &= [A]\{v_i\} - a_{i,i}\{v_i\} - d_i\{v_{i-1}\} \\ \{\bar{\bar{w}}_{i+1}\} &= [A]^T\{\bar{\bar{v}}_i\} - a_{i,i}\{\bar{\bar{v}}_i\} - d_i\{\bar{\bar{v}}_{i-1}\} \\ d_{i+1} &= [\{\bar{\bar{w}}_{i+1}\}^T\{w_{i+1}\}]^{1/2} \end{aligned}\right\} \quad i = 1,m \tag{13a}$$

$$\left.\begin{aligned} \{v_{i+1}\} &= \frac{1}{d_{i+1}}\{w_{i+1}\} \\ \{\bar{\bar{v}}_{i+1}\} &= \frac{1}{d_{i+1}}\{\bar{\bar{w}}_{i+1}\} \end{aligned}\right\} \quad i = 1, m-1 \tag{13b}$$

where the sequence is initialized by choosing random, biorthonormal starting vectors for $\{v_1\}$, $\{\bar{\bar{v}}_1\}$ and by setting $d_1=0; \{v_o\}=\{\bar{\bar{v}}_o\}=\{0\}$.

The final off-diagonal term, d_{m+1} , given by equations (13) is used in establishing error estimates for the computed eigenvalues, as described below. In addition, the above algorithm is modified in the computational scheme as follows:

1. Each pair of vectors $\{v_{i+1}\}$, $\{\bar{\bar{v}}_{i+1}\}$, calculated in equations (13b), is reorthogonalized to all previously computed pairs, before re-entering equations (13a).

2. The size, m , of the reduced problem is a function of the number of accurate eigenvalues requested by the user and is limited to the number of finite physical eigenvalues available.

The eigenvalues, $\bar{\lambda}$, and eigenvectors, $\{y\}$, of equation (9) are extracted using the Q-R iteration algorithm and eigenvector computational scheme described in connection with the Upper Hessenberg method in NASTRAN (ref. 5). They are then converted to physical form.

CRITERIA FOR THE SIZE OF THE REDUCED EIGENVALUE PROBLEM

The maximum number of finite eigensolutions, including any existing rigid body modes, is equal to the rank, r, of the eigenmatrix, [A], in equation (2). Thus, for example, massless degrees of freedom, appearing as zero diagonal terms in the [M] matrix, will result in singularities (rank reduction), which imply infinite physical eigenvalues. These spurious roots are swept out of the problem in the complex FEER process, with a consequent reduction in the available eigensolutions.

A further consideration in limiting the maximum problem size is that the user has the option of requesting eigensolutions in the neighborhood of several shift points $(\lambda_{01},\lambda_{02},\ldots)$ in the complex plane. In the Tridiagonal Reduction method, all eigensolutions, f, obtained for previous shift points are swept out of the problem to prevent their re-genera ion when dealing with the current shift point. This implies that the maximum possible size, m, of the reduced problem is further limited to

$$m_{max} = r - f \quad . \tag{14}$$

On the basis of numerical experiments, similar to those cited in reference 1 for real eigenvalue analysis, it has been found that when $m << m_{max}$, a first grouping of more than $m/2$ computed eigenvalues closest to the shift point are in accurate agreement with the corresponding number of exact eigenvalues, provided that $7 \leq m \leq m_{max}$. The remaining reduced-system roots are spread across the remaining exact eigenspectrum. To enhance the accuracy of the associated eigenvectors, the minimum problem size is further increased to twelve, again assuming that $m << m_{max}$.

Thus, if the user requests a total of $\bar{q}$ eigenvalues closest to a specified point in the complex plane, the order of the reduced problem is initially set to

$$m \begin{cases} = \min[(2\bar{q}+10),(2n-f)]; & [B] \neq [0] \quad , \qquad (15a) \\ = \min[(2\bar{q}+10),(n-f)]; & [B] = [0] \quad . \qquad (15b) \end{cases}$$

Although the total number of eigensolutions requested should not exceed m_{max}, there is usually no simple way to discern this upper limit in complex eigenvalue problems. However, the reorthogonalization tests are designed to automatically establish this upper limit. If the latter tests fail for some vector pair $\{v_{i+1}\}$, $\{\bar{v}_{i+1}\}$, this is an indication that a null vector has

been generated because m_{max} linearly independent vectors have already been obtained. The recurrence algorithm, equations (13), is then terminated and the order of the eigenproblem is further reduced to $m = i$.

ERROR ESTIMATES FOR THE COMPUTED EIGENVALUES

Following a development similar to that of reference 1 for real eigenvalue analysis, it can be shown that

$$||\Lambda_i| - |\bar{\Lambda}_i|| \approx |d_{m+1} y_{mi}| \quad . \tag{16}$$

The above shows that the absolute value of the difference between the computed and true eigenvalue magnitudes is proportional to the magnitude of d_{m+1} (which is the next off-diagonal term that would be generated had the reduced tridiagonal matrix, [H] , been increased from order m to order $m+1$) and y_{mi} , (which is the last term in the reduced-system eigenvector associated with $\bar{\Lambda}$) .

Converting equation (16) to physical eigenvalue form, using equations (3) and (4), yields,

$$\xi_i = \left| \frac{|\bar{p}_i - \lambda_o|}{|p_i - \lambda_o|} - 1 \right| \approx \frac{|d_{m+1} y_{mi}|}{|\bar{\Lambda}_i|} \quad ; \; [B] \neq [0] \quad , \tag{17a}$$

$$\xi_i = \left| \frac{|\bar{p}_i^2 - \lambda_o^2|}{|p_i^2 - \lambda_o^2|} - 1 \right| \approx \frac{|d_{m+1} y_{mi}|}{|\bar{\Lambda}_i|} \quad ; \; [B] = [0] \quad . \tag{17b}$$

The use of the above error estimates as criteria for acceptable eigensolutions is as follows:

(a) If the physical eigenvalue, $\bar{p}_i$, corresponds to a zero root (e.g., a rigid body mode), the above computational scheme is invalid and therefore bypassed. Denoting t as the number of digits carried in the computations, a zero root is assumed to occur whenever

$$\frac{|\bar{p}_i|}{RMS} < 10^{-t/3} \quad , \tag{18}$$

where

$$RMS = \frac{1}{m} \left[|\bar{p}_1^2| + |\bar{p}_2^2| + \ldots\ldots + |\bar{p}_m^2| \right]^{1/2} \quad , \tag{19}$$

and is denoted by setting the error ξ_i to zero.

(b) The eigenvalues are listed in order of increasing distance from the shift point, λ_o , to determine whether their associated estimated errors, ξ_i , meet an acceptable relative error tolerance set by the user on the EIGC bulk data card (the default value is 0.10/n, where n is the order of the stiffness matrix). The first eigenvalue not meeting the tolerance test, as well as all subsequent eigenvalues further removed from the center of interest, are considered to lack sufficient accuracy and are therefore discarded.

(c) Acceptable eigenvalues obtained in the above manner are reordered according to the magnitude of the imaginary part, with positive values considered as a group ahead of all negative values.

NASTRAN USER'S INSTRUCTIONS

Figure 1 shows modifications of the EIGC card in the NASTRAN bulk data deck which accommodate user implementation of the Tridiagonal Reduction method for complex eigenvalue analysis. The modifications consist of additions to the standard user instructions and are underscored for ease in identification.

When the complex Tridiagonal Reduction method is invoked, the E parameter on this card represents the maximum allowable value of the computed absolute relative error in a physical eigenvalue. If this value is exceeded, the associated eigensolution is not accepted for further processing by NASTRAN. A detailed list of the maximum relative errors computed by complex FEER can be obtained by requesting DIAG 12 in the NASTRAN Executive Control Deck.

USER MESSAGES AND OPTIONAL DIAGNOSTICS

Functional Module User Messages

The following is a description of the NASTRAN user messages which may be generated by NASTRAN during the execution of the Complex Tridiagonal Reduction method and which are unique to this method. Explanatory information is provided following the text of each message and, in the case of a fatal message, corrective action is indicated. Refer to the NASTRAN Users' Manual, Section 6 for a complete listing of other system and user messages.

Fatal messages cause the termination of the execution following the printing of the message text. These messages will always appear at the end of the NASTRAN output. Warning and information messages will appear at various places in the output stream. Such messages convey only warnings or information to the user. Consequently, the execution continues in a normal manner following the printing of the message text.

3149 *** USER WARNING MESSAGE 3149, USER SPECIFIED NEIGHBØRHØØD CENTERED AT ØRIGIN NØT ALLØWED, CENTER SHIFTED TØ THE RIGHT .001.

Point of interest in the complex plane $(\alpha_{ai},\omega_{ai})$, closest to which the eigenvalues will be computed, was input as (0.0, 0.0) on an EIGC bulk data continuation card. Since this is an inadmissible choice, the point automatically used was (.001, 0.0).

3150 *** USER WARNING MESSAGE 3150, DESIRED NUMBER ØF EIGENVALUES *** INVALID. SET = 1.

Number of accurate roots desired N_{d1} , was omitted, input as zero or negative on an EIGC bulk data continuation card. The number automatically used was 1.

3151 *** USER WARNING MESSAGE 3151, DYNAMIC MATRIX IS SINGULAR (ØCCURRENCE ****) IN NEIGHBØRHØØD CENTERED AT **** ****

Point of interest in the complex plane $(\alpha_{ai},\omega_{ai})$, closest to which the eigenvalues will be computed, was input too close to an eigenvalue on an EIGC bulk data continuation card. The point is automatically shifted by adding .02 to both the real and imaginary parts. If the dynamic matrix is still singular, the next neighborhood, if any, is searched.

3152 *** USER INFØRMATIØN MESSAGE 3152, SUBRØUTINE ALLMAT ØUTPUT EIGENVALUE **** IS NULL.

When an eigenvalue output from subroutine ALLMAT is exactly zero, the formula for computing the associated theoretical error test fails. The magnitude of the eigenvalue is considered to be 10^{-10} for use in that formula.

3153 *** USER WARNING MESSAGE 3153, ATTEMPT TØ NØRMALIZE NULL VECTØR IN SUBRØUTINE CFEER4. NØ ACTION TAKEN.

An eigenvector output from subroutine ALLMAT is a zero-vector.

3154 *** USER WARNING MESSAGE 3154, SIZE ØF REDUCED PRØBLEM DECREMENTED ØNCE (NØW ****) DUE TØ NULL ERRØR ELEMENT.

If subroutine CFEER4 receives a reduced tridiagonal matrix having error element d_{m+1} exactly (0,0), it is impossible to compute meaningful theoretical error estimates for any of the eigenvalues. The size of the reduced problem is reduced by one, so that d_m becomes the new error element.

3155 *** USER WARNING MESSAGE 3155, REDUCED PRØBLEM HAS VANISHED. NØ RØØTS FØUND.

If decrementing the size of the reduced problem (see message 3154) causes the size to become zero, the program continues to the next neighborhood, if any.

3156 *** USER WARNING MESSAGE 3156, SIZE ØF REDUCED PRØBLEM RESTØRED TØ **** BECAUSE NEXT ERRØR ELEMENT WAS ALSØ NULL. ERRØR ELEMENT SET = **** ****

This message follows message 3154. If d_m is also exactly zero (in addition to d_{m+1} being exactly zero), then the original reduced problem size is restored and d_{m+1} is set to $(\varepsilon,0)$ where $\varepsilon = E/100$ and E is the error tolerance on acceptable eigenvalues input on the EIGC bulk data card.

3157 *** USER WARNING MESSAGE 3157, FEER PRØCESS MAY HAVE CALCULATED FEWER ACCURATE MØDES **** THAN REQUESTED IN THE NEIGHBØRHØØD ØF **** ****

The desired number of eigenvalues specified on the EIGC bulk data continuation card exceeds the additional number that can be calculated by the Complex Tridiagonal Reduction (Complex FEER) method in the current neighborhood.

3158 *** USER WARNING MESSAGE 3158, NØ ADDITIØNAL MØDES CAN BE FØUND BY FEER IN THE NEIGHBØRHØØD ØF **** ****

An initial pseudo-random vector cannot be made orthogonal to the existing set of orthogonal vectors (which come from Restart and from all prior-neighborhood sets of eigensolutions).

3159 *** USER INFØRMATIØN MESSAGE 3159, ALL SØLUTIØNS HAVE BEEN FØUND.

The FEER method has solved the entire problem. Any additional neighborhoods (as specified by the presence of EIGC bulk data continuation cards) are ignored.

3160 *** USER INFØRMATION MESSAGE 3160, MINIMUM ØPEN CØRE NØT USED BY FEER **** WØRDS (**** K BYTES).

This message indicates the amount of open core, in both bytes and words, not used by FEER.

3161 *** USER WARNING MESSAGE 3161, DESIRED NUMBER ØF EIGENSØLUTIØNS **** FØR NEIGHBØRHØØD **** ØF **** CENTERED AT **** **** EXCEEDS THE EXISTING NUMBER ****, ALL EIGENSØLUTIØNS WILL BE SØUGHT.

The desired number of eigenvalues specified on the EIGC bulk data continuation card exceeds the size of the eigenmatrix, which is the maximum possible number of existing eigenvalues.

3162 *** USER WARNING MESSAGE 3162, ATTEMPT TØ NØRMALIZE NULL VECTØR. NØ ACTIØN TAKEN.

The general vector normalization routine (CFNØR1 or CFNØR2) has a zero-vector input to it.

3163 *** USER WARNING MESSAGE 3163, ALL **** SØLUTIØNS HAVE FAILED ACCURACY TEST. NØ RØØTS FØUND.

The number of eigensolutions passing the relative error test is zero. The maximum allowable error for the relative error test is specified in field 7 of the EIGC bulk data card. A detailed list of the computed error bounds could have been obtained by requesting DIAG 12 in the Executive Control Deck.

3164 *** USER INFØRMATIØN MESSAGE 3164, ALL **** SØLUTIONS ARE ACCEPTABLE.

All the eigensolutions obtained in the reduced problem corresponding to the point of interest pass the relative error test. The maximum allowable error for the relative error test is specified in field 7 of the EIGC bulk data card. A detailed list of the computed error estimates could have been obtained by requesting DIAG 12 in the Executive Control Deck.

3165 *** USER INFØRMATIØN MESSAGE 3165, **** SØLUTIØNS HAVE BEEN ACCEPTED AND **** SØLUTIØNS HAVE BEEN REJECTED.

Some eigensolutions passed the relative error test and some did not.

3166 *** USER INFØRMATIØN MESSAGE 3166, **** MØRE ACCURATE EIGENSØLUTIØNS THAN THE **** REQUESTED HAVE BEEN FØUND FØR NEIGHBØRHØØD **** ØF **** CENTERED AT **** ****. USE DIAG 12 TØ DETERMINE ERRØR ESTIMATES.

The number of eigensolutions passing the relative error test is greater than the number requested on the corresponding EIGC bulk data continuation card. The maximum allowable error for the relative error test is specified in field 7 of the EIGC bulk data card. A detailed list of the computed error estimates could have been obtained by requesting DIAG 12 in the Executive Control Deck.

The Eigenvalue Summary Table

The following summary of the eigenvalue analysis performed, using the complex Tridiagonal Reduction (FEER) method, is automatically printed:

1. Number of eigenvalues extracted.

2. Number of starting points used.

   This corresponds to the total number of random starting and restart vectors used by the complex FEER process for all neighborhoods.

3. Number of starting point moves.

   Not used in FEER (set equal to zero).

4. Number of triangular decompositions.

   Always equal to the number of points of interest (neighborhoods) in the complex plane processed by FEER, since ordinarily only one triangular decomposition is required by FEER for each point of interest, unless the dynamic matrix is singular at a given point of interest, in which case an additional decomposition is required (obtained by moving the point of interest slightly).

5. Total number of vector iterations.

   The total number of reorthogonalizations of all the trial vectors employed.

6. Reason for termination.

   (0) All, or more solutions than the number requested by the user, have been determined (normal termination).

   (1) All neighborhoods have been processed, but FEER has not obtained the desired number of roots in each neighborhood, possibly because they have already been found in other neighborhoods.

   (2) Abnormal termination - either no roots found or none pass the FEER error test.

Optional Diagnostic Output

The user can obtain special detailed information relating to the generation of the reduced problem size, the elements of the reduced tridiagonal matrix, vector reorthogonalization iterations, computed error estimates, order of eigenvalue extraction, and distance of extracted

eigenvalue from the center of interest by requesting DIAG 12 in the NASTRAN executive control deck.

The meaning of this information is explained below in the order in which it appears in the DIAG 12 output.

****FEER**** (FAST EIGENVALUE EXTRACTIØN RØUTINE)****

This header is always printed first.

****SINGLE PRECISIØN WØRDS ØF ØPEN CØRE NØT USED (SUBRØUTINE XXXX)

**** - Open core not used by subroutine XXXX, in single-precision words.
XXXX - Either CFCNTL, CFEER3, or CFEER4. This message appears three times.

CFCNTL ACCURACY CRITERIØN * (INPUT VALUE**)

* - Accuracy criterion, used for rejecting eigensolutions (expressed as a percentage).
** - Value of accuracy criterion input by the user on the EIGC bulk data card.

CFCNTL NEICHBØRHØØD * CENTER = ** ** NØ.DES.RTS. = *** NØNSYM = ****

* - Positive integer indicating which neighborhood, or center of interest, is currently being processed.
** ** - Center of interest in the complex plane.
*** - Number of desired roots for the current neighborhood, input by the user on the corresponding EIGC bulk data continuation card.
**** - Indicator which, when nonzero, forces the program to consider the matrices as non-symmetric, even though they may actually be symmetric. This is input by the user in field 7 of each EIGC bulk data continuation card. This input was used during program checkout of the complex FEER process, and it should have no affect on the solution. However, the user should leave field 7 blank on each EIGC continuation card.

REØRTHØGØNALIZATIØN ITERATIØN * TARGET VALUE = **
ERRORS = *** *** *** ***

* - The reorthogonalization iteration-number. This message will appear many times, as the FEER process "cleans up" each trial vector by forcing it to be as orthogonal as possible to the set of vectors already computed.
** - Convergence tolerance, such that the errors must be smaller than this value. In order to avoid taking square roots, the tolerance and errors are all squared.
*** *** *** *** - Four reorthogonalization errors, the first two of which correspond to the orthogonality of the current right and left handed trial vectors, respectively, with respect to all

previously computed vectors in the current neighborhood, and the latter two of which correspond to the orthogonality of the same vectors with respect to all eigenvectors previously computed (restart and prior neighborhoods).

```
REDUCED TRIDIAGØNAL MATRIX ELEMENTS RØW *
ØFF DIAGØNAL = ** **
    DIAGØNAL = *** ***
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*	-	The row number of the (reduced) tridiagonal matrix.
** **	-	Value of the off-diagonal element for that row.
*** ***	-	Value of the diagonal element for that row.

Following the printing of several lines containing reorthogonalization information and reduced tridiagonal matrix elements, when the FEER process has finished its computations for the current point of interest, the header (see above) is printed once again, followed by a table which summarizes all the eigensolutions found by FEER. This table has seven columns, as follows:

(1) Solution number. This is simply a positive integer 1,2,3,... .

(2) Order of extraction. These numbers indicate the order in which the tridiagonal matrix was constructed.

(3) Distance from center. This is the distance from the extracted eigenvalue to the neighborhood center (which is printed above the table) in the complex plane. The tabular values are sorted according to increasing distance from the center.

(4) Real part of the extracted eigenvalue.

(5) Imaginary part of the extracted eigenvalue.

(6) Theoretical error estimate. This value must be smaller than the Accuracy Criterion (see above) for the eigensolution to be acceptable.

(7) Status. A single word, "accept" or "reject", to indicate the result of the accuracy test. A minus sign (-) is added to "reject" so that the eye can more rapidly distinguish between the two words.

Finally, this table is printed a second time, but with the rejected eigensolutions deleted.

For very small problems, there is a Very Detailed Printout (VDP) option. This option was originally used to debug the complex FEER logic, and is no longer required. DIAG 12 must be specified in the executive control deck to invoke the VDP option, and furthermore, field 6 of a given EIGC bulk data continuation card must have a (floating point) value equal to or greater than the size of the stiffness matrix. Thus, field 6 of each EIGC continuation

card should ordinarily be left blank, so that the VDP option will be suppressed. The actual printed output of this option consists of all vectors for each step of the complex FEER process, which is too extensive and detailed for normal user purposes.

NUMERICAL RESULTS AND COMPARISON WITH EXISTING NASTRAN EIGENVALUE METHODS

Test Problem

During the developmental stages of the complex Tridiagonal Reduction method, a simple three degree of freedom model consisting of the rod and dashpot system shown in figure 2, was employed for experimental and check-out purposes. The eigenvalues of this system consist of one complex-conjugate pair, one pure imaginary and three zero roots., the latter corresponding to rigid-body modes.

A comparison of the performance of the complex Tridiagonal Reduction method, the complex Determinant method and the complex Inverse Power method, is summarized for this example in table I. In run 1, all six eigensolutions were successfully found using complex FEER and a single shift point in the second quadrant of the complex plane; only one triangular decomposition of the dynamic matrix was required. In run 2, using the complex Determinant method, 29 triangular decompositions were performed and only two, non-zero, eigenvalues were found, in spite of the fact that multiple search regions were used, including one region encompassing the origin. The reason given for termination in the eigenvalue summary table was that "all predictions for eigenvalues are outside the regions specified", even though this was not the case.

Runs 3-5 were with the complex Inverse Power method, which did not fare too well in this exercise. A large number of iterations were performed, resulting in relatively high CPU and I/O times. In addition, the first run yielded only two roots and the remaining two, none at all, even though the search regions selected, while differing from run to run, encompassed the known eigenvalues.

All the above runs, and those reported below, were performed on the NASA/GSFC IBM 360/95 computer.

Tidal Frequencies and Modes in Closed, Shallow Basins

Concurrent with the complex FEER development, an independent study was conducted (ref. 7) to estimate the tidal frequencies and mode shapes in two of the Great Lakes, namely, Lake Erie and Lake Superior. It was decided to use a finite-element displacement formulation and the complex eigenvalue extraction capabilities in NASTRAN for this purpose. The mathematical modeling was based on the following assumptions:

(a) Negligible convective accelerations and other nonlinear effects.

(b) Inviscid, incompressible flow.

(c) Negligible vertical velocity compared to lateral velocities.

(d) The Boussinesq hypothesis of hydrostatic pressure variations.

(e) Sufficient shallowness to permit integration-averaging through the depth.

As a consequence of the above simplifications, the Navier-Stokes equations reduce to the two-dimensional form,

$$\frac{\partial \bar{u}}{\partial t} + g \frac{\partial \zeta}{\partial x} - f\bar{v} = 0 \tag{20a}$$

$$\frac{\partial \bar{v}}{\partial t} + g \frac{\partial \zeta}{\partial y} + f\bar{u} = 0 \tag{20b}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0 \quad , \tag{20c}$$

where

$\bar{u}, \bar{v}$ = integrated average velocities in the x and y (horizontal) directions
ζ = local water height, measured from the mean surface
h = local depth of water, measured from the mean surface
f = the Coriolis parameter, $2\Omega sin\phi$, with ϕ the latitude, and Ω the earth's rotational rate
g = acceleration due to gravity.

Using the Galerkin method, a finite element representation of the above equations was developed, in which the nodal variables are $\bar{u}$, $\bar{v}$ and ζ . A data-generator code was then written, which generated DMIG card images for use as NASTRAN input.

The two lakes are shown in figures 3 and 4 and their finite-element meshes are given in figures 5 and 6. The mesh of Lake Erie contains 81 nodes and 204 unconstrained degrees of freedom, while the representation of Lake Superior involves 124 nodes and 299 unconstrained degrees of freedom. These selections were based on numerical convergence studies with successively finer meshes, and represent the fineness needed to obtain two or three accurate modes.

In the early stages of the study, the complex Inverse Power method was used, but had to be abandoned because of inconsistencies in the results; the roots obtained seemed to depend on the search region selected and false roots were almost always calculated very close to the starting point in the region. A tightening of the convergence criterion "E" on the EIGC bulk data card was attempted, but this did not resolve the difficulties.

Some limited success was achieved using the complex Determinant method, but here again, difficulties were encountered. Unless the search region could be rather carefully and precisely defined, the chances of finding roots was somewhat remote. The method is apparently quite sensitive in its search pattern, and does not home-in on a root if there is an extensive search area to work on.

At some point approximately mid-way in the study the complex FEER capability was completed and integrated into NASTRAN, whereupon it was applied to the tidal mode problem for the lakes. The results obtained were consistently good. To check their accuracy, changes were made in the shift points, mathematical scaling and the number of requested eigensolutions (i.e., the truncated size of the problem). These variations had only a negligible effect on the calculated frequencies and mode shapes.

Some timing results for Lake Erie, using complex FEER and the Determinant method are given in table II. As noted above, the Determinant method was viable only if the search regions were made very small, implying that the locations of the roots were rather well known in advance.

The Upper Hessenberg method could not be used for these models, since the maximum region available on the IBM 360/95 was 900K, which would only permit a 43 node mesh with this method.

CONCLUDING REMARKS

Initial experiences with the complex Tridiagonal Reduction (FEER) method indicate that it is very effective in extracting any desired number of accurate complex eigensolutions in the neighborhood of a selected shift point on the complex plane. The method automatically computes complex roots at increasing distances from the selected point until the requisite number, specified by the user, is obtained. In this respect, a disadvantage of the complex Determinant and complex Inverse Power methods, namely, a very careful delineation of search regions, is eliminated. In the case of multiple shift points, it has been found that complex FEER successfully sweeps-out eigensolutions obtained for previous shift points and prevents their regeneration when dealing with the current shift point.

Since the mathematical properties and characteristics of complex eigenvalue problems are very broad and varied, it should be recognized that the results reported herein with regard to computational efficiency and timing, are only indicative of a small class of problems. A fuller assessment of the capabilities of this new method can only be obtained following extensive application experiences within the user community.

REFERENCES

1. Newman, M.; and Flanagan, P. F.: Eigenvalue Extraction in NASTRAN by the Tridiagonal Reduction (FEER) Method. NASA CR-2731, Aug. 1976.

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4. Ramsden, R. N.; and Stoker, R. J.: Mass Condensation: A Semi-automatic Method for Reducing the Size of Vibration Problems. Int. J. Num. Meth. Eng., Vol. 1, 1969, pp. 333-349.

5. The NASTRAN Theoretical Manual (Level 17), NASA SP-221, National Aeronautics and Space Administration, 1978, Section 10.5.

6. Wilkinson, J. H.: The Algebraic Eigenvalue Problem. Clarendon Press, Oxford, 1965.

7. Eades, J. B., Jr.: Tidal Frequency Estimation for Closed Basins. AMA Report No. 78-11, NASA Contract NAS5-22927, Feb. 1978.

TABLE I. COMPARISON OF EIGENVALUE METHODS FOR ROD-AND-DASHPOT TEST PROBLEM

Run	Method	Time, minutes CPU	Time, minutes I/O	Region (K)	Comments
1	Complex FEER	0.151	2.262	300	Eigensolutions Found: All 6 No. of Triangular Decompositions: 1 Search Regions: One point in 2nd quadrant of complex plane Reason for Termination: All solutions found.
2	Complex Determinant	0.174	2.858	320	Eigensolutions Found: 2 (3 rigid body modes at origin missed) No. of Triangular Decompositions: 29 Search Regions: 3, around known roots, including origin Reason Given for Termination: All predictions are outside regions specified
3	Complex Inverse Power	0.271	8.997	320	Eigensolutions Found: 2 (3 rigid body modes missed) No. of Triangular Decompositions: 1 in last search region Search Regions: 3, around known roots, including origin Reason Given for Termination: Number of desired roots have been found
4	Complex Inverse Power	0.238	4.100	320	Eigensolutions Found: None No. of Triangular Decompositions: 4 in last search region Search Regions: 1, encompassing all 5 existing roots on and above real axis Reason Given for Termination: Four starting point moves while tracking a single root
5	Complex Inverse Power	0.149	1.566	320	Eigensolutions Found: None No. of Triangular Decompositions: 2 Search Regions: Around origin, in attempt to find rigid-body modes Reason Given for Termination: Two successive singularities found while performing triangular decomposition

TABLE II. TIMING RESULTS FOR LAKE ERIE TIDAL MODE CALCULATIONS - 81 NODE MODEL

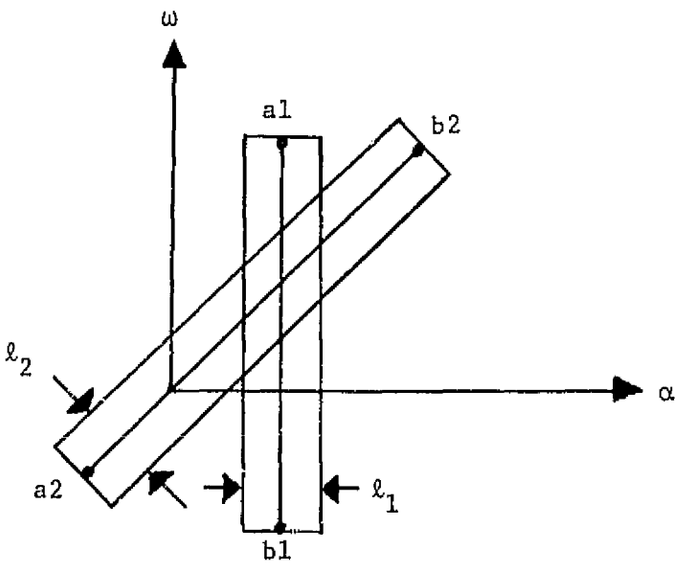
Run	Method	Time, minutes CPU	Time, minutes I/O	Region (K)	Comments
1	Complex FEER	2.164	5.886	700	6 accurate modes requested 4 obtained
2		2.425	8.593	350	15 accurate modes requested 12 obtained
3		7.576	27.751	560	50 accurate modes requested 54 obtained
4	Complex Determinant	6.572	8.618	700	3 modes obtained; insufficient time for more
5		5.315*	7.489	700	6 modes obtained; insufficient time for more

*Less time used due to a more careful choice of the search region.

BULK DATA DECK

Input Data Card EIGC Complex Eigenvalue Extraction Data

Description: Defines data needed to perform complex eigenvalue analysis



Format and Example:

1	2	3	4	5	6	7	8	9	10
EIGC	SID	METHØD	NØRM	G	C	E			+abc
EIGC	14	DET	PØINT	27		1.-8			ABC
+abc	α_{a1}	ω_{a1}	α_{b1}	ω_{b1}	ℓ_1	N_{e1}	N_{d1}		+def
+BC	2.0	5.6	2.0	-3.4	2.0	4	4		DEF
+def	α_{a2}	ω_{a2}	α_{b2}	ω_{b2}	ℓ_2	N_{e2}	N_{d2}		
+EF	-5.5	-5.5	5.6	5.6	1.5	6	3		

(etc.)

Figure 1. Modifications to the EIGC bulk data card for the Tridiagonal Reduction Method.

Field	Contents
SID	Set identification number (unique integer > 0)
METHØD	Method of complex eigenvalue extraction, one of the BCD values, "INV", "DET", "HESS" or "FEER" INV - Inverse power method DET - Determinant method HESS - Upper Hessenberg method FEER - Tridiagonal Reduction Method
NØRM	Method for normalizing eigenvectors, one of the BCD values "MAX" or "PØINT" MAX - Normalize to a unit value for the real part and a zero value for the imaginary part, the component having the largest magnitude PØINT - Normalize to a unit value for the real part and a zero value for the imaginary part the component defined in fields 5 and 6 - defaults to "MAX" if the magnitude of the defined component is zero.
G	Grid or scalar point identification number (Required if and only if NØRM=PØINT)(Integer>0)
C	Component number (Required if and only if NØRM="PØINT" and G is a geometric grid point)(0≤integer≥6)
E	Convergence criterion (optional)(Real≥0.0) For method = "FEER", error-tolerance on acceptable eigenvalues in percent (default value is .10/n, where n is the order of the stiffness matrix)
$(\alpha_{aj}, \omega_{aj})$ $(\alpha_{bj}, \omega_{bj})$	Two complex points defining a line in the complex plane (Real) For method = "FEER", $(\alpha_{aj}, \omega_{aj})$ is a point of interest in the complex plane, closest to which the eigenvalues are computed; $\lvert\alpha_{aj}\rvert + \lvert\omega_{aj}\rvert > 0$. The point $(\alpha_{bj}, \omega_{bj})$ is ignored.
ℓ_j	Width of region in complex plane (Real>0.0) Blank for method = "FEER".
N_{ej}	Estimated number of roots in each region (Integer>0) Ignored for method = "FEER".

Figure 1. Continued

N_{dj} Desired number of roots in each region (Default is $3N_{ej}$) (Integer>0) Desired number of accurate roots for method = "FEER" (Default is 1).

Remarks:

1. Each continuation card defines a rectangular search region. For method = "FEER", the card defines a circular search region, centered at $(\alpha_{aj}, \omega_{aj})$ and of sufficient radius to encompass N_{dj} roots. Any number of regions may be used and they may overlap. Roots in overlapping regions will not be extracted more than once.

2. Complex eigenvalue extraction data sets must be selected in the Case Control Deck (CMETHØD=SID) to be used by NASTRAN.

3. The units of α, ω and ℓ are radians per unit time.

4. At least one continuation card is required.

5. For the determinant method with no damping matrix, complex conjugates of the roots found are not printed.

6. See Section 10.4.4.5 of the Theoretical Manual for a discussion of convergence criteria.

7. For the Upper Hessenberg method, N_{d1} controls the number of vectors computed. Only one continuation card is considered and the (α, ω) pairs, along with the parameters ℓ_1 and N_{e1}, are ignored. Insufficient storage for HESS will cause the program to switch to INV.

8. The error tolerance, E, for the "FEER" method is with regard to

$$\left| \frac{|\tilde{p}_i-(\alpha_{aj}, \omega_{aj})|}{|p_i-(\alpha_{aj}, \omega_{aj})|} - 1 \right| \quad \text{for } [B] \neq [0] \quad \text{and}$$

$$\left| \frac{|\tilde{p}_i^2-(\alpha_{aj}, \omega_{aj})^2|}{|p_i^2-(\alpha_{aj}, \omega_{aj})^2|} - 1 \right| \quad \text{for } [B] = [0] \quad ,$$

where $\tilde{p}_i$ is a computed eigenvalue and p_i an exact eigenvalue.

Figure 1. Concluded.

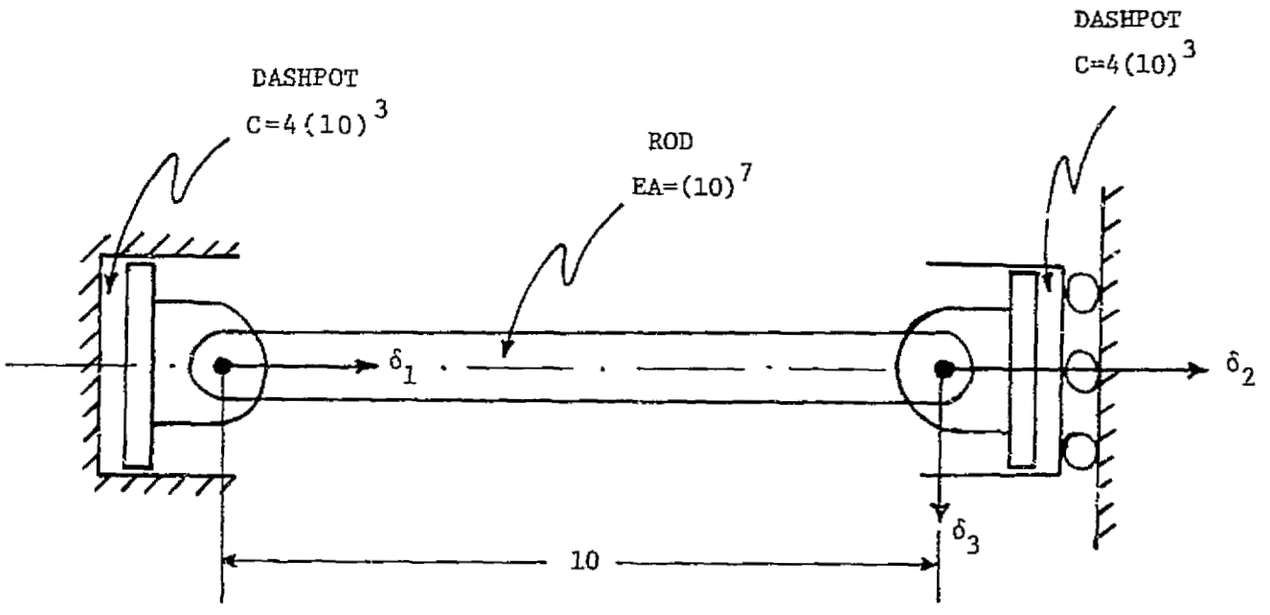


Figure 2. Test Problem -- Rod and Dashpots with 3 Degrees of Freedom

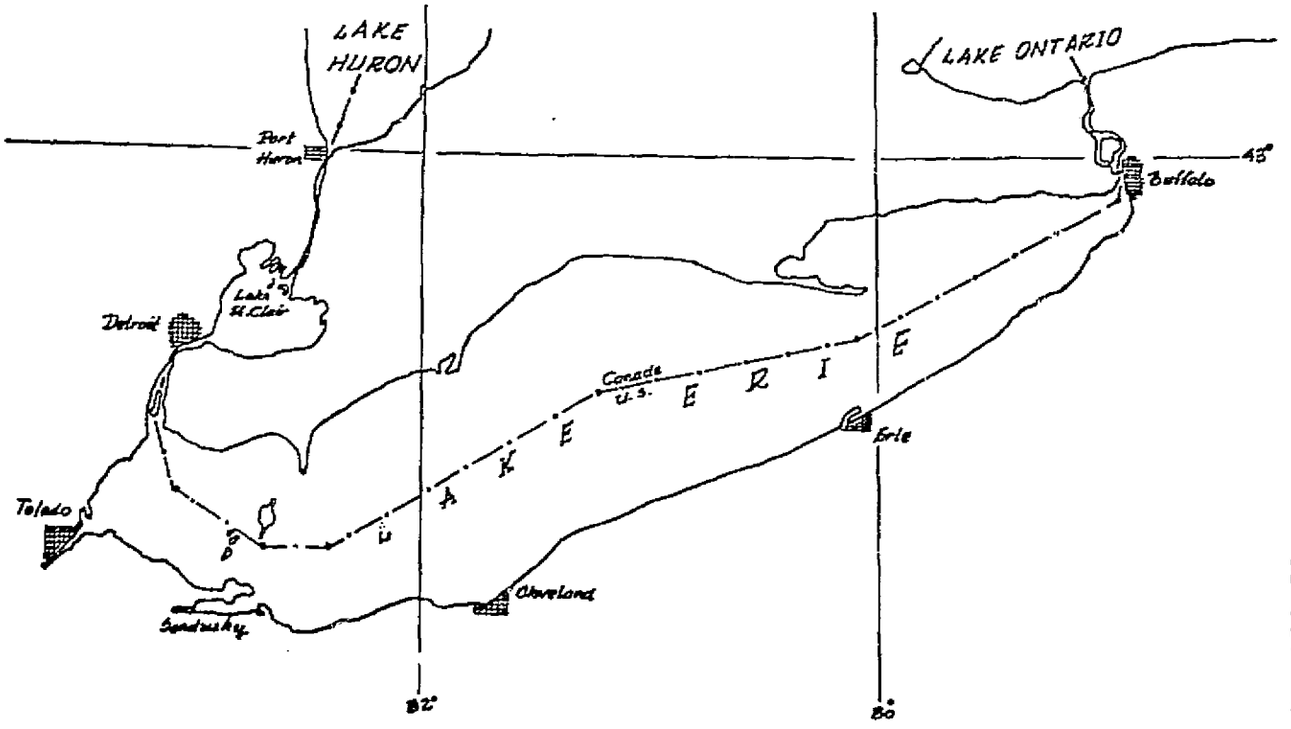


Figure 3. Illustration of Lake Erie and the adjacent geography.

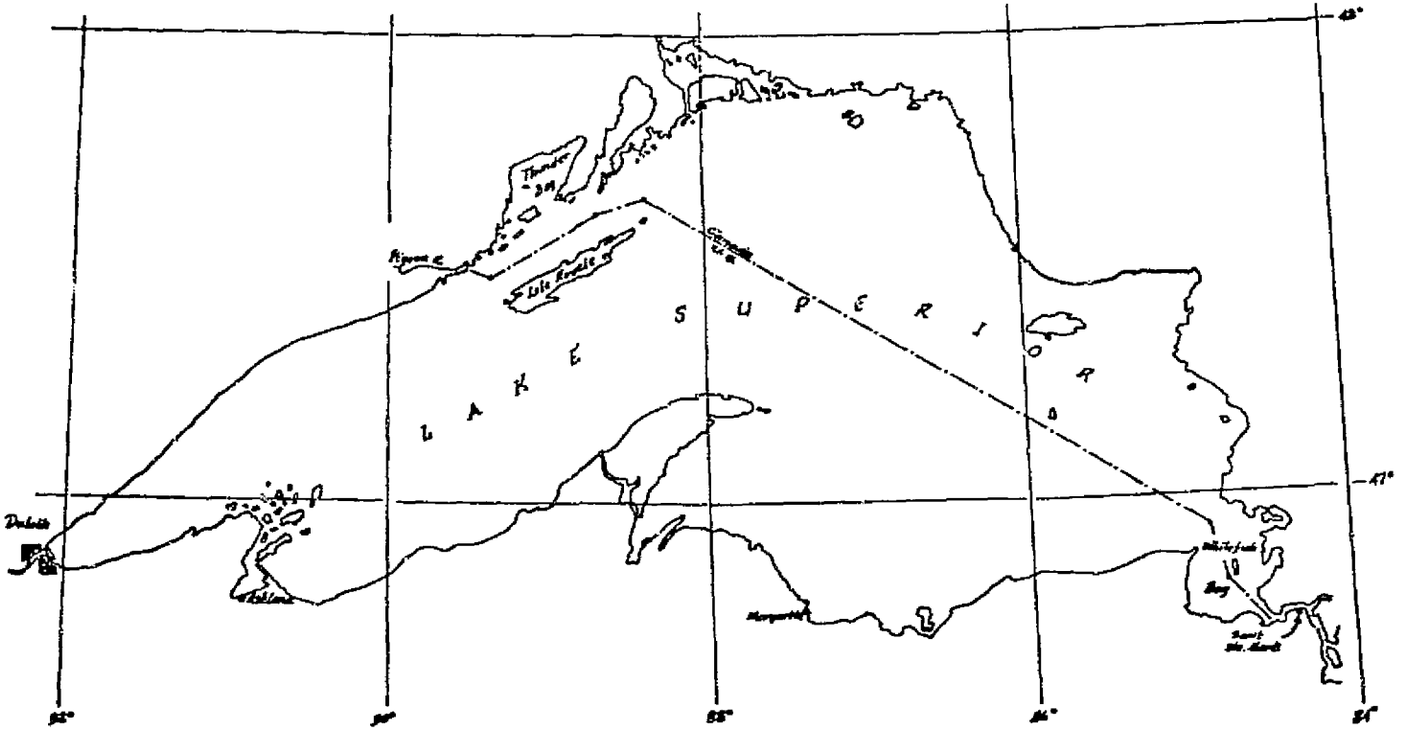


Figure 4. Illustration of Lake Superior and its local geography.

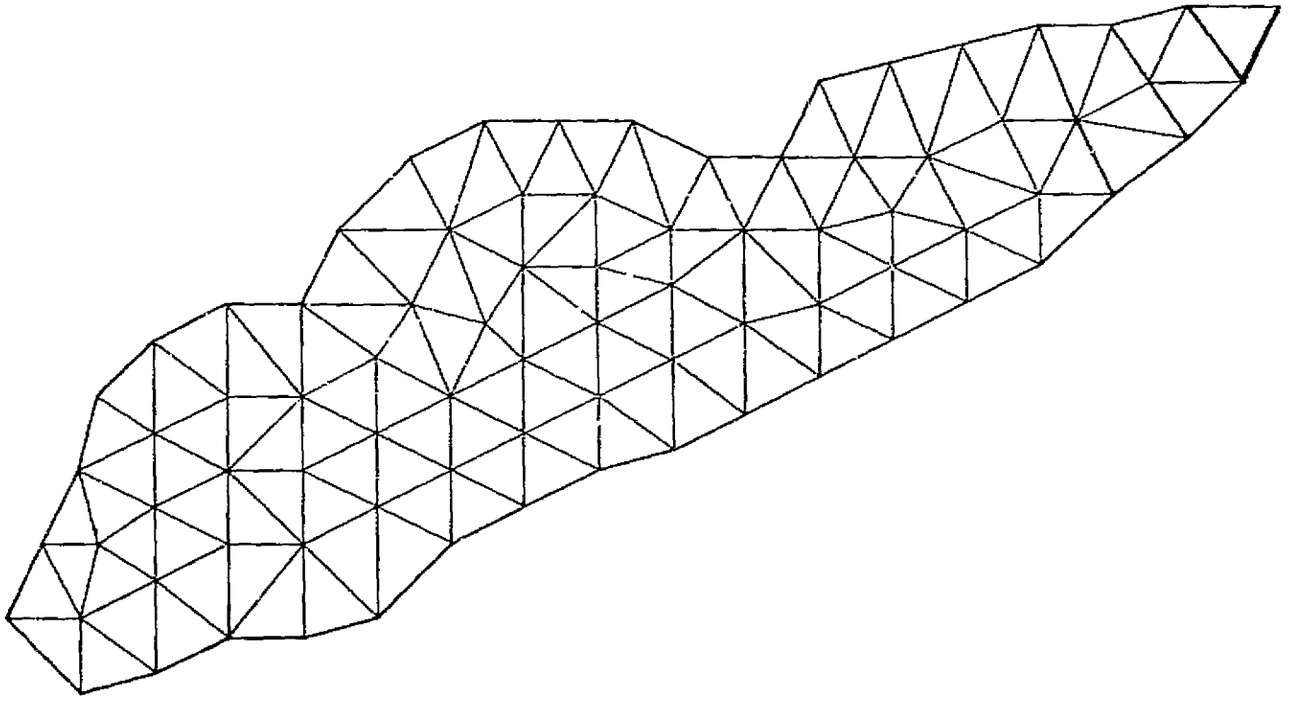


Figure 5. 81 Node Finite Element Model for Lake Erie.

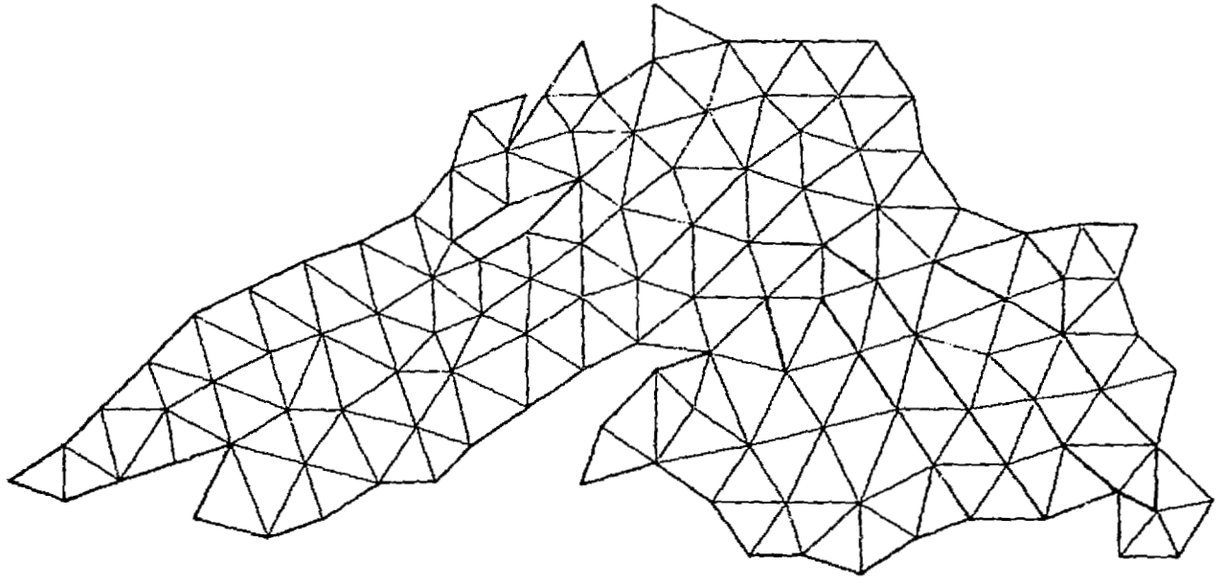


Figure 6. 124 node finite element model for Lake Superior.