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MATHEMATICAL DESCRIPTION
OF THE
ORAN ERROR ANALYSIS PROGRAM

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1.0 INTRODUCTION

The ORAN program simulates a Bayesian least squares data reduction for orbital trajectories. It does not process data, but is intended to compute the accuracy of the results of a data reduction if measurements of a given accuracy are available and processed in a least squares data reduction program. Actual data tape input may be used, but if so, this provides only the time when a measurement was available and the estimated noise on the measurement.

It should be noted that the ORAN program is designed to consider a data reduction process in which a number of satellite data periods are reduced simultaneously. The term arc refers to a specific data period over which one or more satellite orbits are simultaneously integrated and tracked. If there is more than one satellite in an arc, satellite to satellite tracking can be analyzed by ORAN if such tracking is specified on the measurement cards.

In practice, simultaneous reduction of multiple satellite data periods would be done if parameters (such as station positions or geopotential coefficients) were being estimated which had values known to be the same for all data periods and all satellites analyzed. If there are no adjusted parameters common to all arcs, then the results for each arc are completely independent.

Section 2 contains a mathematical description of the error analysis and Section 3 describes the partitioning of the error analysis equations as implemented in ORAN. Section 4 gives a brief description of the type of measurement model contained in the program and Section 5 describes the force model equations which relate the epoch state being estimated to satellite position and velocity at any other time. Section 6 gives the mathematical formulae used to compute each force acting on the satellite and the accompanying variational equation used to relate errors in force model parameters at epoch to errors in satellite position and velocity at any other time.

2.0 MATHEMATICAL DESCRIPTION

The least squares estimator in most orbital determination programs[1] assumes that N measurements can be modeled by the nonlinear regression equation

$$\underline{z} = f(\underline{x},\underline{\gamma}) + \underline{\varepsilon} \tag{2.1}$$

where $\underline{x}$ contains parameters to be estimated (i.e., adjusted), $\underline{\gamma}$ contains errors in parameters which are assumed to be known constants (i.e., unadjusted) and $\underline{\varepsilon}$ is a vector of zero mean measurement noise. It is assumed that the covariance matrix associated with $\underline{\varepsilon}$ is diagonal. The partitioning of parameters into $\underline{x}$ and $\underline{\gamma}$ (i.e., adjusted and unadjusted) is somewhat arbitrary. For any particular problem, the data will be insufficient to adjust all parameters subject to uncertainty, and some reasonable subset of these parameters must be selected for adjustment. The final errors in the adjusted parameters can be decomposed into a component due to measurement noise and a component due to errors in the assumed values of the unadjusted parameters. The error statistics associated with the first are evaluated in the orbital determination program as a noise only covariance matrix. ORAN is used to simulate the orbital determination processing and compute error statistics associated with the second component.

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(1) GEODYN Program Documentation, Volume 1, Section 10.

Bayesian least squares estimation is characterized by the use of a priori information on all parameters being estimated. The estimation technique is otherwise identical to weighted least squares. In the following, no explicit use of a priori information will be made because of the complexity it introduces into the form of the equations. Further, when a priori information is viewed as simply additional observations - as is a perfectly legitimate procedure - then it is not necessary to treat it separately in the mathematical development.

In this section we will consider first the procedure used to derive the least squares estimation equations. The standard nonlinear least squares estimation problem is based on the regression equation given in (2.1).

Since the problem must be solved iteratively using linear theory, for error analysis purposes the adjusted parameters may be expressed as

$$\underline{x} = \underline{x}_N + \delta\underline{x} \tag{2.2}$$

and the unadjusted parameter errors as

$$\delta\gamma = \gamma_N - \gamma \tag{2.3}$$

where $\underline{x}_N$ and $\underline{\gamma}_N$ are nominal values which are known and assumed to be reasonably close to the true values of $\underline{x}$ and $\underline{\gamma}$.

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The linearized version of Equation (2.1) is given by

$$\underline{z} = \underline{f}(\underline{x}_N, \underline{\gamma}_N) + B\, \delta\underline{x} - K\, \delta\underline{\gamma} + \underline{\varepsilon} \tag{2.4}$$

where B and K are the partial derivatives of the measurement with respect to $\underline{x}$ and $\underline{\gamma}$ evaluated at $(\underline{x} = \underline{x}_N,\ \underline{\gamma} = \underline{\gamma}_N)$. The nominal or computed measurement is defined as

$$\underline{z}_N = \underline{f}(\underline{x}_N, \underline{\gamma}_N) \tag{2.5}$$

The weighted least squares estimate of $\delta\underline{x}$ is given by

$$\delta\hat{\underline{x}} = (B^T W B)^{-1}\, B^T W(\underline{z} - \underline{z}_N) \tag{2.6}$$

where W is the weight matrix usually assumed to be given by

$$W^{-1} = E(\underline{\varepsilon}\ \underline{\varepsilon}^T) \tag{2.7}$$

Substituting from (2.4) and (2.5) into (2.6), the error in estimating $\delta\underline{x}$ is given by

$$(\delta\hat{\underline{x}} - \delta\underline{x}) = (B^T W B)^{-1}\, B^T W(\underline{\varepsilon} - K\delta\underline{\gamma}) \tag{2.8}$$

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The covariance of the estimate given by Equation (2.6) is defined as

$$\text{COV } \delta\hat{\underline{x}} = E[(\delta\hat{\underline{x}}-\delta\underline{x})\ (\delta\hat{\underline{x}}-\delta\underline{x})^T] \tag{2.9}$$

Substituting from (2.8) into (2.9), and assuming that $\delta\underline{\gamma}$ is uncorrelated with the measurement noise,

$$E(\delta\underline{\gamma}\ \underline{\varepsilon}^T) = 0, \tag{2.10}$$

it follows that

$$\text{COV } (\delta\hat{\underline{x}}) = (B^TWB)^{-1} + [(B^TWB)^{-1}\ B^TWK]\ \text{COV } \underline{\gamma}[(B^TWB)^{-1}\ B^TWK]^T \tag{2.11}$$

where COV $\underline{\gamma}$ is the covariance matrix associated with $\delta\underline{\gamma}$

$$\text{COV } \underline{\gamma} = E(\delta\underline{\gamma}\ \delta\underline{\gamma}^T) \tag{2.12}$$

The total covariance is decomposed into a noise contribution and an unadjusted parameter contribution. In most practical situations, the noise contribution is negligible compared to the unadjusted parameter effects.

COV $\underline{\gamma}$ is usually assumed to be a diagonal matrix implying that the components of $\delta\underline{\gamma}$ are statistically independent. For notational convenience in the following discussions, the normal matrix is defined as

$$N = B^TW\ B \tag{2.13}$$

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ORAN does not compute the entire systematic error covariance matrix given in (2.11). In order to reduce computational resource requirements only the diagonal elements of this matrix are computed.

SECTION 3.0
PARTITIONING OF THE ERROR ANALYSIS EQUATIONS

Considerable savings in both the number of computations and core storage requirements can be achieved by partitioning the matrices and vectors involved in Equations (2.6) and (2.7). This is done in both GEODYN and ORAN. The ORAN partitioning of the adjusted parameters is given by

$$\Delta \underline{x}_s = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \\ \underline{k} \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1 \\ \vdots \\ \underline{\alpha}_n \\ \hline \underline{\beta}_1 \\ \vdots \\ \underline{\beta}_n \\ \hline \underline{k} \end{bmatrix} \tag{3.1}$$

where $\underline{\alpha}_i$ contains the ith arc orbital elements, $\underline{\beta}_i$ contains all of the other ith arc adjusted parameters, $\underline{k}$ contains the adjusted parameters which are common to all arcs and n is the total number of arcs processed.

In a similar way the unadjusted parameter errors are partitioned as:

$$\underline{\gamma} = \begin{bmatrix} \underline{\gamma}_a \\ \underline{\gamma}_k \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_{a_1} \\ \vdots \\ \underline{\gamma}_{a_n} \\ \underline{\gamma}_k \end{bmatrix} \tag{3.2}$$

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where $\underline{\gamma}_{a_i}$ are the unadjusted parameters associated with the i^{th} arc and $\underline{\gamma}_k$ contains the unadjusted parameters common to all arcs.

It is assumed that the measurement noise is uncorrelated between measurements. Thus the weight matrix can be partitioned into arc components as:

$$W = \begin{bmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_n \end{bmatrix} \tag{3.3}$$

where the individual arc weight matrices W_i are also diagonal.

From the partitioning of $\Delta\underline{x}_s$ in (3.1), the matrix of partial derivatives of the measurements with respect to the adjusted parameters can be partitioned as

$$B=[B_a|B_b|B_k] = \left[\begin{array}{cccc|cccc|c} B_{a_1} & 0 & \cdots & 0 & B_{b_1} & 0 & \cdots & 0 & B_{k_1} \\ 0 & B_{a_2} & \cdots & 0 & 0 & B_{b_2} & \cdots & 0 & B_{k_2} \\ \vdots & & & & \vdots & & & & \vdots \\ 0 & 0 & \cdots & B_{a_n} & 0 & 0 & \cdots & B_{b_n} & B_{k_n} \end{array}\right] \tag{3.4}$$

From the partitioning of $\underline{\gamma}$ in (3.2), the matrix of partial derivatives of the measurements with respect to the unadjusted parameters can be partitioned as

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$$K = [K_a | K_k] = \left[\begin{array}{cccc|c} K_{a_1} & 0 & \cdots & 0 & K_{k_1} \\ & K_{a_2} & \cdots & 0 & K_{k_2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & K_{a_n} & K_{k_n} \end{array}\right] \tag{3.5}$$

From (3.3) and (3.4) it can be shown that the normal matrix defined in equation (2.10) is partitioned as

$$N = \left[\begin{array}{c|c|c} B_a^T W B_a & B_a^T W B_b & B_a^T W B_k \\ \hline B_b^T W B_a & B_b^T W B_b & B_b^T W B_k \\ \hline B_k^T W B_a & B_k^T W B_b & B_k^T W B_k \end{array}\right] \tag{3.6}$$

3.1 FIRST LEVEL PARTITIONING

The first partitioning in ORAN separates the adjusted parameters into orbital elements and all other parameters as

$$\Delta \underline{X}_s = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \\ \underline{k} \end{bmatrix} = \begin{bmatrix} \underline{\alpha} \\ \tilde{\underline{\beta}} \end{bmatrix} \tag{3.1.1}$$

and the corresponding partitioning of the normal matrix is given by

$$N = \left[\begin{array}{c|c} N_1 & N_2 \\ \hline N_2^T & N_4 \end{array}\right] \tag{3.1.2}$$

where

$$N_1 = B_a^T W B_a$$

$$N_2 = [B_a^T W B_b \mid B_a^T W B_k]$$

$$N_4 = \left[\begin{array}{c|c} B_b^T W B_b & B_b^T W B_k \\ \hline B_k^T W B_b & B_k^T W B_k \end{array}\right]$$

The partitioned inverse of the normal equations is given by

$$N^{-1} = \left[\begin{array}{c|c} M_a & M_{ab} \\ \hline M_{ab}^T & M_b \end{array}\right] = M \qquad (3.1.3)$$

Utilizing the relationship NM=1, where I is the identify matrix, it can be shown that the components of M are given by

$$M_a = N_1^{-1} + Q M_b Q^T \qquad (3.1.4)$$

$$M_b = (N_4 - N_2^T Q)^{-1} \qquad (3.1.5)$$

$$M_{ab} = -QM_b \qquad (3.1.6)$$

where

$$Q = N_1^{-1} N_2$$

Rewriting equation (2.6) in partitioned form

$$\Delta \underline{X}_s = \left[\begin{array}{c} \underline{\alpha} \\ \sim \\ \underline{\beta} \end{array}\right] = \left[\begin{array}{c|c} M_a & M_{ab} \\ \hline M_{ab}^T & M_b \end{array}\right] \left[\begin{array}{c} B_a^T W \\ \hline \tilde{B}_b^T W \end{array}\right] K \underline{Y} \qquad (3.1.7)$$

where

$$\tilde{B}_b = [B_b \vdots B_k]$$

and

$$\tilde{\underline{\beta}} = \begin{bmatrix} \underline{\beta} \\ \cdots \\ \underline{k} \end{bmatrix}$$

Performing the matrix multiplications in (3.13) and using equations (3.10), (3.11) and (3.12) the errors in the orbital elements due to errors in unadjusted parameters are given by

$$\underline{\alpha} = N_1^{-1} B_a^T W K \underline{\gamma} - Q \tilde{\underline{\beta}} \tag{3.1.8}$$

And the errors in the rest of the adjusted parameters due to errors in the unadjusted parameters are given as

$$\tilde{\underline{\beta}} = M_b [\tilde{B}_b^T - Q^T B_a^T] W K \underline{\gamma} \tag{3.1.9}$$

3.2 SECOND LEVEL PARTITIONING

In the second level of partitioning the common adjusted parameters are separated from the remaining arc adjusted parameters as

$$\tilde{\underline{\beta}} = \begin{bmatrix} \underline{\beta} \\ \cdots \\ \underline{k} \end{bmatrix} \tag{3.2.1}$$

For notational convenience define the matrix H as

$$H = \left[I - B_a N_1^{-1} B_a^T W \right] \tag{3.2.2}$$

and note that H is a diagonal matrix

$$H = \begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_n \end{bmatrix}$$

with the typical diagonal element corresponding to the ith arc is given by

$$H_i = [I - B_{a_i} (B_{a_i}^T W_i B_{a_i})^{-1} B_{a_i}^T W_i] \qquad (3.2.3)$$

Using the matrix partitioning shown in equations (3.6) and (3.1.2) equation (3.1.9) can be expanded as

$$\left[\begin{array}{c} \underline{\beta} \\ \hline \underline{k} \end{array}\right] = \left[\begin{array}{c|c} M_{b_1} & M_{b_2} \\ \hline M_{b_2}^T & M_{b_4} \end{array}\right] \left[\begin{array}{c} B_b^T W H \\ \hline B_k^T W H \end{array}\right] [K_a \underline{Y}_a + K_n \underline{Y}_k] \qquad (3.2.4)$$

substituting the definitions from equation (3.8) in equation (3.11) it can be shown that M_b is given by

$$M_b = \left[\begin{array}{c|c} N_{b_1} & N_{b_2} \\ \hline N_{b_2}^T & N_{b_4} \end{array}\right]^{-1} = \left[\begin{array}{c|c} M_{b_1} & M_{b_2} \\ \hline M_{b_2}^T & M_{b_4} \end{array}\right] \qquad (3.2.5)$$

where

$$N_{b_1} = B_b^T W H B_b$$

$$N_{b_2} = B_b^T W H B_k$$

$$N_{b_4} = B_k^T W H B_k$$

Again using the partitioned form of the matrix inverse

$$M_{b_1} = N_{b_1}^{-1} + \tilde{Q} M_{b_4} \tilde{Q}^T \qquad (3.2.6)$$

$$M_{b_4} = [N_{b_4} - N_{b_2}^T \tilde{Q}]^{-1} \tag{3.2.7}$$

$$M_{b_2} = -\tilde{Q} M_{b_4} \tag{3.2.8}$$

where

$$\tilde{Q} = N_{b_1}^{-1} N_{b_2}$$

Using the relationships from (3.21), (3.22) and (3.23) along with the partitioned forms of K and $\underline{\gamma}$ equation (3.19) can be written as

$$\underline{\beta} = N_{b_1}^{-1} B_b W H [K_a \underline{\gamma}_\alpha + K_k \underline{\gamma}_k] - \tilde{Q} \underline{k} \tag{3.2.9}$$

$$\underline{k} = M_{b_4} [B_k^T W H - \tilde{Q}^T B_b^T W H] [K_a \underline{\gamma}_a + K_k \underline{\gamma}_k]$$

3.3 ACCUMULATION OF SYSTEMATIC ERRORS AND ERROR SENSITIVITIES

Summarizing the results of the previous two sections the systematic error in the common adjusted parameters is given by

$$\underline{k} = M_{b_4} [B_k^T - \tilde{Q}^T B_b^T] W H [K_a \underline{\gamma}_a + K_k \underline{\gamma}_k] \tag{3.3.1}$$

The error in arc adjusted parameters (exclusive of orbital elements) is given by

$$\underline{\beta} = N_{b_1}^{-1} B_b W H [K_a \underline{Y}_a + K_k \underline{Y}_k] - \tilde{Q} \underline{k} \tag{3.3.2}$$

And the error in the orbital elements is given by

$$\underline{\alpha} = N_1^{-1} B_a^T W [K_a \underline{Y}_a + K_k \underline{Y}_k] - Q \begin{bmatrix} \underline{\beta} \\ \underline{k} \end{bmatrix} \tag{3.3.3}$$

Error sensitivities are computed by taking the partial derivatives of each adjusted parameter with respect to each unadjusted parameter. In order to minimize core storage ORAN processes one arc of data at a time. Various matrices which involve summations over all arcs are accumulated in core. Components of the sensitivity matrices which are indigenous to individual arcs are temporarily stored on a tape or disk scratch file. Therefore equations (3.25), (3.26) and (3.27) will be used to develop these partial derivatives which will be expressed in terms of the individual arc matrices.

From equation (3.25) the partial derivatives of the common adjusted parameters with respect to the common unadjusted parameters are given by

$$\frac{\partial \underline{k}}{\partial \underline{Y}_k} = M_{b_4} [B_k^T - \tilde{Q}^T B_b^T] W H K_k \tag{3.3.4}$$

In terms of the individual arc matrices M_{b_4} can be expressed as

$$M_{b_4} = \left\{ \sum_{i=1}^{n} B_{k_i}^T W_i H_i [I - B_{b_i} (B_{b_i}^T W_i B_{b_i})^{-1} B_{b_i}^T W_i H_i] B_{k_i} \right\}^{-1} \tag{3.3.5}$$

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Expanding equation (3.28) in a similar manner

$$\frac{\partial \underline{k}}{\partial \underline{\gamma}_k} = M_{b_4} \left\{ \sum_{i=1}^{n} B_{k_i}^T W_i H_i [I - B_{b_i} (B_{b_i}^T W_i H_i B_{b_i})^{-1} B_{b_i}^T W_i H_i] K_{k_i} \right\} \tag{3.3.6}$$

The partial derivatives of the common adjusted parameter with respect to the unadjusted parameters of the i-th arc are given by

$$\frac{\partial \underline{k}}{\partial \underline{\gamma}_{a_i}} = M_{b_4} \left\{ B_{k_i}^T W_i H_i \left[I - B_{b_i} (B_{b_i}^T W_i H_i B_{b_i})^{-1} B_{b_i}^T W_i H_i \right] K_{a_i} \right\} \tag{3.3.7}$$

From equation (3.26) the partial derivatives of the i-th arc adjusted parameters (exclusive of the orbital element) with respect to the common unadjusted parameters are given by

$$\frac{\partial \underline{\beta}_i}{\partial \underline{\gamma}_k} = (B_{b_i}^T W_i H_i B_{b_i})^{-1} B_{a_i}^T W_i H_i K_{k_i} - \tilde{Q}_i \frac{\partial \underline{k}}{\partial \underline{\gamma}_k}. \tag{3.3.8}$$

where

$$\tilde{Q}_i = (B_{b_i}^T W_i H_i B_{b_i})^{-1} B_{a_i}^T W_i H_i B_{k_i}$$

Also from equation (3.26) the partial derivatives of the i-th arc adjusted parameters (exclusive of orbital elements) with respect to the i-th arc unadjusted parameters are given by

$$\frac{\partial \underline{\beta}_i}{\partial \underline{\gamma}_{a_i}} = (B_{b_i}^T W_i H_i B_{b_i})^{-1} B_{b_i}^T W_i H_i K_{a_i} - \tilde{Q}_i \frac{\partial \underline{k}}{\partial \underline{\gamma}_{a_i}} \tag{3.3.9}$$

It is necessary to compute the effect of one arc's unadjusted parameters on another arc's adjusted parameters. Thus from (3.26) the partial derivatives of ith arc adjusted parameters (exclusive of orbital elements) with respect to the jth arc unadjusted parameters are given by

$$\frac{\partial \underline{\beta}_i}{\partial \underline{\gamma}_{a_j}} = -\tilde{Q}_i \frac{\partial \underline{k}}{\partial \underline{\gamma}_{a_j}} \tag{3.3.10}$$

In a similar manner from equation (3.27) the partial derivatives of the i-th arc orbital elements with respect to the unadjusted parameters are given by

$$\frac{\partial \underline{\alpha}_i}{\partial \underline{\gamma}_{a_i}} = (B_{a_i}{}^T W_i B_{a_i})^{-1} B_{a_i}{}^T W_i \left\{ K_{a_i} - B_{b_i} \frac{\partial \underline{\beta}_i}{\partial \underline{\gamma}_{a_i}} - B_{k_i} \frac{\partial \underline{k}}{\partial \underline{\gamma}_{a_i}} \right\} \tag{3.3.11}$$

$$\frac{\partial \underline{\alpha}_i}{\partial \underline{\gamma}_{a_j}} = -(B_{a_i}{}^T W_i B_{a_i})^{-1} B_{a_i}{}^T W_i \left\{ B_{b_i} \frac{\partial \underline{\beta}_i}{\partial \underline{\gamma}_{a_j}} + B_{k_i} \frac{\partial \underline{k}}{\partial \underline{\gamma}_{a_j}} \right\} \tag{3.3.12}$$

$$\frac{\partial \underline{\alpha}_i}{\partial \underline{\gamma}_k} = (B_{a_i}{}^T W_i B_{a_i})^{-1} B_{a_i}{}^T W_i \left\{ K_{k_i} - B_{b_i} \frac{\partial \underline{\beta}_i}{\partial \underline{\gamma}_k} - B_{k_i} \frac{\partial \underline{k}}{\partial \underline{\gamma}_k} \right\} \tag{3.3.13}$$

4.0 PROPAGATION OF ERROR SENSITIVITIES

Various ORAN output displays require computation of the effects of unadjusted parameter errors on the satellite position and motion at some time other than epoch. This is given by

$$\Delta \underline{a}_t = \frac{\partial \underline{a}_t}{\partial \underline{x}_0} \Delta \underline{x}_0 + \frac{\partial \underline{a}_t}{\partial \underline{r}} \underline{\gamma} \quad (4.1)$$

where $\underline{a}_t$ is the satellite positon and velocity at time t and $\underline{x}_0$ is the vector of adjusted parameters, including the orbital elements at epoch time. The first term in (4.1) is due to errors in the estimated epoch state and the second term is due to the effect of the unadjusted parameters on propagation of that epoch state to time t. The error in the epoch state can be decomposed into a random component given by

$$\Delta \underline{x}_r = (B^T W B)^{-1} B^T W \underline{\varepsilon} \quad (4.2)$$

and a systematic component due to errors in the unadjusted parameters given by

$$\Delta \underline{x}_s = -(B^T W B)^{-1} B^T W K \underline{\gamma} \quad (4.3)$$

Substituting from (4.2) and (4.3) into (4.1), this gives

$$\Delta \underline{a}_t = \frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)^{-1} B^T W \underline{\varepsilon} + \left[\frac{\partial \underline{a}_t}{\partial \underline{\gamma}} - \frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)^{-1} B^T W K \right] \underline{\gamma} \quad (4.4)$$

The first term in (4.4) is the random error in $\underline{a}_t$ due to measurement noise. The second term in the systematic error in $\underline{a}_t$ is due to the unadjusted parameters. These two components are independent since

$$E(\underline{\gamma}\ \underline{\varepsilon}^T) = 0$$

The partial derivatives $\frac{\partial \underline{a}_t}{\partial x_0}$ and $\frac{\partial \underline{a}_t}{\partial \underline{\gamma}}$ are obtained through integration of the appropriate variational equations as described in Section 5.

The sensitivity of $\underline{a}_t$ to the unadjusted parameters is given by

$$\frac{\partial \underline{a}_t}{\partial \underline{\gamma}} = - \frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)\ B^T W K + \frac{\partial \underline{a}_t}{\partial \underline{\gamma}} \tag{4.5}$$

Equation (4.5) is used by ORAN to compute the sensitivity of each element in $\underline{a}_t$ to each unadjusted parameter. The total covariance matrix associated with $\underline{a}_t$ is given by

$$\begin{aligned} E(\Delta\underline{a}_t \Delta\underline{a}_t^T) = & \left[\frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)^{-1} B^T W\right] E(\underline{\varepsilon}\ \underline{\varepsilon}^T) \left[\frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)^{-1} B^T W\right]^T \\ & + \left[\frac{\partial \underline{a}_t}{\partial \underline{\gamma}} - \frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)^{-1}\ B^T W K\right] \mathrm{COV}\ \underline{\gamma} \left[\frac{\partial \underline{a}_t}{\partial \underline{\gamma}} - \frac{\partial \underline{a}_t}{\partial \underline{x}_0} (B^T W B)^{-1}\ B^T W K\right]^T \end{aligned} \tag{4.6}$$

The first component of (4.6) is the noise only contribution and the second component is the unadjusted parameter contribution. To avoid excessive computations, ORAN computes only the diagonal elements of these covariance matrices. For each element of $\underline{a}_t$ the program computes the noise only sigma, the sigma due to unadjusted parameters and the total sigma. 22<

SECTION 5.0
MEASUREMENT MODELING AND MEASUREMENT ERRORS

5.1 MEASUREMENT MODELING

In order to simulate a least squares data reduction ORAN must compute partial derivatives of the measurements with respect to both adjusted and unadjusted parameters. The basic types of observation in ORAN are:

- right ascension and declination
- range
- range rate
- ℓ and m direction cosines
- X and Y angles
- Azimuth and elevation
- Altimeter height
- Inter-Satellite Range and Range Rate

These measurements are geometric in nature. The computed values for the observations are obtained by applying geometric relationships to the computed values for the relative positions and velocities of the satellite and the observer at the desired time.

5.1.1 Range and Range Rate

Range:

Consider the station-satellite vector:

$$\bar{\rho} = \bar{r} - \bar{r}_{ob} \tag{5.1.1.1}$$

where

$\bar{r}$ is the satellite position vector (x,y,z) in the geocentric Earth-fixed system, and

$\bar{r}_{ob}$ is the station vector in the same system.

The magnitude of this vector, ρ, is the (slant) range, which is one of the measurements.

Range rate:

The time rate of change of this vector $\bar{\rho}$ is

$$\dot{\bar{\rho}} = \dot{\bar{r}} \tag{5.1.1.2}$$

as the velocity of the observer in the Earth-fixed system is zero. Let us consider that

$$\bar{\rho} = \rho\hat{u} \tag{5.1.1.3}$$

where

$\hat{u}$ is the unit vector in the direction of $\bar{\rho}$.

Thus we have

$$\dot{\bar{\rho}} = \dot{\rho}\hat{u} + \rho\dot{\hat{u}} \qquad (5.1.1.4)$$

The quantity $\dot{\rho}$ in the above equation is the computed value for the range rate and is determined by

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{r}} \qquad (5.1.1.5)$$

The partial derivatives of range and range rate with respect to the satellite position and velocity are given below. All are in the geocentric, Earth-fixed system. (The r_i refer to the Earth-fixed components of $\bar{r}$.)

Range:

$$\frac{\partial \rho}{\partial r_i} = \frac{\rho_i}{\rho} \qquad (5.1.1.6)$$

Range rate:

$$\frac{\partial \dot{\rho}}{\partial r_i} = \frac{1}{\rho}\left[\dot{r}_i - \frac{\dot{\rho}\rho_i}{\rho}\right] \qquad (5.1.1.7)$$

$$\frac{\partial \dot{\rho}}{\partial \dot{r}_i} = \frac{\rho_i}{\rho} \qquad (5.1.1.8)$$

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The derivatives of range and range rate with respect to time are presented below. All are in the Earth-fixed system.

Range:

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{r}} \qquad (5.1.1.9)$$

Range Rate:

The range rate derivative deserves special attention. Remembering that

$$\dot{\bar{\rho}} = \dot{\bar{r}} , \qquad (5.1.1.10)$$

We write

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{\rho}} \qquad (5.1.1.11)$$

Thus

$$\ddot{\rho} = \hat{u} \cdot \dot{\bar{\rho}} + \dot{\hat{u}} \cdot \ddot{\bar{\rho}} \qquad (5.1.1.12)$$

Because

$$\dot{\bar{\rho}} = \frac{d}{dt}(\rho\hat{u}) = \rho\dot{\hat{u}} + \dot{\rho}\hat{u} \tag{5.1.1.13}$$

we may substitute in Equation 4 above for $\dot{\hat{u}}$:

$$\ddot{\rho} = \frac{1}{\rho}(\dot{\bar{\rho}} \cdot \dot{\bar{\rho}} - \dot{\rho}\,\hat{u} \cdot \dot{\bar{\rho}}) + \hat{u}\cdot\ddot{\bar{\rho}} \tag{5.1.1.14}$$

or, as

$$\dot{\rho} = \hat{u} \cdot \dot{\bar{\rho}} \tag{5.1.1.15}$$

we may write

$$\ddot{\rho} = \frac{1}{\rho}(\dot{\bar{\rho}} \cdot \dot{\bar{\rho}} - \dot{\rho}^2 + \bar{\rho} \cdot \ddot{\bar{\rho}}) \tag{5.1.1.16}$$

The gradient of this potential with respect to the Earth-fixed position coordinates of the satellite is the part of $\ddot{\bar{\rho}}$ due to the geopotential:

$$\frac{\partial U}{\partial r_i} = -\frac{GM}{r^3}\left[1 - \frac{3\, a_e^2\, C_{20}}{2\, r^2}\left(5 \sin^2\phi - 1 - 2\,\frac{z}{r_i}\right)\right] r_i \qquad (5.1.1.17)$$

We must add to this the effect of the rotation of the coordinate system. (The Earth-fixed coordinate system rotates with respect to the true of date coordinates with a rate $\dot{\theta}_g$, the time rate of change of the Greenwich hour angle.)

The components of $\ddot{\bar{\rho}}$ are then

$$\ddot{\rho}_1 = \frac{\partial U}{\partial r_1} + [\dot{x} \cos \theta_g + \dot{y} \sin \theta_g]\, \dot{\theta}_g + \dot{r}_2\, \dot{\theta}_g \qquad (5.1.1.18)$$

$$\ddot{\rho}_2 = \frac{\partial U}{\partial r_2} + [-\dot{x} \sin \theta_g + \dot{y} \cos \theta_g]\, \dot{\theta}_g - \dot{r}_1\, \dot{\theta}_g \qquad (5.1.1.19)$$

$$\ddot{\rho}_3 = \frac{\partial U}{\partial r_3} = \frac{\partial U}{\partial z} \qquad (5.1.1.20)$$

where $\dot{x}$ and $\dot{y}$ are the true of date satellite velocity components.

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5.1.2 Altimeter Height

The altimeter height is unique in that the satellite is making the observation. While this is actually a measurement from the satellite to the surface of the Earth, it is taken to be a measurement of the spheroid height and the time rate of change of that quantity for obvious reasons. Using the formula for spheroid height determined in Section 5.1 of the GEODYN Program Documentation Volume I,

$$H_{alt} = r - a_e - \frac{3}{2} a_e f^2 \left(\frac{z}{r}\right)^4 + \left(a_e f + \frac{3}{2} a_e f^2\right) \left(\frac{z}{r}\right)^2 \qquad (5.1.2.1)$$

where

a_e is the Earth's mean equatorial radius,

f is the Earth's flattening, and

z is r_3, the z component of the Earth-fixed satellite vector.

For error analysis purposes, the partial derivatives of the altimeter measurement with respect to the satellite position, velocity, and time are needed. These are derived directly from the analytical expression for H_{ALT}.

$$\frac{\partial H_{alt}}{\partial r_i} = \frac{r_i}{r} + \frac{1}{r}\left[\left(2\ a_e\ f + 3\ a_e\ f^2\right)\left(\frac{z}{r}\right) - 6\ a_e\ f^2\left(\frac{z}{r}\right)^3\right] X \left[\frac{\partial z}{\partial r_i} - \frac{z x_i}{r^2}\right] \tag{5.1.2.2}$$

The time derivative of altimeter range is given by

$$\dot{H}_{ALT} = \frac{\partial H_{ALT}}{\partial r_1}\dot{r}_1 + \frac{\partial H_{ALT}}{\partial r_2}\dot{r}_2 + \frac{\partial H_{ALT}}{\partial r_3}\dot{r}_3 \tag{5.1.2.3}$$

The altimeter measurement is actually made to the geoid surface instead of the spheroid surface. A detailed geoid is necessary, however, to model the altimeter measurements to properly exploit their full accuracy.

5.1.3 Right Ascension and Declination

The topocentric right ascension α and declination δ are inertial coordinate system measurements as illustrated in Figure 4.1. ORAN computes these angles from the components of the Earth-fixed station-satellite vector and the Greenwich hour angle θ_g.

$$\alpha = \tan^{-1}\left(\frac{\rho_2}{\rho_1}\right) + \theta_g \tag{5.1.3.1}$$

$$\delta = \sin^{-1}\left(\frac{\rho_3}{\rho}\right) \tag{5.1.3.2}$$

The partial derivatives of these measurements with respect to the Earth-fixed satellite position vector $\bar{r}$ are given by

Right Ascension:

$$\frac{\partial\alpha}{\partial r_1} = \frac{-\rho_2}{\sqrt{\rho_1^{\,2} + \rho_2^{\,2}}} \tag{5.1.3.3}$$

$$\frac{\partial\alpha}{\partial r_2} = \frac{\rho_1}{\sqrt{\rho_1^{\,2} + \rho_2^{\,2}}} \tag{5.1.3.4}$$

$$\frac{\partial\delta}{\partial r_3} = 0 \tag{5.1.3.5}$$

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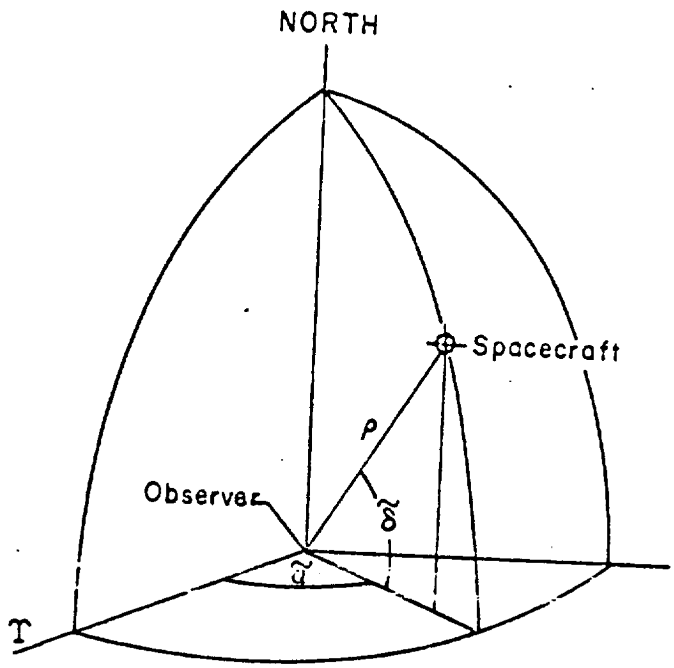

Figure 5.1. Topocentric right ascension & declination angles

Declination:

$$\frac{\partial \delta}{\partial r_1} = \frac{-\rho_1 \rho_3}{\rho^2 \sqrt{\rho_1^2 + \rho_2^2}} \tag{5.1.3.6}$$

$$\frac{\partial \delta}{\partial r_2} = \frac{-\rho_2 \rho_3}{\rho \sqrt{\rho_1^2 + \rho_2^2}} \tag{5.1.3.7}$$

$$\frac{\partial \delta}{\partial r_3} = \frac{\sqrt{\rho_1^2 + \rho_2^2}}{\rho^2} \tag{5.1.3.8}$$

The time derivatives are given by

Right ascension: $$\dot{\alpha} = \frac{u_1 \dot{r}_2 - u_2 \dot{r}_1}{\rho\,(1-u_3^2)} \tag{5.1.3.9}$$

Declination: $$\dot{\delta} = \frac{\dot{r}_3 - \dot{\rho}\, u_3}{\rho \sqrt{1-u_3^2}} \tag{5.1.3.10}$$

where the unit vector $\hat{u}$ is defined as

$$\hat{u} = \frac{\bar{\rho}}{|\bar{\rho}|}$$

5.1.4 Direction Cosines

There are three direction cosines associated with the station-satellite vector in the topocentric system. Description of these measurements requires the $\hat{N}$, $\hat{Z}$, and $\hat{E}$ (north, zenith and east baseline unit vectors which describe the tropocentric system along with the $\hat{u}$). The direction cosines are computed as:

$$\ell = \hat{u} \cdot \hat{E} \tag{5.1.4.1}$$

$$m = \hat{u} \cdot \hat{N} \tag{5.1.4.2}$$

$$n = \hat{u} \cdot \hat{Z} \tag{5.1.4.3}$$

The ℓ and m direction cosines are observation types for ORAN.

The partial derivatives of the direction cosines with respect to the satellite position vector are given by

$$\frac{\partial \ell}{\partial r_i} = \frac{1}{\rho} \left[E_i - \ell u_i \right] \tag{5.1.4.4}$$

$$\frac{\partial m}{\partial r_i} = \frac{1}{\rho} \left[N_i - m u_i \right] \tag{5.1.4.5}$$

$$\frac{\partial n}{\partial r_i} = \frac{1}{\rho} \left[Z_i - n u_i \right] \tag{5.1.4.6}$$

where

E_i = component of $\hat{E}$ in the r_i direction

N_i = component of $\hat{N}$ in the r_i direction

Z_i = component of $\hat{Z}$ in the r_i direction

The time derivatives of the ℓ and m direction cosines are given by

$$\dot{\ell} = \frac{\dot{\bar{\rho}} \cdot \hat{E} - \ell\dot{\rho}}{\rho} \tag{5.1.4.7}$$

$$\dot{m} = \frac{\dot{\bar{\rho}} \cdot \hat{N} - m\dot{\rho}}{\rho} \tag{5.1.4.8}$$

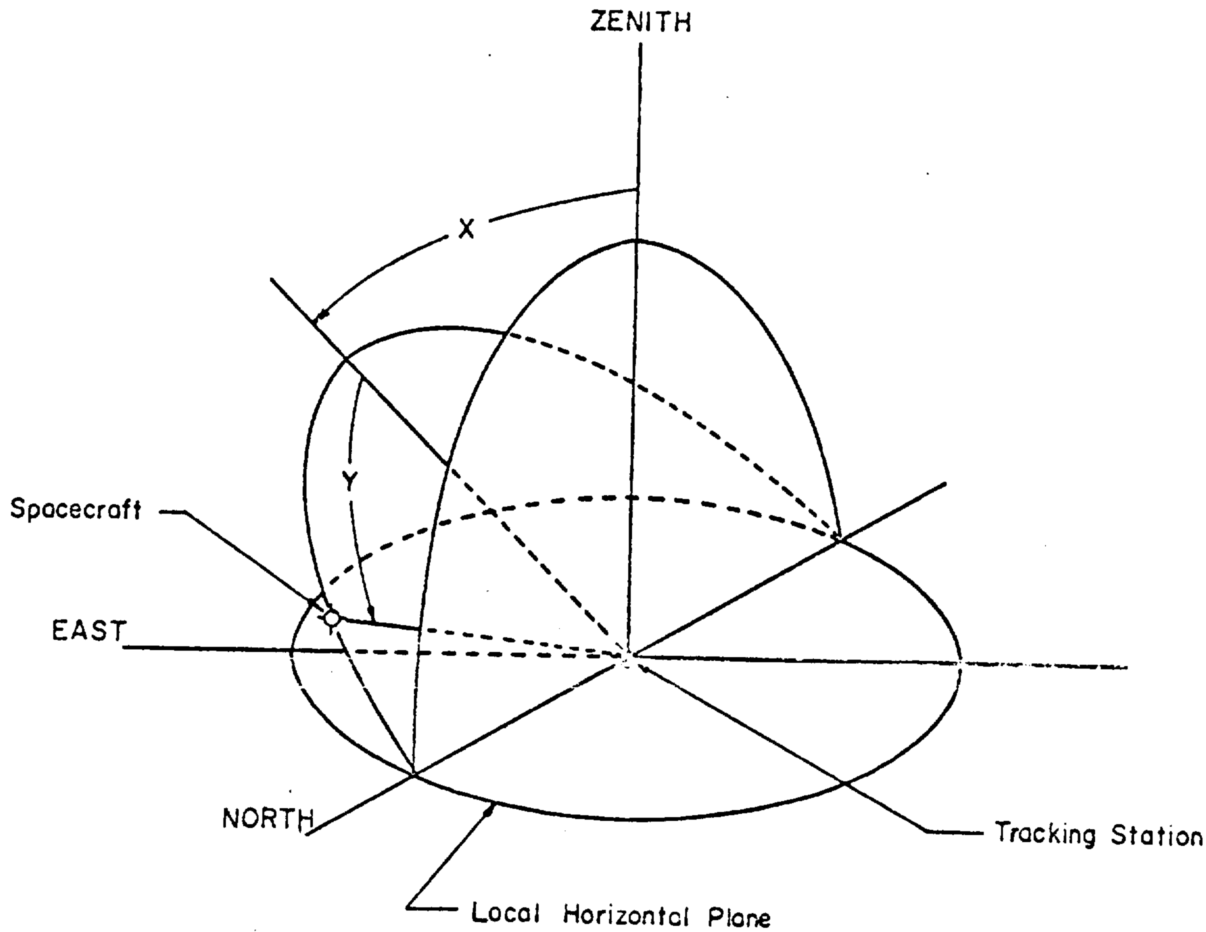

Figure 5.2. X and Y Angles

5.1.5 x and y Angles

The x and y angles, as illustrated in Figure 5.2, are computed in a tropocentric coordinate system as

$$X_a = \tan^{-1}\left(\frac{\ell}{n}\right) \tag{5.1.5.1}$$

$$Y_a = \sin^{-1}(m) \tag{5.1.5.2}$$

The derivatives of the x and y angles with respect to the satellite position vector are

$$\frac{\partial X_a}{\partial r_i} = \frac{nE_i - \ell Z_i}{\rho(1-m^2)} \tag{5.1.5.3}$$

$$\frac{Y_a}{\partial r_i} = \frac{N_i - mu_i}{\rho\sqrt{1-m^2}} \tag{5.1.5.4}$$

and the time derivatives are given by

$$\dot{X}_a = \frac{\dot{\bar{\rho}} \cdot (n\hat{E} - \ell\hat{Z})}{\rho(1-m^2)} \tag{5.1.5.5}$$

$$\dot{Y}_a = \frac{\dot{\bar{\rho}} \cdot \hat{N} - m\dot{\rho}}{\rho\sqrt{1-m^2}} \tag{5.1.5.6}$$

5.1.6 Azimuth and Elevation

Figure 5.3 illustrates the measurement of azimuth and elevation. These angles are computed in the topocentric coordinate system as

$$A_z = \tan^{-1}\frac{\ell}{m} \tag{5.1.6.1}$$

$$E_\ell = \sin^{-1}(n) \tag{5.1.6.2}$$

The partial derivatives with respect to the satellite position vector are given by

$$\frac{\partial A_z}{\partial r_i} = \frac{mE_i - \ell N_i}{\rho\sqrt{1-n^2}} \tag{5.1.6.3}$$

$$\frac{\partial E_\ell}{\partial r_i} = \frac{Z_i - nu_i}{\rho(1-n^2)} \tag{5.1.6.4}$$

and the partial derivatives with respect to time are

$$\dot{A}_z = \frac{\dot{\bar{\rho}} \cdot (m\hat{E} - \ell\hat{N})}{\rho\ (1-m^2)} \tag{5.1.6.5}$$

$$\dot{E}_\ell = \frac{\dot{\bar{\rho}} \cdot \hat{Z} - m\dot{\rho}}{\rho\ \sqrt{1-m^2}} \tag{5.1.6.6}$$

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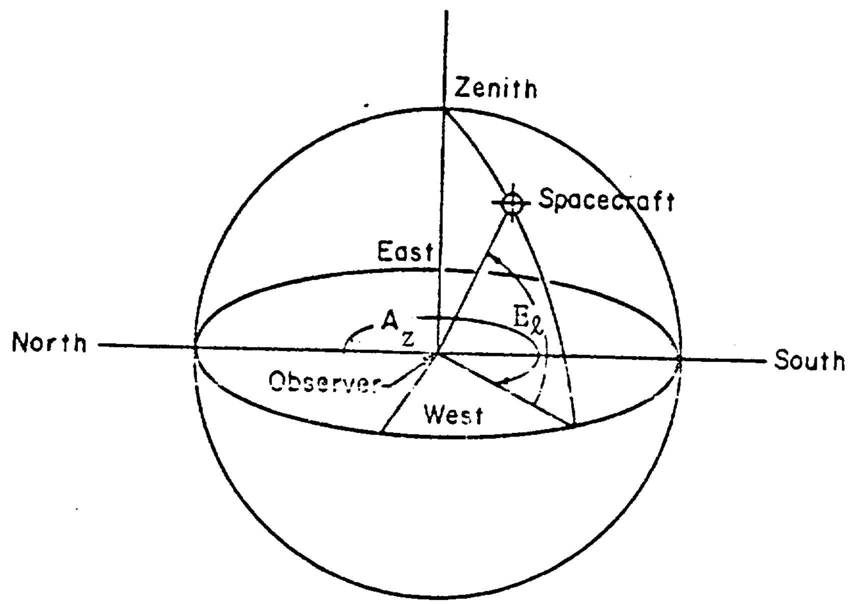

Figure 5.3. Azimuth and Elevation Angles

5.1.7 Satellite-Satellite Range and Range Rate

The range measurement from one satellite to another is computed as follows.

Let $\vec{X}_1$ be the inertial coordinates of the transmitting satellite and $\vec{X}_2$ the inertial coordinates of the receiving satellite. Then the range (or distance) between the two satellites is given by

$$R = \sqrt{(\vec{X}_2 - \vec{X}_1) \cdot (\vec{X}_2 - \vec{X}_1)} \quad . \tag{5.1.7.1}$$

The time rate of change of range, or just range rate, is calculated by differentiating (1) with respect to time:

$$\dot{R} = \frac{(\vec{X}_2 - \vec{X}_1) \cdot (\dot{\vec{X}}_2 - \dot{\vec{X}}_1)}{R} \tag{5.1.7.2}$$

ORAN can also simulate relay range and range rate measurements. Relay range is simply the sum of two range measurements: the range from some transmitting station to a satellite plus the range from that same satellite to another satellite. This configuration is given in Figure 5.4. Thus, according to the notation in Figure 5.4, the relay range is defined as

$$\begin{aligned} R_{\text{Relay}} &= \left|\vec{R}_1\right| + \left|\vec{R}_2\right| . \\ &= R_1 + R_2 \end{aligned} \tag{5.1.7.3}$$

Likewise, the relay range rate is the time derivation of (3), or

$$\dot{R}_{\text{Relay}} = \dot{R}_1 + \dot{R}_2 \ . \tag{5.1.7.4}$$

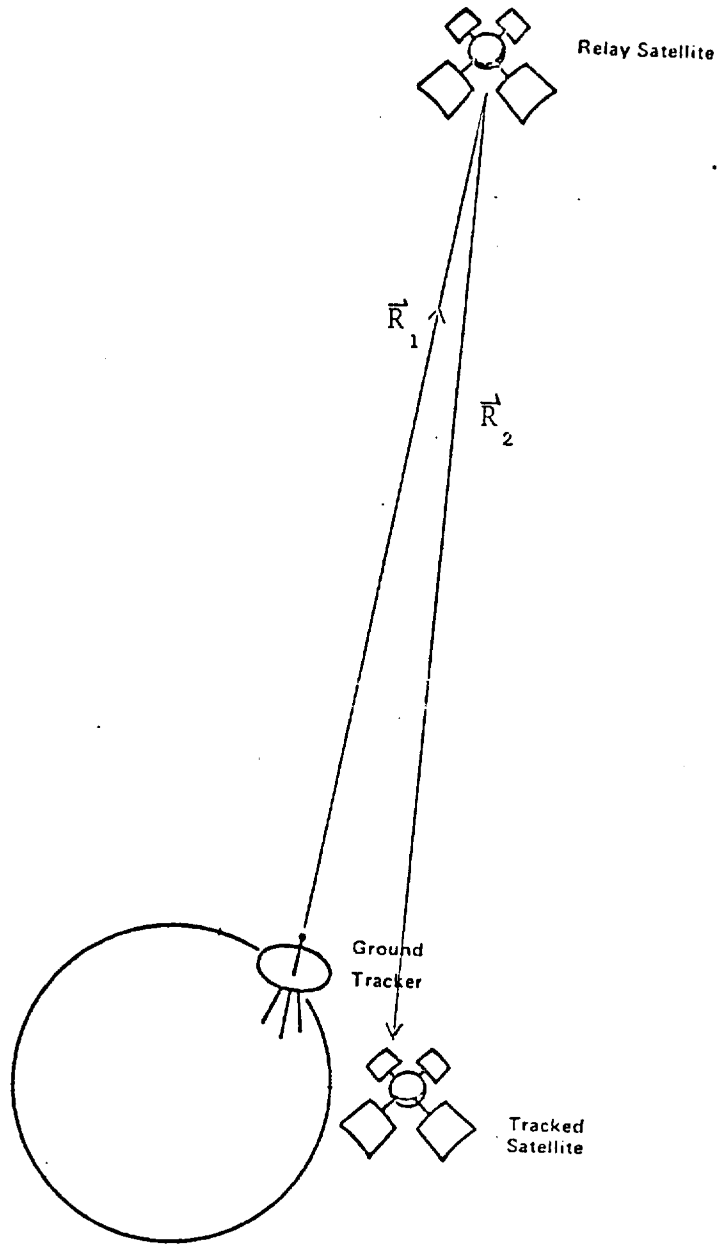

Figure 5.4. Geometry for Satellite-Satellite Tracking

Since the partial derivative of a sum is equal to the sum of the individual partial derivatives, any partial derivative of R_{Relay} or $\dot{R}_{Relay}$ can be found by summing the individual partials of the two quantities in the sum. Specifically, if one wants the partial with respect to some parameter σ, then

$$\frac{\partial R_{relay}}{\partial \sigma} = \frac{\partial R_1}{\partial \sigma} + \frac{\partial R_2}{\partial \sigma} , \tag{5.1.7.5}$$

$$\frac{\partial \dot{R}_{relay}}{\partial \sigma} = \frac{\partial \dot{R}_1}{\partial \sigma} + \frac{\partial \dot{R}_2}{\partial \sigma} . \tag{5.1.7.6}$$

Table 5.1. Measurement Error Parameters

Error Source	Error Parameters	Measurements Affected
Sensor Alignment	b = Bias Error	All
Station Timing	Δt_1 = Timing Error	All
Uncertainty in satellite time corresponding to measurement	Δt_2 = Transit Time Error	All
Uncertainty in converting between UT1 and UTC time	Δt_3 = Time Conversion Error	All
Tropospheric Refraction	η = Refraction Coefficient	All
Ionospheric Refraction (S Band Only)	$\Delta\rho_I, \Delta\dot{\rho}_I$ = Ionospheric corrections for range and range rate	Range, Range Rate
Space Plasma	AR_{sp}=Space Plasma Correction	Range, Range Rate
Station Survey	$\Delta X, \Delta Y, \Delta Z$ = Error in locating station on its local datum	All
Spin Axis Distance	Δd = error in station distance from the Earth's instantaneous spin axis	All
Longitude Error	$\Delta\lambda$ = Station Longitude Error	All
Polar Movement	X_p, Y_p = Angular variations in the position of the Earth axis of rotation relative to values at a fixed epoch	All
Solid Earth Tidal Displacements	η_2, ℓ_2 = Love numbers of the second and third kind	All
Ocean Loading Displacement	L = Distance from Shoreline	All
Location of Station relative to Earth's center of mass	$\Delta X, \Delta Y, \Delta Z$ = Station location errors relative to Earth's center of mass	All

5.2 MEASUREMENT ERRORS

This section discusses the individual measurement error sources and the mathematics used to model them in ORAN. Table 5.1 lists all measurement error sources and the associated error parameters.

5.2.1 Bias

Bias errors are considered as constants which must be added on to the computed value in order to better represent the observed. Therefore,

$$Z_c = Z_c' + b \tag{5.2.1.1}$$

where

Z_c is the computed measurement corrected for any biases

Z_c' is the computed measurement based only on satellite geometry

b is the bias

Thus

$$\frac{\partial Z_c}{\partial b} = 1 \text{ for all measurement types.} \tag{5.2.1.2}$$

5.2.2 Timing

Should the time tag of the measurement be incorrect, then a correction to this time tag is called a correction to timing. Any error in this correction can be found by computing the partial derivative of the computed measurement and multiplying by the time error, or

$$\frac{\partial Z_c}{\partial t} \Delta t = \Delta Z_{c_{Tim}} . \tag{5.2.2.1}$$

where Δt is the timing error, but

$$\begin{aligned} \frac{\partial Z_c}{\partial t} &= \frac{\partial Z_c}{\partial \bar{r}} \cdot \frac{\partial \bar{r}}{\partial t} \\ &= \frac{\partial Z_c}{\partial \bar{r}} \cdot \dot{\bar{r}} \end{aligned} \tag{5.2.2.2}$$

where

$\bar{r}$ is the satellite position vector

$\dot{\bar{r}}$ is the satellite velocity vector.

Thus

$$\Delta Z_{c_{Tim}} = \frac{\partial Z_c}{\partial \bar{r}} \cdot \dot{\bar{r}} \; \Delta t \tag{5.2.2.3}$$

5.2.3 Transit Time

The time tag assigned to the measurement is usually the time at which the station receives the signal. But the satellite retransmitted the signal to the receiving station at some earlier time. Therefore, two times are involved. To simplify matters somewhat, the observed measurements usually have been corrected so the computation process of the computed values can assume the satellite and station at the same time. An error in this transit time correction is similar to the timing error just discussed, but now the systematic error is some fractional part of the estimated transit time, i.e.,

$$\Delta Z_{c_{\text{transit time}}} = \frac{\partial Z_c}{\partial \bar{r}} \cdot \dot{\bar{r}} \quad \Delta T \tag{5.2.3.1}$$

where ΔT is the error in the transit time. If p is the fractional error in transit time correction, then

$$\Delta T = p\left(\frac{R}{c}\right) \tag{5.2.3.2}$$

where

R is range

C is speed of light

5.2.4 STATION LOCATIONS

In the preceding sections measurement equations have been developed for a relative satellite-station geometry. These measurements are used to determine satellite position and motion in an inertial coordinate system at some epoch time. In transforming from the relative coordinate system at the time of measurement to the inertial system at epoch we must account for both the movement of the satellite and the movement of the station in inertial coordinates during the time period between measurement and epoch. The equations of motion for the satellite are given in Sections 5 and 6. The station movement is due to the movement of the Earth (considered as a solid body) and to the movement of the Earth's crust relative to the central mass. Station coordinates are referenced to a particular epoch time (usually 1900.0) and the movement of the station since this time is included in the computation of a station-satellite measurement. The solid body component of station motion is due to the Earth's rotation, nutation and precession. These are very well known and make negligible contributions to station location error.

The effect of an error in station location on the computed measurement can be determined by the following expression

$$Z_{c_{STA}} = - \frac{\partial Z_c}{\partial \bar{r}} \cdot \Delta \bar{r}_{STA} \qquad (5.2.4.1)$$

where

$\Delta Z_{c_{STA}}$	is the error in the computed measurement due to an error in station position
$\bar{r}$	is the satellite position vector
$\Delta \bar{r}_{STA}$	is the error in station position (in same coordinate system as $\bar{r}$)

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This is obvious if one considers, for example, a range measurement from a station to a satellite. If the station height were raised, the same effect on the measurement would occur if the satellite height were lowered.

At any measurement time the total station location error can be expressed as

$$\Delta\bar{r}_{STA} = \Delta\bar{r}_S + \Delta\bar{r}_E + \Delta\bar{r}_{SE} + \Delta\bar{r}_{OL} + \Delta\bar{r}_p \tag{5.2.4.2}$$

where the components are defined as

$\Delta\bar{r}_S$ = survey error. This is the error in a station's location relative to the local datum. Each station on the same local datum will have a different value of $\Delta\bar{r}_s$.

$\Delta\bar{r}_E$ = station location error due to uncertainty in location of the local datum with respect to the center of mass of the Earth. All stations on the same local datum will have the same $\Delta\bar{r}_E$.

The remaining components are due to uncertainties in the movement of the Earth's crust relative to the central mass. These are:

$\Delta\bar{r}_{SE}$ = error due to solid Earth tidal displacements. This is relatively a local error.

$\Delta\bar{r}_{OL}$ = error due to ocean loading displacement. This is also a relatively local error which depends upon the distance of the station from the shoreline.

$\Delta\bar{r}_{p}$ = error due to polar motion

The equations for these last three components are developed in the following sections.

5.2.5 Polar Motion

The changes in station longitude and latitude due to polar motion are defined as

$$\Delta\lambda = \lambda_m - \lambda_c = (X_p \sin\lambda_m - Y_p \cos\lambda_m) \tan \phi_m \quad \text{(SEC ARC)} \qquad (5.2.5.1)$$

$$\Delta\phi = \phi_m - \phi_c = -(X_p \cos\lambda_m + Y_p \sin\lambda_m) \quad \text{(SEC ARC)} \qquad (5.2.5.2)$$

where

subscripts c and 0 denote computed and observed values respectively, and

X_p, Y_p are angular variations (seconds of arc) in the position of the Earth's axis of rotation relative to values at a fixed epoch.

The change in longitude, $\Delta\lambda$, can be related to UT1 by the equation

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Errors in station position due to polar motion are represented by uncertainties in X_p and Y_p, and are obtained in the form

$$\frac{\partial Z_c}{\partial X_p} = \frac{\partial Z_c}{\partial \Delta\lambda} \frac{\partial \Delta\lambda}{\partial X_p} + \frac{\partial Z_c}{\partial \Delta\phi} \frac{\partial \Delta\phi}{\partial X_p} \qquad (5.2.5.3)$$

$$\frac{\partial Z_c}{\partial Y_p} = \frac{\partial Z_c}{\partial \Delta\lambda} \frac{\partial \Delta\lambda}{\partial Y_p} + \frac{\partial Z_c}{\partial \Delta\phi} \frac{\partial \Delta\phi}{\partial Y_p} \qquad (5.2.5.4)$$

where Z_c is the computed measurement.

5.2.6 Solid Earth Tidal Displacements

Let the total displacement of a station on the surface of the Earth due to the solid Earth tide be expressed in spherical coordinates (r,θ,λ):

$$\bar{N}_T = \hat{r}\, N_{Tr} + \hat{\theta}\, N_{T\theta} + \hat{\lambda}\, N_{T\lambda} \,, \tag{5.2.6.1}$$

where θ is the co-latitude and λ if the east longitude. In this coordinate system the gradient operator is:

$$\nabla \equiv \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\lambda} \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} \,. \tag{5.2.6.2}$$

Let

$$\nabla_h \equiv \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\lambda} \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} \,. \tag{5.2.6.3}$$

Then (Diamante and Williamson, 1972):

$$\bar{N}_T = \frac{\ell_2}{g}\, r\, \nabla_h\, U_{T_2} + \frac{h_2}{g}\, U_{T_2}\, \hat{r} \tag{5.2.6.4}$$

$$= \frac{\ell_2 r}{g} \left(\nabla U_{T_2} - \hat{r} \frac{\partial U_{T_2}}{\partial r} \right) + \frac{h_2}{g}\, U_{T_2}\, \hat{r} \tag{5.2.6.5}$$

where h_2 and ℓ_2 are the Love numbers of the second and third kind.

Using

$$U_{T_2} = \frac{GM_d}{2\ R_d^3}\ r^2 \left(3\ (\hat{R}_d \cdot \hat{r})^2 - 1\right) \tag{5.2.6.6}$$

$$\overline{N}_T = \frac{GM_d r^2}{R_d^3\ g} \left\{ \left[3\ell_2 (\hat{R}_d \cdot \hat{r})\right] R_d + \left[3 \left(\frac{h_2}{2} - \ell_2\right) (\hat{R}_d \cdot \hat{r})^2 - \frac{h_2}{2}\right] \hat{r} \right\} . \tag{5.2.6.7}$$

Using $g = \frac{GM_e}{r^2}$; and r=re, the local value of the Earth's radius,

$$\overline{N}_T \bigg|_{r = r_e} = \left(\frac{M_d}{M_e} \frac{r_e^4}{R_d^3}\right) \left\{ \left[3\ell_2\ (\hat{R}_d \cdot \hat{r})\right] \hat{R}_d + \left[3 \left(\frac{h_2}{2} - \ell_2\right) (\hat{R}_d \cdot \hat{r})^2 - \frac{h_2}{2}\right] \hat{r} \right\} . \tag{5.2.6.8}$$

$\overline{N}_T$ acts as a measurement error by introducing a time variation in the position of any station on the Earth's surface. Typically, N_{Tr} is on the order of 0.5 meters, while the tilt $d\xi'$ is on the order of 10^{-2} seconds of arc which is equivalent to horizontal displacements N_{TH} 0.36 meters. Values of h_2 and $_2$ are found in the range:

$$\left.\begin{array}{l} 0.587 \le h_2 \le 0.610 \\ 0.068 \le \ell_2 \le 0.082 \end{array}\right\} . \qquad (5.2.6.9)$$

Again, there are <u>two</u> contributions (5.2.6.8) from the combined effects of the Sun and Moon:

$$\overline{N}_T = \overline{N}_{T_m} + \overline{N}_{T_s}, \qquad (5.2.6.10)$$

where $\overline{N}_{T_m}$ and $\overline{N}_{T_s}$ are obtained from (5.2.6.8) replacing the subscript "d" with "m" and "s", respectively.

5.2.7 Ocean Loading Displacements

Many of the currently active satellite tracking stations are on or near the coast. The tidal motions in the great bodies of water covering 70 percent of the surface of the Earth are now known to produce fluctuations in the observations of the solid Earth tidal effects. These oceanic perturbations account for about 10% of the observed gravity tide, 25% of the observed strain tide and 90% of the observed tilt tide at the Earth's surface near coastlines (Diamante and Williamson, 1972) principal ocean tide contribution to the observed surface gravity, strain and tilt is mainly a result of the deflection of the surface of the Earth under the oceanic loading. It is precisely this contribution to the strain tide (and hence station position) that is of importance in satellite operations.

Unfortunately, difficulties arise in developing useful ocean loading models of the strain tide due to:

(1) The theory of the ocean tides themselves is generally deficient. Complications arise from the fact that the oceans do not cover the entire surface of the planet and the depth of the oceans varies considerably.

(2) Observational data are limited. The state of the tides in the great open oceans is virtually unknown. The regions inland to about 400-500 km from continental coast-lines are dominated by the influence of local and regional tides. At about the 400 km distance from the shoreline,

regional tides provide about an equal contribution to the variations in the solid Earth tide as do the open oceans. However, further inland the open oceans provide the dominant contribution. The ocean loading effect, however, drops off as the distance from the shore, L, increases.

(3) Additional complications arise even where data are available. Even in the case of isolated islands, observations have been found to correlate with the tides of some more distant region, rather than with the immediate regional tide. Furthermore, the most frequently available and reliable observational data of the fluctuations in the solid Earth tide have been made with gravimeters.

In view of these difficulties, we have developed a semi-empirical error model for station position which will incorporate available observational data. These observational data are mainly in the form of variations in the vertical component of the surface gravity which are translated into a model of the vertical strain variation, acceptable for error analysis applications. The horizontal components of the strain are neglected, being of secondary importance in any case. Kuo, et. al. (1970) have made measurements of the fluctuations in the M_2 and O_1 gravity tide constituents. However, the M_2 measurements are generally better and provide better agreement with numerical models. Farrell (1970) provides some data on M_2, O_1, S_2K_2, and P_1K_1 gravity measurements, but the M_2 measurements are the most prevalent.

Gravimetric measurements are generally presented as a <u>fraction</u> of the theoretical solid earth M_2 tides:

$$\delta g_r = -\left(1 - \frac{3}{2} k_2 + h_2\right) \left.\frac{2U_{TM_2}}{r}\right|_{r - r_e} , \qquad (5.2.7.1)$$

Assuming the value of the gravimetric factor $\bar{\delta}$:

$$\bar{\delta}_o = \left(1 - \frac{3}{2} k_2 + h_2\right) = 1.16 , \qquad (5.2.7.2)$$

and zero phase κ. The result is normalized by the theoretical value of the M_2 solid earth tide on a rigid earth:

$$\delta^R_{g_r} = - \left.\frac{2U_{TM_2}}{r}\right|_{r = r_e} . \qquad (5.2.7.3)$$

The measurements provide an amplitude

$$\Delta\bar{\delta}\ (\bar{r}_e)$$

in terms of a percent of $\bar{\delta}$ with a time dependence in terms of a phase angle

$$\kappa(\bar{r}_e) \text{ in degrees.}$$

$\Delta\bar{\delta}$ and κ are indicated as functions of the local position on the surface of the Earth, $\bar{r}_e$.

Kuo et. al. (1970) and Kuo and Jachens (1970) made a series of long-term gravimetric measurements across the continental United States. All of the stations used in that study were within $\pm$ 1° of the 40° parallel of latitude. In employing these results, we will refer all distances from the coast-lines to the 40° parallel. Given the geographical makeup of the United States, we can expect their results to apply reasonably well for any U.S. station position in the latitude range:

$$35° \text{ North} \leq \phi \leq 50° \text{ North.}$$

Using the stations nearest to the shore:

$$\text{New York} \begin{cases} 40°49.0'\text{N} \\ 73°58.0'\text{W} \end{cases}$$

and

$$\text{Point Arena, California} \begin{cases} 38°54.3'\text{N} \\ 123°42.4'\text{W} \end{cases}$$

as the Atlantic coast origin and Pacific coast origins (L=0), respectively, the results of Kuo, et. al. for the M_2-tide (shown graphically in Figure 5.5) have been reduced to tabular form (Table 5.2)[1]. Distances are measured along the 40° parallel line from the Atlantic and Pacific coasts. Additional values of the measured gravimetric factors $\Delta\bar{\delta}$ and phase κ for the M_2 and O_1 tides have been given by Farrell (1970) for a number of isolated stations, including:

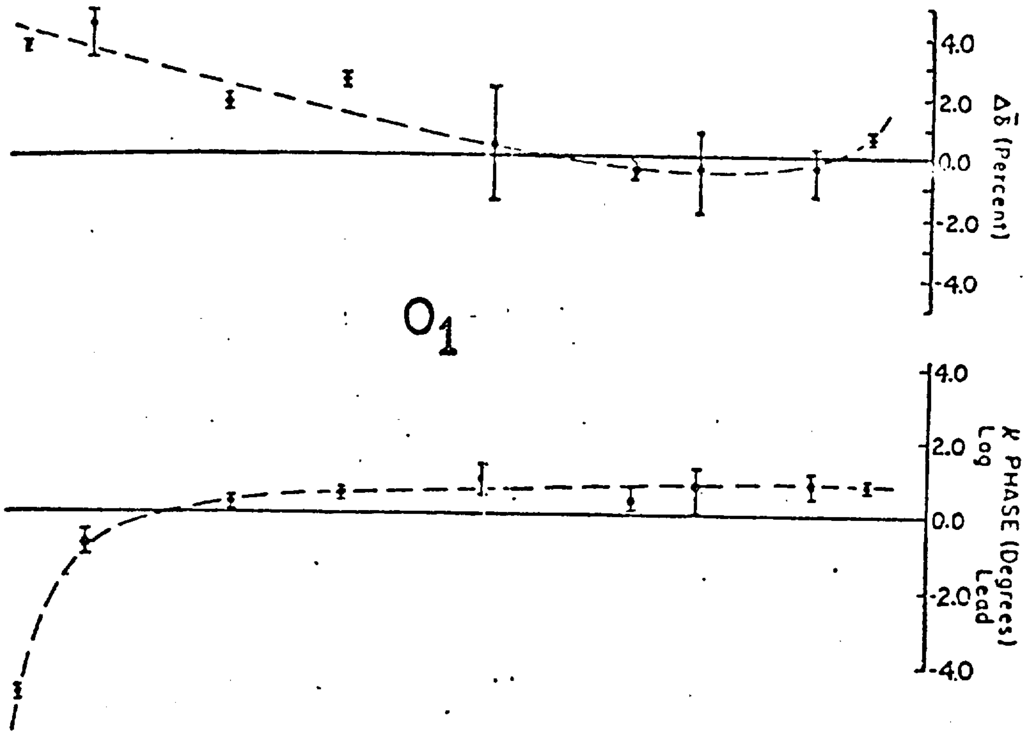
$$\text{Bermuda} \begin{cases} M_2 \text{ tide: } \Delta\bar{\delta} = 1.249, \ \kappa = 5.2° \\ O_1 \text{ tide: } \Delta\bar{\delta} = 1.198, \ \kappa = 0.3° \end{cases}$$

$$\text{Honolulu} \begin{cases} M_2 \text{ tide: } \Delta\bar{\delta} = 1.190, \ \kappa = -2.0° \\ O_1 \text{ tide: } \Delta\bar{\delta} = 1.181, \ \kappa = 6.1°. \end{cases}$$

Kuo et. al. have also made corresponding measurements for the O_1 tidal effects across the United States. These results, shown graphically in Figure 5.6, do not agree as well with the numerical model calculations as do their M_2 tidal measurements, however.

1.

$$\Delta\bar{\delta} = \frac{\bar{\delta}-\bar{\delta}_o}{\bar{\delta}_o} \text{ x } 100 \text{ (in percent)}.$$



THE RELATIVE DIFFERENCES OF OBSERVED GRAVIMETRIC FACTOR $\bar{\delta}$ (IN TERMS OF $\Delta\bar{\delta}$ IN PERCENT) AND OF THE PHASE ANGLE κ IN DEGREES FOR THE O_1 TIDAL CONSTITUTENT. THE DASHED CURVE WAS OBTAINED BY A LEAST SQUARES ANALYSIS.

(—FROM KUO ET AL, 1970)

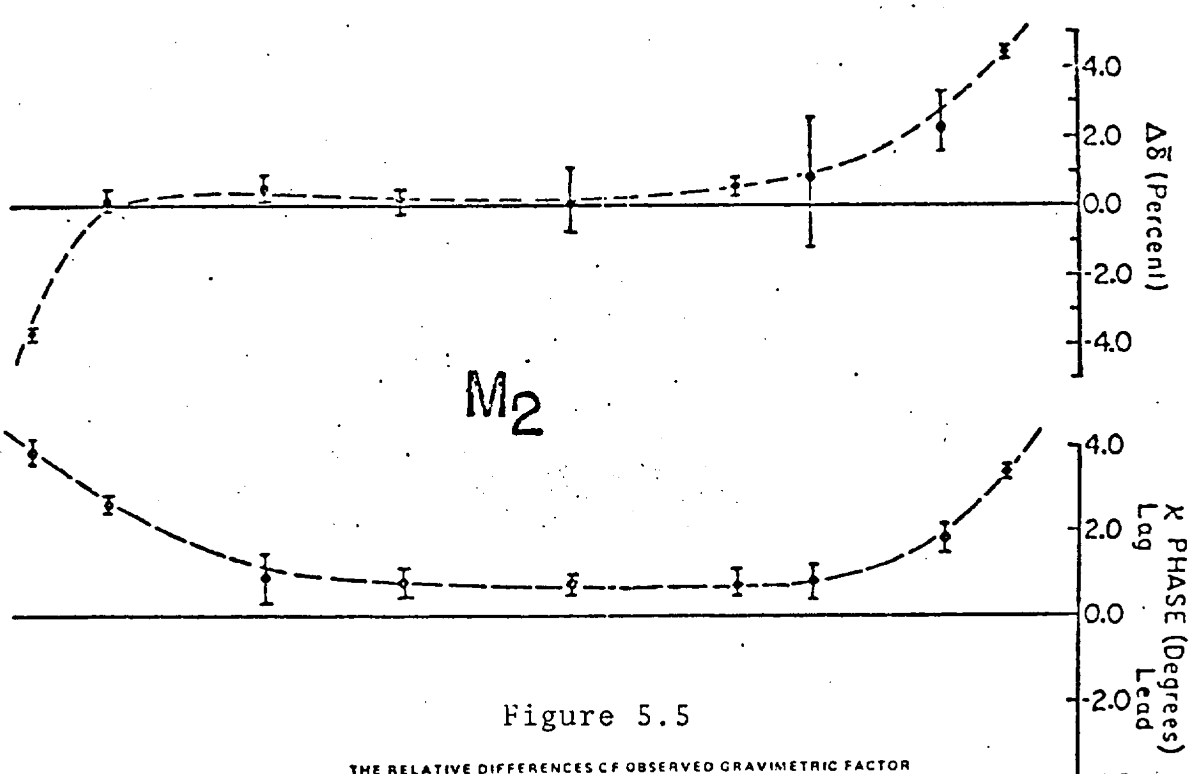


Figure 5.5

THE RELATIVE DIFFERENCES OF OBSERVED GRAVIMETRIC FACTOR $\bar{\delta}$ (IN TERMS OF $\Delta\bar{\delta}$ IN PERCENT) AND OF THE PHASE ANGLE κ IN DEGREES FOR THE M_2 TIDAL CONSTITUENT. THE DASHED CURVE WAS OBTAINED BY A LEAST SQUARES ANALYSIS.

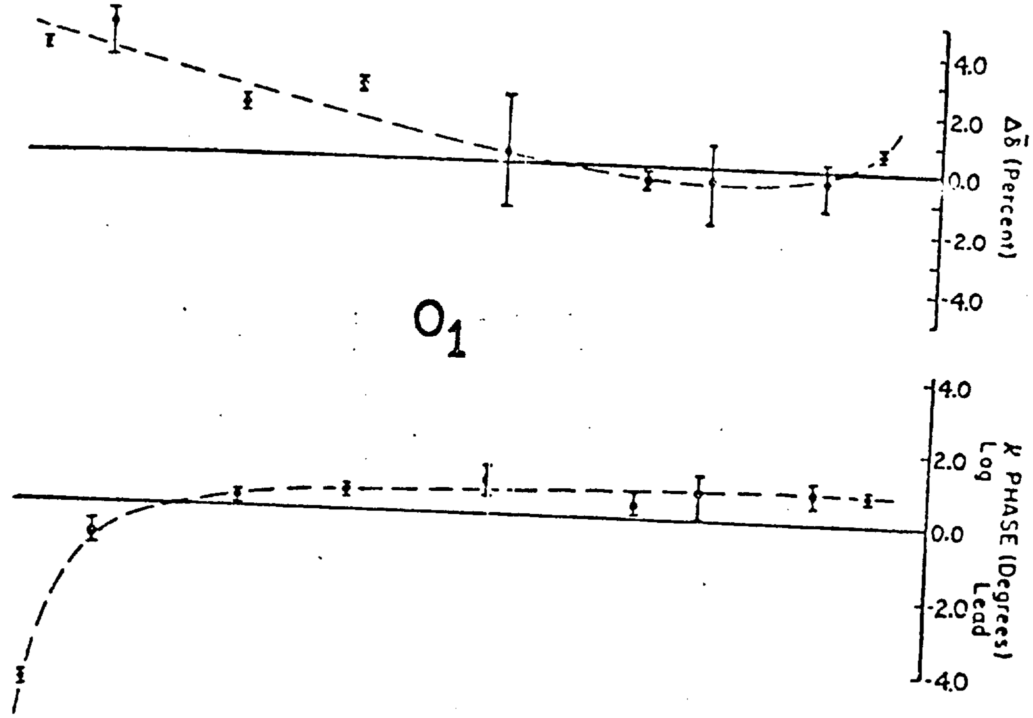
(—FROM KUO ET AL, 1970)

TABLE 5.2

Values of $\Delta\bar{\delta}(L)$ and $\kappa(L)$ the M_2-tide for the Continental United Stated Referred to the 40° Parallel of Latitude and with New York City as the Origin for Atlantic Coast Values (L=0) and Point Arena as the Origin for Pacific Coast Values (L=0)***

	Distance From Pac. Coast(km)	Distance From Atl. Coast(km)	$\Delta\bar{\delta}$ %	κ Degrees
1	~ 0	4242	-3.560	3.87
2	202	4040	-0.925	3.21
3	404	3838	0.150	2.54
4	606	3636	0.308	1.94
5	808	3434	0.385	1.51
6	1010	3232	0.423	1.21
7	1212	3030	0.308	0.968
8	1414	2828	0.270	0.848
9	1616	2626	0.193	0.786
10	1818	2424	0.160	0.755
11	2020	2222	0.150	0.725
12	2222	2020	0.150	0.725
13	2424	1818	0.160	0.725
14	2626	1616	0.231	0.725
15	2828	1414	0.308	0.725
16	3030	1212	0.463	0.725
17	3232	1010	0.655	0.755
18	3434	808	1.000	0.847
19	3636	606	1.385	1.09
20	3838	404	2.159	1.45
21	4040	202	3.159	2.18
22	4242	~ 0	4.540	3.45

***Reduced From Kuo et. al., 1970.



THE RELATIVE DIFFERENCES OF OBSERVED GRAVIMETRIC FACTOR $\bar{\delta}$ (IN TERMS OF $\Delta\bar{\delta}$ IN PERCENT) AND OF THE PHASE ANGLE κ IN DEGREES FOR THE O_1 TIDAL CONSTITUTENT. THE DASHED CURVE WAS OBTAINED BY A LEAST SQUARES ANALYSIS.

(—FROM KUO ET AL, 1970)

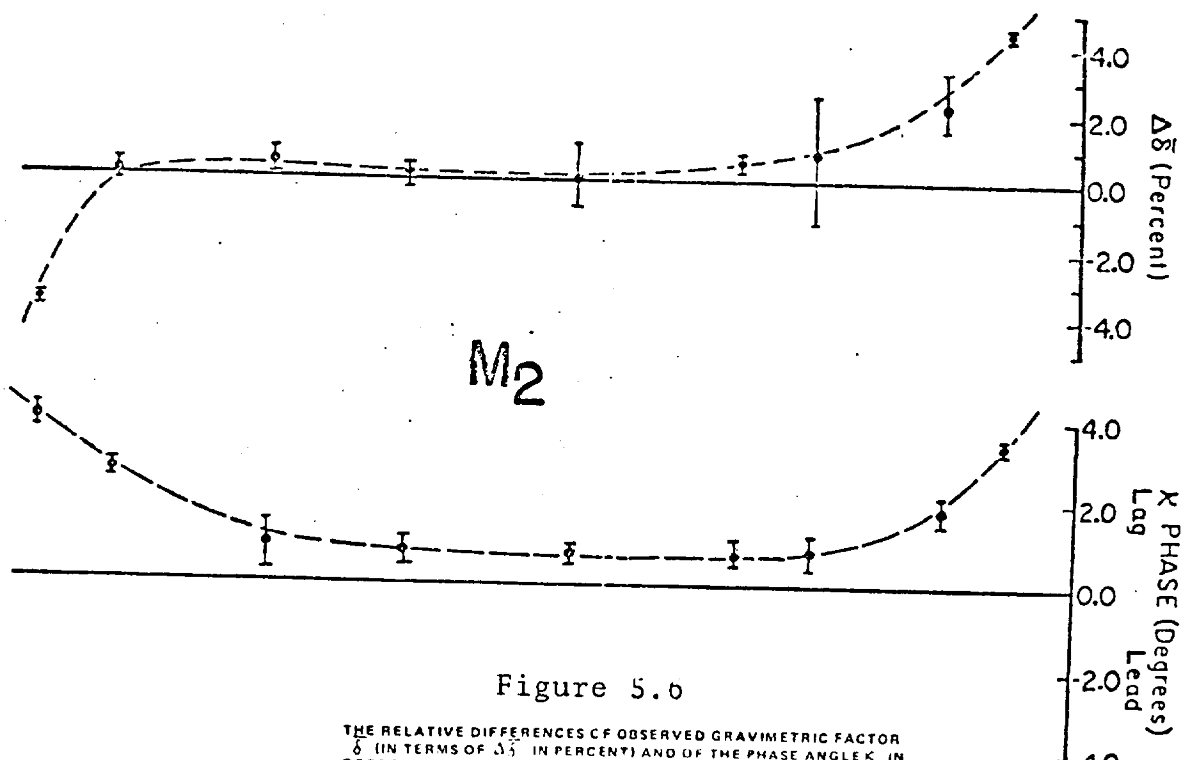


Figure 5.6

THE RELATIVE DIFFERENCES OF OBSERVED GRAVIMETRIC FACTOR $\bar{\delta}$ (IN TERMS OF $\Delta\bar{\delta}$ IN PERCENT) AND OF THE PHASE ANGLE κ IN DEGREES FOR THE M_2 TIDAL CONSTITUENT. THE DASHED CURVE WAS OBTAINED BY A LEAST SQUARES ANALYSIS.

(—FROM KUO ET AL, 1970)

TABLE 5.3

Values of $\Delta\delta(L)$ and $\kappa(L)$ the M_2-tide for the Continental United Stated Referred to the 40° Parallel of Latitude and with New York City as the Origin for Atlantic Coast Values (L=O) and Point Arena as the Origin for Pacific Coast Values (L=O)***

	Distance From Pac. Coast(km)	Distance From Atl. Coast(km)	$\Delta\delta$ %	κ Degrees
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17	3232	1010	0.655	0.755
18	3434	808	1.000	0.847
19	3636	606	1.385	1.09
20	3838	404	2.159	1.45
21	4040	202	3.159	2.18
22	4242	~ 0	4.540	3.45

***Reduced From Kuo et. al., 1970.

To good approximation the tidal potential for the M_2 lunar tidal component (principal lunar semi-diurnal tide) is (Diamante and Williamson, 1972):

$$U_{TM_2} = \frac{3}{4} \frac{GM_m}{d_m} \left(\frac{r}{d_m}\right)^2 \cos^2\phi \cos^2\delta_m \cos 2t \qquad (5.2.7.4)$$

where ϕ is the geocentric latitude of the station, t is the hour angle of the Moon and δ_m is the lunar declination. d_m is the mean distance of the Moon. Let $\hat{N}$ be a unit vector along the north polar axis of the Earth, then:

$$U_{TM_2} = \frac{3}{4} \frac{GM_m}{d_m} \left(\frac{r}{d_m}\right)^2 \left\{ 2[(\hat{N}\cdot\hat{r})(\hat{N}\cdot\hat{R}_m) - \hat{R}_m\cdot\hat{r}]^2 - [1-(\hat{N}\cdot\hat{R}_m)^2][1-(\hat{N}\cdot\hat{r})^2] \right\} . \qquad (5.2.7.5)$$

Similarly, to good approximation the O_1 tidal constituent (principal lunar diurnal tide is:

$$U_{TO_1} = -\frac{3}{4} \frac{GM_m}{d_m} \left(\frac{r}{d_m}\right)^2 \left(\frac{d_m}{R_m}\right)^3 \sin 2\phi \sin 2\delta_m \cos t . \qquad (5.2.7.6)$$

or

$$U_{TO_1} = -3 \frac{GM_m}{d_m} \left(\frac{r}{d_m}\right)^2 \left(\frac{d_m}{R_m}\right)^3 \left\{ (\hat{N}\cdot\hat{r})^2 (\hat{N}\cdot\hat{R}_m)^2 - (\hat{N}\cdot\hat{r})(\hat{N}\cdot\hat{R}_m)(\hat{R}_m\cdot\hat{r}) \right\} . \tag{5.2.7.7}$$

The measurements of Kuo et. al., (1970) and other investigators are of the form:

$$\delta^{SO}_{g_r} = \left\{ \frac{\Delta\bar{\delta}}{100} \left(\frac{-2U_{T_2}\,\bar{\delta}}{r} \right) \right\}_{r = r_e} , \tag{5.2.7.8}$$

for specific lunar tidal components of U_{T_2}. Equating (4) and (8) the solid Earth surface displacement in the radial direction is given by

$$N^{SO}_{Tr} = \left\{ \frac{4}{3} \left(\frac{\Delta\delta}{100}\right) \frac{\bar{\delta}_o U_{T_2}}{g} \right\}_{r = r_e} . \tag{5.2.7.9}$$

In terms of the contributions of the M_2 and O_1 tidal effects (Diamante and Williamson, 1972):

$$\begin{aligned} N^{SO}_{Tr} &\simeq N^{SO}_{Tr}(M_2) + N^{SO}_{Tr}(O_1) \\ &= \left\{ \frac{4}{3} \frac{r^2\,\bar{\delta}_o \times 10^{-2}}{GM_e} \left[\Delta\delta(M_2)\, U_{TM_2} + \Delta\delta(O_1)\, U_{TO_1} \right] \right\}_{r = r_e} , \end{aligned} \tag{5.2.7.10}$$

or

$$N_{Tr}^{SO}(L) \approx N_{Tr}^{SO}(M_2;L) + N_{Tr}^{SO}(O_1;L) \quad , \tag{5.2.7.11}$$

$$N_{Tr}^{SO}(M_2;L) = 1.16\times10^{-2}\ \Delta\bar{\delta}\ (M_2;L) \left(\frac{R_e}{d_m}\right)^3 \left(\frac{M_m}{M_e}\right) R_e$$

$$\left\{2[(\hat{N}\cdot\hat{r})(\hat{N}\cdot\hat{R}_m^*) - (\hat{R}_m^*\cdot\hat{r})]^2 - [1-(\hat{N}\cdot\hat{R}_m^*)^2][1-(\hat{N}\cdot\hat{r})^2]\right\} , \tag{5.2.7.12}$$

$$N_{Tr}^{SO}(O_1;L) = 1.16\times10^{-2}\ \Delta\bar{\delta}(O_1;L) \left(\frac{R_e}{d_m}\right)^3 \left(\frac{M_m}{M_e}\right) R_e$$

$$\left\{4(\hat{N}\cdot\hat{r})(\hat{N}\cdot\hat{R}_m^{**})[(\hat{R}_m^{**}\cdot\hat{r}) - (\hat{N}\cdot\hat{r})(\hat{N}\cdot\hat{R}_m^{**})]\right\} \quad , \tag{5.2.7.13}$$

where L is the distance from the shoreline and the local phase angle $\kappa(L)$ has been taken into account by the relations:

$$(\hat{R}_m^*)_x = (\hat{R}_m)_x \cos\kappa\ (M_2;L) - (\hat{R}_m)_y \sin\kappa\ (M_2;L) \ , \tag{5.2.7.14}$$

$$(\hat{R}_m^*)_y = (\hat{R}_m)_y \cos\kappa\ (M_2;L) + (\hat{R}_m)_x \sin\kappa\ (M_2;L) \ , \tag{5.2.7.15}$$

$$(\hat{R}_m^{**})_x = (\hat{R}_m)_x \cos\kappa\ (O_1;L) - (\hat{R}_m)_y \sin\kappa\ (O_1;L) \ , \tag{5.2.7.16}$$

$$(\hat{R}^{**}_m)_y = (\hat{R}_m)_y \cos \kappa \, (O_1;L) + (\hat{R}_m)_x \sin \kappa \, (O_1;L) \quad , \qquad (5.2.7.17)$$

$$(\hat{R}^{*}_m)_z = (\hat{R}^{**}_m)_z = (\hat{R}_m)_z \quad . \qquad (5.2.7.18)$$

For most error analysis applications, however, the phase lag can be assumed equal to zero and the O_1 component (equation 3.28) may be neglected. In any case, the O_1 component can generally be neglected for stations within the latitude band:

$$-40° \leq \phi \leq 40° \quad .$$

In ORAN the O_1 component is neglected. The values of κ and $\Delta\bar{\delta}$ are computed by fitting a polynomial in L to the curves of Figure 4.5. Then Equations 12, 14, 15 and 18 are used to compute the change in station height.

5.2.8 Measurement Errors Due to Medium Distortion

In ORAN uncertainties in range and range rate due to the following transmission medium effects are treated.

- Tropospheric refraction
- Ionospheric refraction
- Space plasma

These are discussed in detail in the following sections.

5.2.9 Errors in Tropospheric Refraction

Except at very low elevation angles, (<5°), the primary effect of the troposphere is a decrease in the velocity of propagation. At the Earth's surface, this decrease is about 300 parts per million, decreasing to about 1 part per million at a height of 30 km. Considering the Earth's atmosphere to be horizontally stratified, as is almost always done in data reduction, a good approximation to the integrated tropospheric effect on range measurements is

$$\Delta R_T = \frac{2.77(N_s/328.5)}{.026 + \sin E} \text{ meters} \tag{5.2.9.1}$$

where

$R_{OBSERVED} = R_{COMPUTED} + \Delta R_T$,

N_s is the deviation of surface index of refraction from unity in parts per million, and

E is the elevation angle.

The most serious error in applying this correction to data is due to errors in the surface index of refraction at the tracking site. For this reason, tropospheric refraction errors are modeled in ORAN as

$$\frac{\partial(\Delta R_T)}{\partial N_s} = \frac{(2.77/328.5)}{.026 + \sin E} \tag{5.2.9.2}$$

The systematic effect of tropospheric refraction on range rate errors is obtained by differentiating the range error with respect to time,

$$\frac{\partial(\Delta \dot{R}_T)}{\partial N_s} = \frac{(-2.77/328.5)}{(.026 + \sin E)^2} \cos E \, \dot{E} \tag{5.2.9.3}$$

Elevation:

For elevation observations, the partial with respect to refraction is

$$\frac{\partial E}{\partial N_s} = \frac{10^3}{16.44+930} \tan E \tag{5.2.9.4}$$

Azimuth is not affected by refraction.

Direction Cosines:

$$\frac{\partial \ell}{\partial n_s} = -\sin A_z \sin E \frac{\partial E}{\partial n_s} \tag{5.2.9.5}$$

$$\frac{\partial m}{\partial m_s} = -\cos A_z \sin E \frac{\partial E}{\partial m_s} \tag{5.2.9.6}$$

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X and Y angles:

$$\frac{\partial X}{\partial n_s} = \frac{\sin A_z}{(\sin^2 E + \sin^2 A_z \cos^2 E)} \frac{\partial E}{\partial n_s} \tag{5.2.9.7}$$

$$\frac{\partial Y}{\partial n_s} = - \frac{\cos A_z \sin E}{\sqrt{1 - \cos^2 A_z \cos^2 E}} \frac{\partial E}{\partial n_s} \tag{5.2.9.8}$$

5.2.10 Errors in Ionospheric Refraction

The effect of the ionosphere on a range measurement is evaluated by considering

$$R_{OBSERVED} = R_{COMPUTED} + \Delta R_I$$

The correction ΔR_I is modeled by fitting a polynomial to the curves in Figure 4.6, which were taken from JPL SPS 37-41, Volume III, page 8. The polynomial takes the form

$$\Delta R_I = C_0 + C_1 \text{ SIN } E + C_2 \text{ SIN}^2 E + C_3 \text{ SIN}^3 E \tag{5.2.10.1}$$

where E is the elevation angle and the C_i are obtained by a least squares fit to selected points from the curves. Errors in range rate are obtained by differentiating ΔR_I.

An estimate of the error in ΔR_I is given by the above reference to be 10% on a day-to-day basis for a particular location.

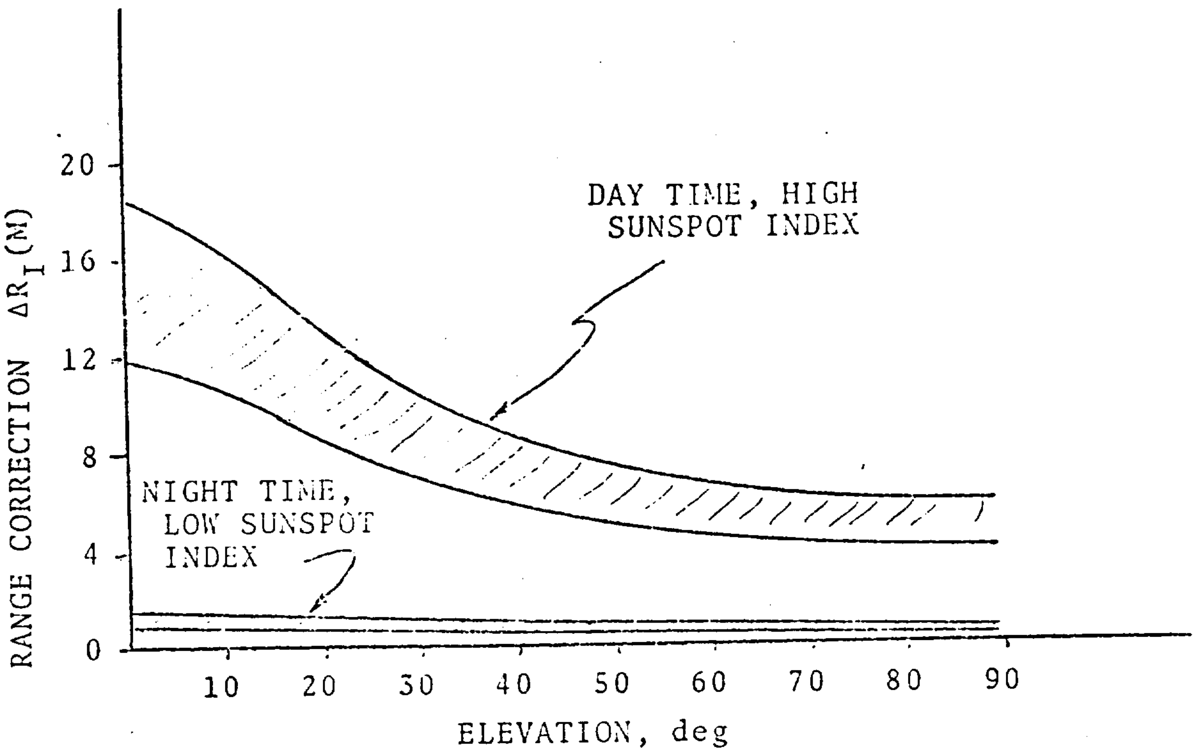

Figure 5.7 Effects of Ionosphere on Range Measurements DSIF S-band System ($f = 2.3 \times 10^9$ Hz)

5.2.11 Errors in Space Plasma

Space plasma represents another type of propagation error. Unlike the ionosphere, which is assumed to terminate somewhere near 600 KM above the Earth's surface, space plasma continues ad-infinitum, and is reasonably represented by a $1/r^2$ law. Therefore, no closed-form solution exists for its effect on measurements, and an integration process must be performed. Let

$$R_{OBSERVED} = R_{COMPUTED} + \Delta R_{SP}$$

where ΔR_{SP} is modeled by the relationship

$$\Delta R_{SP} = \frac{44.3}{f^2} \int N_p \, ds \qquad (5.2.11.1)$$

and

f = frequency of wave (Hz)

N_p = proton density per cubic centimeter

s = ray path

Figure 5.7, which was taken from the JPL SPS 37-41, Vol. III, page 6, gives the assumed proton density representation as a function of the distance from the sun.

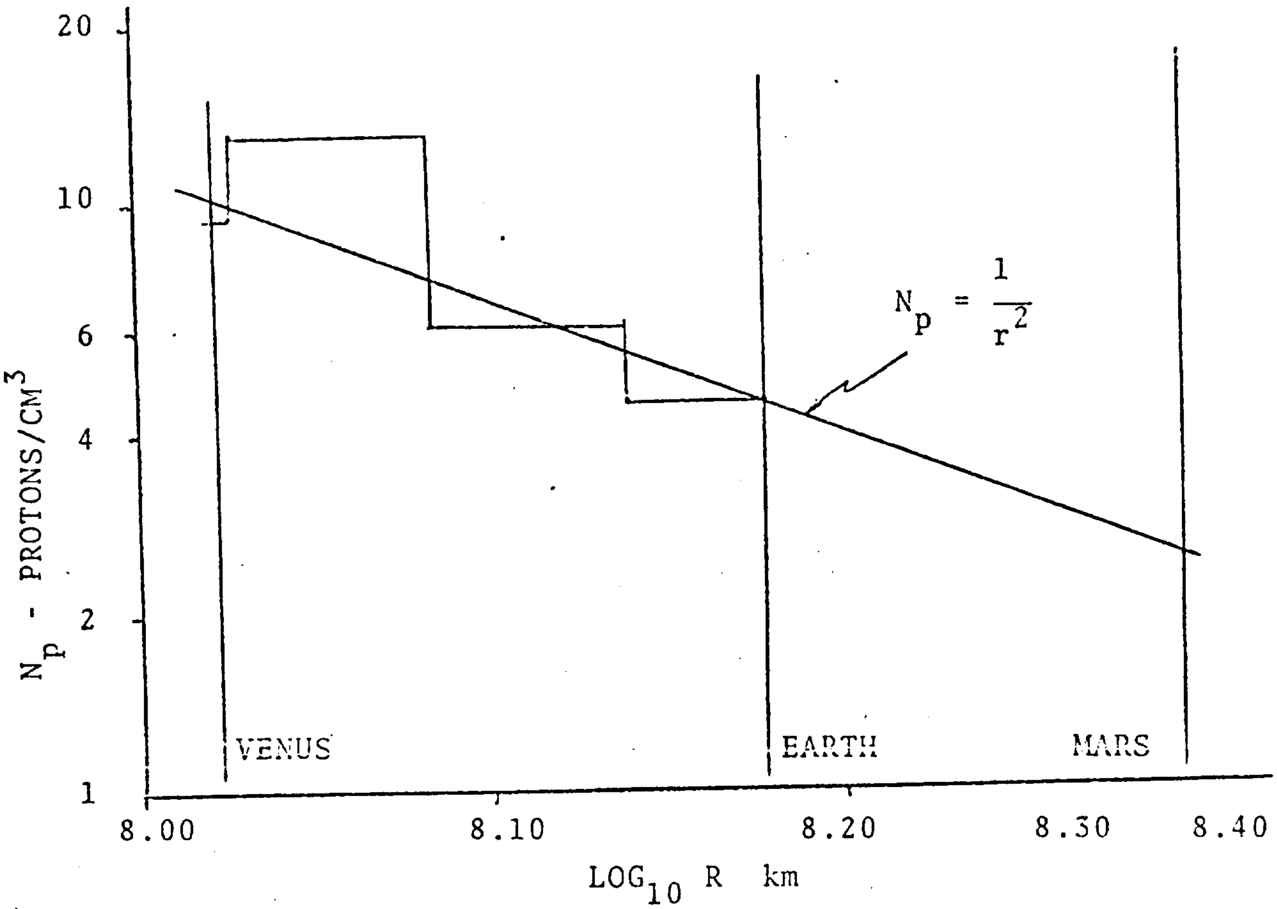

Figure 5.7 Logarithmic Plot of
Proton Density and Flux vs Distance from the Sun

The proton density was taken to be

$$N_p = A[\log R] + B$$

which was converted into a 4th degree polynomial

$$N_p = K_0 + K_1R + K_2R^2 + K_3R^3 + K_4R^4$$

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The coefficients K_i were obtained by a least squares fit to selected points from Figure 5.7.

The integration process is numerical (Simpsons Rule) in which the total ray path is divided in half. This halving process is necessary because of the possible situation shown in Figure 5.8 in which the proton density curve along the ray path would be bell shaped (Figure 5.9)

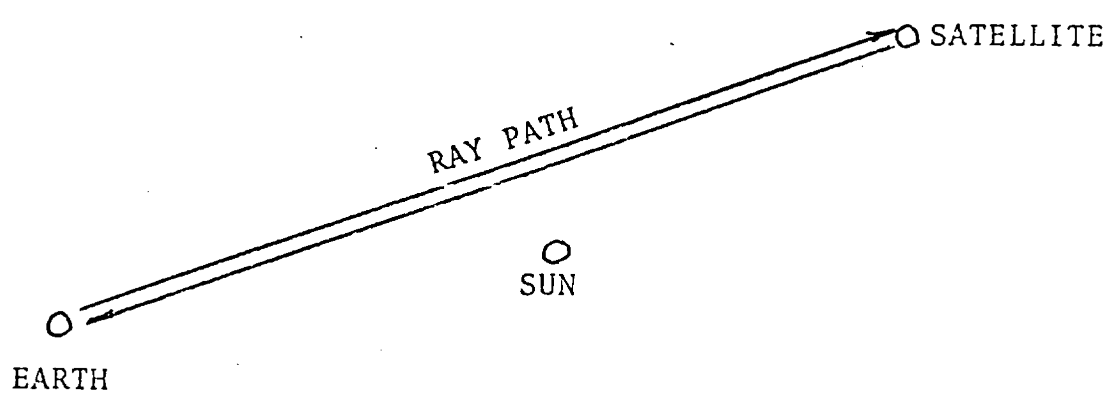

Figure 5.8 Earth-Sun-Satellite Configuration which Requires Halving the Ray Path to Accurately Represent Proton Density

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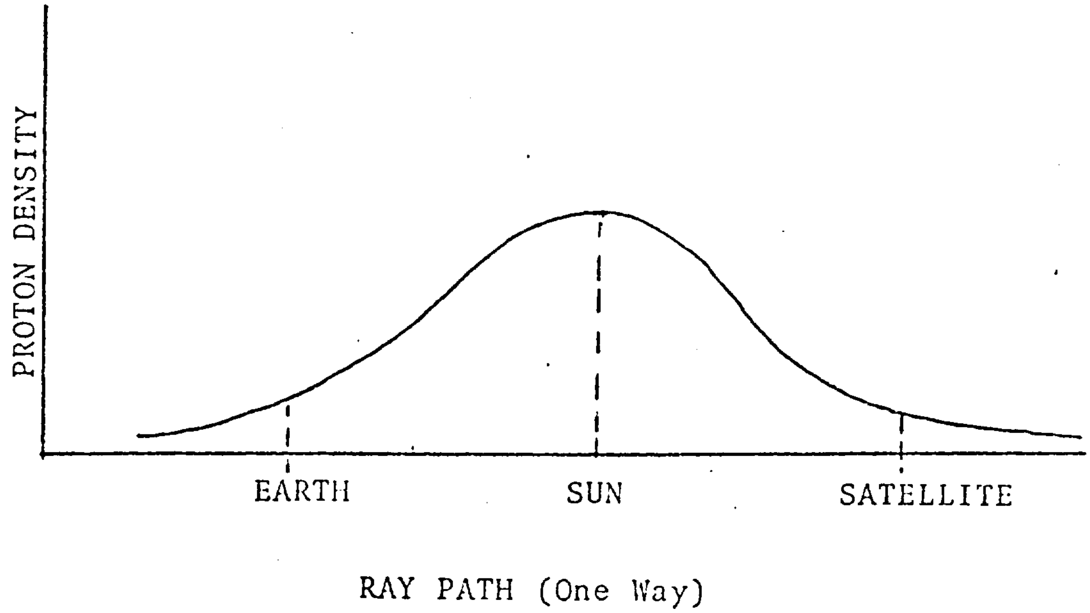

Figure 5.9. Proton Density Corresponding to Figure 5.8

Figure 5.9 shows that a quadrature expression which does not consider an intermediate value of the proton density on the ray path would give a totally erroneous density representation.

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SECTION 6.0
FORCE MODEL AND VARIATIONAL EQUATIONS

A fundamental part of the ORAN Program requires computing positions and velocities of the spacecraft at each observation time. The dynamics of the situation are expressed by the equations of motion, which provide a relationship between the orbital elements at any given instant and the initial conditions of epoch. There is an additional requirement for variational partials, which are the partial derivatives of the instantaneous orbital elements with respect to the parameters at epoch. These partials are generated using the variational equations, which are analogous to the equations of motion.

6.1 EQUATIONS OF MOTION

In a geocentric inertial rectangular coordinate system, the equations of motion for a spacecraft are of the form

$$\ddot{\overline{r}} = - \frac{\mu \overline{r}}{r^3} + \overline{A} \qquad (6.1.1)$$

where

$\overline{r}$ is the position vector of the satellite.

μ is the GM, where G is the gravitational constant and M is the mass of the Earth.

$\overline{A}$ is the acceleration caused by the asphericity of the Earth, extraterrestrial gravitational forces, atmospheric drag, and solar radiation.

This provides a system of second order equations which, given the epoch position and velocity components, may be integrated to obtain the position and velocity at any other time. This direct integration of these accelerations in Cartesian coordinates is known as Cowell's method and is the technique used in ORAN's orbit generator. This method was selected for its simplicity and its capacity for easily incorporating additional perturbative forces.

There is an alternative way of expressing the above equations of motion:

$$\ddot{\bar{r}} = \nabla U + \bar{A}_D + \bar{A}_R \tag{6.1.2}$$

where

- U is the potential field due to gravity,
- $\bar{A}_D$ contains the accelerations due to drag, and
- $\bar{A}_R$ contains the accelerations due to solar radiation pressure.

This is, of course, just a regrouping of terms coupled with a recognition of the existence of a potential field. This is the form used in ORAN.

The inertial coordinate system in which these equations of motion are integrated in ORAN is that system corresponding to the true of date system of the epoch time. The complete definitions for these coordinate systems (and the Earth-fixed system) are presented in the GEODYN Program Documentation, Volume 1, Section 3.0.

The evaluation of the accelerations for $\ddot{\bar{r}}$ is performed in the true of date system. Thus there is a requirement that the inertial position and velocity output from the integrator be transformed to the true of date system for the evaluation of the accelerations, and a requirement to transform the computed accelerations from the true of date system to the inertial system.

6.2 THE VARIATIONAL EQUATIONS

The variational equations have the same relationship to the variational partials as the satellite position vector does to the equations of motion. The variational partials are defined as the $\frac{\partial \bar{x}(t)}{\partial \bar{\beta}(t_o)}$ where $\bar{x}(t)$ spans the true of date position and velocity of the satellite at a given time; i.e.,

$$\bar{x}(t) = x, y, z, \dot{x}, \dot{y}, \dot{z} \quad ;$$

and $\bar{\beta}(t_o)$ spans the epoch parameters; i.e.,

x_o, y_o, z_o	the satellite position vector at epoch
$\dot{x}_o, \dot{y}_o, \dot{z}_o$	the satellite velocity vector at epoch
C_D	the satellite drag factor
C_R	the satellite emissivity factor
C_{nm}, S_{nm}	gravitational harmonic coefficients for each n, m pair
χ	surface density coefficients

GM	Earth gravitational constant
Δg	lumped effect of gravitational model differences. The differences between several built-in gravity models may be propagated as a single parameter which approximates errors in low degree and order models. Logically, this parameter should never be considered adjusted.
γ_D	ratio of Earth reflected radiation to incoming solar flux
GM_s	Sun gravitational constant
GM_m	Moon gravitational constant
K_2	Love Number

Also, the ORAN program can treat the perturbations in the Earth gravitational field due to mascon of mass m at an arbitrary location in the Earth.

Let us first realize that the variational partials may be partitioned according to the satellite position and velocity vectors at the given time. Thus the required partials are

$$\frac{\partial \bar{r}(t)}{\partial \bar{\beta}(t_o)} , \quad \frac{\partial \dot{\bar{r}}(t)}{\partial \bar{\beta}(t_o)}$$

where

> $\bar{r}(t)$ is the satellite position vector (x,y,z) in the true of date system, and
>
> $\dot{\bar{r}}(t)$ is the satellite velocity vector $(\dot{x},\dot{y},\dot{z})$ in the same system.

The first of these, $\frac{\partial \bar{r}(t)}{\partial \bar{\beta}(t_o)}$ can be obtained by the double integration of

$$\frac{\partial \ddot{\bar{r}}(t)}{\partial \bar{\beta}(t_o)} \tag{6.2.1}$$

or rather, since the order of differentiation may be exchanged,

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial \bar{r}(t)}{\partial \bar{\beta}(t_o)} \right) \tag{6.2.2}$$

Note that the second set of partials, $\frac{\partial \dot{\bar{r}}}{\partial \bar{\beta}(t_o)}$ may be obtained by a first order integration of $\frac{\partial \ddot{\bar{r}}(t)}{\partial \bar{\beta}(t_o)}$. Hence we recognize that the quantity to be integrated is $\frac{\partial \ddot{\bar{r}}(t)}{\partial \bar{\beta}(t_o)}$. Using the first form given for the equations of motion in the previous subsection, the variational equations are given by

$$\frac{\partial \ddot{\bar{r}}(t)}{\partial \bar{\beta}(t_o)} = \frac{\partial}{\partial \bar{\beta}(t_o)} \left[\Delta U(t) + \bar{A}_R(t) + \bar{A}_D(t) \right] \qquad (6.2.3)$$

where

$U(t)$ is the potential field due to gravitational effects at time t.

$\bar{A}_R(t)$ is the acceleration due to radiation pressure at time t.

$\bar{A}_D(t)$ is the acceleration due to drag at time t.

The similarity to the equations of motion is now obvious. When the app priate partial derivatives are computed, Equation (6.2.3) is of the form

$$\ddot{\bar{y}}(t) = A(t)\ \dot{\bar{y}}(t) + B(t)\ \bar{y}(t) + C(t) \qquad (6.2.4)$$

where

$$A(t) = \frac{\ddot{\bar{r}}(t)}{\partial \bar{r}(t)}$$

$$B(t) = \frac{\partial \ddot{\bar{r}}(t)}{\partial \dot{\bar{r}}(t)}$$

$$C(t) = \frac{\partial \ddot{\overline{r}}(t)}{\partial \overline{\beta}(t_o)}$$

$$\overline{y}(t) = \frac{\partial \overline{r}(t)}{\partial \overline{\beta}(t_o)}$$

$$\dot{\overline{y}}(t) = \frac{\partial \dot{\overline{r}}(t)}{\partial \overline{\beta}(t_o)}$$

This is a linear differential equation with coefficients A(t), B(t) and C(t) which are known functions of time. In ORAN a Cowell predictor-corrector integrator is used to compute $\overline{r}(t)$, $\dot{\overline{r}}(t)$ and the coefficients A(t), B(t) and C(t). Then an Adams-Cowell corrector only integrator uses these coefficients to solve the variational equations (6.2.4) for $\overline{y}(t)$ and $\dot{\overline{y}}(t)$.

SECTION 7.0
FORCE MODEL COMPONENTS

Equation (6.1.1) expresses the instantaneous satellite acceleration as the sum of individual acceleration due to the gravitational field, atmospheric drag, and solar radiation pressure. This section describes how each of these accelerations and the associated variational equations corresponding to Equation (6.2.4) are evaluated. Table 6.1 lists the contributions to each of these acceleration components and the associated error parameters.

7.1 THE EARTH'S POTENTIAL

In ORAN the Earth's potential is described by a spherical harmonic expansion with surface densities used to model local irregularities.

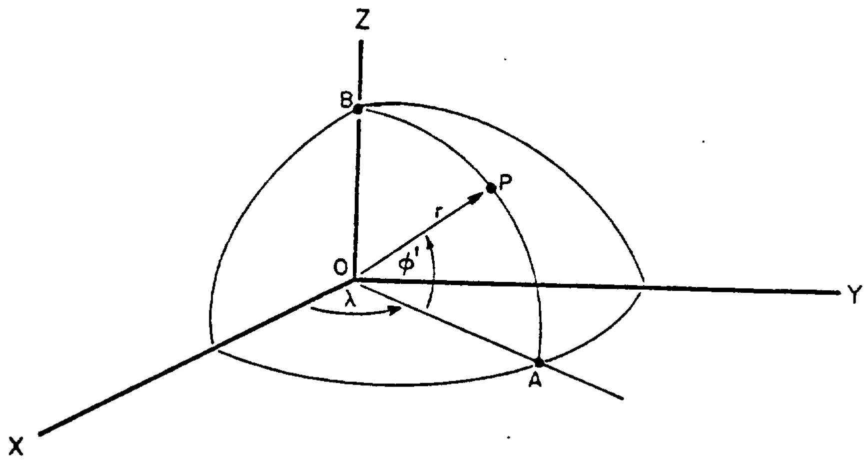

Figure 7.1 : Spherical Coordinates

7.1.1 Spherical Harmonic Expansion

The Earth's potential is most conveniently expressed in a spherical coordinate system as is shown in Figure 7.1. By inspection:

- ϕ', the geocentric latitude, is the angle measured from $\overline{OA}$, the projection of $\overline{OP}$ in the X-Y plane, to the vector $\overline{OP}$.

- λ, the east longitude, is the angle measured from the positive direction of the X axis to $\overline{OA}$.

- r is the magnitude of the vector $\overline{OP}$.

Let us consider the point P to be the satellite position. Thus, $\overline{OP}$ is the geocentric Earth-fixed satellite vector corresponding to $\overline{r}$, the true of date satellite vector, whose components are (x,y,z). The relationship between the spherical coordinates (Earth-fixed) and the satellite position coordinates (true of date) is then given by

$$r = \sqrt{x^2 + y^2 + z^2} \tag{7.1.1}$$

$$\phi' = \sin^{-1} \frac{z}{r} \tag{7.1.2}$$

$$\lambda = \tan^{-1} \frac{y}{x} - \theta_g \tag{7.1.3}$$

where θ_g is the rotation angle between the true of date system and the Earth-fixed system (see GEODYN Program Documentation, Volume 1, Section 3.4).

The Earth's gravity field is represented by the normal potential of an ellipsoid of revolution and small irregular variations, expressed by a sum of spherical harmonics. This formulation, used in ORAN is

$$U = \frac{GM}{r} \left\{ 1 + \sum_{n=2}^{nmax} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n P_n^m \left(\sin \phi'\right) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right] \right\} \tag{7.1.4}$$

where

G is the universal gravitational constant,

M is the mass of the Earth,

r is the geocentric satellite distance,

nmax is the upper limit for the summation (highest degree),

a_e is the Earth's mean equatorial radius,

ϕ' is the satellite geocentric latitude,

λ is the satellite east longitude,

$P_n^m(\sin\phi)$ indicate the associated Legendre functions, and

C_{nm} and S_{nm} are the denormalized gravitational coefficients.

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The gravitational accelerations in true of date coordinates $(\ddot{x}, \ddot{y}, \ddot{z})$ are computed from the geopotential, $U(r, \phi', \lambda)$, by the chain rule; e.g.,

$$\ddot{x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \phi'}\frac{\partial \phi'}{\partial x} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial x} \tag{7.1.5}$$

The accelerations $\ddot{y}$ and $\ddot{z}$ are determined likewise. The partial derivatives of U with respect to r, ϕ', and λ are given by

$$\frac{\partial U}{\partial r} = \frac{GM}{r^2}\left[1 + \sum_{n=2}^{nmax}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda \right. \tag{7.1.6}$$

$$\left. + \; S_{nm} \sin m\lambda) \; (n + 1) \; P_n^m \; (\sin \phi')\right]$$

$$\frac{\partial U}{\partial \lambda} = \frac{GM}{r} \sum_{n=2}^{nmax}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} (S_{nm} \cos m\lambda - C_{nm} \sin m\lambda) \tag{7.1.7}$$

$$m \; P_n^m \; (\sin \phi')$$

$$\frac{\partial U}{\partial \phi'} = \frac{GM}{r} \sum_{n=2}^{nmax}\left(\frac{a_e}{r}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \tag{7.1.8}$$

$$\left[P_n^{m+1} \; (\sin \phi') - m \tan \phi' \; P_n^m \; (\sin \phi')\right]$$

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The partial derivatives of r, ϕ', and λ with respect to the true of date satellite position components are

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r} \tag{7.1.9}$$

$$\frac{\partial \phi'}{\partial r_i} = \frac{1}{\sqrt{x^2+y^2}} \left[-\frac{z r_i}{r^2} + \frac{\partial z}{\partial r_i} \right] \tag{7.1.10}$$

$$\frac{\partial \lambda}{\partial r_i} = \frac{1}{\sqrt{x^2+y^2}} \left[\frac{\partial y}{\partial r_i} - \frac{y}{x} \frac{\partial x}{\partial r_i} \right] \tag{7.1.11}$$

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The Legendre functions are computed via recursion formulae:

Zonals: $m=0$

$$P_n^o\ (\sin \phi') = \frac{1}{n} \left[(2n-1) \sin \phi' P_{n-1}^o\ (\sin \phi') - (n-1)\ P_{n-2}^o\ (\sin \phi') \right] \qquad (7.1.12)$$

$$P_1^o\ (\sin \phi') = \sin \phi' \qquad (7.1.13)$$

Tesserals: $m \neq 0$ and $m \leq n$

$$P_n^m\ (\sin \phi') = P_{n-2}^m\ (\sin \phi') + (2n-1) \cos \phi' P_{n-1}^{m-1}\ (\sin \phi') \qquad (7.1.14)$$

$$P_1^1\ (\sin \phi') = \cos \phi' \qquad (7.1.15)$$

Sectorals: $m=n$

$$P_n^m = (2n-1) \cos \phi' P_{n-1}^{n-1}\ (\sin \phi') \qquad (7.1.16)$$

The derivative relationship is given by

$$\frac{d}{d\phi'} P_n^m (\sin \phi') = P_n^{m+1} (\sin \phi') - m \tan \phi' P_n^m (\sin \phi') \tag{7.1.17}$$

It should also be noted that multiple angle formulas are used for evaluating the sine and cosine of $m\lambda$.

The variational equations require the computation of the matrix U_{2c}, whose elements are given by

$$\left. U_{2c} \right)_{i,j} = \frac{\partial^2 U}{\partial r_i \, \partial r_j} \tag{7.1.18}$$

where

$r_i = \{x, y, z\}$, the true of date satellite position.

U is the geopotential.

Because the Earth's field is in terms of r, $\sin \phi'$, and λ, we write

$$U_{2c} = C_1^T U_2 C_1 + \sum_{k=1}^{3} \frac{\partial U}{\partial e_k} C_{2k} \tag{7.1.19}$$

where

e_k ranges over the elements r, $\sin \phi'$, and λ

U_2 is the matrix whose i, j^{th} element is given by $\frac{\partial^2 U}{\partial e_i \, \partial e_j}$

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C_1 is the matrix whose i, jth element is given by $\dfrac{\partial e_i}{\partial r_j}$

and

C_{2k} is a set of three matrices whose i, jth elements are given by $\dfrac{\partial^2 e_k}{\partial r_i \, \partial r_j}$

We compute the second partial derivatives of the potential U with respect to r, ϕ', and λ:

$$\frac{\partial^2 U}{\partial r^2} = \frac{2GM}{r^3} + \frac{GM}{r^3} \sum_{n=2}^{nmax} (n+1)\,(n+2) \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n}$$

$$(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\; P_n^m\,(\sin \phi') \qquad (7.1.20)$$

$$\frac{\partial^2 U}{\partial r \partial \phi'} = - \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n} (C_{nm} \cos m\lambda$$

$$+ S_{nm} \sin m\lambda)\; \frac{\partial}{\partial \phi'}\,(P_m^n\,(\sin \phi')) \qquad (7.1.21)$$

$$\frac{\partial^2 U}{\partial r \partial \lambda} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n} m$$

$$(-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda)\; P_n^m\,(\sin \phi') \qquad (7.1.22)$$

$$\frac{\partial^2 U}{\partial \phi'^2} = \frac{GM}{r} \sum_{n=2}^{nmax} \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)$$

$$\frac{\partial^2}{\partial \phi^2} \left(P_n^m (\sin \phi')\right) \tag{7.1.23}$$

$$\frac{\partial^2 U}{\partial \phi' \partial \lambda} = \frac{GM}{r} \sum_{n=2}^{nmax} \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n} m \, (-C_{nm} \sin m\lambda \tag{7.1.24}$$

$$+ S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} \left(P_n^m (\sin \phi')\right)$$

$$\frac{\partial^2 U}{\partial \lambda^2} = - \frac{GM}{r} \sum_{n=2}^{nmax} \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n} m^2 \, (C_{nm} \cos m\lambda \tag{7.1.25}$$

$$+ S_{nm} \sin m\lambda) \, P_n^m (\sin \phi')$$

where

$$\frac{\partial}{\partial \phi'} \left(P_n^m (\sin \phi')\right) = P_n^{m+1} (\sin \phi') - m \tan \phi' \, P_n^m (\sin \phi') \tag{7.1.26}$$

$$\frac{\partial^2}{\partial \phi'^2}\left(P_n^m\ (\sin\phi')\right) = P_n^{m+2}(\sin\phi') - (m+1)\tan\phi'\ P_n^{m+1}(\sin\phi')$$

$$- m\tan\phi'\left[P_n^{m+1}\ (\sin\phi') - m\tan\phi'\ P_n^m(\sin\phi')\right]$$

$$- m\sec^2\phi'\ P_n^m\ (\sin\phi') \qquad (7.1.27)$$

The elements of U_2 have almost been computed. What remains is to transform from (r, ϕ', λ) to $(r, \sin\phi', \lambda)$. This affects only the partials involving ϕ':

$$\frac{\partial U}{\partial \sin\phi'} = \frac{\partial U}{\partial \phi'}\ \frac{\partial \phi'}{\partial \sin\phi'} \qquad (7.1.28)$$

$$\frac{\partial^2 U}{\partial \sin\phi'^2} = \frac{\partial \phi'}{\partial \sin\phi'}\left(\frac{\partial^2 U}{\partial \phi'^2}\right)\ \frac{\partial \phi'}{\partial \sin\phi'} + \frac{\partial U}{\partial \phi'}\ \frac{\partial^2 \phi'}{\partial \sin\phi'^2}$$

$$(7.1.29)$$

where

$$\frac{\partial \phi'}{\partial \sin\phi'} = \sec\phi' \qquad (7.1.30)$$

$$\frac{\partial^2 \phi'}{\partial \sin\phi'^2} = \sin\phi'\sec^3\phi' \qquad (7.1.31)$$

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For the C_1 and C_{2k} matrices, the partials of r, sin ϕ', and λ are obtained from the usual formulas:

$$r = \sqrt{x^2+y^2+z^2} \tag{7.1.32}$$

$$\sin \phi' = \frac{z}{r} \tag{7.1.33}$$

$$\lambda = \tan^{-1}\left(\frac{y}{x}\right) - \theta_g \tag{7.1.34}$$

We have for C_1:

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r} \tag{7.1.35}$$

$$\frac{\partial \sin \phi'}{\partial r_i} = \frac{-z\, r_i}{r^3} + \frac{1}{r}\,\frac{\partial z}{\partial r_i} \tag{7.1.36}$$

$$\frac{\partial \lambda}{\partial r_i} = \frac{1}{x^2+y^2}\left[x\,\frac{\partial y}{\partial r_i} - y\,\frac{\partial x}{\partial r_i}\right] \tag{7.1.37}$$

The C_{2k} are symmetric. The necessary elements are given by

$$\frac{\partial^2 r}{\partial r_i \partial r_j} = \frac{r_i \ r_j}{r^3} + \frac{1}{r} \frac{\partial r_i}{\partial r_j} \tag{7.1.38}$$

$$\frac{\partial^2 \sin \phi'}{\partial r_i \ \partial r_j} = \frac{3z \ r_i \ r_j}{r^5} - \frac{1}{r^3}\left[r_j \frac{\partial z}{\partial_{\ i}} + r_i \frac{\partial z}{\partial_{\ j}} + z \frac{\partial r_i}{\partial_{\ j}} \right] \tag{7.1.39}$$

$$\frac{\partial^2 \lambda}{\partial r_i \ \partial r_j} = \frac{-2r_j}{(x^2+y^2)^2} \left[x \frac{\partial y}{\partial r_i} - y \frac{\partial x}{\partial r_i} \right] \tag{7.1.40}$$

$$+ \frac{1}{x^2+y^2} \left[\frac{\partial x}{\partial r_j} \frac{\partial y}{\partial r_j} - \frac{\partial y}{\partial r_j} \frac{\partial x}{\partial r_j} \right]$$

If gravitational constants, C_{nm} or S_{nm} are being treated, we require their partials in the f matrix for the variational equations computations. These partials are

$$\frac{\partial}{\partial C_{nm}} \left(- \frac{\partial U}{\partial r} \right) = (n+1) \frac{GM}{r^2} \left(\frac{a_e}{r} \right)^n \cos (m\lambda) \ P_n^m(\sin \phi') \tag{7.1.41}$$

$$\frac{\partial}{\partial C_{nm}} \left(- \frac{\partial U}{\partial \lambda} \right) = m \frac{GM}{r} \left(\frac{a_e}{r} \right)^n \sin (m\lambda) \ P_n^m (\sin \phi') \tag{7.1.42}$$

$$\frac{\partial}{\partial C_{nm}}\left(-\frac{\partial U}{\partial \phi'}\right) = -\frac{GM}{r}\left(\frac{a_e}{r}\right)^n \cos(m\lambda)\left[P_n^{m+1}(\sin\phi')\right.$$

$$\left.- m \tan\phi' \, P_n^m(\sin\phi')\right] \qquad (7.1.43)$$

The partials for S_{nm} are identical with cos (mλ) replaced by sin (mλ) and with sin (mλ) replaced by -cos (mλ).

These partials are converted to inertial true of date coordinates using the chain rule; e.g.,

$$\frac{\partial}{\partial C_{nm}}\left(-\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial C_{nm}}\left(\frac{-\partial U}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial C_{nm}}\left(\frac{-\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial x}$$

$$+ \frac{\partial}{\partial C_{nm}}\left(\frac{-\partial U}{\partial \phi'}\right)\frac{\partial \phi'}{\partial x} \qquad (7.1.44)$$

The logic and appropriate array dimensions of the ORAN program have been modified to accommodate uncertainties in the coefficients of the zonal and tesseral harmonic terms in the geopotential up to and inclduing (30,30). This is near to the practical limit for the current 360 series machines. Beyond (30,30), the accuracy of the computations of the perturbations and partials becomes poor for double precision and excessive amounts of computing time are involved in calculating the Legendre polynomials. However, the highly localized gravitational anomalies that require higher order spherical harmoncis for adequate representation can be simulated by using the surface density model described in Section 7.2.

Table 7.1. Force Model Error Components

Force Type	Force Model	Error Parameters
Gravitational Field	Spherical Harmonic Expansion of Earth's Gravitational Potential	GM_E = Earth's gravitational constant C_{mn}, S_{mn} = Coefficients of spherical harmonic expansion
	Surface Density	ϕ_i i=1, 2..p density for surface density blocks where p=number of blocks
	Lunar and Solar Perturbations	GM_m = Lunar Gravitational Constant GM_s = Solar Gravitational Constant
	Solid Earth Tides	K_2 = Love Number
	Lumped Gravitational Error	GM_E, C_{mn}, S_{mn} = Same as above - only those terms which are not treated individually are included in the lumped error.
Solar Radiation	Direct Solar Radiation Pressure	C_R = Solar radiation coefficient
	Earth Reflected Radiation Pressure	γ_D = Ratio of Earth reflected radiation to incoming solar flux
Other	Atmospheric Drag	C_D = Atmospheric drag coefficient
	Micrometeorite Impact	$\ddot{\bar{r}}_t$ = Acceleration due to impact

7.2 MASCONS AND SURFACE LAYER DENSITY

The surface layer density represents the acceleration due to a localized gravity anomaly. This acceleration is proportional to the area of the particular block in question and inversely proportional to the square of the distance from the spacecraft. The surface layer density block can be represented as a mascon by equating the errors in the spacecraft acceleration due to a mascon and local density layers. Let $\ddot{\bar{R}}_{DEN}$ be this acceleration. Then:

$$\ddot{\bar{R}}_{DEN_j} = \sum_i \left(\frac{\phi_i \; A_i \; x_{ij}}{\rho_i^3} \right) \qquad (7.2.1)$$

where ϕ_i is the density of block i; A_i is the area of block i; x_i is the distance along the j axis; and ρ_i^3 is the cube of distance from block i to the spacecraft.

ORAN was modified to account for both propagations of errors in this model and adjustments of the local density values. Any number of blocks can be generated by the program. However, the user is cautioned not to specify so many blocks that an excessive amount of core is required.

For integration of these force model partials, the differential equation

$$\frac{d^2 \left(\dfrac{\partial \bar{x}}{\partial \phi_i} \right)}{dt^2} = \frac{\partial \bar{f}}{\partial \bar{x}} \; \frac{\partial x}{\partial \phi_i} + \frac{\partial \bar{f}}{\partial \phi_i} \qquad (7.2.2)$$

must be solved.

To do this, ORAN required the second term on the right-hand side. It is found by differentiating $\ddot{\bar{R}}_{DEN}$ with respect to ϕ_i,

$$\frac{\partial \bar{F}}{\partial \phi_i} = \frac{\partial \ddot{\bar{R}}_{DEN}}{\partial \phi_i} = \frac{A_i X_{ij}}{\rho_i^2} . \tag{7.2.3}$$

The complete details of the surface layer density model in the ORAN program have been given by Martin (1972).

MASCONS are modeled as point masses which may be located at any point in the earth. These point masses perturb the satellite orbit in the same manner as do geopotential harmonics and are consequently treated similarly in the ORAN program. The gravitational potential of a MASCON produces a satellite force which can be integrated numerically along with the variational equations for epoch element and geopotential coefficient partial derivatives. Thus, as far as the EASST program is concerned, the only thing unique about a MASCON is its forcing function in the variational equations. The appropriate forcing function will now be derived.

The potential of a mass m at a point which is a distance ρ from the mass is, by definition

$$\frac{Gm}{\rho} \tag{7.2.4}$$

where G is the gravitational constant. Similarly, the potential of a mass (M-m) at a distance ρ_1 is

$$\frac{G(M-m)}{\rho_1} \tag{7.2.5}$$

The potential of both masses at the same point, P, is

$$V = \frac{G(M-m)}{\rho_1} + \frac{Gm}{\rho} \tag{7.2.6}$$

This expression for the potential V can be used to derive the disturbing potential of a MASCON. Let m be the mass of the MASCON and (M-m) be the mass of the body containing the MASCON, so that the total mass within the primary body is M. Then the potential of the central force is GM/r where r is the distance of the point P from the center of gravity of the MASCON and the primary body. If the MASCON (m) is fixed with respect to the mass (M-m), then we can write the disturbing potential as

$$\Delta V = V - GM/r$$

$$= \frac{G(M-m)}{\rho_1} + \frac{Gm}{\rho} - \frac{GM}{r} \,. \tag{7.2.7}$$

This expression can be simplified if we are prepared to neglect terms of order $\left(\frac{m}{M}\right)^2$. Even for the moon $\left(\frac{m}{M}\right)^2$ is of order 10^{-5}. For the earth, for which $\left(\frac{m}{M}\right)$ is much smaller, we are completely justified in neglecting $\left(\frac{m}{M}\right)^2$. With this approximation the disturbing potential can be written as:

$$\Delta V = \frac{Gm}{\rho} - \frac{Gm}{r^3}\,(xX + yY + zZ) - \frac{Gm}{r} \tag{7.2.8}$$

where (x,y,z) are the coordinates of P, and (X,Y,Z) are the coordinates of the MASCON relative to the center of gravity of the system.

From the disturbing potential we can derive the components of the disturbing acceleration by differentiation. The x-component is

$$\frac{\partial(\Delta V)}{\partial x} = -\frac{Gm}{\rho^3}\,(x - X) + \frac{Gm}{r^3}\left\{(x-X) + \frac{3x}{r^2}\,(xX + yY + zZ)\right\}, \tag{7.2.9}$$

with identical forms for the other two components.

7.3 LUNAR AND SOLAR PERTURBATIONS

Consider a system of n point masses m_i $(i = 0,1,2,\ldots n-1)$. The equations of motion of the i^{th} mass relative to the mass m_0 can be written:

$$\ddot{\bar{r}}_{0i} = -\frac{G(m_0+m_i)}{r_{0i}^3}\bar{r}_{0i} + G\sum_{\substack{j=1\\ j\neq i}}^{n-1} m_j \left[\frac{\bar{r}_{ij}}{r_{ij}^3} - \frac{\bar{r}_{0j}}{r_{0j}^3}\right] . \tag{7.3.1}$$

Let m_0 denote the Earth and m_i the satellite. The satellite's motion relative to the Earth is then (m_i is negligible with respect to the other masses)

$$\ddot{\bar{r}}_{01} = -\frac{Gm_0}{r_{01}^3}\bar{r}_{01} + G\sum_{j=2}^{n-1} m_j \left[\frac{\bar{r}_{1j}}{r_{1j}^3} - \frac{\bar{r}_{0j}}{r_{0j}^3}\right] . \tag{7.3.2}$$

The variational equation for an error in the gravitational constant Gm_j $(j = 2,3,\ldots n-1)$ is then

$$\frac{\partial\ddot{\bar{r}}_{01}}{\partial(Gm_j)} = \frac{d^2\left(\dfrac{\partial\bar{r}_{01}}{\partial(Gm_j)}\right)}{dt^2} = \frac{\bar{r}_{1j}}{r_{1j}^3} - \frac{\bar{r}_{0j}}{r_{0j}^3} . \tag{7.3.3}$$

These variational equations are integrated numerically to give $\partial(r_{01})/\partial(Gm_j)$.

Recent determinations of GM for the Moon give a value of about 4902.6 kg^3/sec^2 with a standard deviation of about 0.25 kg^3/sec^2. Consequently a reasonable estimate of the uncertainty in GM for the Moon is given by $0.25/4902.6 \approx 5.0 \times 10^{-5}$.

7.4 SOLID EARTH TIDES

The tidal displacement of mass produces a perturbation in the gravitational field of the Earth, U_D. Following the small response assumption, Love (1911) represented the disturbance potential, at the surface $r=R_e$, by

$$U_D(R_e) = k\, U_T(R_e) \tag{7.4.1}$$

This can be generalized (Kaula, 1968) as:

$$U_D(R_e) = \sum_{n=2}^{\infty} k_n\, U_{Tn}(R_e) \tag{7.4.2}$$

Since $\frac{R_e}{R_d} << 1$, terms involving $n>2$ can be neglected to first approximation, and

$$U_D(R_e) \approx k_2\, U_{T2}(R_e). \tag{7.4.3}$$

There is an unresolved discrepancy with the best theoretical value of k_2 which is equal to 0.290 and the best estimates based on satellite measurements, which now center on 0.25.

The second order third body disturbing potential at the surface of the earth is given by

$$U_{T_2} = \frac{GM_d R_e^2}{R_d^3} P_2 (\cos \theta) \tag{7.4.4}$$

where G is the gravitational constant, M_d is the mass of the disturbing body, R_e is the radius of the Earth, and R_d is the magnitude of the vector $\overline{R}_d$ from the Earth to the disturbing body. The angle θ is the angle enclosed by $\overline{R}_d$ and the vector $\overline{r}$ from the earth to the satellite. Evaluating the Legrendre polynomial and using the vector formulation for cos θ yields

$$U_{T_2} = \frac{GM_d R_e^2}{2 R_d^3} \left[3 \left[\frac{\overline{R}_d \cdot \overline{r}}{R_d r} \right]^2 - 1 \right] . \tag{7.4.5}$$

Letting the "^" notation refer to the unit vector,

$$U_D(r) = \frac{k_2}{2} \frac{GM_d}{R_d^3} \frac{R_e^5}{r^3} \left[3 (\hat{R}_d \cdot \hat{r})^2 - 1 \right] \tag{7.4.6}$$

The acceleration produced at position $\overline{r}$ by the solid Earth tidal bulge is (Diamante and Williamson, 1972):

$$\ddot{\overline{r}} = \nabla U_D = \left(- \frac{3}{2} k_2 \frac{GM_d}{R_d^3} \frac{R_e^5}{r^4} \right) \left(3 (\hat{R}_d \cdot \hat{r})^2 - 1 \right) \hat{r}$$

$$+ 6 \left(\frac{1}{2} k_2 \frac{GM_d}{R_d^3} \frac{R_e^5}{r^3} \right) \left(\frac{(\hat{R}_d \cdot \hat{r})}{r} \hat{R}_d - \frac{(\hat{R}_d \cdot \hat{r})^2}{r} \hat{r} \right) . \tag{7.4.7}$$

The variational equation is:

$$\frac{d^2}{dt^2}\left(\frac{\partial \bar{r}}{\partial k_2}\right) = \frac{1}{2}\ \frac{GM_d}{R_d^3}\ \frac{R_e^5}{r^4}\left\{[3-15(\hat{R}_d\cdot\hat{r})^2]\hat{r} + 6(\hat{R}_d\cdot\hat{r})\hat{R}_d\right\}$$

$$+\ \frac{k_2}{2}\ \frac{GM_d}{R_d^3}\ \frac{R_e^5}{r^5}\left\{\left[105(\hat{R}_d\cdot\hat{r})^2\left(\hat{r}\cdot\frac{\partial\bar{r}}{\partial k_2}\right) - 30(\hat{R}_d\cdot\hat{r})\left(\hat{R}_d\cdot\frac{\partial\bar{r}}{\partial k_2}\right)\right.\right.$$

$$\left. -15\left(\hat{r}\cdot\frac{\partial\bar{r}}{\partial k_2}\right)\right]\hat{r} + \left[6\left(\hat{R}_d\cdot\frac{\partial r}{\partial k_2}\right) - 30(\hat{R}_d\cdot\hat{r})\left(\hat{r}\cdot\frac{\partial\bar{r}}{\partial k_2}\right)\right]\hat{R}_d$$

$$\left. +\left[3-15(\hat{R}_d\cdot\hat{r})^2\right]\frac{\partial\bar{r}}{\partial k_2}\right\} + \left(\frac{\partial\bar{r}}{\partial k_2}\cdot\nabla\right)\bar{F}\quad . \qquad (7.4.8)$$

The variational equation is actually given by the sum of the expression above for the Moon and Sun, i.e., with the subscript "d" replaced by "s" and "m". These equations were derived under the assumption that the response time of the Earth is negligible. Actually, the tide will be displaced by some angle $\delta = n_o \cdot \Delta t$ where n_o is the angular velocity of the Earth and Δt is the time lag of the tide. This phase lag may be accounted for by replacing $\bar{R}_d$ in the above Equation with a vector $\bar{R}_d^*$ which corresponds to $\bar{R}_d$ rotated through an angle δ in the direction of Earth's rotation (Diamante and Williamson, 1972).

7.5 LUMPED GEOPOTENTIAL ERROR DUE TO GRAVITY MODEL DIFFERENCES

The ORAN program possesses the capability of modeling the total set of errors in the set of spherical harmonic coefficients used to perform an orbital data reduction. The rationale for the use of this form is basically as follows. Let x_{APL} represent a satellite orbit generated or estimated using a set of geopotential coefficients estimated by an investigator, or group of investigators, which we have denoted by APL. Similarly, let x_{SAO} represent the same orbit ephemeris obtained (estimated) in the same way except that geopotential coefficients estimated by a group denoted by SAO were used. Let us further assume that group APL and group SAO operated completely independently using different tracking data on different satellites. On this basis, we deduce that errors in their geopotential models and thus geopotential dependent errors in x_{APL} and x_{SAO} are independent. It is not necessary to assume that the two models are equally accurate.

Let us then consider the variance of the difference between x_{APL} and x_{SAO}. We can obtain this variance in terms of the variance of the individual model errors by first writing the difference as

$$x_{APL} - x_{SAO} = (x_{APL} - x_t) - (x_{SAO} - x_t) = \delta x_{APL} - \delta x_{SAO} \quad (7.5.1)$$

where

x_t is the true ephemeris and δx_{APL} and δx_{SAO} are the errors in the two orbits using the two different geopotential models. Then

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$$\mathrm{Var}(x_{APL}-x_{SAO}) = E[(x_{APL}-x_{SAO})(x_{APL}-x_{SAO})^T] \tag{7.5.2}$$

$$= \mathrm{Var}\ \delta x_{APL} + \mathrm{Var}\ \delta x_{SAO}$$

For the special case in which the models are of equal variance (accuracy), we have

$$\mathrm{Var}\ (\delta x_{APL}) = \mathrm{Var}\ (\delta x_{SAO}) = \frac{1}{2}\ \mathrm{Var}\ (x_{APL}-x_{SAO}) \tag{7.5.3}$$

If the models are not of equal accuracy, but we can relate the variances of the two models by a constant factor,

$$\mathrm{Var}\ \delta x_{APL} = k\ \mathrm{Var}\ \delta x_{SAO}, \tag{7.5.4}$$

we then have the relation

$$\mathrm{Var}\ (\delta x_{SAO}) = \frac{1}{1+k}\ \mathrm{Var}\ (x_{APL}-x_{SAO}) \tag{7.5.5}$$

If the SAO model is the more accurate, then

$$\frac{1}{1+k} < \frac{1}{2} \tag{7.5.6}$$

and we have an upper limit on the coefficient of the gravity model difference variance which is needed to obtain the variance of the SAO model.

In practice, we would not expect k to be exactly a constant, nor would we expect to find geopotential models with completely independent errors, but we would expect to be able to choose models which are nearly independent and with a suitable definition of the variances of δx_{APL} and δx_{SAO} (e.g., sampled over all possible orbits), then k is a constant. In any event, we will use the above relation obtained using the SAO gravity model, in terms of the gravity model difference effect.

It is possible to compute the effects on an orbit of a gravity model difference using the integration of a set of variational equations. Consider first the variational equation for a spherical harmonic which we can write as

$$\frac{d^2}{dt^2}\left(\frac{\partial \bar{r}(t)}{\partial C_{nm}}\right) = \frac{\partial \ddot{\bar{r}}(t)}{\partial \bar{r}(t)} \frac{\partial \bar{r}(t)}{\partial C_{nm}} + \frac{\partial \ddot{\bar{r}}(t)}{\partial \dot{\bar{r}}(t)} \frac{\partial \dot{\bar{r}}(t)}{\partial C_{nm}} + \frac{\partial \ddot{\bar{r}}(t)}{\partial C_{nm}} \tag{7.5.6}$$

Where $\bar{r}(t)$ is the satellite position (x,y,z) at time t and $\ddot{\bar{r}}(t)$ is the acceleration. Note that the acceleration due to gravity is dependent only on position. Therefore, the second term on the right hand side of the above equation is zero. The effect of a small error δC_{nm} on the orbital parameters at time t is given by

$$\frac{d^2}{dt^2}(\delta \bar{r}) = \left\{\frac{\partial \ddot{\bar{r}}}{\partial \bar{r}} \frac{\partial \bar{r}}{\partial C_{nm}} + \frac{\partial \ddot{\bar{r}}}{\partial C_{nm}}\right\} \delta C_{nm} \tag{7.5.7}$$

A corresponding equation holds for the S_{nm} coefficient of the same degree and order

$$\frac{d^2}{dt^2}(\delta\bar{r}(t)) = \left\{ \frac{\partial\ddot{\bar{r}}(t)}{\partial\bar{r}(t)} \frac{\partial\bar{r}(t)}{\partial S_{nm}} + \frac{\partial\ddot{\bar{r}}(t)}{\partial S_{nm}} \right\} \delta S_{nm} \tag{7.5.8}$$

If we sum (7.5.7) and (7.5.8) for a range of values of n and m we get the variational equation for the total error in the orbital parameters at time t due to errors in the spherical harmonic coefficients

$$\frac{d^2}{dt^2}(\delta\bar{r}(t)) = \sum_{n,m} \left[\frac{\partial\ddot{\bar{r}}(t)}{\partial\bar{r}(t)} \left(\frac{\partial\bar{r}(t)}{\partial C_{nm}} \delta C_{nm} + \frac{\partial\bar{r}(t)}{\partial S_{nm}} \delta S_{nm} \right) + \frac{\partial\ddot{\bar{r}}(t)}{\partial S_{nm}} \delta S_{nm} + \frac{\partial\ddot{\bar{r}}(t)}{\partial C_{nm}} \delta C_{nm} \right] \tag{7.5.9}$$

Within the limits of linearity (and ORAN is a linear error analysis program), the quantities in brackets are the orbit differences due to a set of differences (or errors) in the spherical harmonic coefficients. We specialize this set of differences to be obtained by differencing geopotential model sets which are both as accurate and independent as possible. Equation (7.5.9) then gives the variational equation for the effects which the set of coefficient differences produce on the estimated orbit.

In ORAN each spherical harmonic coefficient may be 1) adjusted, 2) treated as an individual unadjusted parameter or 3) included in the lumped parameter model of equation (7.5.9).

7.6 DIRECT SOLAR RADIATION PRESSURE

For a spherical satellite with an integrated reflectivity $\bar{\rho}_R$, the acceleration is (Diamante, 1972b):

$$\bar{a} = \ddot{\bar{r}} = -\nu(1+\bar{\rho}_R) \frac{S}{c} \left(\frac{A}{m}\right) d_o^2 \frac{(\bar{R}_S - \bar{r})}{|\bar{R}_S - \bar{r}|^3} , \tag{7.6.1}$$

where

$$\nu = \begin{cases} 0, & \text{when the satellite is in the Earth's shadow,} \\ 1, & \text{when the satellite is illuminated by Sun,} \end{cases}$$

$\bar{r}$ = satellite position vector (Earth centered coordinate system),

$\bar{R}_S$ = position vector of the sun (Earth centered coordinate system),

A = cross-sectional area of the satellite[1],

m = mass of the satellite.

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[1] The cross-sectional area is in a plane normal to the direction of the radiation flux. A is therefore constant for spherical satellites. However, for other geometries, variable cross-section along the orbit must be taken into account or an average value of A adopted.

Clearly, when $f_1 \geq 0$, the satellite is always in full sunlight. If $f_1 < 0$, the satellite will be in shadow when:

$$\bar{r}\cdot\bar{r} - (\bar{r}\cdot\hat{R}_S)^2 \leq R_e^{\,2} \tag{7.6.2}$$

R_e is the radius of the Earth. Let:

$$f_2 = r^2 - (\hat{R}_S\cdot\bar{r})^2 - R_e^{\,2}. \tag{7.6.3}$$

Then

$$\nu = \begin{cases} 1, & f_1 \geq 0 \\ 1, & f_1 < 0 \text{ and } f_2 > 0 \\ 0, & f_1 < 0 \text{ and } f_2 \leq 0 \end{cases} \tag{7.6.4}$$

An error in the direct solar radiation pressure force model can be expressed as an uncertainty in $\bar{\rho}_R$, or as an uncertainty in C_R. Then

$$\frac{\partial}{\partial C_R}\,\bar{r} = \frac{d^2}{dt^2}\left(\frac{\partial\bar{r}}{\partial C_R}\right) = -\nu\ \frac{S}{c}\left(\frac{A}{m}\right)\ d_o^{\,2}\left[1 + C_R\,\frac{\partial\bar{r}}{\partial C_R}\cdot\nabla\right]\frac{(\bar{R}_S-\bar{r})}{|\bar{R}_S-\bar{r}|^3}$$

$$+\left(\frac{\partial\bar{r}}{\partial C_R}\cdot\nabla\right)\bar{F}, \tag{7.6.5}$$

where ∇ is the gradient operator and $\bar{F}$ includes the effects of all the other forces acting on the satellite. The variational equation for $\frac{\partial \bar{r}}{\partial C_R}$ reduces to (Diamante, 1972b):

$$\frac{\partial}{\partial C_R}\ddot{\bar{r}} = \frac{d^2}{dt^2}\left(\frac{\partial \bar{r}}{\partial C_R}\right)$$

$$= -\nu \frac{S}{c}\left(\frac{A}{m}\right) d_o^{\,2}\left\{\left(\frac{\bar{R}_S-\bar{r}}{|\bar{R}_S-\bar{r}|^3}\right)\left(1 + 3\,\frac{C_R(\bar{R}_S-\bar{r})}{|\bar{R}_S-\bar{r}|^2}\cdot\frac{\partial \bar{r}}{\partial C_R}\right)\right.$$

$$\left. - \frac{C_R}{|\bar{R}_S-\bar{r}|^3}\,\frac{\partial \bar{r}}{\partial C_R}\right\} + \left(\frac{\partial \bar{r}}{\partial C_R}\cdot\nabla\right)\bar{F}\,. \qquad (7.6.6)$$

7.7 EARTH REFLECTED RADIATION

Over the long term, the radiation budget of the Earth must be in equilibrium. A large fraction of the solar radiation received by the Earth is reflected back, almost immediately, at nearly the same wavelengths at which it is received. The ratio of the reflected radiation to the incoming solar flux is known as the <u>albedo</u>, γ_R. The best estimate of the long-term mean global albedo for the Earth has been obtained from TIROS VII measurements:

$$\gamma_R = 0.32 \qquad (7.7.1)$$

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The remaining fraction of the incoming solar flux is absorbed by the atmosphere, oceans and surface of the Earth and is eventually re-emitted as the far-infrared planetary radiation. From equilibrium requirements, this fraction is:

$$\gamma_{IR} = 1 - \gamma_R = 0.68 \tag{7.7.2}$$

Part of the reflected radiation is scattered diffusely and part is scattered specularly. The specular reflection is mostly from small bodies of water scattered over the surface of the Earth and occasional calm areas of the oceans. Therefore,

$$\gamma_R = \gamma_D + \gamma_S, \tag{7.7.3}$$

where the subscripts D and S stand for diffuse and spectral. Using a value of 4/3 for the index of refraction of water, the refractivity at normal incidence is 0.02. Furthermore, although 71% of the Earth's surface is covered by water, only a small fraction of that can be expected to produce specular reflection at any one instant in time. Therefore, to good approximation,

$$\gamma_D \approx \gamma_R = 0.32 \tag{7.7.4}$$

Since for the purposes of error analysis we need only consider the radial accelerations, we have employed a simplified analytic model for the Earth reflected radiation (Diamante, 1972b).

Danjon (1954) made observations of the Earth-shine reflected from the dark hemisphere of the Moon. He found that the phase function of the Earth is better represented by a non-Lambert Law expression. At great distances, the Danjon relation can be approximated by a $(1-\cos\beta_s)^2$ dependence. The Earth receives solar energy at a rate:

$$\pi R_e^{\,2}\; S\left(\frac{d_o}{R_s}\right)^2 \text{ ergs/sec,}$$

and reflects:

$$\gamma_D\; \pi R_e^{\,2}\; S\left(\frac{d_o}{R_s}\right)^2 \text{ ergs/sec}$$

away. If this is distributed at great distances according to the function:

$$C(r)\;(1-\cos\beta_s)^2,$$

then the total energy flux through a spherical surface of radius r, assuming that the flux is radially outward, is:

$$2\pi r^2\, C(r) \int_0^{\pi} \sin\ _s(1-2\cos\beta_s + \cos^2\beta_s)\,d\beta_s$$

$$= \frac{16}{3}\,\pi\, r^2\, C(r).$$

Setting this equal to $\gamma_D \pi R_e^2 S$:

$$C(r) = \frac{3}{16} \gamma_D \left(\frac{R_e}{r}\right)^2 S\left(\frac{d_o}{R_s}\right)^2 \qquad (7.7.5)$$

Therefore, the Wyatt-Danjon expression for the radiation pressure is

$$P = 3/16 \; \gamma_D\left(\frac{R_e}{r}\right)^2 \frac{S}{c} \left(\frac{d_o}{R_s}\right)^2 (1-\cos\beta_s)^2, \qquad (7.7.6)$$

and the radial acceleration is:

$$\hat{r}A_R = 3/16 \; \gamma_D(1+\bar{\rho}_R) \frac{A}{m} \frac{S}{c} \left(\frac{d_o}{R_s}\right)^2 \left(\frac{R_e}{r}\right)^2 (1-\cos\beta_s)^2 \; \hat{r}. \qquad (7.7.7)$$

A force error in the Earth-reflected radiation pressure can be modeled as an error in γ_D. Noting that:

$$\cos\beta_s = \hat{R}_s \cdot \hat{r} \qquad (7.7.8)$$

where the "^" notation again denotes the unit vector, the variational equation is then:

$$\frac{\partial}{\partial\gamma_D}(A_R\,\hat{r}) = \frac{3}{16}(1+\bar{\rho}_R)\,\frac{A}{m}\,\frac{S}{c}\left(\frac{d_o}{R_s}\right)^2 R_e^{\,2}\left(1+\gamma_D\,\frac{\partial\bar{r}}{\partial\gamma_D}\cdot\nabla\right)\left(1-\frac{R_s\,\bar{r}}{r}\right)^2\frac{\bar{r}}{r^3}$$

$$+\left(\frac{\partial\bar{r}}{\partial\gamma_D}\cdot\nabla\right)\bar{F} \qquad (7.7.9)$$

or

$$\frac{\partial\ddot{\bar{r}}}{\partial\gamma_D} \approx \frac{\partial}{\partial\gamma_D}(A_R\,\hat{r}) = \frac{3}{16}(1+\bar{\rho}_R)\,\frac{A}{m}\,\frac{S}{c}\left(\frac{d_o}{R_s}\right)^2 R_e^{\,2}\,G(\bar{r}) + \left(\frac{\partial\bar{r}}{\partial\gamma_D}\cdot\nabla\right)\bar{F}$$

$$(7.7.10)$$

where

$$G(\bar{r}) = \left\{\frac{1}{r^2}\left(1-\frac{\hat{R}_s\cdot\bar{r}}{r}\right)^2 - \frac{2\gamma_D}{r^3}\left(1-\frac{\hat{R}_s\cdot\bar{r}}{r}\right)\left[\hat{R}_s\cdot\frac{\partial\bar{r}}{\partial\gamma_D} - \left(\hat{R}_s\cdot\frac{\bar{r}}{r}\right)\left(\frac{\partial\bar{r}}{\partial\gamma_D}\cdot\frac{\bar{r}}{r}\right)\right]\right.$$

$$\left. - \frac{3\gamma_D}{r^3}\left(1-\frac{\hat{R}_s\cdot\bar{r}}{r}\right)^2\left(\frac{\partial\bar{r}}{\partial\gamma_D}\cdot\frac{\bar{r}}{r}\right)\right\}\frac{\bar{r}}{r} + \left\{\frac{\gamma_D}{r^3}\left(1-\frac{\hat{R}_s\cdot\bar{r}}{r}\right)^2\right\}\frac{\partial\bar{r}}{\partial\gamma_D}.$$

$$(7.7.11)$$

7.8 ATMOSPHERIC DRAG

The acceleration $\ddot{\bar{r}}$ on a satellite due to the atmospheric drag force is given by the expression:

$$\ddot{\bar{r}} = -1/2\ C_D\ \frac{A_S}{M_s}\ \rho \left| \dot{\bar{r}} \right| \dot{\bar{r}} \qquad (7.8.1)$$

where

C_D is the coefficient of drag for satellite,

A_S is the cross-section area of the satellite, (m^2)

m_S is the mass of the satellite (kg)

$\dot{\bar{r}}$ is the velocity vector of the satellite, relative to the atmosphere (under the assumption that the atmosphere rotates with the Earth) (m/sec),

$\bar{r}$ is the position vector from the center of the Earth to the satellite (m),

ρ is the mass density of the atmosphere at position r (kg/m^3).

Strictly speaking, A_S is the area of the satellite projected in the $\dot{\bar{V}}_r$ direction. However, in practice the mean cross-section is used (constant) and C_D is also taken to be a satellite dependent constant.

The major source of error in determining the drag accelerations arises from uncertainties in the density, ρ. Since these uncertainties can usually be expected to exceed drag coefficient uncertainties, density errors can conveniently be represented as errors in C_D. Let

$$\bar{f} = -\frac{1}{2}\left(\frac{A_S}{m_S}\right)\rho\,|\dot{\bar{r}}|\,\dot{\bar{r}} \tag{7.8.2}$$

The variational equations for $\frac{\partial \bar{r}}{\partial C_D}$ are then:

$$\frac{\partial \ddot{\bar{r}}}{\partial C_D} = \frac{d^2}{dt^2}\left(\frac{\partial \bar{r}}{\partial C_D}\right) = \bar{f} + \left(C_D \frac{\partial \bar{r}}{\partial C_D} \cdot \nabla\right)\bar{f} + \left(C_D \frac{\partial \dot{\bar{r}}}{\partial C_D} \cdot \nabla_{\dot{\bar{r}}}\right)\bar{f} + \left(\frac{\partial \bar{r}}{\partial C_D} \cdot \nabla\right)\bar{F}$$

$$\frac{\partial \dot{\bar{r}}}{\partial C_D} = \frac{d}{dt}\left(\frac{\partial \bar{r}}{\partial C_D}\right) \tag{7.8.3}$$

where $\bar{F}$ involves the effects of all other forces (assumed independent of velocity) and ∇ is the gradient operator.

The righthand side of equation (2.3) may be expanded, remembering that ρ is not a function of velocity. The expressions for the partial derivatives have been given by Williamson, Martin and Dutcher, 1971. All derivatives of ρ, however, must be evaluated numerically for a given satellite orbit using the atmospheric density model described below. An error in the satellite position due to an uncertainty in C_D (drag error) is then obtained by integrating the above variational equations for $\frac{\partial \overline{r}}{\partial C_D}$.

The Jacchia 1965 Model Atmosphere, with revisions through 1968, has been selected to represent the atmospheric density in the drag error calculations of the ORAN program.

The atmospheric density calculation in the ORAN program is performed by subroutine DENSTY. This subroutine is a modification of the one currently employed in the GEODYN programs and is based on the static diffusion models of Jacchia (1964, 1968).

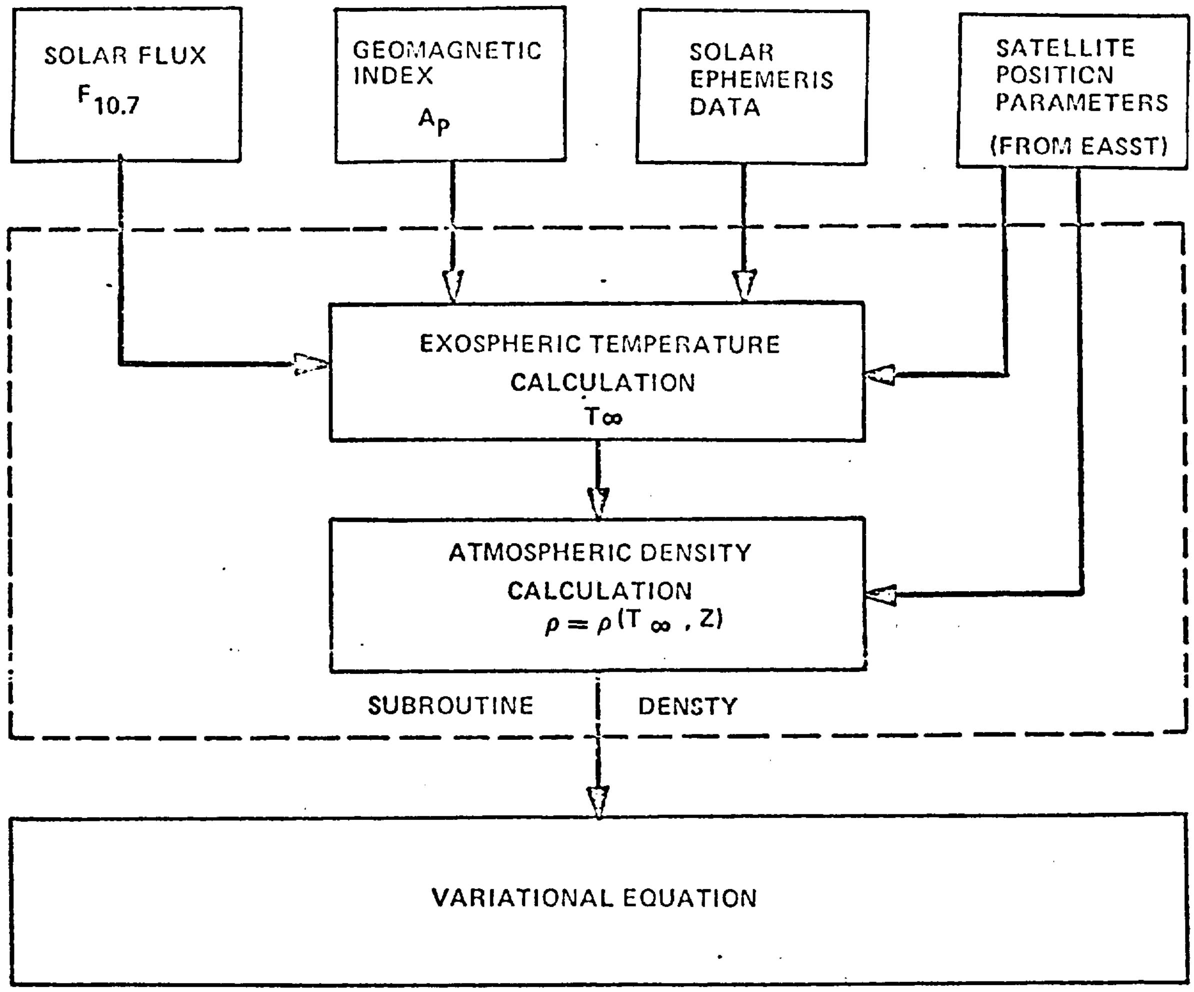
The 1968 Jacchia Model Atmosphere allows the calculation to be structured into two major steps. In the first step, the exospheric temperature is calculated from data on solar flux, geomagnetic index, solar ephemeris and satellite position information. In the second step, the density is calculated as a function of the exospheric temperature T_∞, and the satellite altitude above the surface of the Earth, z. This second step is accomplished by using a polynomial fit to tabulated solutions of differential equations. The complete procedure is summarized in Figure 2.1.

The daily value of the 10.7 cm solar flux strength, $F_{10.7}$, is in units of 10^{-22} watts/m^2/cycle/second of bandwidth. In the GEODYN program, the daily values of $F_{10.7}$ are data input and $\overline{F}_{10.7}$ is obtained by averaging the daily values over the preceding 54 days (two solar rotations). In the ORAN program, it is anticipated that for most error analysis applications and mission planning, the daily values will either be unavailable (before the fact) or represent an unnecessary burden on storage and computation for the level of accuracy required. Consequently, in the ORAN program,

$$F_{10.7} = \overline{F}_{10.7}.$$

Average values of $\overline{F}_{10.7}$ have been obtained by averaging the daily record of $F_{10.7}$ over intervals of the 11-year solar cycle corresponding to periods of high, medium and low solar activity. These values are 218.80, 130.62 and 124.35, respectively. Commonly accepted values of $F_{10.7}$ for very high, medium and very low levels of solar activity are 250-275, 150, and 65-75, respectively. Appropriate values may be selected for the anticipated solar activity levels in the period of interest.

The correction for geomagnetic activity was provided by Jacchia, Slowey and Verniani, 1967 and is based on ap, the three-hour planetary geomagnetic index. In applying this result in the ORAN program, the three-hour equivalent planetary amplitude ap was replaced by the daily equivalent planetary AP in order to minimize data input and storage. The index Ap is defined as the arithmetic average of the eight values of ap for a given day. For application of the geomagnetic correction in ORAN, the same remarks apply here as for input data on $F_{10.7}$. Average values of Ap corresponding to the <u>same</u> averaging periods over the 11 year solar cycle employed in the calculation of average values of $\overline{F}_{10.7}$ for periods of high, medium and low solar activity were found. These are respectively Ap=20.24, 13.63 and 10.98, corresponding to the values $\overline{F}_{10.7}$=218.80, 130.62 and 124.35 for high, medium and low solar activity. The indices ap and Ap can range from 0 to 400. Days when $Ap \leq 2$ are considered to be geomagnetically quiet and slightly disturbed days are those for $4 \leq Ap \leq 10$. Moderate disturbances are in the range Ap=40-50 and intense disturbances range Ap_100.



THE ORAN DRAG MODEL ERROR CALCULATION

Figure 7.2

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7.9 MICROMETEORITE IMPACT AND THRUST ERRORS

Collision of a spacecraft with micrometeorites can be modeled by uncertainties in spacecraft thrusting. If $\ddot{\bar{r}}_T$ is the acceleration due to the thrusting, then the error in position and velocity due to an uncertainty in $\ddot{\bar{r}}_T$ at the time of application is

$$\Delta\bar{r} = \frac{\partial\bar{r}}{\partial\ddot{\bar{r}}_T}\Delta\ddot{\bar{r}}_T = \frac{1}{2}(\Delta\ddot{\bar{r}}_T)(\Delta t)^2, \tag{7.9.1}$$

$$\Delta\dot{\bar{r}} = \frac{\partial\dot{\bar{r}}}{\partial\ddot{\bar{r}}_T}\Delta\ddot{\bar{r}}_T = (\Delta\bar{r}_T)(\Delta t), \tag{7.9.2}$$

where Δt is the duration of the thrust and $\Delta\bar{r}_T$ is the uncertainty in the magnitude. These relations provide the initial values for integrating variational equations to provide the effects on the orbit as a function of time.

Information on the meteoric environment near to the Earth and in interplanetary space, useful for application of the ORAN error model, is provided in <u>Space and Planetary Environment Criteria Guidelines For Use In Space Vehicle Development, 1971 Revision</u>.

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