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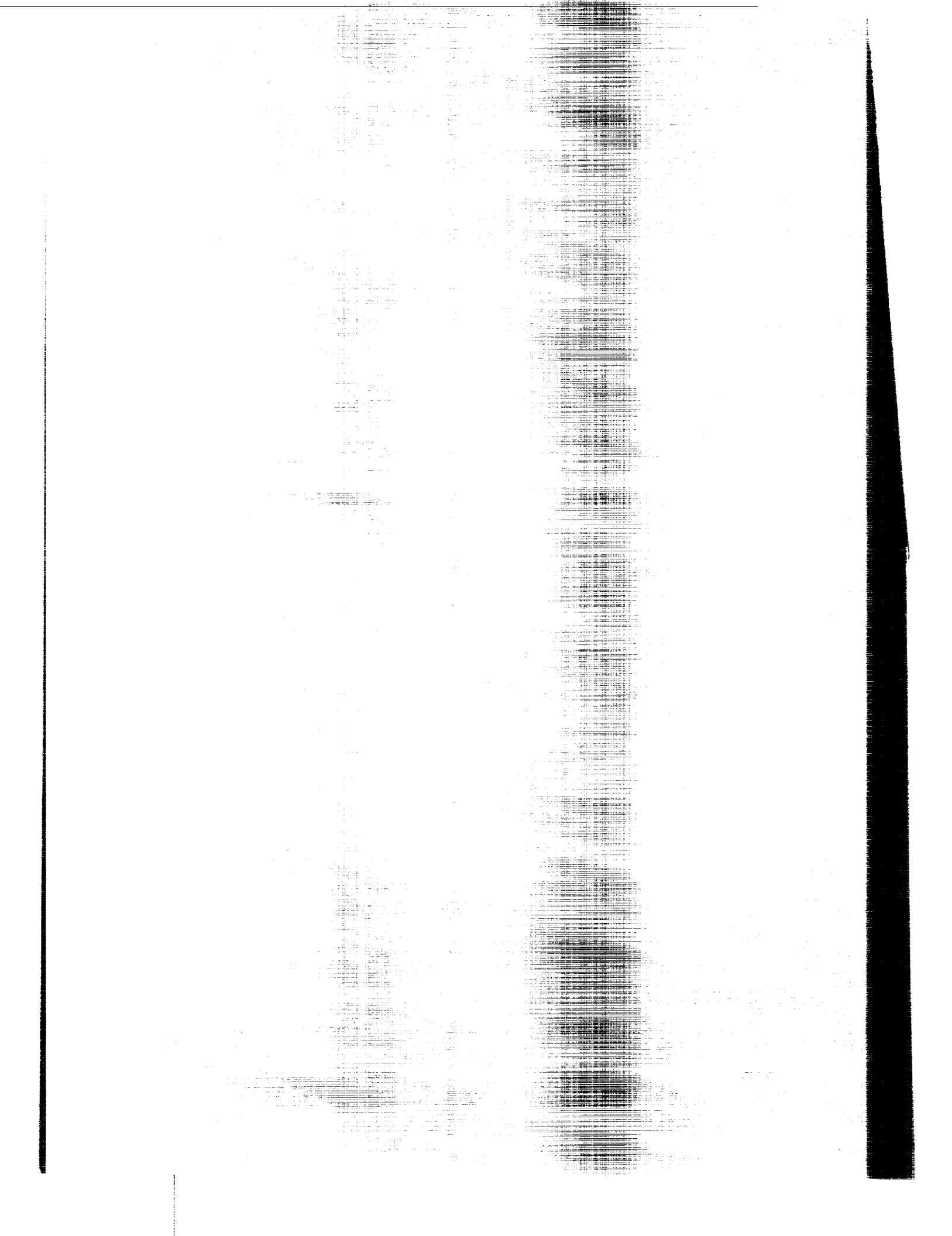
Methods of Separation of Variables in Turbulence Theory

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Methods of Separation of Variables in Turbulence Theory

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METHODS OF SEPARATION OF VARIABLES IN TURBULENCE THEORY

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SUMMARY

Two schemes of closing turbulent moment equations are proposed both
of which make double correlation equations separated into single-point
equations. The first is based on neglected triple correlations. Grid-
produced turbulence is described in this light as time-independent,
cylindrically-isotropic turbulence, showing the existence of two modes
in fluctuations, one obeying and the other disobeying Taylor's hypothesis,
respectively. Application to wall turbulence guided by a new asymptotic
method for the Orr-Sommerfeld equation suggests existence of a neutrally
stable mode of essentially three dimensional nature. The second closure
scheme is based on an assumption of identity of the separated variables
through which triple and quadruple correlations are formed. The result-
ing equation adds, to its equivalent of the first scheme, an integral of
nonlinear convolution describing a direct energy-cascading.

I. INTRODUCTION

A. Separability of Double Correlation Equations

The first sound basis on which equations governing double and higher-order turbulent correlations are formulated is due to Kármán and Howarth (ref. 1). Among other correlation or Reynolds' stress equations constructed in various ways on the common basis of Navier-Stokes framework this equation has an additional kinematical attribute that the Kármán-Howarth formalism is consistent also with the BBGKY hierarchy (ref. 2). In other words, it alone is concurrent with Liouville's equation; a master equation of nonequilibrium statistical mechanics. This fact has been revealed in reference 2 by starting with the two-point BBGKY equation transformed into fluid-dynamic moment equations through 13-moment expansion, leading for incompressible turbulence to an equation identified a posteriori with

$$<\Delta u_j \widehat{(NS)}_\ell + \Delta \hat{u}_\ell (NS)_j> = 0 \tag{1}$$

where we have defined the operator $(NS)_j$ by

$$(NS)_j = \left[\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} - \nu \frac{\partial}{\partial x_r^2} \right] u_j + \frac{1}{\rho} \frac{\partial p}{\partial x_j} \tag{2}$$

and an averaging procedure by the symbol $< >$. In these expressions quantities without and with caret ($\wedge$) denote those at space points $\vec{x}$ and $\vec{\hat{x}}$, respectively, u_j and p are the fluctuating fluid-dynamic velocity and pressure, respectively, and Δu_j is the instantaneous velocity fluctuation

$$\Delta u_j = u_j - u_j \tag{3}$$

namely, instantaneous deviation around the average velocity

$$u_j = <u_j> \tag{4}$$

Equation (1) gives us a 'recipe' of how to cook fluid-dynamic equations correctly for the correlation equation to be consistent with the BBGKY formalism. It requires that the velocity fluctuation to be multiplied on the Navier-Stokes operator is at a different space-point and at the same time, so that it prohibits using the same space-points $\vec{x} = \vec{\hat{x}}$ in Eq. (1) in contrast to what is often done in constructing model equations. Because of the seven-dimensional structure in $(\vec{x}, \vec{\hat{x}}, t)$ space

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this equation, known to fluid-dynamicists without its having the firmer basis recognized, has been thought as "totally intractable" (ref. 3) for flows with more realistic geometry. Applications, therefore, have been limited so far to homogeneous turbulence, where physical quantities depend only on relative location of the two points. If, however, an assumption is made that triple correlation in Eq. (1) is negligibly small, the equation is seen to be separable independently of its flow geometry. Actually this observation has rendered the equation separated out of (1) an Orr-Sommerfeld equation (ref. 2).

The feasibility of the variable separation has been found originally by a heuristic observation, therefore, it has been believed for some time that such a separability would be restricted to incompressible flows. Actually, allowance for the density fluctuation in the case of compressible turbulence makes it difficult to judge its feasibility on intuitive basis. We will see, however, that the separability of the equation is examined more easily through the kinetic equation rather than its moment (fluid-dynamic) versions. Following this guideline we will examine the structure of the kinetic equation of the second BBGKY hierarcy governing a correlation function between two phase-space points $z = (\vec{x}, \vec{v})$ and $\hat{z} = (\vec{\hat{x}}, \vec{\hat{v}})$. The correlation function as meant here is defined by

$$g(z, \hat{z}) \equiv \langle \Delta\delta \, \Delta\hat{\delta} \rangle \tag{5}$$

where $\Delta\delta$ denotes instantaneous fluctuation

$$\Delta\delta \equiv \delta - f \tag{6}$$

of the microscopic density (ref. 4) around its average

$$f = \langle \delta \rangle \tag{7}$$

namely, the Boltzmann function. There are direct relationships between the kinetic mean and fluctuation variables given here and those in fluid-dynamic space as defined loosely by (3) and (4). These are given consistently with the classical kinetic theory as

$$u_j = \rho^{-1} \int v_j \, \delta \, d\vec{v} \tag{8}$$

$$\Delta u_j = \rho^{-1} \int v_j \, \Delta\delta \, d\vec{v} \tag{9}$$

where ρ is the density, and where the ad hoc assumption of incompressibility is made for brevity of the expressions. By taking product of the

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instantaneous velocity fluctuations (9) at different space points $\vec{x}$ and $\vec{\hat{x}}$, and then taking average, we have an expression for turbulent velocity correlation tensor

$$\langle \Delta u_j \Delta \hat{u}_\ell \rangle = \rho^{-2} \int v_j \, \hat{v}_\ell \; g(z,\hat{z}) d\vec{v} \; d\vec{\hat{v}} \tag{10}$$

where the definition (5) of kinetic correlation function g has been incorporated. The function g of form (5) includes, by its nature, two types of fluctuations in gases; namely, the thermal and the turbulent fluctuations (ref. 2). In what follows, the first effect is ignored since it is by far the smaller in magnitude, also we are interested only in fluid mechanics. If g = 0 the gas is in (binary) molecular chaos.

B.   A Perspective of the Work

Examination of the structure of the two-point kinetic equation reveals a fact that the variables are separable into two groups (z,t) and (ẑ,t) under any conditions of geometry or compressibility of a flow if, in the equations, the term of triple correlation is negligible compared with those of binary correlations,

$$\langle \Delta\!\!\!/\, \Delta\hat{\!\!\!/}\, \Delta\tilde{\!\!\!/} \rangle = 0 \tag{11}$$

in other words, if the gas is in "ternary" molecular chaos. (First separability condition, section II.). The separated equation in (z,t) space forms a closed set together with the Boltzmann equation in which g of (5) is retained in the collision integral. Each of the coupled equations is moment-expanded by means of 13-moment method, providing with 26-moment fluid-dynamic equations. Coupling between mean and fluctuating quantities is effected through only two terms representing the Reynolds stress and the turbulent heat flux density in the mean momentum and the energy equations (section III.). This method gives us a means for a more direct formulation of grid-produced turbulence as time-independent and cylindrically-isotropic phenomena, enabling us to find a solution hidden behind the conventional assumptions. Also the method provides us with dispersion relationships, showing that the grid turbulence consists of two modes, the one stationary relative to the grids, the other frozen to the fluid; the latter alone obeys Taylor's hypothesis. An initial-value problem is posed in such a form that complete knowledge of the velocity correlation is provided in terms of

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frequency-analyzed transverse correlation data prescribed at an initial plane (section IV.). Application of the proposed formalism to incompressible wall turbulence is also undertaken. The separated fluctuation variable is, in this case, governed by Orr-Sommerfeld's equation (ref. 2). Difficulties in solving the equation, caused by anomalous velocity variation near the wall is circumvented by devising a new asymptotic method. Essential role of three-dimensionality of fluctuations is maintaining Reynolds stress is elucidated using the tool (section V.). In section VI. the first closure condition is discarded and is replaced with the second closure condition

$$\phi^\dagger = \phi^{\dagger\dagger} \tag{12}$$

where $\phi^\dagger$ and $\phi^{\dagger\dagger}$ denote, respectively, separated variables through which triple and quadruple correlations are formed. This closure retains terms of the triple correlations in the two-point equation, deducing this and three-point equations in an identical form through the variable-separation. The resulting separated equation describes contribution from the triple correlation as a convolution integral, indicating primary role of the term to be energy-cascading that is far reaching in the frequency space.

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SYMBOLS

A	magnitude of total wave number defined by (101)
A_L	complex amplitude of Laplace mode defined by (82)
A_O	complex amplitude of Oseen mode defined by (82)
a_O	quantity defined by (B.13)
$B(f)$	operator defined by (124)
B_O	complex amplitude of Oseen mode defined by (81)
b	velocity gradient at the wall nondimensionalized by uniform flow velocity u and geometrical characteristic length ℓ
C	isothermal speed of sound (39)
c	wave velocity defined by (101)
D	differential operator d/dy
$F_\iota(\iota;1,2,3,4)$	four independent solutions of Orr-Sommerfeld equation given by (115)
f	Boltzmann function
$\mathfrak{f}(z)$	microscopic density, or instantaneous number density in the phase space
$\Delta \mathfrak{f}$	defined by (6)
$f_{II}(z,\hat{z})$	two-particle distribution function
G	quantity defining Φ_2 through (B.15)
$g(z,\hat{z})$	correlation in the phase space
$H_{ij\ldots}^{(n)}$	threedimensional Hermite polynomials
$I_\iota(\iota=2,3)$	integrals given by (B.30)
$J(z|\hat{z})$	Boltzmann collision operator defined by (15)
$J_\iota^{(n)}$	quantity given by (B.36)
k_1	complex wave number in stream direction
$\vec{k}$	$=(k_1,k_2,k_3)$ wave number vector
L	operator defined by (54)
$L^{(o)}$	linear operator defined by (113)
$L^{(1)}$	linear operator defined by (114)

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ℓ geometrical characteristic length

m   mass of a particle

m_j mean mass flux density

$(NS)_j$ nonlinear operator defined by (2)

p   mean pressure; defined by (40) for compressible flows

p instantaneous pressure

P_{jk} turbulent stress deviator tensor defined by (46)

Q_j turbulent heat flux density defined by (47)

q_j separated velocity fluctuation variable for incompressible flows

$q_{j\ell...}^{(n)}$ expansion coefficient of ϕ defined by (48)

R   Reynolds number defined by (79)

R designating real part

$R_{ij..,\ell m..}^{(J,K)}$ turbulent correlation in physical space

r   defined by (84)

S   quantity defined by (B.16)

U   velocity of a uniform flow

u_j j-component of mean velocity

u_j j-component of instantaneous velocity

Δu_j j-component of instantaneous veloicty fluctuation defined by (3)

V   quantity defined by (B.7)

$\vec{v}$ coordinate in the (molecular) velocity space

W   quantity defined by (B.7)

W Wronskian given by (B.23)

W_{III} Wronskian given by (B.27)

W_{IV} Wronskian given by (B.34)

$\vec{x}$ coordinate in the physical space

Y   auxiliary variable defining Y_j

Y_j	defined by (95) and (97)
$Y^{(n)}$	n-th order eigenfunction in the successive approximation (109) obeying (110) through (112)
z	auxiliary variable defining Y_j through (97) and obeying (100)
z	$z = (\vec{x}, \vec{v})$
α	wave number in streamwise (x) direction (defined by (95))
$\Gamma_{\iota(\iota;2,3)}$	quantities defining ϕ_4 through (B.21)
γ	wave number in spanwise (z) direction (defined by (95))
δ	designating deterministic perturbation
ε	scaling parameter defined by (105)
$\Phi_{\iota(\iota;1,2,3,4)}$	four independent solutions of (107)
$\phi(z)$	separated fluctuation variables
η	defined by (70)
Λ	linear operator defined by (133)
λ	thermal conductivity coefficient (in the main text). ah hoc variable defined by (B.5) (in Appendix B)
μ	viscosity coefficient (in the main text) ah hoc variable defined by (B.5) (in Appendix B)
ν	kinematic viscosity
ρ	density
τ	quantity defined by (B.13)
Θ	threedimensionality of the fluctuation defined by (98)
χ	defined by (24)
ξ_j	nondimensional molecular velocity defined by (38)
η	independent variable defined by (105)
Ω	operator defined by (18)
Ω_a	operator defined by (127)
ω	constant separating variables (see (16) and (17))
$\omega(\xi)$	local Maxwellian distribution defined by (37)

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SYMBOLS (Concluded)

< >	taking average
^	symbol designating a second point
~	symbol designating a third point

Superscripts

*	complex conjugate
+	pertaining to quantities nondimensionalized by the friction velocity and the kinematic viscosity

II. KINETIC EQUATION IN SEPARATED FORM

A. BBGKY Hierarchy Closed at Two-Particle Equation

According to the standard BBGKY hierarchy formalism the one- and two-particle equations with the latter being truncated by closure condition (11) of ternary molecular chaos, are written, respectively, as

$$\left(\frac{\partial}{\partial t} + \vec{v} \frac{\partial}{\partial \vec{x}} \right) f(z) = J(z|\hat{z}) [f(z) f(\hat{z}) + g(z,\hat{z})]_{x=\hat{x}} , \tag{13}$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \frac{\partial}{\partial \vec{x}} + \vec{\hat{v}} \frac{\partial}{\partial \vec{\hat{x}}} \right) g(z,\hat{z}) = J(z|\tilde{z}) [f(z) g(\tilde{z},\hat{z})$$

$$+ f(\tilde{z}) g(z,\hat{z})]_{x=\tilde{x}} + J(\hat{z}|\tilde{z}) [f(\hat{z}) g(\tilde{z},z) + f(\tilde{z}) g(\hat{z},z)]_{\hat{x}=\tilde{x}} \tag{14}$$

where f is the Boltzmann function defined by (7), g is the kinetic correlation function given by (5) and connected with two-particle distribution function f_{II} of the BBGKY formalism through

$$g(z,z) \equiv \langle \Delta \oint \Delta \hat{\oint} \rangle$$

$$= f_{II}(z,\hat{z}) - f\hat{f}$$

In this formula and in what follows an abbreviated expression $\hat{f} = f(\hat{z})$ is employed. The operator $J(z|\tilde{z})$ in Eqs. (13) and (14) is the Boltzmann integral operator which acts on the field particle $\tilde{z}$ in the manner

with

$$\left. \begin{array}{l} J(z|\tilde{z}) [g(\tilde{z})]_{x=\tilde{x}} \equiv \frac{1}{m} \int [g(z',\tilde{z}') - g(z,\tilde{z})] d\vec{K} d\vec{v} \\[2em] d\vec{K} \equiv 2\pi |\vec{v} - \vec{\tilde{v}}| b db . \end{array} \right\} \tag{15}$$

In order for the two particles z and $\tilde{z}$ to execute a molecular collision we need to impose a condition $\vec{x} = \vec{\tilde{x}}$, which is indicated as a subscript to the collision integral. In this collision integral the quantities with prime denote the state before a collision leading to unprimed state upon collision, b is the impact parameter, and m is the mass of a particle.

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B. Separation of Variables

The form of equation (14) suggests possibility of the six-dimensional equation to be separated into two groups of variables z and $\hat{z}$ by assuming

$$g(z,\hat{z},t) \equiv \langle \Delta \mathfrak{z} \, \Delta \hat{\mathfrak{z}} \rangle = \int \phi(z,t;\omega) \, \hat{\phi}(\hat{z},t;\omega) \, d\omega \qquad (16)$$

This expression, substituted into (14) and rearranged properly leads to the equation of the following form

$$\frac{\Omega\phi}{\phi} = - \frac{\hat{\Omega}\hat{\phi}}{\hat{\phi}} = i\omega \qquad (17)$$

with

$$\Omega\phi \equiv \left[\frac{\partial}{\partial t} + \vec{v} \, \frac{\partial}{\partial \vec{x}} \right] \phi(z) - J(z|\tilde{z})[f(z)\phi(\tilde{z}) + f(\tilde{z})\phi(z)]_{x=\tilde{x}} \qquad (18)$$

and with $\hat{\Omega}$ defined likewise, namely, with variable z replaced by $\hat{z}$. Note that the time t is the only variable common in both Ω and $\hat{\Omega}$. In line with the general separability theorem, the left-hand side of (17) is only a function of z, whereas the middle term depends only on $\hat{z}$; therefore $i\omega$ is necessarily a constant separating the variables. The governing equations for ϕ and $\hat{\phi}$ take, accordingly, the following forms

$$\left. \begin{array}{l} \Omega\phi - i\omega\phi = 0 \\[2mm] \hat{\Omega}\hat{\phi} + i\omega\hat{\phi} = 0 \end{array} \right\} \qquad (19)$$

In view of expression (18) we see that ϕ and $\hat{\phi}$ obey a linearized Boltzmann equation in a generalized sense ($\omega \neq 0$). It is also seen that the separation constant ω has the dimension of the frequency. It would be natural to postulate that correlation function g be symmetric with respect to its arguments $(z,\hat{z})$. As is easily seen from (16), this condition is met either with $\hat{\phi} = \phi$ or with

$$\hat{\phi} = \phi^* \qquad (20)$$

where the symbol (*) denotes the complex conjugate. This latter requirement is satisfied, in view of Eqs. (19) and reality of operator Ω, if and only if

$$\omega^* = \omega, \text{ or } \omega; \text{ real} \qquad (21)$$

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In contrast, the former condition ($\hat{\phi} = \phi$) together with Eqs. (19) requires necessarily $\omega = 0$, resulting in a trivial solution (ref. 2) $\phi \equiv 0$. These observations indicate that ϕ is essentially complex.

It seems to be worthwhile to remark at this point certain parallelism of function ϕ to Schrödinger's function Ψ, and of Eq. (19) to Schrödinger's equation

$$H\Psi - i\hbar\partial\Psi/\partial t = 0 \qquad (22)$$

where H and $\hbar$ are Hamiltonian operator and Planck's constant, respectively. Both of the equations have the imaginary factors associated with temporal operators based on essentially physical basis, not as a result of mathematical method of solution (c.f. Eq. (35) below). Also, both of the dependent variables ϕ and Ψ are related through the same bilinear forms to the same quantity, namely, the probability density. Its actual form in the case of the Schrödinger function is well-known: the quantity $\Psi^*\Psi d\vec{x}$ gives the probability of finding a quantum particle within a three-dimensional volume $d\vec{x}$ in the physical space $[\vec{x},\vec{x}+d\vec{x}]$. The counterpart expression for ϕ follows from (16) and (20), namely,

$$g(z,\hat{z}) \equiv <\Delta\hat{\phi}\; \Delta\hat{\hat{\phi}}> = \int \chi(z,\hat{z};\omega)\,d\omega \qquad (23)$$

with

$$\chi(z,\hat{z};\omega) = R\{\phi(z;\omega)\phi^*(\hat{z};\omega)\} \qquad (24)$$

where the symbol $R\{\;\}$ denotes taking the real part. In fact χ is bound to be a real quantity because expression (23) defines its physical meaning as the spectral density (in the frequency-space) of turbulent correlations in the phase space. More precisely, expression $<\Delta\hat{\phi}\; \Delta\hat{\hat{\phi}}>d\vec{v}d\vec{\hat{v}}$ gives the probability of finding a particle located at $\vec{x}$ in a cell $[\vec{v},\vec{v}+d\vec{v}]$ of the (molecular) velocity space and another one that is located at $\vec{\hat{x}}$, interacting with the first one, to be found in a cell $[\vec{\hat{v}},\vec{\hat{v}}+d\vec{\hat{v}}]$. Therefore, the quantity

$$R\{\phi(z;\omega)\phi^*(\hat{z};\omega)\}d\vec{v}\;d\vec{\hat{v}}\;d\omega \qquad (25)$$

gives the probability of finding such two particles in the frequency subspace $[\omega,\omega+d\omega]$, in addition to their being in the cells $d\vec{v}$ and $d\vec{\hat{v}}$, respectively. Since the function ϕ is attached physical meaning only via (25) as in its equivalent $\Psi^*\Psi d\vec{x}$ for the Schrödinger function, the following transformation

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$$\phi \rightarrow e^{i\alpha}\phi \tag{26}$$

where α stands for any real quantity, keeps observable quantities invariant as in quantum mechanics.

Expression (25) constitutes a basis on which any two-point quantities of fluid turbulence can be calculated. For example, the turbulent correlation between velocities at two points $\vec{x}$ and $\hat{\vec{x}}$ is, from (23) and (10),

$$<\Delta u_j \Delta \hat{u}_\ell> = R\{\int d\omega (\int v_j \phi d\vec{v}) (\int \hat{v}_\ell \hat{\phi}^* d\hat{\vec{v}})\} \tag{27}$$

and likewise for any other macroscopic quantities definable in the classical kinetic theory.

With these physical backgrounds the Boltzmann function f and the separated fluctuation variables ϕ form a closed set consisting of Eqs. (13) and (19). These are written, in view of (23) and (24), as .

$$\left[\frac{\partial}{\partial t} + \vec{v} \frac{\partial}{\partial \vec{x}}\right] f = J(z|\tilde{z}) [f\tilde{f} + R\{\int \phi(z) \phi^*(\tilde{z}) d\omega\}]_{x=\tilde{x}} \tag{28}$$

$$\left[-i\omega + \frac{\partial}{\partial t} + \vec{v} \frac{\partial}{\partial \vec{x}}\right] \phi = J(z|\tilde{z}) [f(z)\phi(\tilde{z}) + f(\tilde{z})\phi(z)]_{x=\tilde{x}} \tag{29}$$

Needless to say that Eq. (28) is exact in the present context, whereas Eq. (29) which is a result of separation of variables is subject to, but only to assumption (11) of 'ternary' molecular chaos. We see, therefore, that the separability is not affected by compressibility or geometry of the flows, proving the point at issue stated at the beginning in Section I.A.

C.   Implications of Dual Time-Dependency of Eq. (29)

As a result of introduction of the separation constant ω having the dimension of the frequency, Eq. (29) is seen to have dual time dependencies through explicit ($\partial/\partial t$) and implicit ($-i\omega$) temporal operators. We will show that the former acts on deterministic unsteadiness associated with its mean motion, whereas the latter reflects stochastic unsteadiness of the turbulent motions that is averaged out, so is not to appear explicitly, in the one-point regime (Eq. (28)). Regarding the explicit one an explanation will be given through an assertion that if f is

time-independent, so is ϕ. The contrapositive of the statement is; if ϕ turns out to be time-dependent, it is caused by time-dependency of f. In other words, explicit time-dependency $(\partial/\partial t \neq 0)$ is due only to unsteadiness in mean motion of the turbulent flow.

Proof of the assertion, in its first expression is the following: For a time-independent f, Eq. (29) allows a formally time-dependent solution

$$\phi(z,t) = \exp(-i\omega_0 t)\,\phi_0(z) \tag{30}$$

The invariance, however, of physical quantities under transformation (26) applied here assures that the temporal factor in (30) does not affect on physical quantities, so does not generate unsteadiness of a new class to be superimposed on the steady background flow. (Note that ω_0 of (30) be real in order to keep $g(z,\hat{z})$ of Eq. (23) symmetric with respect to z and $\hat{z}$.)

We have thus proved that for steady mean flows temporal factors are involved in ϕ only through implicit dependence $(-i\omega)$. This is a fundamental property of turbulent fluctuations as distinct from deterministic perturbations such as Tollmien-Schlichting waves that are bound to be explicitly time-dependent.

It seems to be difficult to draw a clearcut picture as to the implication of the implicit time-dependency at this stage. In a subsequent section (Section IV.C), however, the quantity ω introduced in the theory as a separation constant will be given a firmer physical basis through a dispersion relationship derived for grid turbulence, where ω/k_1 (k_1; the wave number in stream direction of the turbulent fluctuation) is interpreted as the propagation velocity of a turbulent vortex. In this connection the implicit time-dependency may be thought as having its physical origin in stochastic elements of coherent turbulent vortices; for example, the lifetime of retaining their identity as observed in a free shear layer (ref. 5), or stochastically repeated up- and down-washes in the wall-shear layer considered as being caused by aligned longitudinal vortices (ref. 6).

D. Turbulent Fluctuation ϕ as Distinct from Deterministic Perturbation δf

Suppose a small time-dependent disturbance $\delta f(z,t)$ superimposed on a steady basic flow described by the Boltzmann function f which is time-

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independent otherwise. The small disturbance δf as meant here is of fluid-dynamic nature, regardless of its kinetic-theoretical expression, as would be seen from its definition

$$\delta f = \sum_{\lambda=1}^{13} (\partial f / \partial A_\lambda) \delta A_\lambda \tag{31}$$

where A_λ stands for 13 mean fluid-dynamic variables specifying the Boltzmann function (ref. 7).

Let us observe turbulent phenomena in which the two fluctuations δf and ϕ are coexistent. One such example is a boundary-layer undergoing natural (ref. 8) or forced (ref. 9) transitions where δf corresponds to naturally or artificially generated sinusoidal perturbations. Under this condition the function ϕ is necessarily responsible for turbulent part of the flow observed as bursts and/or turbulent spots in the experiments. In these experimental observations where both deterministic and turbulent fluctuations are operating, we are especially interested in the situation where condition

$$O(|\phi|) \sim O(|\delta f|)$$

holds. For superimposed perturbation onto a basic distribution which, otherwise, is time-independent;

$$f \to f + \delta f$$

we see from Eq. (28) that f and δf obey the following equations;

$$\left(\frac{\partial}{\partial t} + \vec{v} \, \frac{\partial}{\partial \vec{x}}\right) f = J(z|\tilde{z}) \, [f\tilde{f} + \delta f \delta \tilde{f} + R\{\int \phi \tilde{\phi}^* d\omega\}]_{x=\tilde{x}} \tag{32}$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \, \frac{\partial}{\partial \vec{x}}\right) \delta f = J(z|\tilde{z}) \, [f\delta\tilde{f} + \tilde{f}\delta f]_{x=\tilde{x}} \tag{33}$$

where term $\delta\phi$ induced responding to δf is ignored because of its second order smallness. Since Eq. (33) is linear in δf and depends on t only through the derivative it reduces, by putting

$$\delta f = R\{\exp(-i\omega_o t) \delta h\} \tag{34}$$

to

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$$\left(-i\omega_o + \vec{v}\,\frac{\partial}{\partial\vec{x}}\right)\delta h = J(z|\tilde{z})\,[f\delta\tilde{h} + \tilde{f}\delta h]_{x=\tilde{x}} \tag{35}$$

The requirement for reality of δf in (34) comes from the fact that the fluid-dynamic perturbation

$$\delta A = \int \alpha(z)\,\delta f\, d\vec{v}$$

where A is a macroscopic variable and α is its particle-dynamic equivalent, is an observable quantity such as the Tollmien-Schlichting wave.

It is noteworthy coincidence that Eqs. (29) and (35) governing ϕ and δh, respectively, are identical in form under steady flow condition ($\partial/\partial t = 0$) in the former. The coincidence, however, breaks down at the point of their different ways of interaction with the mean flow, reflecting their difference in physical nature. It is readily seen from Eq. (32) which reads

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial\vec{x}}\right)f = J(z|\tilde{z})\,[f\tilde{f} + R\{\exp(-i\omega_o t)\times\delta h\}R\{\exp(-i\omega_o t)\delta\tilde{h}\}$$

$$+ R\{\int\phi\tilde{\phi}^*d\omega\}]_{\tilde{x}=x} \tag{36}$$

where Eq. (34) has been used. An obvious distinction is generation, on one hand, of (temporal) higher harmonics, namely, of the factor $\exp(\pm 2i\omega_o t)$ to appear in the quadratic term of the deterministic perturbation δh, whereas the temporal factors cancel out from the term $\phi\tilde{\phi}^*$ of the turbulent fluctuation as discussed in Section II.C.

This feature of nongeneration of higher harmonics, spatially or temporally, has been originally found in the physical space for the Reynolds stress (ref. 10), in contrast to the similar terms of nonlinear stability theory (refs. 11,12) which give rise to these effects. It will be seen (Section III.A) that the Reynolds stress in the physical space emerges from the bilinear term of Eq. (36), proving the identity of the two assertions. A direct observation by Roshko (ref. 5) on coherent vortices in a free turbulent shear layer exhibits evidence of no generation of such higher harmonics in spite of their characteristically nonlinear patterns.

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III. A CLOSED SET OF FLUID EQUATIONS

A. Review of One-Particle Turbulence Formalism

Fluid-dynamic version of the 'turbulent' Boltzmann equation (28) has been derived in ref. 13 using the 13-moment expansion carefully extended to turbulent cases: We expand the Boltzmann function in the following form

$$f(z) = \rho\omega(\xi)\left[1 + \frac{P_{j\ell}}{2p}H_{j\ell}^{(2)} + \frac{Q_j}{5pC}H_j^{(3)}\right]$$

with

$$\omega(\xi) = \frac{1}{(2\pi C^2)^{3/2}}\exp\left(-\frac{\xi^2}{2}\right)$$

$\left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\}$ (37)

where ρ is the average density, and H's are the Hermite polynomials defined in terms of nondimensional velocity variable $\vec{\xi}$ given by

$$\xi_j = \frac{v_j - \rho^{-1}m_j}{C}$$

(38)

In order to preserve the original formality of this scheme (ref. 7), it is required that m_j be taken as the average mass-flux density and C be chosen as

$$C = (p/\rho)^{1/2}$$

(39)

where p is a 'pseudo' pressures, distinct from the average thermodynamic pressure π by the added contribution due to turbulence;

$$p = \pi + (3\rho)^{-1}R_{j,j}^{(1,1)}$$

(40)

For exact definition of the term $R_{j,j}^{(1,1)}$ see Eq. (41) below.

On the same background the kinetic correlation function g which appears in the collision integral of Eq. (28) is expanded in a double series of $H(\vec{\xi})$ and $\tilde{H} = H(\vec{\tilde{\xi}})$ as

$$g(z,\tilde{z}) = \omega(\xi)\omega(\tilde{\xi})\sum_{(J,K)=(0,0)}^{(3,3)}\frac{R_{ij\cdots,\ell m\cdots}^{(J,K)}}{C^J\tilde{C}^K J!K!}H_{ij\cdots}^{(J)}\tilde{H}_{\ell m\cdots}^{(K)}$$

(41)

A word must be mentioned at this point of truncating expansions (37) and (41) at third-order Hermite polynomials in line with the 13-moment regime. In laminar (classical) cases this is perfectly legitimate on fluid-dynamics

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point of view, because the neglected terms are of $O(\varepsilon^2)$ (ε: the mean free path), no information on fluid-dynamics are lost by this truncation. For turbulent cases, on the other hand, the neglected terms include fluid-dynamic quantities of $O(\Delta_2)$, namely, terms of double correlations

$$\Delta_2 \sim <\Delta A \ \Delta B>$$

where A and B are any gasdynamic variables. Neglecting these terms, however, is justified only for Maxwellian molecules because these terms are not to intervene in the gasdynamic equations due to a special structure of the collision integral peculiar to this type of molecules (ref. 6).

Once the moment expansion for f and g have been assumed in forms (37) and (41) fluid moment equations out of one-particle kinetic Eq. (28) are determined uniquely through a standard procedure, leading to

$$\frac{\partial \rho}{\partial t} + \frac{\partial m_k}{\partial x_k} = 0 \tag{42}$$

$$\frac{\partial m_j}{\partial t} + \frac{\partial}{\partial x_k}\left(\frac{m_k m_j}{\rho} + \pi\delta_{jk} + p_{jk}\right) = 0 \tag{43}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{m_j^2}{\rho} + e\right] + \frac{\partial}{\partial x_k}\left[\frac{m_j^2 m_k}{2\rho^2} + \frac{m_k h}{\rho} + \frac{m_j}{\rho}p_{jk} + Q_k\right] = 0 \tag{44}$$

with

$$e \equiv \frac{3}{2}\pi + \frac{1}{2\rho}R_{r,r}^{(1,1)}, \quad h \equiv e + \pi, \tag{45}$$

$$p_{jk} = (p_{jk})_{NS} + \frac{1}{\rho}R_{j,k}^{(1,1)} \tag{46}$$

$$Q_k = (q_k)_F + \frac{5}{6\rho}R_k^{(1,2)} \tag{47}$$

where $(p_{jk})_{NS}$ and $(q_k)_F$ are the viscous stress tensor and the heat flow vector connected with the average gasdynamic quantities through classical Navier-Stokes' and Fourier's laws, respectively. Note that turbulence corrections to the stress tensor and the heat flow vector comprise only one term in each case; the Reynolds stress $\rho^{-1}R_{j,k}^{(1,1)}$ and turbulent heat flux density $(5/6)R_k^{(1,2)}$. If the gas is not monatomic, the factor 5/6 is to be replaced with $\gamma/3(\gamma-1)$, where γ is the adiabatic index. A noteworthy remark at this point is that all the equations (42) through (47) are written in compressibility-invariant form, so that the transport

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relationships (46) and (47), each expressed in a form with single-term turbulence correction, is valid for compressible flows as well with no recourse to mass averaging (ref. 13). Also, deduction up to this point is exact within the framework of fluid description for Maxwell molecules, independent of any closure hypothesis to be effected at a subsequent hierarchy. We note that the key to the compressibility-invariant expressions of turbulence equations is to employ variables proportional to density, namely, the mass-flux density and the pressure to replace the velocity and the temperature, respectively. With this code in mind, all the equations derived in this section are reproducible using a priori gasdynamic equations (ref. 13).

B. Fluid Moments from Separated Two-Particle Equation

Our task in this paragraph is to derive from Eq. (29) equations governing the Reynolds stress $R_{j,k}^{(1,1)}$ and the turbulent heat-flux $R_k^{(1,2)}$ appearing in turbulent transport relationships (46) and (47), respectively. For this purpose we expand the dependent variable ϕ in the form:

$$\phi(z) = \omega(\xi)\left[q^{(0)} + \frac{q_j^{(1)}}{C}H_j^{(1)} + \frac{q_{jk}^{(2)}}{2C^2}H_{jk}^{(2)} + \frac{q_j^{(3)}}{5C^3}H_j^{(3)}\right] \qquad (48)$$

This form of expansion assimilates the 13-moment expansion (37) for the Boltzmann function except that the coefficient of the first order Hermite polynomial is nonvanishing, also the tensor coefficient of the second order has a nonvanishing trace;

$$q_{jj}^{(2)} = q^{(2)} \qquad (49)$$

This situation arises from the fact that we have no degrees of freedom left out to suppress these quantities because these have been depleted in defining the expansion form of the one-particle distribution. Expansion coefficients q's of (48) are related to those R's of (41) through (23) and (24). For example, the Reynolds stress and the turbulent heat flux are given in bilinear forms of q's through putting $\hat{x} = x$;

$$R_{j,k}^{(1,1)} = R\{\int q_j^{(1)}(\vec{x})\, q_k^{(1)*}(\vec{x})\,d\omega\}$$

$$R_j^{(1,2)} = R\{\int q_j^{(1)}(\vec{x})\, q^{(2)*}(\vec{x})\,d\omega\}$$

$$\left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\}\ 50)$$

Equations governing q's in the fluid-dynamic space are obtained from (29) through the standard moment expansion procedure familiar in the 13-moment theory and are detailed elsehwere (ref. 14). Only the final form of the equations would suffice for what follows:

$$Lq^{(0)} + \frac{\partial q_r^{(1)}}{\partial x_r} = 0 \tag{51}$$

$$Lq_j^{(1)} + q_r^{(1)} \frac{\partial w_j}{\partial x_r} + \frac{\partial p^{(2)}}{\partial x_j} - \frac{\partial}{\partial x_r} \left\{ \mu \left[\frac{\partial}{\partial x_j} \left(\frac{q_r^{(1)}}{\rho} \right) \right. \right.$$
$$\left. \left. + \frac{\partial}{\partial x_r} \left(\frac{q_j^{(1)}}{\rho} \right) - \frac{2}{3} \delta_{jr} \frac{\partial}{\partial x_k} \left(\frac{q_k^{(1)}}{\rho} \right) \right] \right\} + q^{(0)} \frac{Dw_j}{Dt} = 0 \tag{52}$$

$$L \left[\frac{3}{2} p^{(2)} \right] + p^{(2)} \frac{\partial w_r}{\partial x_r} + \frac{\partial}{\partial x_r} \left(\frac{5}{2} c^2 q_r^{(1)} \right) - \frac{\partial}{\partial x_r} \left[\frac{\lambda}{R} \frac{\partial}{\partial x_r} \left(\frac{q^{(2)}}{\rho} \right) \right]$$

$$+ q_r^{(1)} \frac{Dw_r}{Dt} - \mu \frac{\partial w_j}{\partial x_r} \left[\frac{\partial}{\partial x_j} \left(\frac{q_r^{(1)}}{\rho} \right) + \frac{\partial}{\partial x_r} \left(\frac{q_j^{(1)}}{\rho} \right) \right.$$

$$\left. - \frac{2}{3} \delta_{jr} \frac{\partial}{\partial x_k} \left(\frac{q_k^{(1)}}{\rho} \right) \right] = 0 \tag{53}$$

with the following definitions of the symbols,

$$Lq \equiv - i\omega q + \frac{\partial q}{\partial t} + \frac{\partial}{\partial x_r} w_r q \tag{54}$$

$$p^{(2)} \equiv \frac{1}{3} q^{(2)} + c^2 q^{(0)} \tag{55}$$

$$w_r \equiv \frac{m_r}{\rho}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + w_r \frac{\partial}{\partial x_r}$$

where μ and λ denote the molecular viscosity and the molecular heat conductivity, respectively.

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The system of Eqs. (51) through (53) comprises five equations to determine five unknowns $q^{(0)}$, $q_j^{(1)}$, and $q^{(2)}$ (or $p^{(2)}$) that correspond to $\delta\rho$, $\rho\delta w_j$ and $\rho R\delta T$ (or δp), respectively, of deterministic perturbations. Actually the system reduces to that of fluid-dynamic stability for compressible flows (ref. 15) if we put $\omega = 0$ in definition (54) for operator L together with the envisaged substitution for q's. This proves in the fluid-dynamic space the formal similarity existing between kinetic equations (29) and (33).

IV. GRID-PRODUCED TURBULENCE IN THE LIGHT OF SEPARATION OF VARIABLES

A. Steady-Flow Formalism Without Isotropy or Taylor's Hypotheses

Nowadays, a widely used means of understanding experimental data on wind tunnel- or grid-produced turbulence is to utilize the theory of isotropic turbulence with proper interpretation for obtained solution. This method of approach, however, is subject to two assumptions in addition to the one for closure. The first of these is that grid-produced turbulence is (spherically) isotropic. Speaking more exactly, this is an approximation, in nature, rather than an assumption; in fact, a grid-produced turbulence is not spherically but cylindrically (or axially) isotropic (ref. 16) because the presence of a constant field vector $\vec{u} = (U,0,0)$ (U; the mean velocity in the wind tunnel) distorts the tensor field to be so. A consequence of the isotropy assumption is to suppress the pressure-velocity correlation identically (ref. 17):

$$R_j^{(1,2)} = <\Delta u_j \Delta \hat{p}> = 0 \tag{56}$$

In the lack of this quantity (and under homogeneity condition) the triple correlation Δ_3 is the only term causing effects other than simple translation or molecular diffusion of turbulence. (See Eq. (138) of Section VI.). The danger is; if a slight anisotropy (in the spherical sense) exists in the actual grid flow, causing a slight response in the form of nonvanishing $R_j^{(1,2)}$, it may well dominate effects due to the terms of $O(\Delta_3)$ because it is of $O(\Delta_2)$; a lower order correlation. This means that the isotropy postulate needs to be checked carefully before the quality of a closure condition to be effected on Δ_3 is discussed on the basis of comparison with experiments done for a grid-produced turbulence.

The second assumption is Taylor's assumption claiming that the turbulence fluctuation is carried with the mean flow;

$$t = \frac{x}{U} \tag{57}$$

where U is the (constant) velocity of the wind-tunnel flow. This assumption is needed to interpret space (x)-dependent experimental data in terms of time-dependent solution of the isotropy theory. This assumption has no precarious effects as the isotropy assumption does. It would, however, be avoidable to invoke if a more direct formalism is available such that decay of turbulence is described in terms of space coordinate x instead of time t. We will show in what follows how these hypotheses are eliminated, and will see what are hidden behind.

B. A Hidden Solution

With a frame of reference fixed to a grid or turbulence generating apparatus, Eqs. (51) and (52) under uniform (u = constant) and steady ($\partial/\partial t = 0$) flow conditions read

$$Lq^{(0)} + \frac{\partial q_r^{(1)}}{\partial x_r} = 0 \tag{58}$$

$$Lq_j^{(1)} + \frac{\partial p^{(2)}}{\partial x_j} - \nu\left(\frac{\partial^2 q_j^{(1)}}{\partial x_r^2} + \frac{1}{3}\frac{\partial}{\partial x_j}\frac{\partial q_r^{(1)}}{\partial x_r}\right) = 0 \tag{59}$$

with

$$L \equiv -i\omega + U\frac{\partial}{\partial x_1} \tag{60}$$

where ν denotes the kinematic viscosity. Taking divergence of Eq. (59) and eliminating the pressure $p^{(2)}$ from the two equations, we have

$$\left(L - \nu\frac{\partial^2}{\partial x_r^2}\right)\left(\frac{\partial^2 q_j^{(1)}}{\partial x_k^2} - \frac{\partial}{\partial x_j}\frac{\partial q_r^{(1)}}{\partial x_r}\right) = 0 \tag{61}$$

or for incompressible flows ($q^{(0)} = 0$)

$$\frac{\partial q_j^{(1)}}{\partial x_j} = 0 \tag{62}$$

$$\left(-i\omega + U\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x_r^2}\right)\frac{\partial^2 q_j^{(1)}}{\partial x_k^2} = 0 \tag{63}$$

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Note that Eq. (63) is the Orr-Sommerfeld equation under a uniform flow condition. The solution of Eq. (63) is written in the form:

$$q_j^{(1)} = q_{O,j} + q_{L,j} \tag{64}$$

where $q_{O,j}$ and $q_{L,j}$ obey, respectively, a generalized Oseen equation and Laplace's equation;

$$\left[-i\omega + U \frac{\partial}{\partial x_1} - \nu \frac{\partial^2}{\partial x_r^2} \right] q_{O,j} = 0 \tag{65}$$

$$\frac{\partial^2 q_{L,j}}{\partial x_r^2} = 0 \tag{66}$$

Before proceeding further it would be relevant to see at this point how Eq. (63) compares with the classical counterpart under the same approximation $\Delta_3 = 0$: In the classical theory this case has been called the final stage of decay, and the governing equation is

$$\left(\frac{\partial}{\partial t} - 2\nu \frac{\partial^2}{\partial r_k^2} \right) R_{j,\ell}^{(1,1)} = 0 \tag{67}$$

where r_k denotes $x_k - \hat{x}_k$. Equation (67) is apparently a diffusion equation, parabolic and of the second order, whereas, in contrast, our equation (63) is elliptic and of the fourth order. This difference is mainly ascribed to the a priori lack of the pressure-velocity correlation (56) that is an immediate consequence of the isotropy postulate, causing thereby a difference by the factor of Laplacean.

C. Dispersion Relationships

The two types of mode given by $q_{O,j}$ and $q_{L,j}$ of (65) and (66), respectively, are attached simple physical interpretations through dispersion relationships; namely, relationships between the frequency ω and the wave number k_j. To obtain this we assume q in the form

$$q = \exp ik_j x_j \tag{68}$$

where k_2 and k_3 are taken to be real reflecting invariance of correlations with respect to Galileian transformation in (x_2, x_3) plane normal to the flow direction. Then for the Oseen mode q_O, we have an algebraic version

of differential equation (65) in view of (68):

$$-i\omega + i(k_1)_O U + \nu[(k_1)_O^2 + \kappa^2] = 0 \tag{69}$$

with

$$\kappa^2 = k_2^2 + k_3^2 \tag{70}$$

Expressions (69) determines complex wave number

$$(k_1)_O = (k_1)_{O,r} + i(k_1)_{O,i} \tag{71}$$

in the form;

$$\frac{\omega}{(k_1)_{O,r}} = U \tag{72}$$

$$(k_1)_{O,i} = \frac{\nu}{U} [(k_1)_{O,r}^2 + \kappa^2] \tag{73}$$

Since the quantity on the left-hand side of (72) means the wave-propagation velocity, this relation shows that the wave propagates at the fluid velocity; namely, the turbulence is frozen to the fluid. In other words, this mode obeys Taylor's hypothesis. Although (72) as the solution of Eq. (69) is subject to an approximation, correction terms to appear on the right-hand side is shown to be of $O(\nu^2)$, proving that this mode obeys the Taylor rule exactly within fluid-dynamic regime. Condition (73), on the other hand, describes how this mode decays spatially.

In a similar way we are led to the dispersion relationship for the Laplace mode: Substituting (68) into (66) we obtain

$$(k_1)_{L,r} = 0 \tag{74}$$

$$(k_1)_{L,i} = \kappa \tag{75}$$

claiming that this mode is stationary in space, namely, relative to the grid, and spatial rate of decay is κ. It shows us that this mode decays faster than the previous mode (73) ($\sim O(\nu)$), so that it is observed immediately downstream of the grid, then is followed by the Oseen mode, corresponding to the final stage of decay. Thus we have learned that the turbulent field consists of superposition of two modes, the one obeying and the other disobeying the Taylor hypothesis, respectively, and that the hypothesis is met better as we go downstream because of faster decay of the Laplace mode.

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D. Determination of Axially-Isotropic Correlation Tensor with Given Data at an Initial Plane through a Bessel Inversion

As contrast to the classical theory which is time-dependent and spherically isotropic, we aim at formulating the velocity correlation tensor as space-dependent and axially-isotropic. For this purpose it seems more convenient to write our governing equations (62) and (63) in the form

$$\left[-i\omega + \frac{\partial}{\partial x} - \frac{1}{R} \nabla^2 \right] \nabla^2 q_1^{(1)} = 0 \tag{76}$$

$$\left[-i\omega + \frac{\partial}{\partial x} - \frac{1}{R} \nabla^2 \right] \left(\frac{\partial q_2^{(1)}}{\partial z} - \frac{\partial q_3^{(1)}}{\partial y} \right) = 0 \tag{77}$$

$$\frac{\partial q_1^{(1)}}{\partial x} + \frac{\partial q_2^{(1)}}{\partial y} + \frac{\partial q_3^{(1)}}{\partial z} = 0 \tag{78}$$

with

$$\left. \begin{aligned}
(x,y,z) &= \frac{1}{\ell} (x_1, x_2, x_3) \\
\nabla^2 &= \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \\
R &= U \frac{\ell}{\nu} \\
\frac{\omega \ell}{U} &\rightarrow \omega, \text{ (redefinition)}
\end{aligned} \right\} \tag{79}$$

where R is the Reynolds number, and the grid mesh size ℓ is used as a characteristic length for nondimensionalization. If the solution is sought in form (68), where we have redefined

$$\ell k_j \rightarrow k_j$$

Eqs. (77) and (78) are solved for $q_2^{(1)}$ and $q_3^{(1)}$ as

$$\left. \begin{aligned}
q_2^{(1)} &= i\kappa^{-2} \left[k_3 \tilde{q} + k_2 \frac{\partial q_1^{(1)}}{\partial x} \right] \\
q_3^{(1)} &= i\kappa^{-2} \left[-k_2 \tilde{q} + k_3 \frac{\partial q_1^{(1)}}{\partial x} \right]
\end{aligned} \right\} \tag{80}$$

where $\tilde{q}$ is an auxiliary function obeying eq. (65). We can, therefore, put

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$$\tilde{q} = B_O q_O \tag{81}$$

with complex constant B_O. As for $q_1^{(1)}$ we have already obtained a general solution

$$q_1^{(1)} = A_O q_O + A_L q_L \tag{82}$$

where q_O and q_L obey (65) and (66), respectively.

Our task here is to show the following: the complete knowledge of the correlation tensor

$$R_{j,\ell}^{(1)}(\vec{x},\hat{\vec{x}}) = \int q_j^{(1)}(\vec{x}) q_\ell^{(1)*}(\hat{\vec{x}}) d\omega dk_2 dk_3 \tag{83}$$

at any set of points $(\vec{x},\hat{\vec{x}})$ is provided if a complete knowledge of the tensor element is given at the initial plane

$$x = \hat{x} = 0$$

as functions of the transverse separation

$$r = [(y - \hat{y})^2 + (z - \hat{z})^2]^{1/2} \tag{84}$$

and in the form of frequency-analyzed correlation

$$R_{j,\ell}^{(1,1)} = \int (R_{j,\ell}^{(1,1)})_\omega d\omega \tag{85}$$

as appearing on the right-hand side of the integral. In mathematical language, six independent initial data of the symmetric tensor $[R_{j,\ell}^{(1,1)}(r)]_\omega$ in the transverse plane $x = \hat{x} = 0$ are necessary and sufficient for unique determination of six constants; A_O, A_L, and B_O in (81) and (82).

The way in which these complex constants are calculated in terms of the initial data is described in some detail in Appendix A. A brief sketch of the solution through a functional inversion and the meaning of the obtained results are discussed here: From Eqs. (83), (68), and (82) we have

$$R_{1,1}^{(1,1)}(\vec{x},\hat{\vec{x}}) = 2\pi \int (A_O e^{ik_1 x} + A_L e^{-\kappa x})(A_O^* e^{-ik_1^* \hat{x}}$$

$$+ A_L^* e^{-\kappa \hat{x}}) J_0(\kappa r) \kappa d\omega d\kappa \tag{86}$$

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where k_1 is given by (71), J_0 denotes Bessel's function, and an integration with respect to the azimuth angle arctan k_2/k_3 has been carried out. At the initial plane $x = \hat{x} = 0$, and in terms of the frequency-analyzed initial data of (85), Eq. (86) reads

$$[R^{(1,1)}_{1,1}]_\omega = 2\pi \int_0^\infty (A_O + A_L)(A_O^* + A_L^*) J_0(\kappa r)\kappa d\kappa$$

This relationship is inverted by means of an integral formula (A.9) to give

$$(A_O + A_L)(A_O^* + A_L^*) = \frac{1}{2\pi} \int_0^\infty [R^{(1,1)}_{1,1}]_\omega r J_0(\kappa r) dr \tag{87}$$

Six independent relationships of similar forms are obtained likewise. These determine constants A_O, A_L, and B_O uniquely depending on two parameters; the frequency ω and the transverse wave number κ.

Approach along this line, in principle, to the grid-produced turbulent phenomena has been initiated by Nakagawa (ref. 18). The formulation described here is essentially the same except that in the earlier work the Laplace mode was suppressed ($A_L = 0$), so that the initial-value problem with the <u>full</u> tensor components as posed here would have caused overdeterminacy.

E. Pressure-Velocity Correlation as a Prime Order Quantity

Once the amplitudes of the two modes have been determined uniquely ((A.10) through (A.15)), we can show that the pressure-velocity correlation

$$R^{(1,2)}_j = \int q^{(1)}_j(\vec{x}) p^{(2)*}(\hat{\vec{x}}) d\omega\, dk_2 dk_3 \tag{88}$$

is necessarily nonvanishing and behaves as a correlation of prime order of magnitude as contrast to $R^{(1,2)}_j \equiv 0$ of the isotropic turbulence, also to $R^{(1,2)}_j \sim O(\Delta_3)$ of some other cases of homogeneous turbulence.

Solutions (82) and (80) for $q^{(1)}_j$, and Eq. (59) solved for the separated pressure fluctuation $p^{(2)}$ give

$$p^{(2)} = -(1 + i\omega/\kappa) A_L q_L \tag{89}$$

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An important implication of this formula is that only the Laplace mode contributes to the pressure fluctuation, namely to generation of noise, so that the Oseen mode is interpreted as representing quiescent component independent of its turbulence intensity. Since the quantity $p^{(2)}$ is proportional to noise amplitude human timpanic membranes perceive, this observation serves to give an intuitive criterion as to whether the Laplace mode is important under an experimental condition: It would be non-negligible compared with the Oseen mode if noise is sensible in the process of generation of a grid-turbulence. According to this criterion, the Laplace mode seems to play a role even for relatively low-speed flows through a grid such as an outlet flow of a room airconditioner.

In view of expression (89), the pressure-velocity correlation (88) is seen to consist of terms of $O(A_0 A_L)$, $O(A_L^2)$, and $O(B_0 A_L)$. Since none of these terms vanish other than exceptional cases of no particular interest we may conclude that the pressure-velocity correlation has the same order of magnitude as other double correlations. This conclusion, as it contrasts drastically with the formula $R_j^{(1,2)} \sim O(\Delta_3)$ used for homogeneous turbulence, will need a supplementary explanation to reconcile the seeming gap: If a flow is homogeneous but not spherically isotropic, $R_j^{(1,2)}$ is shown to obey Poisson's equation of the form

$$\frac{\partial^2}{\partial \hat{x}_r^2} R_j^{(1,2)} = O(\Delta_3)$$

where the source term comprises only the triple correlations. Its general solution is given, accordingly, by

$$R_j^{(1,2)} = P + H$$

where $P(\sim O(\Delta_3))$ is a particular solution of the Poisson equation, and H is a harmonic function to be responsible for the boundary conditions. It should be borne in mind that the assertion $R_j^{(1,2)} \sim O(\Delta_3)$ above holds only under the condition $H \sim 0$ or $O(\Delta_3)$, which is the case with the homogeneous turbulence with boundary conditions given at infinity. The fact that we are dealing with a flow which is semi-infinite ($x \geq 0$) in extent rules out the possibility, because the function H should cope with a boundary condition that is imposed at a finite point ($x = 0$), and is of $O(\Delta_2)$ as has been assured.

V. APPLICATION TO INCOMPRESSIBLE WALL TURBULENCE

A. The Reynolds Stress as Coupling Function

In this section we will focus on incompressible shear turbulence with special reference to wall turbulence, and will see how the equations governing fluctuations are coupled with those describing the mean motions by demonstrating the actual expression of the coupling, namely, the Reynolds stress.

Under the condition of incompressibility (ρ: const., $q^{(0)} = 0$) equations governing average and fluctuating quantities constitute a closed set with Eqs. (42), (43), (51), and (52);

$$\partial u_r / \partial x_r = 0 \tag{90}$$

$$(NS)_j = -R\{\frac{\partial}{\partial x_r} \int q_j \, q_r^* \, d\omega\} \tag{91}$$

$$\partial q_r / \partial x_r = 0 \tag{92}$$

$$(-i\omega + \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} - \nu \frac{\partial^2}{\partial x_r^2})q_j + \frac{\partial u_j}{\partial x_r} q_r + \frac{1}{\rho} \frac{\partial p^{(2)}}{\partial x_j} = 0 \tag{93}$$

where $(NS)_j$ is the Navier-Stokes' operator (2) replaced with the average quantities, and q_j is defined by

$$q_j = \rho^{-1} q_j^{(1)}$$

If, in addition, the flow is considered to be a parallel flow

$$u_j = U\delta_{j1} \, u(x_2) \tag{94}$$

with x_2-axis taken normal to the wall surface, we can assume, without loss of generality, as

$$q_j = Y_j(y)\exp i(\alpha x + \gamma z) \tag{95}$$

where

$$(x,y,z) = (x_1, x_2, x_3)/\ell \tag{96}$$

denotes a cartesian coordinate nondimensionalized by a fluid-dynamic characteristic length ℓ. In expression (95), α and γ are wave numbers in the directions of the velocity (x) and the (minus) vorticity (z), respectively, made nondimensional in terms of ℓ.

Since, in our system, the quantities u_j and q_j are dependent on each other through Eq. (91), the parallel flow assumption (94) prohibits the Reynolds stress to appear on its right-hand side from depending on x and/or z explicitly. It leads necessarily to the requirements that both α and γ be real. In fact, if α is a complex number, the Reynolds stress would be x-dependent like $\exp[-2I(\alpha)x]$, where $I(\ )$ stands for the imaginary part. Of the two requirements, reality of γ is understood as a postulate for invariance of the Reynolds stress to a Galileian transformation in the y-z plane. Reality of α, on the other hand, limits permissible classes of solutions to those u(y)'s which have neutral stability under a certain condition. It will be discussed in Section V.D.

Provided, for the time being, that the condition is fulfilled, Eqs. (90) through (93) reduce to a system of ordinary differential equations in y in the following fashion: First we introduce two auxiliary variables Y and Z to represent Y_j of (95) obeying constraint (92) of the continuity condition;

$$Y_1 = \frac{i}{\alpha} \left(DY + \frac{\Theta^2}{1+\Theta^2} Z\right)$$

$$Y_2 = Y \tag{97}$$

$$Y_3 = -\frac{i\Theta}{\alpha(1+\Theta^2)} Z$$

where we have introduced the notations

$$D = d/dy$$

and

$$\Theta = \gamma/\alpha \tag{98}$$

the latter representing a three dimensional effect of the fluctuations. Equations governing Y and Z are, then, obtained from (93) by eliminating the pressure fluctuations, yielding (ref. 10)

$$\left[(u-c)(D^2-A^2) - \frac{1}{i\alpha R}(D^2-A^2)^2 - u''\right]Y = 0 \tag{99}$$

$$\left[u-c - \frac{1}{i\alpha R}(D^2-A^2)\right]Z = -A^2 \int^y \left[u-c - \frac{1}{i\alpha R}(D^2-A^2)\right]Y \, dy \tag{100}$$

where u' stands for Du, and

$$R = U\ell/\nu$$

$$A = (1 + \Theta^2)^{1/2} \alpha \qquad (101)$$

$$c = \omega/\alpha$$

respectively, give the Reynolds number, the magnitude of the wave number vector, the streamwise propagation velocity of the coherent wave as discussed in Section IV.C. In the latter expression ω is the nondimensional frequency subject to redefinition (79).

Boundary conditions for Y and Z are obtained through the fact that the fluctuation amplitude (97) must vanish at the solid boundary or at an asymptotic limit where the flow tends to equilibrium;

$$Y = DY = 0$$
$$\qquad \qquad (102)$$
$$Z = 0$$

Equations (99) and (100) for Y and Z constitute a closed set together with the Navier-Stokes equation (91), namely,

$$(NS)_1 = -dR_{12}/dy \qquad (103)$$

where

$$R_{12}(y) = R\{i \int \alpha^{-1} (DY + \frac{\Theta^2}{1+\Theta^2} Z) Y^* d\omega\} \qquad (104)$$

is the shearing component of the Reynolds stress nondimensionalized properly. Without solving for Y and Z, it is assured from condition (102) that $Y \sim y^2$ and $Z \sim y$ for $y \ll 1$, so that the Reynolds stress (104) varies like y^3 in agreement with the known dependence (ref. 19) in the viscous sublayer.

B. A New Scaling of the Orr-Sommerfeld Equation Appropriate for Wall Turbulence

The fact that the amplitude Y for the y-component of the turbulent fluctuation obeys the Orr-Sommerfeld equation as evidenced by Eq. (99) has been pointed out in reference 2. For a fully-developed wall turbulence, however, the equation has a peculiar difficulty in the method of solution caused by the fact that the curvature term u" of the equation has an extraordinary magnitude, not to be neglected as is disposed of through the classical scaling (refs. 20,21). A key to the solution is provided (ref. 10) by employing a new scaling parameter ε in stretching

coordinate from the physical (y) to the scaled (η) ones;

$$\eta = y/\varepsilon$$

$$\varepsilon = (\alpha R)^{-1/2} \tag{105}$$

Note that the classical scaling is $\varepsilon \sim R^{-1/3}$, so that the rate of stretching is greater for the present case. With this scaling law the Orr-Sommerfeld equation (99) reduces to

$$\ddddot{Y} - i(u-c)\ddot{Y} + i\ddot{u}Y = 0 \tag{106}$$

with

$$\dot{Y} = dy/d\eta, \text{ etc.}$$

retaining the curvature term as a leading one, and in a form as if added to the two-term equation of Heisenberg (ref. 20) widely used in the stability theory.

Because of the non-negligible third term intervening in the equation the classical asymptotic method which solves the two-term equation using universal (Hankel) functions is no longer available here. It turns out, however, that an alternative asymptotic method is workable through an observation that owing to the presence of the third term Eq. (106) is integrable once, yielding

$$\dddot{Y} - i(u-c)\dot{Y} + i\dot{u}Y = iC_1 \tag{107}$$

Four independent solutions Φ_1 of this equation are obtained with some similarity to and more essential difference from the corresponding solutions of the classical equation. For the method of solution and their actual expressions see Appendix B. A few remarks are in order: Solutions Φ_1 and Φ_2, the counterparts of slowly varying solutions of the classical equation, behave actually so in and only in asymptotic limit ($\eta \gg 1$),

$$\Phi_1 \sim -(u-c) \int^{\eta} (u-c)^{-2} d\eta$$

$$\tag{108}$$

$$\Phi_2 \sim (u-c)$$

In the region of high shearing rate ($\eta \sim O(1)$), these are subject to rapid variation in contrast to 'uniformly' slow variation of the classical solutions throughout the field. Two other solutions Φ_3 and Φ_4

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represent, respectively, rapidly decaying and growing functions with the asymptotic expressions in agreement with those of classical solutions given in terms of Hankel functions of first and second kinds. The solution procedures are: functions Φ_3 and Φ_2 are obtained by solving first order equations of which the critical layer $(u-c \doteq 0)$ is a regular point. No multiplevaluedness as encountered in the classical theory occurs in the quadrature through the point. Once these functions are determined through a straightforward numerical method, Φ_4 and Φ_1 are solved analytically and exactly in terms of them.

C. An Eigenvalue Problem

A salient feature of our scaling law (105) is that the scaled or 'inner' equation (106) serves also as the starting equation in a systematic method of successive approximation solving the full Orr-Sommerfeld equation (99). In fact, expansion of the eigenfunction in the following form;

$$Y = Y^{(0)} + (\varepsilon A)^2 Y^{(1)} + (\varepsilon A)^4 Y^{(2)} + \ldots$$

$$y = \varepsilon \eta \tag{109}$$

$$\varepsilon = (\alpha R)^{-1/2}$$

substituted in the Orr-Sommerfeld equation yields the following set of equations determining $Y^{(n)}$ successively;

$$L^{(0)} Y^{(0)} = 0 \tag{110}$$

$$L^{(0)} Y^{(1)} = L^{(1)} Y^{(0)} \tag{111}$$

$$L^{(0)} Y^{(n)} = L^{(1)} Y^{(n-1)} - Y^{(n-2)}, \quad (n \geq 2) \tag{112}$$

where operators $L^{(0)}$ and $L^{(1)}$ are defined by

$$L^{(0)} = d^4/d^4 - i(u-c)d^2/d\eta^2 + i\ddot{u} \tag{113}$$

$$L^{(1)} = 2d^2/d\eta^2 - i(u-c) \tag{114}$$

Equation (110), together with (113), confirms the envisaged assertion not to be expected for its classical equivalent. This property enables us to reach the full solution F_ι simply through advancing the successive approximation starting with the inner solution Φ_ι:

$$F_\iota = \Phi_\iota + \sum_{n=1}^{\infty} (\varepsilon A)^{2n} \Phi_\iota^{(n)} \qquad (\iota:1,2,3,4) \tag{115}$$

The higher order approximation $\phi_\iota^{(n)}$ is obtained by a method of finding a particular solution of Eq. (111) or (112) as given in Appendix B. Asymptotic expressions ($\eta \gg 1$) of F_1 and F_2 thus obtained are of particular interest,

$$F_1 \sim -(u-c)\left[\int_{\hat{y}}^{y}(u-c)^{-2}dy + A^2 \int_{\hat{y}}^{y}(u-c)^{-2}dy \times\right.$$

$$\left.\int_{\hat{y}}^{y}(u-c)^{2}dy \int_{\hat{y}}^{y}(u-c)^{-2}dy + A^4 \dots\right]$$

$$F_2 \sim (u-c)\left[1 + A^2 \int_{\hat{y}}^{y}\frac{dy}{(u-c)^2} \int_{\hat{y}}^{y}(u-c)^{2}dy + A^4 \dots\right] \quad \text{(for } y \gg \epsilon)$$

These expressions which follow immediately from Eqs. (108) and (B.37) of Appendix prove exact coincidence with Heisenberg's outer (inviscid) solution (ref. 20). This fact indicates that F_1 of (115) thus obtained is valid for the outer- as well as the inner-layers, enabling us to by-pass the conventional procedure of matching the two solutions.

With these F's having so derived the eigenfunction Y is written as

$$Y = \sum_{\iota=1}^{4} C_\iota F_\iota$$

Since, as seen in Section V.B, F_4 grows drastically with η, we need to put $C_4 = 0$ to cope with boundary condition (102) at $\eta = \infty$. On the other hand, condition $Y(0) = 0$ imposes $C_3 = 0$, because $F_3(0) = 1 \neq 0$, whereas $F_1(0) = F_2(0) = 0$. Thus we are led to the final expression of the eigenfunction;

$$Y = C_1 F_1 + C_2 F_2 \tag{116}$$

depending only on two functions F_1 and F_2 throughout the region in question. Note, for comparison, that its equivalent of the classical asymptotic theory comprises three terms ($C_3 \neq 0$). Imposition of two

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remaining conditions

$$Y'(0) = 0 \tag{117}$$

$$Y' + AY = 0, \qquad (\text{at } y = \tilde{y}) \tag{118}$$

yet to be satisfied by Y of (116) leads to an eigenvalue condition in the following form,

$$E \equiv \begin{vmatrix} F_1'(0) & F_2'(0) \\ F_1'(\tilde{y}) + AF_1(\tilde{y}) & F_2'(\tilde{y}) + AF_2(\tilde{y}) \end{vmatrix} = 0 \tag{119}$$

Adequacy of using the outer boundary condition of form (118) in turbulent cases is discussed in Appendix B.

VI. THE SECOND METHOD OF SEPARATION OF VARIABLES: NONLINEAR CONVOLUTION

In this section we will seek a condition which leaves the triple correlation, as it appears in the double correlation equation, untouched and under which the double correlation equation is still accessible to the method of separating variables.

It has been shown in reference 28 that Eqs. (13) and (14) which, according to the BBGKY formalism, are derived from Liouville's equation through multiple integrations are also derivable on the basis of the following equation:

$$B(\delta) \equiv \left[\frac{\partial}{\partial t} + \vec{v}\,\frac{\partial}{\partial \vec{x}}\right]\delta(z) - J(z|\tilde{z})\,[\delta(z)\,\delta(\tilde{z})] = 0 \tag{120}$$

This equation is the Boltzmann equation in which the Boltzmann function f is replaced by the microscopic density δ, namely, its fluctuating equivalent. This equation may well be called the 'master' Boltzmann equation in the sense that the whole system of the BBGKY hierarchy is derived also from this equation through a more straightforward procedure. In fact, it is easily confirmed that Eq. (13) is expressed in terms of Eq. (120) as

$$\langle B(\delta_a)\rangle = 0 \tag{121}$$

with $\delta_a \equiv \delta(z_a)$. Similarly, the two-particle equation (14) with the triple correlation retrieved is written as

$$\langle \Delta\delta_b B(\delta_a) + \Delta\delta_a B(\delta_b)\rangle = \langle \Delta\delta_b \Omega_a(\Delta\delta_a) + \Delta\delta_a \Omega_b(\Delta\delta_b)\rangle = 0 \tag{122}$$

with operator Ω_a defined by

$$\Omega_a(\Delta\delta_a) \equiv \left[\frac{\partial}{\partial t} + \vec{v}_a\,\frac{\partial}{\partial \vec{x}_a}\right]\Delta\delta_a - J(z_a|z_{\tilde{a}})\,(\delta_a\Delta\delta_{\tilde{a}} + \delta_{\tilde{a}}\Delta\delta_a + \Delta\delta_a\Delta\delta_{\tilde{a}}) \tag{123}$$

Obviously Ω_a differs from Ω of (18) by the presence of terms quadratic in $\Delta\delta$. Because of this term Eq. (122) includes triple correlation of the form $\Delta\delta_a\Delta\delta_{\tilde{a}}\Delta\delta_b$, therefore recourse is made to the equation of the succeeding hierarchy for its determination. This equation is given by

$$\left\langle \sum_{\begin{pmatrix}a,b,c:\\ \text{cycl.permut.}\end{pmatrix}} \Delta\delta_b \Delta\delta_c B(\delta_a)\right\rangle = 0$$

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or, alternatively, in terms of operator Ω_a of (123);

$$\sum_{\begin{pmatrix} a,b,c: \\ cycl.permut. \end{pmatrix}} \left\{ \begin{aligned} &<\Delta\delta_b \Delta\delta_c \Omega_a (\Delta\delta_a)> \\ &- <\Delta\delta_b \Delta\delta_c> J(z_a|z_{\tilde{a}}) (<\Delta\delta_a \Delta\delta_{\tilde{a}}>) \end{aligned} \right\} = 0 \tag{124}$$

where the summation is over cyclic permutation (a→ b→ c→ a), each term
yielding three of them. Equation (124) comprises three kinds of corela-
tion quantities; triple and quadruple correlations in linear forms and
products of double correlations.

Let us search for the solution of Eq. (122) again in the form of
separation of variables

$$<\Delta\delta_a \Delta\delta_b> = \int \phi(z_a;\omega_a) \phi(z_b;\omega_b) \delta(\omega_a + \omega_b) d\omega_a d\omega_b \tag{125}$$

which is an alternative expression of (16) with supplementary condition
(20). In the same way, we assume the triple correlation governed by
(124) to be separable,

$$<\Delta\delta_a \Delta\delta_b \Delta\delta_c> = \int \delta(\Sigma\omega_a) \Pi[\phi^{\dagger}(z_a;\omega_a) d\omega_a] \tag{126}$$

where the sum (Σ) and the product (Π) are over three indices a, b, and c.
An integral expression of the like is assumed for the quadruple correla-
tion with the integrand given by a quadruple product of $\phi_a^{\dagger\dagger}$. In general,
a function ϕ of (125) will differ from $\phi^{\dagger}$ of (126), etc.

Consider an iteration procedure in which the product of double
correlation, namely, the last term of (124) is neglected at the outset,
whereas no approximation is invoked in (122). Then, it is readily seen
that Eqs. (122) and (124) reduce to the same equation if we put

$$\phi = \phi^{\dagger} = \phi^{\dagger\dagger} \tag{127}$$

with the resulting equation

$$\left(-i\omega + \frac{\partial}{\partial t} + \vec{v} \frac{\partial}{\partial \vec{x}}\right)\phi - J(z|\tilde{z})[f\tilde{\phi} + \tilde{f}\phi + \int \phi(\omega-\omega')\tilde{\phi}(\omega')d\omega'] = 0 \tag{128}$$

where we have dropped the subscripts (a,b,c) because there would occur
no confusion from this point on by so doing. If the nonlinear term

(with the integral) is deleted in Eq. (128) in line with the ternary chaos hypothesis, it reduces to Eq. (18) as it should.

Looking back the procedure leading to the closed Eq. (128) for ϕ, we see that the second equality of condition (127) serves as the closure condition. In fact, only under this condition, the governing equations for ϕ and $\phi^\dagger$ prove to be identical, both yielding Eq. (128), therefore the first equality of (127) is justified a posteriori. Since it is intuitively reasonable to interpret mathematical realization of a physical concept of taking average < > by the integral of form (125), (126), etc. then our closure assumption that triple and quadruple correlations are expressed using a common variable $\phi^\dagger = \phi^{\dagger\dagger}$ seems to have a sound basis.

Physical implications of the nonlinear term may be easier to understand by looking at the fluid-dynamic version of Eq. (128). The moment expansion to lead to fluid equations was described in Section III, so will not be repeated. Only the final forms of the equations governing the correlation tensor for incompressible flow are given here:

$$R_{j\ell}^{(1,1)}(\vec{x},\hat{\vec{x}}) = \int q_j(\vec{x};\omega)q_\ell(\hat{\vec{x}};-\omega)\,d\omega \tag{129}$$

$$\frac{\partial q_j}{\partial x_j} = 0 \tag{130}$$

$$\left(-i\omega + \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial x_r} - \nu\frac{\partial^2}{\partial x_r^2}\right)q_j + \frac{\partial u_j}{\partial x_r}q_r$$

$$+ \frac{1}{\rho}\frac{\partial p^{(2)}}{\partial x_j} + \frac{\partial}{\partial x_r}\int q_j(\omega-\omega')q_r(\omega')\,d\omega' = 0 \tag{131}$$

where q_j has been defined in Section V.A, the q_j's except those inside the integral of (131) stand for $g_j(\vec{x};\omega)$. This equation is to be contrasted with Eq. (93) where the triple correlation has been ignored.

For a closer look at Eq. (131) we assume the mean flow to be a parallel shear flow [$u_j = \delta_{j1}u(x_2)$], and apply an operator $\delta_{2j}\partial^2/\partial x_r^2 - \partial^2/\partial x_j\partial x_2$ to the equation to eliminate the pressure term. We have, then,

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$$\Lambda q_2 = -\frac{\partial^2}{\partial x_r^2} \int \frac{\partial q_2(\omega-\omega')}{\partial x_j} \, q_j(\omega') \, d\omega'$$

$$+ \frac{\partial}{\partial x_2} \int \frac{\partial q_r(\omega-\omega')}{\partial x_j} \frac{q_j(\omega')}{\partial x_r} \, d\omega' \tag{132}$$

where the operator Λ on the left-hand side is given by

$$\Lambda \equiv \left(-i\omega + \frac{\partial}{\partial t} + u \frac{\partial}{\partial x_1} - \nu \frac{\partial^2}{\partial x_r^2} \right) \frac{\partial^2}{\partial x_j^2} - u'' \frac{\partial}{\partial x_1} \tag{133}$$

that is, the Orr-Sommerfeld operator with added term of $-i\omega$ reflecting the effects of the variable separation. Thus we see that the velocity variable q_2 obeys a (generalized) Orr-Sommerfeld equation, 'driven' by terms due to the triple correlation that are quadratic in fluctuation velocities with different internal variable, namely, different frequencies. Thus, the role of the triple correlation has turned out to be the following: For not too weak turbulence in which the terms of $O(q)^2$ still stand, it acts as controlling the interaction between turbulent fluctuations with different frequencies; the form of the nonlinear interaction is mathematically a convolution integral and physically a continuous cascade of turbulent energy in the frequency space. This is in contrast with the role of non-linear terms of the Orr-Sommerfeld equation of classical stability theory, where the energy cascading is through discrete higher harmonics (refs. 11,12).

VII. CONCLUDING REMARKS

In concluding this report, a brief review from a different angle will be given to help clarify physical implications of the two closure assumptions we have employed; the neglected triple correlation (Section II) and the identity of separated variables in triple and quadruple correlations (Section VI). One way of doing this would be through Eq. (1) for incompressible flows written in the following form:

$$\underbrace{\left[\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} + \hat{u}_r \frac{\partial}{\partial \hat{x}_r}\right] R_{j\ell}^{(1,1)}}_{(0)} \underbrace{- \nu\left(\frac{\partial^2}{\partial x_r^2} + \frac{\partial^2}{\partial \hat{x}_r^2}\right) R_{j\ell}^{(1,1)}}_{(1)}$$

$$\underbrace{+ \frac{\partial u_j}{\partial x_r} R_{r\ell}^{(1,1)} + \frac{\partial \hat{u}_\ell}{\partial \hat{x}_r} R_{jr}^{(1,1)}}_{(2)} \underbrace{+ \frac{\partial R_{,\ell}^{(2,1)}}{\partial x_j} + \frac{\partial R_{j,}^{(1,2)}}{\partial \hat{x}_\ell}}_{(3)}$$

$$\underbrace{+ \frac{1}{\rho} \frac{\partial R_{jr,\ell}^{(1,1,1)}}{\partial x_r} + \frac{1}{\rho} \frac{\partial R_{j,r\ell}^{(1,1,1)}}{\partial \hat{x}_r}}_{(4)} = 0 \qquad (134)$$

We note that this equation is exact within this context, but is not closed because of the triple correlations

$$R_{jr\ell}^{(1,1,1)}(\vec{x}, \tilde{x}, \hat{\vec{x}}) \equiv \rho^3 \langle \Delta u_j \Delta \tilde{u}_r \Delta \hat{u}_\ell \rangle$$

intervening in the form

$$R_{jr,\ell}^{(1,1,1)} \equiv [R_{jr\ell}^{(1,1,1)}]_{\tilde{x}=x}$$

also $R_{j,r\ell}^{(1,1,1)}$ defined likewise, depending on two space points. Equation (134) describes how the double correlation $R_{j\ell}^{(1,1)}$, in the course of convection and diffusion (terms (0) and (1), respectively), interacts with the mean shear flow (2), with the pressure-velocity correlation (3), and with the triple correlation (4).

For the classical isotropic turbulence terms (2) and (3) vanish; thus, the triple correlation is the only factor responsible for active mechanism in turbulence evolution as contrast to passive ones through the convection and the diffusion prevailing in the final stage of decay. Grid-produced turbulence as formulated in Section IV., in contrast, keeps another active term (3) finite, lessening the danger of making assumption on term (4), since the latter term is higher order in correlations. Its finiteness leads to finding of a pre-final stage where turbulence decays more quickly than downstream (the Laplace mode), within the same regime of ignored triple correlation.

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For shear turbulent flows where terms (2) are also finite, the danger of neglecting the triple correlation is diluted more than in the grid-turbulence case.

The attempt to incorporate effects due to finiteness of the triple correlation through the second assumption is of exploratory nature. Term-by-term comparison of Eq. (134) with Eq. (131) tells us that term (4) is represented by an integral with respect to the frequency in the form of a nonlinear convolution, describing a peculiar mechanism of direct cascading of energy from one turbulent component to the other with the frequency arbitrarily far apart. This is to be contrasted with in-direct ones due to term (2), where the interaction between two modes are via a mediator, namely, the shearing motion of the mean flow.

VIII. ACKNOWLEDGMENTS

APPENDIX A

UNIQUE DETERMINATION OF COMPLEX AMPLITUDES A_O, A_L, AND B_O OF
GRID-TURBULENCE FLUCTUATIONS IN TERMS OF CORRELATION DATA
AT AN INITIAL PLANE

The problem to be solved separately here is to determine complex
amplitudes A_O, A_L, and B_O appearing in solutions (81) and (82) of the
separated fluctuation variables $q_j^{(1)}$ in terms of a set of correlation
data given at an initial plane;

$$x = \hat{x} = 0 \qquad\qquad (A.1)$$

To work out this task we first write $q_j^{(1)}$ given by (82) and (80) in
the form

$$\left.\begin{aligned}
q_1^{(1)} &= (A_O X_O + A_L X_L)\exp\, i(k_2 y + k_3 z) \\[2mm]
q_2^{(1)} &= \frac{i}{\kappa^2}\,[(k_3 B_O + i k_2 k_1 A_O) X_O - \kappa k_2 A_L X_L]\exp\, i(k_2 y + k_3 z) \\[2mm]
q_3^{(1)} &= \frac{i}{\kappa^2}\,[(-k_2 B_O + i k_3 k_1 A_O) X_O - \kappa k_3 A_L X_L]\exp\, i(k_2 y + k_3 z)
\end{aligned}\right\} \quad (A.2)$$

with

$$\left.\begin{aligned}
X_O(x) &= \exp(i k_1 x) \\[2mm]
X_L(x) &= \exp(-\kappa x)
\end{aligned}\right\} \quad (A.3)$$

where we have employed Eqs. (81), (68), (72) through (75). With these
formulae expression (83) with (85) for frequency-analyzed velocity
correlation tensor is

$$[R_{j,\ell}]_\omega = \int q_j^{(1)}(\vec{x})\, q_\ell^{(1)*}(\vec{x})\,\kappa\, d\kappa\, d\beta \qquad\qquad (A.4)$$

where we have suppressed superscript (1,1) in Eq. (83) and β is defined
by

$$\left.\begin{aligned}
k_2 &= \kappa\,\cos\,\beta \\[2mm]
k_3 &= \kappa\,\sin\,\beta
\end{aligned}\right\} \quad (A.5)$$

We introduce a cylindrical coordinate system in the fluid-dynamic space;

$$\left.\begin{aligned}
y - \hat{y} &= r\,\cos\,\theta \\[2mm]
z - \hat{z} &= r\,\sin\,\theta
\end{aligned}\right\} \quad (A.6)$$

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Since we may assume constants A_O, A_L, and B_O not to depend on θ because of cylindrical isotropy of the phenomena, integration with respect to variable $\beta - \theta$ in (A.4) is seen to be practicable: To carry out the integration we make use of the following formulae;

$$\int_0^{2\pi} \binom{k_2}{k_3} \exp [i\kappa r \cos (\beta - \theta)] d(\beta - \theta) = 2\pi i\kappa \binom{\cos \theta}{\sin \theta} J_1(\kappa r)$$

$$\int_0^{2\pi} \binom{k_2^2}{k_3^2} \exp [i\kappa r \cos (\theta - \beta)] d(\beta - \theta)$$

$$= \pi\kappa^2 [J_0(\kappa r) \mp \cos 2\theta\, J_2(\kappa r)]$$

$$\int_0^{2\pi} k_2 k_3 \exp [i\kappa r \cos (\theta - \beta)] d(\beta - \theta) = -\pi k^2 \sin 2\theta\, J_2(\kappa r)$$

(A.7)

where J's are the Bessel functions and r has been defined by (84). In obtaining these formulae we have made use of Hansen's integral representation for Bessel's function,

$$J_n(x) = \frac{1}{\pi i^n} \int_0^\pi e^{ix \cos\theta} \cos n\theta\, d\theta$$

Assume that we have a set of measured data for six (independent) tensor components $[R_{j\ell}]_\omega$ at $x = \hat{x} = 0$ and $\theta = 0$. These measured data given as functions of r are shown to be necessary and sufficient for unique determination of the complex amplitudes in the following way: Substituting (A.2) into (A.4) with integral formulae (A.7) into account, we have the following expressions:

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$$R_{11\omega} = 2\pi \int_0^\infty (A_O + A_L)(A_O^* + A_L^*) \, J_0(\kappa r)\kappa d\kappa,$$

$$R_{22\omega} = \pi \int_0^\infty \kappa^{-1}d\kappa\{B_O B_O^*[J_0(\kappa r) + J_2(\kappa r)] + k_1 k_1^* A_O A_O^*[J_0(\kappa r)$$

$$- J_2(\kappa r)] + i\kappa k_1^* A_L A_O^*[J_0(\kappa r) - J_2(\kappa r)] - i\kappa k_1 A_O A_L^*[J_0(\kappa r)$$

$$- J_2(\kappa r)] + \kappa^2 A_L A_L^*[J_0(\kappa r) - J_2(\kappa r)]\},$$

$$R_{12\omega} = -2\pi \int_0^\infty d\kappa \, J_1(\kappa r)(A_O + A_L)(ik_1^* A_O^* + \kappa A_L^*),$$

$$R_{13\omega} = -2\pi \int_0^\infty d\kappa \, J_1(\kappa r)(A_O + A_L) \, B_O^*,$$

$$R_{33\omega} = \pi \int_0^\infty \kappa^{-1}d\kappa\{B_O B_O^*[J_0(\kappa r) - J_2(\kappa r)] + k_1 k_1^* A_O A_O^*[J_0(\kappa r) + J_2(\kappa r)]$$

$$+ i\kappa(k_1^* A_L A_O^* - k_1 A_L^* A_O)[J_0(\kappa r) + J_2(\kappa r)] + \kappa^2 A_L A_L^*[J_0(\kappa r)$$

$$+ J_2(\kappa r)]\},$$

$$R_{23\omega} = -\pi \int_0^\infty \kappa^{-1}d\kappa\{B_O(ik_1^* A_O^* + \kappa A_L^*) + B_O^*(-ik_1 A_O + \kappa A_L)\} \, J_2(\kappa r).$$

$$\left.\rule{0pt}{12em}\right\} \text{(A.8)}$$

On the right-hand side of these expressions, the real part should be taken because of the reason mentioned in Section II. If we are reminded of the fact that $R_{j\ell\omega}$ is the only functions of r, also that the variable r appears only through the Bessel functions on the right-hand side, we see that these equations can be inverted through the formula

$$\int_0^\infty J_{n-1}(ar)J_n(br)dr = \begin{cases} \dfrac{a^{n-1}}{b^n}, & (\text{for } b > a > 0) \\[2ex] \dfrac{1}{2b}, & (\text{for } a = b) \\[2ex] 0. & (\text{for } a > b > 0) \end{cases} \qquad \text{(A.9)}$$

In fact, multiplying the first equation of (A.8) with $J_1(br)$ integrating with respect to r, and applying (A.9) we have

$$(A_O + A_L)(A_O^* + A_L^*) = \frac{1}{2\pi} \int_0^\infty R_{11\omega} r J_0(\kappa r)\, dr \tag{A.10}$$

In a similar way the following equations are obtained:

$$\kappa^{-1}(A_O + A_L)(ik_1^* A_O^* + \kappa A_L^*) = -\frac{1}{2\pi} \int_0^\infty R_{12\omega} r J_1(\kappa r)\, dr \tag{A.11}$$

$$-\kappa^{-1}(A_O + A_L)B_O^* = \frac{1}{2\pi} \int_0^\infty R_{12\omega} r J_1(\kappa r)\, dr \tag{A.12}$$

$$i(B_O k_1^* A_O^* - k_1 A_O B_O^*) + \kappa(A_L B_O^* + B_O A_L^*) =$$

$$-\frac{2\kappa}{\pi} \int_0^\infty R_{23\omega} J_1(\kappa r)\, dr + \frac{\kappa^2}{\pi} \int_0^\infty R_{23\omega} r J_0(\kappa r)\, dr \tag{A.13}$$

$$B_O B_O^* = \frac{1}{4\pi} \int_0^\infty (R_{22\omega} - R_{33\omega})[-2\kappa J_1(\kappa r) + \kappa^2 r J_0(\kappa r)]\, dr$$

$$+ \frac{\kappa^2}{4\pi} \int_0^\infty (R_{22\omega} + R_{33\omega}) r J_0(\kappa r)\, dr \tag{A.14}$$

$$(-ik_1 A_O + \kappa A_L)(ik_1^* A_O^* + \kappa A_L^*)$$

$$= \frac{1}{4\pi} \int_0^\infty (R_{22\omega} - R_{33\omega})[2\kappa J_1(\kappa r) - \kappa^2 r J_0(\kappa r)]\, dr$$

$$+ \frac{\kappa^2}{4\pi} \int_0^\infty (R_{22\omega} + R_{33\omega}) r J_0(\kappa r)\, dr \tag{A.15}$$

where, as before, the real part should be taken on the left hand side. We are, thus, led to six relationships (A.10) through (A.15) for unique determination of the complex quantities A_O, A_L, and B_O depending on two parameters ω and κ and in terms of experimental values of $R_{j\ell\omega}$ given as a function of r.

APPENDIX B

A NEW ASYMPTOTIC METHOD OF SOLVING ORR-SOMMERFELD EQUATION

This appendix is mainly concerned with obtaining four independent solutions Φ_i (i:1,2,3,4) for Y;

$$Y = C_1 \Phi_1 + C_2 \Phi_2 + C_3 \Phi_3 + C_4 \Phi_4 \qquad (B.1)$$

obeying Eq. (107), namely,

$$\ddot{Y} - i(u - c)\dot{Y} + i\dot{u}Y = iC_1 \qquad (B.2)$$

For the procedure of deriving this equation from the full-term Orr-Sommerfeld equation, see the main text (Section V.B). In numbering the four independent solutions, we have followed the conventional usage, for example that employed in reference 21, so that each counterpart have the same asymptotic behavior; Φ_1 and Φ_2 tend to slowly varying functions, whereas Φ_3 and Φ_4 exhibit exponential decay and growth, respectively. Our procedure of obtaining the solutions will be carried out in the order of $\Phi_3 \rightarrow \Phi_2 \rightarrow \Phi_4 \rightarrow \Phi_1$ in such a form that each solution will be worked out using knowledge of Φ's at previous stages, and Φ_4 and Φ_1 will be solved analytically in terms of Φ_3 and Φ_2.

Solution Φ_3

First we will consider homogeneous equation ($C_1 = 0$) of (B.2) which provides with solutions Φ_2 through Φ_4;

$$\ddot{Y} - i(u - c)\dot{Y} + i\dot{u}Y = 0 \qquad (B.3)$$

This equation has a structure such that the critical layer (u = c) is an ordinary point, so that no artifice as needed in the classical treatment is necessary in integrating the equation across the point. In view of this feature, also of the prospective exponential decay, the solution Φ_3 may be assumed in the form

$$\Phi_3 = \exp \int_0^\eta \lambda \; d\eta \qquad (B.4)$$

Then Eq. (B.3), subject to this transformation of the dependent variable, reads

ꞁr, alternatively and more conveniently,

$$\dot{\lambda} = \mu - i(u - c),$$

$$\dot{\mu} = -3\lambda \mu + 4i(u - c)\lambda - \lambda^3 \qquad \Bigg\} \qquad (B.5)$$

Since we are seeking a solution of (B.5) which <u>decays</u> exponentially in the asymptotic limit $(\dot{\lambda}, \dot{\mu} \sim 0)$, the root for λ with negative real part;

$$\lambda \sim -e^{\frac{\pi}{4}i}(u - c)^{1/2}$$

$$\mu \sim i(u - c) \qquad \Bigg\} \quad (\text{for } \eta \gg 1) \qquad (B.6)$$

is seen to meet the purpose. In solving Eq. (B.5) for λ and μ with asymptotic conditions (B.6) it is advisable to introduce a variable-transformation

$$V = -\mu + 2i(u-c) - \lambda^2$$

$$W = -2\mu + 3i(u-c) - \lambda^2 \qquad \Bigg\} \qquad (B.7)$$

and to work with the equations in the new dependent variables (V,W),

$$\dot{V} + \lambda V - 2i\dot{u} = 0$$

$$\dot{W} + 2\lambda W - 3i\dot{u} = 0 \qquad \Bigg\} \qquad (B.8)$$

subject to asymptotic conditions

$$V, W \sim 0, \qquad \text{as } \eta \to \infty \qquad (B.9)$$

The actual integration may be started at a point $\eta = \hat{\eta}(\gg 1)$ where initial values

$$V = 2i\dot{u}/\lambda$$

$$W = 3i\dot{u}/2\lambda \qquad \Bigg\} \qquad (B.10)$$

with λ given by (B.6) are moderately small. The quadrature marches inward from this point on with nonlinearity of Eq. (B.8) appearing in the form

$$\lambda = -e^{\frac{\pi}{4}i}(u-c + 2iV - iW)^{1/2} \qquad (B.11)$$

taken into account for $\eta < \hat{\eta}$.

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It should be remarked here that this method of numerical integration of original equation (B.5) works successfully only for the decaying solution Φ_3, and that a formally identical procedure for the growing solution Φ_4 suffers from drastic numerical instabilities. The difference has its origin in the fact that the point $\eta = \infty$ is a saddle singularity of Eq. (B.5) for the case treated, whereas in the Φ_4 case it proves to be a nodal singularity from which an infinite number of solutions emerge.

It can be demonstrated that our solution (B.4) has the same asymptotic form as its classical equivalent for $\eta \gtrsim \hat{\eta} \gg 1$. In fact, a straightforward calculation from (B.4), (B.11) and (B.10) leads to

$$\Phi_3(\eta) = \Phi_3(\hat{\eta}) \left(\frac{\hat{u}-c}{u-c}\right)^{5/4} \exp\left[-e^{\frac{\pi}{4}i} \int_{\hat{\eta}}^{\eta} (u-c)^{1/2} \, d\eta\right] \tag{B.12}$$

$$(\text{for } \eta > \hat{\eta} \gg 1)$$

On the other hand, the classical counterpart of Φ_3 is given by (ref. 20)

$$\left[\Phi_3(\tau)\right]_{Cl.} = \int_{\infty}^{\tau} d\tau \int_{\infty}^{\tau} d\tau \, \tau^{1/2} \, H_{1/3}^{(1)} \left[\frac{2}{3} (ia_0\tau)^{3/2}\right] \tag{B.13}$$

where $H_{1/3}^{(1)}$ denotes Hankel function of the first kind, and τ and a_0 defined by

$$\tau = (y - y_c)(\alpha R)^{1/3}$$
$$a_0 = (u_c')^{1/3}$$

with subscript c signifying the value at the critical layer. For $\tau \gg 1$, (B.13) takes the form

$$\left[\Phi_3(\tau)\right]_{Cl.} \sim \tau^{-5/4} \exp\left[- e^{\frac{\pi}{4}i} \frac{2}{3} (a_0\tau)^{3/2}\right] \tag{B.14}$$

in qualitative agreement with (B.12).

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Solution Φ_2

Once we have obtained one of the solutions of the third order equation (B.2) difficulties are considerably lessened as to solving for the rest of them because of the following theorem: If n independent solutions of a m-th order linear differential equation are at hand, the (n + 1)-th solution is obtained through solving an equation of (m-n)th order. This theorem applied to the current case for the second solution Φ_2 warrants an equation of the order 3 - 1 = 2 to be obtained by putting

$$\Phi_2 = \Phi_3 \int_0^\eta G \, d\eta \qquad (B.15)$$

and by substituting into Eq. (B.2). This, in turn, is equivalent to claiming that if G is assumed in the form

$$G = \exp \int_0^\eta (-\lambda + S) d\eta \qquad (B.16)$$

the transformed variable S obeys a first order (nonlinear) equation. A simple calculation actually confirms the assertion, leaving with a Riccati equation

$$\dot{S} + S^2 + \lambda S - W = 0 \qquad (B.17)$$

where λ and W have been solved from (B.8) and (B.11), respectively. Of the two asymptotic roots of (B.17) for $\dot{S} \sim 0$, the one that vanishes as $\eta \to \infty$, namely

$$S = W/\lambda \qquad (B.18)$$

turns out to be the correct choice. For this root alone gives rise to the solution Φ_2 meeting the requirement of slow variation as $\eta \to \infty$. In fact, then

$$\Phi_2 = \Phi_3 \int_0^\eta d\eta \, \exp\left(\int_0^\eta (-\lambda + S) d\eta\right)$$

$$= \hat{B}_2(u-c), \qquad (\text{for } \eta > \hat{\eta} \gg 1) \qquad (B.19)$$

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with

$$\hat{B}_2 = (\hat{u}-c)^{-3/2} \exp\left[-\frac{\pi}{4}i + \int_0^{\hat{\eta}} S \, d\eta\right]$$

where use is made of the fact that the integrand in (B.19) is a
rapidly growing function, so that only the portion in the vicinity of
the upper bound contributes to the integral, then the integral may be
replaced with

$$(-\lambda + S)^{-1} \times \exp\left[\int_0^{\eta} (-\lambda + S) \, d\eta\right]$$

It seems to be interesting to compare Φ_2 with its classical counter-
parts

$$\left.\begin{array}{l} \left(\Phi_2\right)_{Cl.,in} = 1 + O(\alpha R)^{-1/3} \\[2em] \left(\Phi_2\right)_{Cl.,out} = (u-c) + O(A^2) \end{array}\right\} \tag{B.20}$$

representing inner (viscous) and outer (inviscid) solutions due to
Heisenberg (ref. 20), respectively. We note that the classical Φ_2 is
<u>uniformly</u> slowly varying, whereas Φ_2 as given here is moderately varying
for $\eta \sim O(1)$ (so is rapidly varying as a function of the physical
coordinate $y = \varepsilon\eta$), and tends only asymptotically ($\eta \gg 1$) to a slowly
varying function (B.19). Also to be noted is the coincidence of (B.19)
with the leading term of the classical outer solution, proving the sound
basis of the asymptotic scheme adopted here.

Solution Φ_4

Repeated use of the foregoing theorem assures that the third
solution Φ_4 satisfies a first order linear differential equation to be
deduced from (B.2). This assertion can be materialized by means of the
method of variation of constants;

$$\Phi_4 = \Gamma_2 \Phi_2 + \Gamma_3 \Phi_3 \tag{B.21}$$

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where Φ_2 and Φ_3 are functions of η to be determined. Having eliminated Φ_2 between original equation (A.2) and the supplementary condition

$$\dot{\Gamma}_2 \Phi_2 + \dot{\Gamma}_3 \Phi_3 = 0$$

standard of this method, we are led to an equation for Φ_3 that is essentially of the first order;

$$\frac{\ddot{\Gamma}_3}{\dot{\Gamma}_3} = \frac{\dot{\Phi}_2}{\Phi_2} - 2\,\frac{\Phi_2 \ddot{\Phi}_3 - \Phi_3 \ddot{\Phi}_2}{\Phi_2 \dot{\Phi}_3 - \Phi_2 \dot{\Phi}_3}$$

This equation is analytically integrated to give

$$\Phi_4 = -\Phi_2 \int_0^\eta \Phi_3 W^{-2}\, d\eta + \Phi_3 \int_0^\eta \Phi_2 W^{-2}\, d\eta \qquad \text{(B.22)}$$

where W is the Wronskian formed by Φ_2 and Φ_3

$$W = \begin{vmatrix} \Phi_2 & \Phi_3 \\ \dot{\Phi}_2 & \dot{\Phi}_3 \end{vmatrix} \qquad \text{(B.23)}$$

It is easily confirmed from analytical solution (A.22) for Φ_4 that it has an asymptotic form of exponential growth with η. In fact, using the same approximation as has been utilized in deriving (B.19) we have for (A.22),

$$\left.\begin{aligned} \Phi_4 &\sim \frac{1}{2}\,\frac{1}{\lambda^3 \Phi_2 \Phi_3} \\[2mm] &\sim \hat{B}_4 (u-c)^{-5/4} \exp\left[e^{\frac{\pi}{4}i} \int_{\hat{\eta}}^\eta (u-c)^{1/2}\, d\eta \right] \end{aligned}\right\} \qquad \text{(B.24)}$$

with

$$\hat{B}_4 = \frac{1}{2}\,\frac{e^{\frac{\pi}{4}i}}{\hat{B}_2 \hat{\Phi}_3}\,\frac{1}{(\hat{u}-c)^{5/4}}$$

where, in deriving the second row, asymptotic expressions (B.12) and (B.19) for Φ_2 and Φ_3 have been used. Relationship (A.24) is again in qualitative agreement with its classical equivalent as it is in the case of decaying solution Φ_3. The classical equivalent is (ref. 21)

$$\left[\Phi_4(\tau)\right]_{Cl.} = \int_{-\infty}^{\tau} d\tau \int_{-\infty}^{\tau} d\tau \; \tau^{1/2} \; H_{1/3}^{(2)} \left[\frac{2}{3} (ia_0\tau)^{3/2}\right] \tag{B.25}$$

with the same nomenclatures as before and with the Hankel function of the second kind $H_{1/3}^{(2)}$. The asymptotic expression of (A.25) for $\tau \gg 1$ is

$$\left[\Phi_4(\tau)\right]_{Cl.} \sim \tau^{-5/4} \; \exp\left[e^{\frac{\pi}{4}i} \; \frac{2}{3} \; (a_0\tau)^{3/2}\right] \tag{B.26}$$

in qualitative agreement with (B.24).

The set of solutions (Φ_2, Φ_3, Φ_4) thus constructed has noteworthy characteristics for the Wronskian that serves to simplify the analyses to follow;

$$W_{III} \equiv \begin{vmatrix} \Phi_2 & \Phi_3 & \Phi_4 \\ \dot{\Phi}_2 & \dot{\Phi}_3 & \dot{\Phi}_4 \\ \ddot{\Phi}_2 & \ddot{\Phi}_3 & \ddot{\Phi}_4 \end{vmatrix} = 1 \tag{B.27}$$

This formula is easily checked by noting the following relationship

$$\frac{d^\ell \Phi_4}{d\eta^\ell} = - \frac{d^\ell \Phi_2}{d\eta^\ell} \int_0^\eta \frac{\Phi_3}{W^2} \, d\eta + \frac{d^\ell \Phi_3}{d\eta^\ell} \int_0^\eta \frac{\Phi_2}{W^2} \, d\eta + \frac{\delta_{\ell 2}}{W}$$

holding for ℓ; 0, 1, 2.

Solution Φ_1

Since all the homogeneous solutions have been exhausted the fourth solution Φ_1 of Eq. (B.2) needs to be sought from particular solutions of the equation with $C_1 = 1$;

$$\dddot{\Phi}_1 - i(u-c)\dot{\Phi}_1 + i\dot{u}\Phi_1 = i \tag{B.28}$$

The method of variation of constants provides a workable tool also here, since we have all the homogeneous solutions Φ_2 through Φ_4 at hand. According to the theorem cited no differential equation need to be solved, and a manipulation using key property (A.27) leads to the following from of the solution;

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$$\Phi_1 = i\Phi_2 \int_{\hat{\eta}}^{\eta} W\ I_3\ d\eta - i\Phi_3 \int_0^{\eta} W'\ I_2\ d\eta + i\Phi_4 \int_{\infty}^{\eta} W\ d\eta \qquad (B.29)$$

with

$$I_\iota = \int^{\eta} \Phi_\iota\ W^{-2}\ d\eta \qquad (\iota; 2,\ 3) \qquad (B.30)$$

where $\eta = \hat{\eta} \gg 1$ is a point beyond which the asymptotic expressions for Φ_2 through Φ_4 are valid. Taking derivatives successively we have,

$$\frac{d^\ell \Phi_1}{d\eta^\ell} = i\frac{d^\ell \Phi_2}{d\eta^\ell} \int_{\hat{\eta}}^{\eta} W\ I_3\ d\eta - i\frac{d^\ell \Phi_3}{d\eta^\ell} \int_0^{\eta} W'\ I_2\ d\eta$$

$$+ i\frac{d^\ell \Phi_4}{d\eta^\ell} \int_{\infty}^{\eta} W\ d\eta + i\delta_{3\ell} \qquad (\ell = 1,2,3) \qquad (B.31)$$

a relationship easily checked using (A.26), (A.20) and (A.21).

The function Φ_1 in form of (A.29) is shown to tend to a slowly varying function for $\eta \gg 1$ under the same conditions as invoked in deriving asymptotic expression (B.19) for Φ_2; the resulting expression is

$$\Phi_1 \sim -(u-c) \int_{\hat{\eta}}^{\eta} (u-c)^{-2} d\eta \qquad (\eta > \hat{\eta} \gg 1) \qquad (B.32)$$

For comparison the classical equivalent of this function is noticed;

$$(\Phi_1)_{Cl,in} = \tau + O(\alpha R)^{-1/3}$$

$$\qquad (B.33)$$

$$(\Phi_1)_{Cl,out} = (u-c) \int^{y} (u-c)^{-2} dy + O(A^2)$$

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where τ has been defined in (B.13). Coincidence of leading terms of our asymptotic expression with classical <u>outer</u> solution is observed also here.

In view of (A.31) we can easily show the following relationship to hold regarding the Wronskian formed by the four solutions Φ_1 through Φ_4

$$
\mathcal{W}_{IV} = \begin{vmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\ \dot{\Phi}_1 & \dot{\Phi}_2 & \dot{\Phi}_3 & \dot{\Phi}_4 \\ \ddot{\Phi}_1 & \ddot{\Phi}_2 & \ddot{\Phi}_3 & \ddot{\Phi}_4 \\ \dddot{\Phi}_1 & \dddot{\Phi}_2 & \dddot{\Phi}_3 & \dddot{\Phi}_4 \end{vmatrix}
$$

$$
= -i\mathcal{W}_{III} = -i \tag{B.34}
$$

Higher Order Approximation

As described in the main text the full solution of the Orr-Sommerfeld equation is constructed through a successive approximation starting with Eq. (B.2) whose solutions are now at hand. In practice, the n-th order correction $\Phi_1^{(n)}$ is obtained by solving inhomogeneous equations (112), following the same procedure as has led Eq. (B.28) to solution (B.29). We have, then,

$$
\Phi_1^{(n)} = \Phi_2 \int_{\hat{\eta}}^{\eta} W I_3 J_1^{(n)} d\eta - \Phi_3 \int_0^{\eta} W I_2 J_1^{(n)} d\eta
$$

$$
+ \Phi_4 \int_{\infty}^{\eta} W J_1^{(n)} d\eta \tag{B.35}
$$

with

$$
J_1^{(n)} \equiv 2\dot{\Phi}_1^{(n-1)} - i \int_{\hat{\eta}}^{\eta} (u-c)\Phi_1^{(n-1)} d\eta - \int_{\hat{\eta}}^{\eta} \Phi_1^{(n-2)} d\eta \tag{B.36}
$$

Utilizing, in these formula, the same approximation as employed in deriving asymptotic expressions for Φ_1 and Φ_2, we get

$$
\Phi_1^{(n)} \sim (u-c) \int_{\hat{\eta}}^{\eta} \frac{d\eta}{(u-c)^2} \int_{\hat{\eta}}^{\eta} (u-c)\Phi_1^{(n-1)} d\eta . \quad (\eta > \hat{\eta} \gg 1) \tag{B.37}
$$

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his expression, being substituted into (115) with $\phi_1^{(0)} = \phi_1$ (ι;1,2) given by (B.32) and (B.19), respectively, is readily seen to have exact coincidence with Heisenberg's outer (inviscid) solutions (ref. 20). The series solutions for F_1 and F_2 are provied to be uniformly convergent if the lower bound $\hat{y}(= \epsilon\hat{\eta})$ is taken such that u-c varies only slowly for $\hat{y} < y$. In fact, then, the series sum up to yield

$$F_1 \sim -(u-c) \sinh A(y-\hat{y}) \qquad (B.38)$$

$$F_2 \sim (u-c) \cosh A(y-\hat{y}) \qquad (B.39)$$

Asymptotic Boundary Condition

It is advisable from convergence relief point of view to replace the boundary condition of type (102) at a far boundary $(y = \tilde{y})$ by an equivalent asymptotic one;

$$AY + Y' = 0 \qquad (B.40)$$

This is permissible if, at $y = \tilde{y}$ properly chosen,

$$A^2 > \left|\frac{u''}{u-c}\right| > O(\alpha R)^{-1} \qquad (B.41)$$

a condition accessible to most of the laminar flows and the wall-turbulent flow discussed in the main text. Imposing (B.40) instead of (102) enables us to choose a smaller value for $\tilde{y}$ in favor of convergence in the successive approximation.

Comparison with Existing Stability Calculations

Although the method described here is designed primarily for wall-turbulent flows, we have invoked no assumptions to rule out its applicability to the classical (laminar) solutions. In fact, the classical case is simply a limiting case ($\ddot{u} \to 0$) in Eq. (106) of the present formalism. The whole set of laminar solutions forms, in this sense, a subset of solutions achieved by the present method. Reproduction, therefore, of some of the classical stability characteristics will serve for checkout of the program.

Figure A1 shows the neutral stability curve for Blasius' flow, where the comparison is made with those by Lin (ref. 21) and Wazzan et al.

(ref. 29) obtained through the classical asymptotic method and purely numerical one, respectively. The classical asymptotic method is subject to inaccuracy of $O(\alpha R)^{-/3}$, whereas the present calculation includes error of $O(\alpha^4)$ which is caused by truncating successive approximation at α^2. These limitations are reflected on the behavior of each curve deviating from the computational one with a decrease of R and an increase of α, respectively. Agreement, however is considerably improved in the present case. Experimental data due to Schubauer and Skramstad (ref. 20) are also shown for comparison. Their seeming agreement with Lin's curve at lower Reynolds numbers is due to a fortuitous reason: Good agreement is not to be expected for lower Reynolds numbers where the stability curve becomes sensitive to factors arising from nonparallel streamlines of the Blasius flow. It is properly corrected by Saric et al. incorporating these elements (ref. 31). A more precise comparison among the three methods are seen in Figure A2, where calculated longitudinal fluctuation $|Y'|$ is shown together with those by classical asymptotic method due to Schlichting (ref. 32) and by the computational method (ref. 29), also with the experiment (ref. 30). Prediction by the present method is seen to have the best correlation with the measurement in the case tested.

Euler's Transformation for Improved Convergence

Our main objective of computing the wall-turbulence stability characteristics faces a difficulty not encountered in the laminar case; the problem of convergence in successive approximation (109). It appears that the lower bound $\hat{\eta}$ of the integral in (B.36) was not chosen so as to secure uniform convergence. Since, however, the actual computation has shown that expansion (109) proves to be a (complex) <u>alternating</u> series, one can apply Euler's transformation to speed up convergence. The transformation is well known as a means of finding the sum $\sum\limits_{n=0}^{\infty} a_n$ of an alternating series which is not absolutely convergent, a tool invented by Euler (1755) and applied in its full length to problems of the boundary layer by Meksyn (ref. 33). Operation of the transformation converts the sum to

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n / 2^{(n+1)}$$

with

$$b_n = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!}\, a_r$$

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securing much faster convergence owing to alternating structure of $\{a_r\}$. The method applied to sum an asymptotic series is utilized here: First we find a_m having the minimum absolute value

$$a_m = \min.\{a_n\}$$

If m = 1, the Euler transformation is applied to the whole series. If m > 1, we apply the transformation only to the partial sum; $\{a_n\}$ with $n \geq m + 1$. Repeated use of relevant alternatives at each stage is continued for a given finite sum of N = 10 terms until $\left| b_N / \overset{N}{\Sigma} b_n \right|$ settles down within a prescribed error bound.

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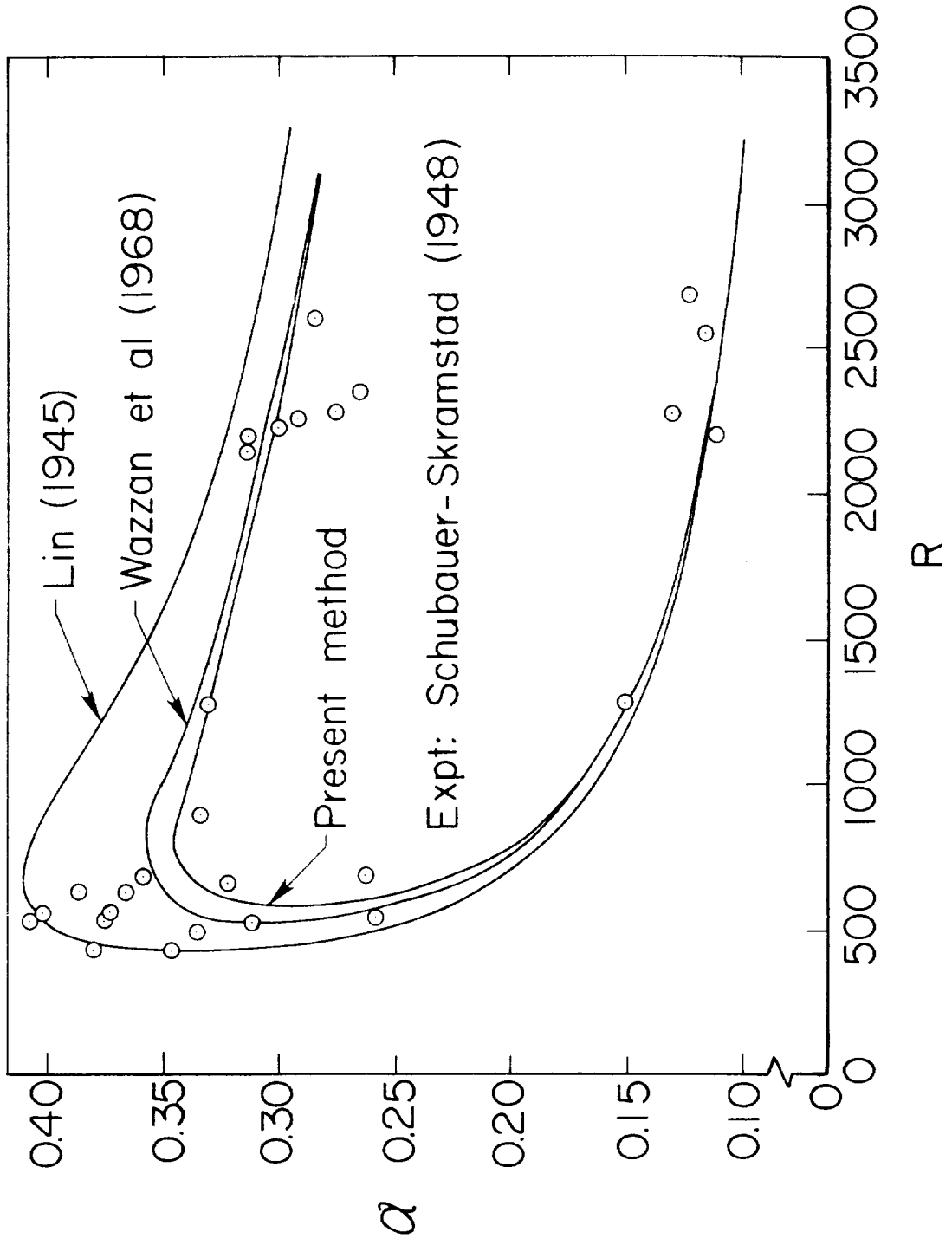33. Meksyn, D.: New Methods in Laminar Boundary-Layer. Pergamon Press, 1961.

Figure A1. Neutral Stability for the Blasius Flow: Comparison of the Classical Asymptotic Method
(Lin, ref. 21), the Numerical Method (Wazzan et al., ref. 29) and the Present Method
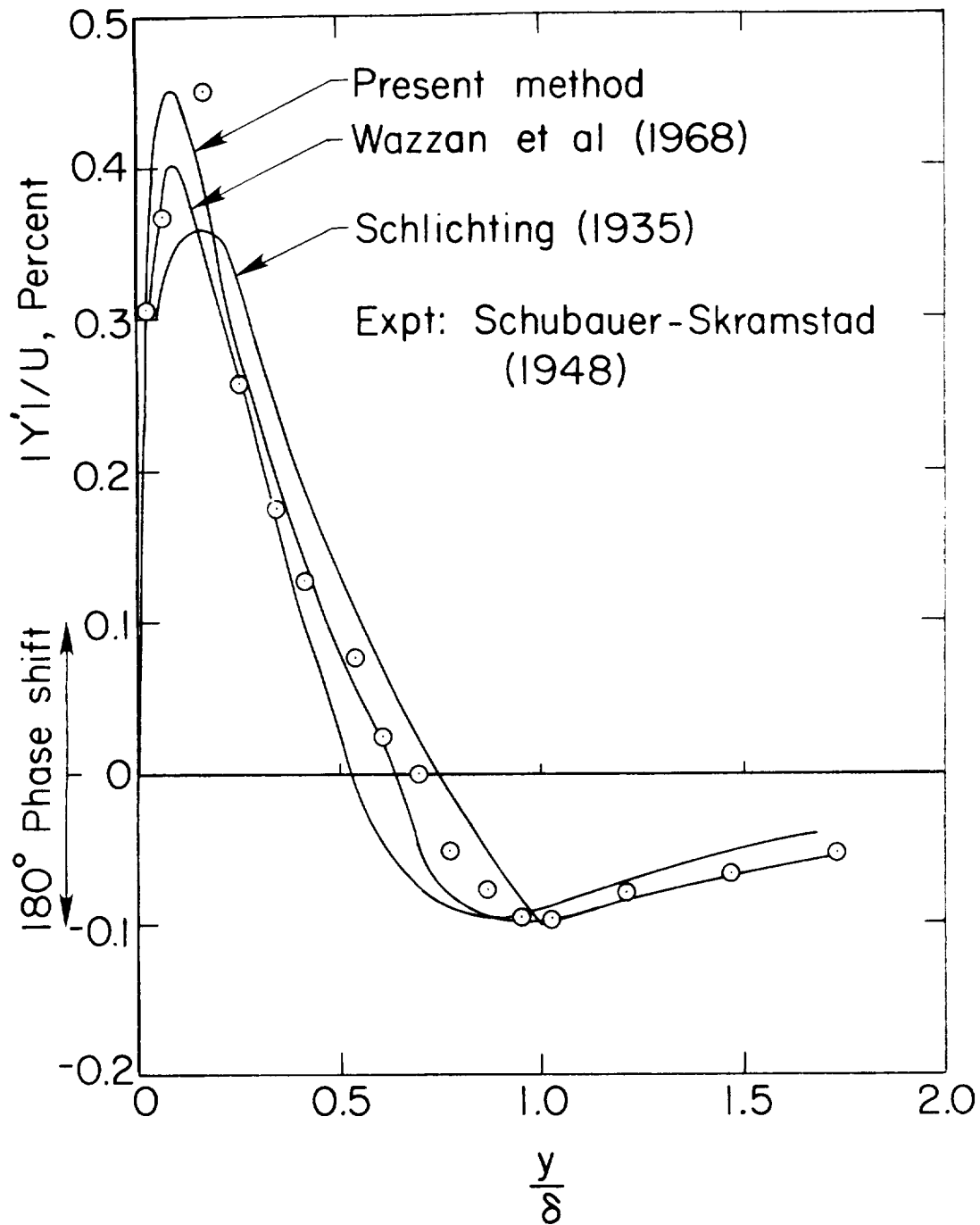with Experiment due to Schubauer and Skramstad (ref. 30).

Figure A2. The Longitudinal Fluctuation, or the First Derivative of the Eigenfunction Y' Computed or Measured at (R = 2080, α = .307) on the Upper Branch of the Blasius Stability Curve: Comparison of the Classical Asymptotic Method (Schlichting, ref. 32), the Numerical Method (Wazzan et al., ref. 29) and the Present Method with Experiment due to Schubauer and Skramstad (ref. 30).

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16. Abstract

Two schemes of closing turbulent moment equations are proposed both of which make double correlation equations separated into single-point equations. The first is based on neglected triple correlation, leading to an equation differing from small-perturbed gasdynamic equations where the separation constant appears as the frequency. Grid-produced turbulence is described in this light as time-independent, cylindrically-isotropic turbulence. Application to wall turbulence guided by a new asymptotic method for the Orr-Sommerfeld equation reveals a neutrally stable mode of essentially three dimensional nature. The second closure scheme is based on an assumption of identity of the separated variables through which triple and quadruple correlations are formed. The resulting equation adds, to its equivalent of the first scheme, an integral of nonlinear convolution in the frequency describing a role due to triple correlation of direct energy-cascading.

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